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Stochastic description for open quantum systems

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Abstract

A linear quantum Brownian motion model with a general spectral density function is considered. In the framework of the influence functional formalism, a Langevin equation can be introduced to describe the system's fully quantum properties even beyond the semiclassical regime. In particular, we show that the reduced Wigner function for the system can be formally written as a double average over both the initial conditions and the stochastic source of the Langevin equation. This is exploited to provide a derivation of the master equation for the reduced density matrix alternative to those existing in the literature. Furthermore, we prove that all the correlation functions obtained in the context of the stochastic description associated to the Langevin equation actually correspond to quantum correlation functions for system observables. In doing so, we also compute the closed time path generating functional of the open system.

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Keywords: Open quantum systems; Langevin equation; Quantum Brownian motion

1. Introduction

Feynman observed long ago that the dynamics of an open quantum system may be described in terms of an equivalent stochastic problem [1]. In this paper, we elaborate on this insight both by showing that a certain class of quantum correlation functions may be obtained directly as ensemble averages in the stochastic formulation and also by giving an alternative derivation of the master equation. Unlike earlier treatments of

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the origin of stochasticity in the semiclassical limit of quantum theories [2] we do not assume decoherence; rather, we are concerned with the fully quantum dynamics of the system.

Our stochastic treatment follows from the observation that the Wigner function of the open quantum system may be represented as the formal distribution function resulting from averaging the solutions of an appropriate Langevin equation over both the initial conditions and the stochastic source. This representation, which is exact for linear systems, may be extended to nonlinear cases through perturbative methods. In fact, this formal stochastic description was exploited in Refs. [3,4] to obtain the master equation for a nonlinear open quantum system with a nonlinear potential, which was then applied to the study of vacuum decay in quantum field theory using real time, as is commonly done in activation processes [5–7]. In addition, the main conclusions reached in the present work for simple quantum Brownian motion (QBM) models can also be very useful to elucidate the precise relationship existing between the correlation functions for the spacetime metric perturbations which have been obtained by means of the so-called Einstein–Langevin equation [8–10] in the framework of stochastic gravity [11,12] and the results which would follow from a purely quantum treatment [13].

We compare the main approaches to the analysis of the stochastic dynamics, namely the Langevin equation and its associated Fokker–Planck equation, to the corresponding quantum approaches, namely the master equation and the Wigner function. We show that the Fokker–Planck equation is the transport equation for the full Wigner function, and as such it is equivalent to the master equation. The Langevin equation, on the other hand, provides a more detailed description of the dynamics, in the sense that the class of quantum correlation functions which may be retrieved from the Langevin equation is larger than the corresponding class for the master or Fokker–Planck equations unless the dynamics is Markovian.

The relationship between these different approaches is summarized in the diagram of Fig. 1. The most fundamental description of the open quantum system is that provided by the Feynman–Vernon influence functional [1]. From this, we may derive the master equation, which gives the dynamics of the reduced density matrix. The integral transform linking the reduced density matrix to the reduced Wigner function allows us to convert the master equation for the former into a transport or Fokker–Planck equation for the latter. Feynman's insight that the influence functional may be thought as well as an ensemble average over an equivalent stochastic noise allows us to retrieve some correlation functions of the quantum problem directly in terms of stochastic averages by introducing a suitable functional change involving a formal Langevin equation. Moreover, the Fokker–Planck equation associated to the Langevin equation gives back the master equation. Thus, the Langevin equation is a very useful tool to gain information on the quantum properties of the system even beyond the semiclassical regime (i.e., in the absence of decoherence), when it no longer describes the actual trajectories of the system.

The plan of the paper with a summary of the relevant results is the following. In Section 2 we briefly review the (QBM) model describing a harmonic oscillator coupled bilinearly to a bath of infinite harmonic oscillators initially in a Gaussian state and with an arbitrary spectral distribution function. We summarize the main formulas that

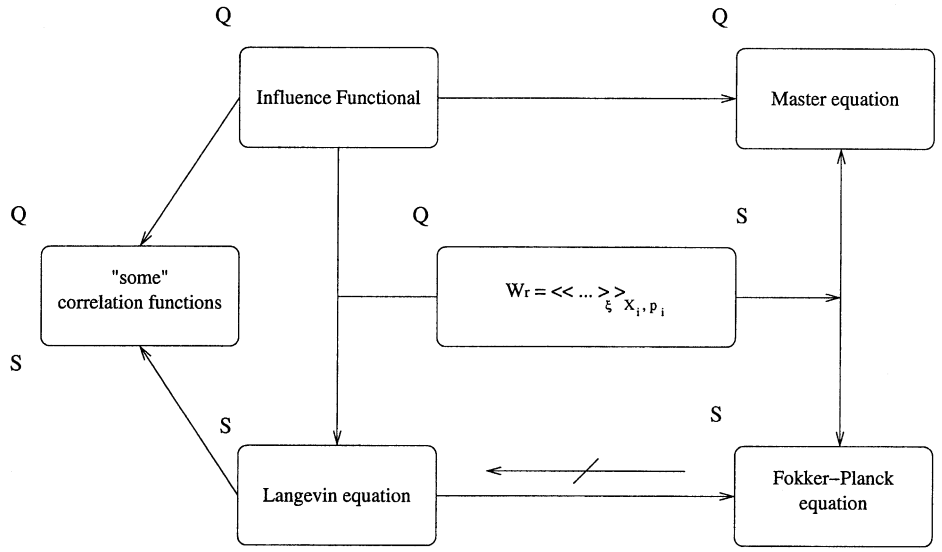

Fig. 1. Diagram showing the interconnections between different quantum properties of an open quantum system on the one hand, and between the elements of the stochastic description on the other hand, as well as the connection between both levels of description. Labels Q and S stand for quantum and stochastic objects, respectively.

beginning with the influence functional lead to the reduced density matrix operator and to the master equation. We also give the equivalent evolution equation for the reduced Wigner function. These results are well known and the computational details, which can be found in the references provided, are omitted.

In Section 3 we derive one of the main results of this paper. We show that the reduced Wigner function for a linear QBM model can be written as a formal distribution function for the system variables defined as follows. We consider the ensemble of system trajectories in phase space which obey the formal Langevin equation for each realization of the stochastic force and different initial conditions; the average over both the initial conditions (which involves the initial reduced Wigner function) and the different realizations of the stochastic source gives just the reduced Wigner function. The key technical point that makes this result relatively easy to derive is the computation of the path integral defining the reduced Wigner function in terms of variables which include the initial conditions and the stochastic force; this is achieved by introducing a functional change which involves the formal Langevin equation. The details of this derivation can be found in Appendix A. As far as we know this result is new. We are only aware of a related result by Halliwell and Zoupas [14] in the limit of large times; see also Ref. [15].

Having shown that the reduced Wigner function is a formal distribution function, it is clear that the dynamical equation for the reduced Wigner function can be deduced using the usual techniques employed to derive the Fokker–Planck equation [16]. The details

of this derivation are left to Appendix B. This constitutes an alternative derivation of the master equation to those given by Hu et al. [17] (see also Ref. [18]), and Halliwell and Yu [19]. We also note that this process is not reversible in the sense that there may be many Langevin equations that lead to the same master equation. This just reflects the fact that when the dynamics is not Markovian (in the sense of Ref. [20]), more information can be extracted from the Langevin equation than from the master equation.

In Section 4 we obtain the second main result of this paper. We show that the correlation functions obtained within the stochastic description provided by the Langevin equation correspond exactly to the quantum correlation functions for the system variables. This is done by explicitly computing the closed time path (CTP) generating functional for the system. It turns out that this generating functional can be written as an average over the initial conditions times a term that depends on the noise kernel, which contains the information on the fluctuations induced by the environment on the system.

We also show that quantum correlation functions cannot be obtained using the propagators for the reduced density matrix unless the system is Markovian, a fact which is discussed in Appendix C. Note that this is in clear contrast with the situation for a closed system, where the unitary propagators, which are solutions of the Schrödinger equation, are sufficient to obtain all the information about the existing quantum correlations.

Finally, in Section 5 we summarize and briefly discuss our results.

Throughout the paper we use units in which $\hbar = 1$ except for Section 2.

2. Influence functional formalism for open quantum systems

2.1. A brief survey of open quantum systems

Open quantum systems [21] are of interest in condensed matter physics [22], quantum optics [23], quantum measurement theory [24], nonequilibrium field theory [25–28], quantum cosmology [29] and semiclassical gravity [30]. Among the most widely used examples of open quantum system is the QBM model, which consists of a single massive particle in a potential (usually quadratic) interacting with an infinite set of independent harmonic oscillators which are initially in a Gaussian state (most often a thermal equilibrium state) [31]. The coupling may be linear both in the system and environment variables or may be nonlinear in some or all of these variables. The frequencies of the environment oscillators are distributed according to a prescribed spectral density function, the simplest case corresponding to the so-called ohmic environment. The linear coupling provides a good description of many open quantum systems in condensed matter physics [32,22], but in field theory [33], quantum cosmology [29] and semiclassical gravity [11,34] the coupling is usually nonlinear. Part of the interest of the linear systems is that they are in many cases exactly solvable and detailed studies of different aspects of open quantum systems can be performed. One of the issues that has received much attention in recent years is environment-induced decoherence

as a mechanism to understand the transition from the quantum to the classical regime [35,36].

Concepts such as the Feynman–Vernon influence functional method, the reduced density matrix, the reduced Wigner function, the master equation, the Fokker–Planck equation and the Langevin equation are some of the key words associated to the study of open quantum systems. One of the purposes of this paper is to review the place of these concepts in the QBM model and to establish their often subtle interrelations. Thus, let us first review some of those concepts and recall their main features.

The *reduced density matrix* is defined from the density matrix of the whole closed system by tracing out the environment. Its dynamical evolution may be given in terms of the Feynman and Vernon influence functional [1]. The influence functional is defined from a path integral involving the action of the system and the environment and an integration of the environment degrees of freedom. Its use in the QBM model is widespread especially since Caldeira and Leggett were able to compute in closed form the propagator for the reduced density matrix in the case of linear coupling with an ohmic environment [32].

The *master equation* is a differential equation describing the evolution of the reduced density matrix. The master equation for linear coupling and ohmic environment at high temperature was first deduced by Caldeira and Leggett [32], it was extended to arbitrary temperature by Unruh and Zurek [37], and it was finally obtained for a general environment (i.e., for an arbitrary spectral density function) by Hu et al. [17]. This result can be extended to the case of nonlinear coupling by treating the interaction perturbatively up to quadratic order [38].

Closely related to the reduced density matrix is the *reduced Wigner function* (in fact one goes from one to the other by an integral transform) [39,40]. The reduced Wigner function is similar in many aspects to a distribution function in phase space, although it is not necessarily positive definite, and the dynamical equation it satisfies is similar to the *Fokker–Planck equation* for classical statistical systems [41,42]. This equation is, of course, entirely equivalent to the master equation for the reduced density matrix and, sometimes, we also refer to it as the master equation. The reduced density matrix has been used to study decoherence induced by the environment [43–46,37,17,38]. The Wigner function has also been used in studies of emergence of classicality induced by an environment [47], especially in quantum cosmology [29].

The *Langevin equation* [48,16] is another relevant equation for open quantum systems. This equation has either been introduced phenomenologically [42] to describe the effect of the environment into a classical system (Brownian motion) or it has been derived within the functional approach (see, however, Refs. [49,50] for a quantum version of the Langevin equation in operator language) as a classical or semiclassical limit. Thus, Gell-Mann and Hartle [2,51] in the framework of the consistent histories approach to quantum mechanics [52] considered the decoherence functional, which is closely related to the influence functional in the case of open quantum systems, to measure the degree of classicality of the system. They were able to show that under certain conditions there exists a semiclassical limit in an open system which may be suitably described by a Langevin equation with the self-correlation of the stochastic source given by the noise kernel which appears in the influence functional.

Langevin-type equations as a suitable tool to study the semiclassical limit have been used recently in semiclassical gravity and cosmology [11,12,34]. In inflationary cosmology they have been used to describe the stochastic effect on the inflaton field [53–59] or the stochastic behavior of large-scale gravitational perturbations [60,10], which is important for cosmological structure formation.

2.2. A linear QBM model

Here we briefly review the main features of a QBM model as an example of a linear open quantum system. Let us consider a harmonic oscillator of mass M, the "system", coupled to a bath of independent harmonic oscillators of mass m, the "environment". For simplicity, let us assume that the system and environment are linearly coupled. The action for the whole set of degrees of freedom is defined by

$$S[x,\{q_j\}] = S[x] + S[\{q_j\}] + S_{int}[x,\{q_j\}]\,, \tag{2.1}$$

where the terms on the right-hand side, which correspond to the action of the system, the environment and the interaction term, respectively, are given by

$$S[x] = \int \mathrm{d}t \left(\frac{1}{2}M\dot{x}^2 - \frac{1}{2}M\Omega^2 x^2\right)\,, \tag{2.2}$$

$$S[\{q_j\}] = \sum_j \int \mathrm{d}t \left(\frac{1}{2}m\dot{q}_j^2 - \frac{1}{2}m\omega_j^2 q_j^2\right)\,, \tag{2.3}$$

$$S_{int}[x,\{q_j\}] = \sum_j c_j \int \mathrm{d}t\; x(t)q_j(t) = \int_0^\infty \mathrm{d}\omega\, \frac{2m\omega}{\pi c(\omega)} I(\omega) \int \mathrm{d}t\; x(t)q(t;\omega)\,, \tag{2.4}$$

where Ω and ω_j are, respectively, the system and environment oscillator frequencies, and we introduced the spectral density function $I(\omega) = \sum_j \pi c_j^2 (2m\omega_j)^{-1}\delta(\omega-\omega_j)$ in the last equality; $c(\omega)$ and $q(t;\omega)$ are functions such that $c(\omega_j)=c_j$ and $q(t;\omega_j)=q_j(t)$, with c_j being system–environment coupling parameters.

The reduced density matrix for an open quantum system is defined from the density matrix ρ of the whole system by tracing out the environment degrees of freedom:

$$\rho_r(x_f,x_f',t_f) = \int \prod_j \mathrm{d}q_j \rho(x_f,\{q_j\},x_f',\{q_j\},t_f)\,. \tag{2.5}$$

The evolution for the reduced density matrix, which is nonunitary and in general not even Markovian, can be written as

$$\rho_r(x_f,x_f',t_f) = \int \mathrm{d}x_i\,\mathrm{d}x_i' J(x_f,x_f',t_f;x_i,x_i',t_i)\rho_r(x_i,x_i',t_i)\,, \tag{2.6}$$

where the propagator J is defined in a path integral representation by

$$J(x_f,x_f',t_f;x_i,x_i',t_i) = \int_{x(t_i)=x_i}^{x(t_f)=x_f} \mathscr{D}x \int_{x'(t_i)=x_i'}^{x'(t_f)=x_f'} \mathscr{D}x' \mathrm{e}^{i(S[x]-S[x']+S_{IF}[x,x'])/\hbar}\,, \tag{2.7}$$

where $S_{IF}[x,x']$ is the influence action introduced by Feynman and Vernon [1]. When the system and the environment are initially uncorrelated, i.e., when the initial density matrix factorizes ($\hat{\rho}(t_i)=\hat{\rho}_r(t_i)\otimes\hat{\rho}_e(t_i)$, where $\hat{\rho}_r(t_i)$ and $\hat{\rho}_e(t_i)$ mean, respectively, the density matrix operators of the system and the environment at the initial time) and the initial density matrix for the environment $\rho_e(\{q_j^{(i)}\},\{q'_j(i)\},t_i)$ is Gaussian, one obtains [1,32]

$$S_{IF}[x,x'] = -2\int_{t_i}^{t_f} \mathrm{d}s \int_{t_i}^{s} \mathrm{d}s'\, \Delta(s)D(s,s')X(s') + \frac{\mathrm{i}}{2}\int_{t_i}^{t_f} \mathrm{d}s \int_{t_i}^{t_f} \mathrm{d}s'\, \Delta(s)N(s,s')\Delta(s')\,, \tag{2.8}$$

where $X(s)\equiv(x(s)+x'(s))/2$ and $\Delta(s)\equiv x'(s)-x(s)$. The kernels $D(s,s')$ and $N(s,s')$ are called the dissipation and noise kernel, respectively. When the bath is initially in thermal equilibrium these kernels are related by the usual fluctuation–dissipation relation [61]. The situation in which no special form is assumed for the spectral density $I(\omega)$, is usually referred to as a general environment. One of the most common particular cases is the so-called Ohmic environment, characterized by $I(\omega)\sim\omega$ (some natural high frequency cut-off is often introduced, otherwise an infinite renormalization is required).

Starting with Eqs. (2.5) and (2.7) a differential equation for the system's reduced density matrix, known as the master equation, can be derived. The expression for a general environment was first obtained by Hu et al. [17] (see also Ref. [19] for an alternative derivation)

$$\begin{aligned}\mathrm{i}\hbar\frac{\partial\rho_r}{\partial t} = &-\frac{\hbar^2}{2M}\left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial x'^2}\right)\rho_r + \frac{1}{2}M\Omega^2(x^2-x'^2)\rho_r \\ &+\frac{1}{2}M\delta\Omega^2(t)(x^2-x'^2)\rho_r - \mathrm{i}\hbar A(t)(x-x')\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)\rho_r \\ &+\hbar B(t)(x-x')\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial x'}\right)\rho_r - \mathrm{i}MC(t)(x-x')^2\rho_r\,,\end{aligned} \tag{2.9}$$

where the functions $\delta\Omega^2(t)$, $A(t)$, $B(t)$ and $C(t)$ represent a frequency shift, a dissipation factor and two diffusive factors, respectively. For explicit expressions of these functions see Appendix B. An alternative representation for the system reduced density matrix is the reduced Wigner function $W_r(X,p,t)$ defined as

$$W_r(X,p,t) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} \mathrm{d}\Delta \mathrm{e}^{\mathrm{i}p\Delta/\hbar}\rho_r(X-\Delta/2, X+\Delta/2, t)\,. \tag{2.10}$$

It follows immediately that the master equation (2.9) can be written in the following equivalent form:

$$\frac{\partial W_r}{\partial t} = \{H_R, W_r\}_{PB} + 2A(t)\frac{\partial(pW_r)}{\partial p} + \hbar B(t)\frac{\partial^2 W_r}{\partial X\,\partial p} + \hbar MC(t)\frac{\partial^2 W_r}{\partial p^2}\,, \tag{2.11}$$

where $\{H_R, W_r\}_{PB} \equiv -(p/M)\partial W_r/\partial X + M\Omega_R^2(t)X\partial W_r/\partial p$ with $\Omega_R^2(t) = \Omega^2 + \delta\Omega^2(t)$. It is worth noticing that this equation is formally analogous to the Fokker–Planck equation for a distribution function.

3. Stochastic description of the system's quantum dynamics

In this section, we show that the reduced Wigner function can be written as a formal distribution function for some stochastic process, and using this result we deduce the corresponding Fokker–Planck equation.

3.1. Reduced density matrix and Wigner function

To find an explicit expression for the reduced density matrix (2.5) at a time t_f, we need to compute the path integrals which appear in Eq. (2.7) for the reduced density matrix propagator. On the other hand, the reduced Wigner function W_r is related to the reduced density matrix by the integral transform (2.10). In Appendix A, we show that W_r can be written in the following suggestive form:

$$W_r(X_f, p_f, t_f) = \left\langle \left\langle \delta(X(t_f) - X_f)\delta(M\dot{X}(t_f) - p_f) \right\rangle_\xi \right\rangle_{X_i, p_i} , \tag{3.1}$$

where

$$\langle \cdots \rangle_\xi \equiv [\det(2\pi N)]^{-1/2} \int \mathscr{D}\xi \ldots \mathrm{e}^{-(1/2)\xi \cdot N^{-1} \cdot \xi} , \tag{3.2}$$

$$\langle \cdots \rangle_{X_i, p_i} \equiv \int_{-\infty}^{\infty} \mathrm{d}X_i \int_{-\infty}^{\infty} \mathrm{d}\, p_i \ldots W_r(X_i, p_i, t_i) \tag{3.3}$$

and $X(t)$ is the solution with initial conditions X_i, p_i of the Langevin-like equation

$$(L \cdot X)(t) = \xi(t) , \tag{3.4}$$

where $L(t,t') \equiv M(\mathrm{d}^2/\mathrm{d}t'^2 + \Omega_{ren}^2)\delta(t - t') + H(t,t')$. Here the functions Ω_{ren} and $H(t,t')$ are defined below Eq. (A.1), and we have also used the notation $A \cdot B \equiv \int_{t_i}^{t_f} \mathrm{d}t\, A(t)B(t)$.

Thus, the reduced Wigner function can be interpreted as an average over a Gaussian stochastic process $\xi(t)$ with $\langle \xi(t) \rangle_\xi = 0$ and $\langle \xi(t)\xi(t') \rangle_\xi = N(t,t')$ as well as an average over the initial conditions characterized by a distribution function $W_r(X_i, p_i, t_i)$. It is only after formally interpreting $\xi(t)$ as a stochastic process characterized by Eq. (3.2) that Eq. (3.4) can be regarded as a Langevin equation. Note that, in general, Eq. (3.4) is not meant to describe the actual trajectories of the system (a brief discussion on this point is given in the last section).

Note, in addition, that although $W_r(X_i, p_i, t_i)$ is real, which follows from the hermiticity of the density matrix, and properly normalized, in general it is not positive everywhere (except for Gaussian states) and, thus, cannot be considered as a probability distribution. The fact that the Wigner function can acquire negative values is crucial since some of the nonclassical features of the quantum state are tightly related to the Wigner function having negative values. For instance, a coherent superposition state is typically characterized by the Wigner function presenting strong oscillations with

negative values in the minima [47,46], which are closely connected to interference terms.

Eq. (3.1) is the main result of this Section and shows that the reduced Wigner function can be interpreted as a formal distribution in phase space. This result will now be used to derive the corresponding Fokker–Planck equation.

3.2. From Langevin to Fokker–Planck: derivation of the master equation

As mentioned above there is a simple one-to-one correspondence between any density matrix and the associated Wigner function introduced in Eq. (2.10). Taking this correspondence into account, the equation satisfied by the reduced Wigner function is equivalent to the master equation satisfied by the reduced density matrix. By deriving Eq. (3.1) with respect to time and using the Langevin-type equation in (3.4), one can obtain a Fokker–Planck differential equation describing the time evolution of the system's reduced Wigner function. The details of the calculation can be found in Appendix B. Our result is the transport equation (B.16) which is written in terms of the time dependent coefficients $A(t)$, $B(t)$ and $C(t)$ defined, respectively, by Eqs. (B.10), (B.17) and (B.18). These coefficients are, of course, in agreement with those previously derived in Refs. [17,19]; see also [18]. Thus, this is yet another alternative way to derive the master or transport equation (2.11).

This new road to the transport equation highlights the fact that while one can derive the Fokker–Planck equation from the Langevin equation, the opposite is not possible in general. One can always consider Langevin equations with stochastic sources characterized by different noise kernels which, nevertheless, lead to the same Fokker–Planck equation and, thus, the same master equation. This can be argued from the expressions obtained in the derivation of the Fokker–Planck equation. Let us consider, for simplicity, the situation corresponding to local dissipation. A local contribution to the noise gives no contribution to $B(t)$, but it does contribute to $C(t)$ as can be seen from Eqs. (B.17) and (B.18) taking into account that $G_{ret}(t,t)=0$ and $\partial G_{ret}(t',t)/\partial t'|_{t'=t}=M^{-1}$. Thus, one can always choose any noise kernel that gives the desired $B(t)$ and then add the appropriate local contribution to the noise kernel to get the desired $C(t)$ keeping $B(t)$ fixed. Note that changing the noise kernel does not change $A(t)$. To illustrate the fact that there exist different noise kernels giving the same $B(t)$, as was stated above, one may consider the particular case corresponding to the weak dissipation limit so that $G_{ret}(t,t') \sim (M\Omega)^{-1}\sin\Omega(t-t')\theta(t-t')$. To see that a different $\tilde{N}(t,t')$ giving the same $B(t)$ as $N(t,t')$ exists reduces then to show that there is at least one non-trivial function $v(s,t)=\tilde{N}(t,t')-N(t,t')$ (with $s=t-t'$) such that for any t we have $\int_0^t \mathrm{d}s\,\sin(\Omega s)v(s,t)=0$, which can be shown to be the case.

The fact that different Langevin equations lead to the same master equation [1] reflects that the former contains more information than the latter. This fact can be qualitatively

[1] In fact, what we showed was that a Langevin equation contains in general more information that the corresponding Fokker–Planck equation. To extend this assertion to the master equation, one should make sure that the different Langevin equations leading to the same Fokker–Planck equation can be obtained from an influence functional. This fact seems plausible provided that one considers general Gaussian initial states for the environment.

understood in the following way. In the influence functional it is only the evolution of the environment degrees of freedom that is traced out. Of course, having integrated over all the possible quantum histories for the environment, no correlations in the environment can be obtained. Nevertheless, since the system is interacting with the environment, non-Markovian correlations for the system at different times may in general persist. On the other hand, when considering either the reduced density matrix or its propagator, also the system evolution, except for the final state, is integrated out. Consequently, information on non-Markovian correlations for the system is no longer available. Thus, only when the system's reduced dynamics is Markovian, i.e., the influence functional is local in time, we expect that the Langevin equation and the master equation contain the same information. In particular, for a Gaussian stochastic source, as in our case, the Langevin equation contains the information about the system correlations at different times which the Fokker–Planck equation cannot in general account for. Only in the case in which the dynamics generated by the Langevin equation is Markovian one can compute the correlation functions just from the solutions of the Fokker–Planck equation or, equivalently, the master equation for the propagator $J(x_2,x_2',t_2;x_1,x_1',t_1)$; see Eq. (2.7). The key point is the fact that the propagator for the reduced density matrix only factorizes when the influence action is local. In Appendix C we give a detailed argument on this point.

It is important to note that for a closed quantum system the evolution determined by the time evolution operators $U(t_2,t_1)$ obtained from the Schrödinger equation is always unitary and, thus, also Markovian. That is why the Schrödinger equation suffices to get the correlation functions for a closed quantum system. On the contrary, for an open quantum system the evolution is nonunitary and, provided that the influence action is nonlocal, not even Markovian.

4. Correlation functions

We have seen that the reduced Wigner function, or equivalently the reduced density matrix, and the master equation governing these functions can be obtained from a formal stochastic description provided by the Langevin equation (3.4). In this Section we show that also entirely quantum correlation functions for the system can be obtained by means of the stochastic description developed in the previous section.

4.1. CTP generating functional for the system and n-point quantum correlation functions

All the relevant quantum correlation functions for the system can be obtained from the CTP generating functional, which is expressed, after integrating out the environment, as [62,63]

$$Z_{CTP}[J,J'] = \int \mathrm{d}x_f \int \mathrm{d}x_i\,\mathrm{d}x_i' \int_{x(t_i)=x_i}^{x(t_f)=x_f} \mathscr{D}x \int_{x'(t_i)=x_i'}^{x'(t_f)=x_f} \mathscr{D}x'\,\mathrm{e}^{\mathrm{i}J\cdot x-\mathrm{i}J'\cdot x'} \times \mathrm{e}^{\mathrm{i}(S[x]-S[x']+S_{IF}[x,x'])}\rho_r(x_i,x_i',t_i)\,, \tag{4.1}$$

where use was made of the influence action introduced in Eq. (2.7). Equivalently, we may rewrite the previous equation changing to semisum and difference variables with $J_\Sigma = (J(t) + J'(t))/2$ and $J_\Delta = J'(t) - J(t)$, integrate the system action by parts and proceed analogously as we did in Appendix A to obtain

$$Z_{CTP}[J_\Sigma, J_\Delta] = \langle e^{-i J_\Delta \cdot X_0} \rangle_{X_i, p_i} e^{-(1/2) J_\Delta \cdot G_{ret} \cdot N \cdot (J_\Delta \cdot G_{ret})^T} e^{-i J_\Delta \cdot G_{ret} \cdot J_\Sigma} , \tag{4.2}$$

where $X_0(t)$ is the solution to the homogeneous equation $(L \cdot X)(t) = 0$ with the initial conditions X_i, p_i. It is interesting to note that the first factor in Eq. (4.2) contains all the information about the initial conditions of the system, whereas the information about the fluctuations induced on the system by the environment is essentially contained in the second factor through the noise kernel. This is the key result of this section which will allow to relate the quantum with the stochastic correlation functions.

Any n-point quantum correlation function for the system position operators can be obtained from the CTP generating functional according to the equation

$$\begin{aligned} &\mathrm{Tr}[(T\hat{x}(t_1)\ldots\hat{x}(t_m))\hat{\rho}(t_i)(\tilde{T}\hat{x}(t_{m+1})\ldots\hat{x}(t_{m+n}))] \\ &\quad = i^{n-m} \left(\frac{\delta}{\delta J}\right)^m \left(\frac{\delta}{\delta J'}\right)^n Z_{CTP}[J, J'] \Big|_{J, J'=0} . \end{aligned} \tag{4.3}$$

Since one can always write J and J' in terms of J_Σ and J_Δ, the right-hand side of Eq. (4.3) can be expressed as a linear combination of terms of the type

$$i^{r+s} \left(\frac{\delta}{\delta J_\Sigma}\right)^r \left(\frac{\delta}{\delta J_\Delta}\right)^s Z_{CTP}[J_\Sigma, J_\Delta] \Big|_{J_\Sigma, J_\Delta = 0} \tag{4.4}$$

with $0 \leqslant r \leqslant n+m$, $0 \leqslant s \leqslant n+m$ and $r+s=n+m$. To obtain an explicit expression one must evaluate Eq. (4.4) with the final result for the CTP generating functional (4.2).

4.2. Quantum correlation functions from stochastic averages

Using expression (4.4) for the case $r = 0$, a connection can be established between the correlation functions for the Gaussian stochastic process associated to $\xi(t)$ via the Langevin-type equation (3.4) with $W_r(X_i, p_i, t_i)$ as the distribution function for the initial conditions, and some quantum correlation functions corresponding to quantum expectation values of products of Heisenberg operators at different instants of time. Any correlation function for the former stochastic process can be obtained from its characteristic functional in the usual way:

$$\Big\langle \langle X(t_1)\ldots X(t_s) \rangle_\xi \Big\rangle_{X_i, p_i} = i^s \left(\frac{\delta}{\delta K}\right)^s \Big\langle \langle e^{-i K \cdot X} \rangle_\xi \Big\rangle_{X_i, p_i} \Big|_{K=0} . \tag{4.5}$$

The generating functional for the aforementioned stochastic process is, in turn, related to the full CTP generating functional previously introduced as follows:

$$\Big\langle \langle e^{-i K \cdot X} \rangle_\xi \Big\rangle_{X_i, p_i} = Z_{CTP}[J_\Sigma = 0, J_\Delta = K] . \tag{4.6}$$

Substituting Eq. (4.6) into Eq. (4.5), rewriting J_Δ in terms of J and J', and using expression (4.3), we can express the correlation functions for the stochastic process in terms of quantum correlation functions for the system observables. In particular, for $s=2$ we have

$$\left\langle \langle X(t_1)X(t_2)\rangle_\xi \right\rangle_{X_i,p_i} = \tfrac{1}{4}[\langle T\hat{x}(t_1)\hat{x}(t_2)\rangle + \langle \hat{x}(t_1)\hat{x}(t_2)\rangle + \langle \hat{x}(t_2)\hat{x}(t_1)\rangle + \langle \tilde{T}\hat{x}(t_1)\hat{x}(t_2)\rangle] = \tfrac{1}{2}\langle\{\hat{x}(t_1),\hat{x}(t_2)\}\rangle \,, \tag{4.7}$$

where, as usual, we used $\langle\cdots\rangle$ to denote the quantum expectation value $\mathrm{Tr}[\ldots\hat{\rho}(t_i)]$.

On the other hand, concentrating on the stochastic description provided by the left-hand side of Eq. (4.7) and elaborating a little bit on it by using Eq. (A.6) and taking into account that $\xi(t)$ is a Gaussian stochastic process characterized by $\langle\xi(t)\rangle_\xi = 0$ and $\langle\xi(t_1)\xi(t_2)\rangle_\xi = N(t_1,t_2)$, we can write

$$\langle\langle X(t_1)X(t_2)\rangle_\xi\rangle_{X_i,p_i} = \langle\langle[X_0(t_1)+(G_{ret}\cdot\xi)(t_1)][X_0(t_2)+(G_{ret}\cdot\xi)(t_2)]\rangle_\xi\rangle_{X_i,p_i} = \langle X_0(t_1)X_0(t_2)\rangle_{X_i,p_i} + (G_{ret}\cdot N\cdot(G_{ret})^{\mathrm{T}})(t_1,t_2)\,. \tag{4.8}$$

Hence, the final result is

$$\tfrac{1}{2}\langle\{\hat{x}(t_1),\hat{x}(t_2)\}\rangle = \langle X_0(t_1)X_0(t_2)\rangle_{X_i,p_i} + (G_{ret}\cdot N\cdot(G_{ret})^{\mathrm{T}})(t_1,t_2)\,. \tag{4.9}$$

The left-hand side of Eq. (4.9) is the quantum correlation function, which can therefore be described within the stochastic scheme in terms of two separate contributions: the first term on the right-hand side corresponds entirely to the *dispersion* in the initial conditions, whereas the second term is due to the fluctuations induced by the stochastic source appearing in the Langevin-type equation (3.4). It should be remarked that, as discussed in Appendix C, for the general case of a nonlocal influence action no quantum correlation functions (except for the trivial case of $n=1$) can be expressed in terms of the propagators for the reduced density matrix, which can be obtained from the master equation.

It is clear from Eqs. (4.5) and (4.6) that only those quantum correlation functions which are obtained by functionally differentiating the CTP generating functional with respect to J_Δ (but not J_Σ) an arbitrary number of times can be related to the stochastic correlation functions (4.5). Let us, therefore, see what is the general expression for all the quantum correlation functions that can be directly obtained from the stochastic description. We begin with the classical correlation functions (4.5) for the stochastic processes $X(t)$ which are solutions of the Langevin-type equation with stochastic source $\xi(t)$ and initial conditions averaged over the initial reduced Wigner function. Then we write these correlation functions in terms of path integrals and use the results of Sections 3 and 4 to relate them to a subclass of quantum correlation functions for the system:

$$\langle\langle X(t_1)\ldots X(t_n)\rangle_\xi\rangle_{X_i,p_i} = [\det(2\pi N)]^{-1/2}\int_{-\infty}^{\infty}\mathrm{d}X_f\int_{-\infty}^{\infty}\mathrm{d}X_i\int_{-\infty}^{\infty}\mathrm{d}\,p_i$$

$$\times \int \mathscr{D}\xi \mathrm{e}^{-(1/2)\xi\cdot N^{-1}\cdot\xi}\,\delta(X(t_f)-X_f)X(t_1)\ldots X(t_n)W_r(X_i,p_it_i)$$

$$=\mathrm{Tr}^*[\hat{X}(t_1)\ldots\hat{X}(t_n)\hat{\rho}(t_i)] \tag{4.10}$$

with $\hat{X}(t_j)=(\hat{x}(t_j)+\hat{x}'(t_j))/2$ and

$$\begin{aligned}\mathrm{Tr}^*[\hat{x}(t_1)\ldots\hat{x}'(t_r)\ldots\hat{x}(t_s)\ldots\hat{x}'(t_u)\ldots\hat{\rho}(t_i)]\\ \equiv\ \mathrm{Tr}\left[\{T\hat{x}(t_1)\ldots\hat{x}(t_{r-1})\hat{x}(t_s)\ldots\hat{x}(t_{u-1})\}\right.\\ \left.\times\,\hat{\rho}(t_i)\{\tilde{T}\hat{x}(t_r)\ldots\hat{x}(t_{s-1})\hat{x}(t_u)\ldots\}\right]\ ,\end{aligned} \tag{4.11}$$

where both the initial density matrix and the trace correspond to the whole closed quantum system (i.e., system plus environment) and T and $\tilde{T}$ denote time and anti-time ordering, respectively. It is then straightforward to show that

$$\begin{aligned}\langle\langle X(t_1)\ldots X(t_n)\rangle_\xi\rangle_{X_i,p_i}=2^{-n}\sum_{m=0}^{n}\frac{1}{m!(n-m)!}\sum_{\sigma\in S_n}\mathrm{Tr}\left[\left\{T\prod_{j=\sigma(1)}^{\sigma(m)}\hat{x}(t_j)\right\}\hat{\rho}(t_i)\right.\\ \left.\times\left\{\tilde{T}\prod_{k=\sigma(m+1)}^{\sigma(n)}\hat{x}(t_k)\right\}\right]\ ,\end{aligned} \tag{4.12}$$

where $\sigma\in S_n$ are all the possible permutations for a set consisting of n elements.

5. Summary and discussion

In this paper, we have considered the stochastic description of a linear open quantum system. We have shown that the reduced Wigner function can be written as a formal distribution function for a stochastic process given by a Langevin-type equation. The master equation has then been deduced as the corresponding Fokker–Planck equation for the stochastic process. We have also shown that a subclass of quantum correlation functions for the system variables can be written in terms of stochastic correlation functions. Our results are summarized in the diagram of Fig. 1 which show all the interconnections between the influence functional, the Langevin equation, the Fokker–Planck equation, the master equation and the correlation functions.

It is important to stress that in general the system trajectories which correspond to solutions of the Langevin equation should be regarded just as a mathematical tool to obtain the time evolution for the reduced Wigner function as well as to compute a class of quantum correlation functions for system observables. The situation is somewhat analogous to what happens with the trajectories of a quantum system in Feynman path integrals. In the latter case probabilities cannot be consistently assigned due to the existence of interference between different trajectories, but nevertheless path integrals can be used to compute quantum correlation functions or the time evolution of the

wave function. On the other hand, in the case of the Langevin equation solutions, probabilities can be assigned, but they do not correspond to quantum histories, i.e., sequences of quantum properties (projectors) at different instants of time.[2] However, as we have shown, they can still be a useful tool to compute both quantum correlation functions and the time evolution of the reduced Wigner function for open quantum systems.

The previous comments are not unrelated to the fact that strictly speaking the Wigner function cannot be interpreted as the probability density for the system position and momentum at a given time (this would be in conflict with Heisenberg's uncertainty principle). Furthermore, depending on the state of a quantum system, its Wigner function, although being properly normalized, may acquire negative values, a fact which is closely related to the existence of coherent quantum superpositions.

We end up emphasizing that despite concentrating on a simple linear QBM model, the main results presented in this paper can be extended to quantum field theory, including nonlinear system-environment interaction and even nonlinear system potentials, which have been applied to study vacuum decay [3,4], as well as stochastic gravity and cosmology [13]. In particular, the generalization of the basic ideas presented here can be very illuminating in order to establish the precise relationship between the correlations of the metric perturbations around a given spacetime background computed in the framework of stochastic gravity [8–10], and the results for the quantum fluctuations of the metric perturbations induced by its interaction with quantum matter fields which would follow from a purely quantum mechanical treatment [65,66]. Therefore, we feel that a rather detailed exposition for a simple model, as given here, is indeed justified as a basis for further developments.

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[2] Otherwise the fact that both the position and the momentum at any instant of time are completely specified for the Langevin equation solutions associated to a given realization of the stochastic source would be in conflict with Heisenberg's uncertainty principle. However, it should be noted that a sufficiently coarse-grained version of the system phase-space trajectories can actually characterize to a very good approximation well-defined quantum histories for the system [64], which under certain conditions (mainly that the decoherence condition is fulfilled) satisfy a Langevin equation formally analogous to the one discussed in this work [2].

Appendix A. Derivation of the stochastic representation for the Wigner function

Let us begin by rewriting the influence action (2.8) as

$$\begin{aligned} S_{IF}[x,x'] &= \int_{t_i}^{t_f} \mathrm{d}s \int_{t_i}^{t_f} \mathrm{d}s' \Delta(s) H_{bare}(s,s') X(s') \\ &\quad + \frac{\mathrm{i}}{2} \int_{t_i}^{t_f} \mathrm{d}s \int_{t_i}^{t_f} \mathrm{d}s' \Delta(s) N(s,s') \Delta(s') \\ &\equiv X \cdot H_{bare} \cdot \Delta + \frac{\mathrm{i}}{2} \Delta \cdot N \cdot \Delta \,, \end{aligned} \tag{A.1}$$

where we used the notation $A \cdot B \equiv \int_{t_i}^{t_f} \mathrm{d}t\, A(t) B(t)$, and defined $H_{bare}(s,s')$ as formally equivalent to $-2D(s,s')\theta(s-s')$. Being the product of two distributions, the latter expression is not well defined in general and suitable regularization and renormalization may be required; see Ref. [67] for details. The local divergences present in $H_{bare}(s,s') = H(s,s') + H_{div}(s)\delta(s-s')$ can be canceled by suitable counterterms Ω_{div} in the bare frequency of the system $\Omega = \Omega_{ren} + \Omega_{div}$. From now on we will consider that this infinite renormalization, if necessary, has already been performed so that both Ω_{ren} and $H(s,s')$ are free of divergences. Now we perform three main steps.

First, we integrate the system action by parts:

$$\begin{aligned} S[x] - S[x'] &= -M \int_{t_i}^{t_f} \mathrm{d}t (\dot{X}(t)\dot{\Delta}(t) - \Omega_{ren}^2 X(t)\Delta(t)) \\ &= -M\dot{X}\Delta\big|_{t_i}^{t_f} + M \int_{t_i}^{t_f} \mathrm{d}t \Delta(t) \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \Omega_{ren}^2 \right) X(t) \,. \end{aligned} \tag{A.2}$$

Second, we perform the Gaussian path integral for $\Delta(t)$. Taking into account that the value of the Jacobian determinant for the change of integration variables $\int_{x_i}^{x_f} \mathscr{D}x \int_{x_i'}^{x_f'} \times \mathscr{D}x' \to \int_{X_i}^{X_f} \mathscr{D}X \int_{\Delta_i}^{\Delta_f} \mathscr{D}\Delta$ is one, the Gaussian path integral for $\Delta(t)$ with Δ_i and Δ_f fixed is performed:

$$\begin{aligned} &\int_{X_i}^{X_f} \mathscr{D}X \int_{\Delta_i}^{\Delta_f} \mathscr{D}\Delta \mathrm{e}^{\mathrm{i}\Delta \cdot L \cdot X} \mathrm{e}^{-(1/2)\Delta \cdot N \cdot \Delta} \\ &= \left(\det \frac{N}{2\pi} \right)^{-1/2} \int_{X_i}^{X_f} \mathscr{D}X \mathrm{e}^{-1/2(L \cdot X) \cdot N^{-1} \cdot (L \cdot X)} \,, \end{aligned} \tag{A.3}$$

where $L(t,t') \equiv M(\mathrm{d}^2/\mathrm{d}t'^2 + \Omega_{ren}^2)\delta(t-t') + H(t,t')$. We note that the final result of Eq. (A.3) does not depend on the final points Δ_i and Δ_f of the integration path. This is true provided that the noise kernel is well-behaved enough: even a noise kernel of the form $N(t,t') \sim \delta(t-t')$ is allowed, whereas only those whose support is too

concentrated on the initial or the final time, such as $N(t,t') \sim \delta(t-t_i)$, should be discarded. Taking into account Eq. (A.3) and the boundary terms coming from the integration by parts of the system action, the integration over Δ_i gives

$$\rho_r(X_f-\Delta_f/2, X_f+\Delta_f/2, t_f) = \left(\det\frac{N}{2\pi}\right)^{-1/2} \int_{-\infty}^{\infty} \mathrm{d}\Delta_i \int_{-\infty}^{\infty} \mathrm{d}X_i \int_{X_i}^{X_f} \times \mathscr{D}X e^{-1/2(L\cdot X)\cdot N^{-1}\cdot(L\cdot X)} e^{-iM\dot{X}_f\Delta_f} e^{iM\dot{X}_i\Delta_i} \rho_r(X_i-\Delta_i/2, X_i+\Delta_i/2, t_i)$$

$$= 2\pi\left(\det\frac{N}{2\pi}\right)^{-1/2} \int_{-\infty}^{\infty} \mathrm{d}X_i \int_{X_i}^{X_f} \mathscr{D}X\, e^{-1/2(L\cdot X)\cdot N^{-1}\cdot(L\cdot X)} \times e^{-iM\dot{X}_f\Delta_f} W_r(X_i, M\dot{X}_i, t_i)\,, \tag{A.4}$$

where in the last step we used Eq. (2.10), which defines the reduced Wigner function.

Third, we carry out the following functional change:

$$X(t) \to \{X_i = X(t_i),\ p_i \equiv M\dot{X}_i = M\dot{X}(t_i),\ \ \xi(t) = (L\cdot X)(t)\}\,. \tag{A.5}$$

Note that with this change the function $X(t)$ gets substituted by the initial conditions (X_i, p_i) and the function $\xi(t)$ in the functional integration. It is important to note that at this point the function $\xi(t)$ is not a stochastic process but just a function over which a path integral is performed. The functional change (A.5) is invertible as can be explicitly seen:

$$\{X_i, p_i,\ \ \xi(t)\} \to X(t) = X_0(t) + \int_{t_i}^{t} \mathrm{d}t' G_{ret}(t,t')\xi(t')\,, \tag{A.6}$$

where $G_{ret}(t',t'')$ is the retarded (i.e., $G_{ret}(t',t'')=0$ for $t' \leqslant t''$) Green function for the linear integro-differential operator associated to the kernel $L(t,t')$, and $X_i(t) = \int_{t_i}^{t} \mathrm{d}t' G_{ret}(t,t')\xi(t')$ is a solution of the inhomogeneous equation $(L\cdot X_i)(t) = \xi(t)$ with initial conditions $X_i(t_i)=0$ and $\partial X_i(t')/\partial t'|_{t'=t_i} = 0$. On the other hand, $X_0(t)$ is a solution of the homogeneous equation $(L\cdot X_0)(t)=0$, with initial conditions $X_0(t_i)=X_i$ and $\dot{X}(t_i) = p_i/M$. Since the change is linear, the Jacobian functional determinant will be a constant (this can be clearly seen by skeletonizing the path integral). After performing the functional change, we obtain

$$\rho_r(X_f-\Delta_f/2, X_f+\Delta_f/2, t_f) = K \int_{-\infty}^{\infty} \mathrm{d}X_i \int_{-\infty}^{\infty} \mathrm{d}\,p_i \int \mathscr{D}\xi\,\delta(X(t_f)-X_f) \times e^{-(1/2)\xi\cdot N^{-1}\cdot\xi} e^{-iM\dot{X}(t_f)\Delta_f} W_r(X_i, p_i, t_i)\,, \tag{A.7}$$

where the delta function $\delta(X(t_f)-X_f)$ was introduced to restrict the functional integral $\int\mathscr{D}\xi$ with free ends, in order to take into account the restriction on the final points of the allowed paths for the integral $\int^{X_f}\mathscr{D}X$ appearing in Eq. (A.4). The contribution from the Jacobian has been included in the constant K. In order to determine this constant,

we demand the reduced density matrix to remain normalized, i.e., that $\mathrm{Tr}\,\rho_r(t_f)=1$ if $\mathrm{Tr}\,\rho_r(t_i)=1$

$$\begin{aligned}1 &= \int_{-\infty}^{\infty} \mathrm{d}X_f \rho_r(X_f,X_f,t_f)\\ &= K\int \mathrm{d}X_f \int \mathscr{D}\xi\delta(X(t_f)-X_f)\mathrm{e}^{-(1/2)\xi\cdot N^{-1}\cdot\xi}\int_{-\infty}^{\infty}\mathrm{d}X_i\int_{-\infty}^{\infty}\mathrm{d}\,p_i W_r(X_i,p_i,t_i)\\ &= K\int \mathscr{D}\xi\mathrm{e}^{-(1/2)\xi\cdot N^{-1}\cdot\xi}\int_{-\infty}^{\infty}\mathrm{d}X_i\int_{-\infty}^{\infty}\mathrm{d}\,p_i W_r(X_i,p_i,t_i)\,. \end{aligned} \tag{A.8}$$

Now, from Eq. (2.10) it can be checked that the normalization of the reduced density matrix $\mathrm{Tr}\,\rho_r(t_i)=1$ implies $\int_{-\infty}^{\infty}\mathrm{d}X_i\int_{-\infty}^{\infty}\mathrm{d}\,p_i W_r(X_i,p_i,t_i)=1$. The constant K is thus determined to be

$$K=\left[\int \mathscr{D}\xi\mathrm{e}^{-(1/2)\xi\cdot N^{-1}\cdot\xi}\right]^{-1}=[\det(2\pi N)]^{-1/2}\,. \tag{A.9}$$

Finally, using definition (2.10) for the Wigner function and the fact that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\Delta_f\mathrm{e}^{\mathrm{i}p_f\Delta_f}\mathrm{e}^{-\mathrm{i}M\dot{X}(t_f)\Delta_f}=\delta(M\dot{X}(t_f)-p_f)$$

we get an expression for the reduced Wigner function

$$\begin{aligned}W_r(X_f,p_f,t_f)=K\int_{-\infty}^{\infty}\mathrm{d}X_i\int_{-\infty}^{\infty}\mathrm{d}\,p_i\int\mathscr{D}\xi\delta(X(t_f)-X_f)\delta(M\dot{X}(t_f)-p_f)\\ \times\mathrm{e}^{-(1/2)\xi\cdot N^{-1}\cdot\xi}W_r(X_i,p_i,t_i)\,,\end{aligned} \tag{A.10}$$

which can be written as Eq. (3.1).

Appendix B. Derivation of the Fokker–Planck equation

The derivation of a Fokker–Planck equation from a Langevin equation with local dissipation is well understood, see Ref. [16]. However, in our case the existence of nonlocal dissipation makes it convenient to review the main steps. Let us begin by computing $\partial W_r/\partial t$ from expression (3.1)

$$\begin{aligned}\frac{\partial W_r(X,p,t)}{\partial t} &= \left\langle\left\langle\dot{X}(t)\delta'(X(t)-X)\delta(M\dot{X}(t)-p)\right\rangle_\xi\right\rangle_{X_i,p_i}\\ &\quad+\left\langle\left\langle\delta(X(t)-X)M\ddot{X}(t)\delta'(M\dot{X}(t)-p)\right\rangle_\xi\right\rangle_{X_i,p_i}\end{aligned}$$

$$= -\frac{p}{M}\frac{\partial W_r(X,p,t)}{\partial X} - \frac{\partial}{\partial p} \times \left\langle \left\langle \delta(X(t)-X)M\ddot{X}(t)\delta(M\dot{X}(t)-p)\right\rangle_{\xi}\right\rangle_{X_i,p_i}, \tag{B.1}$$

where the fact that $\dot{X}(t)$, $\partial/\partial X(t)$ and $\partial/\partial\dot{X}(t)$ may be replaced by p/M, $-\partial/\partial X$ and $-\partial/\partial p$, respectively, since they are multiplying the delta functions, was used in the second equality. Let us now concentrate on the expectation value appearing in the last term and recall the expectation values defined in (3.2)–(3.3). We will consider the Langevin-type equation

$$(L\cdot X)(t') = \xi(t'), \tag{B.2}$$

corresponding to the functional change (A.5) and substitute the corresponding expression for $M\ddot{X}(t)$ so that the last expectation value in (B.1) can be written as

$$-M\Omega_{ren}^2 X W_r(X,p,t) + \left\langle\left\langle\left(-\int_{t_i}^{t} \mathrm{d}tH(t,t')X(t') + \xi(t)\right) \times \delta(X(t)-X)\delta(M\dot{X}(t)-p)\right\rangle_{\xi}\right\rangle_{X_i,p_i}. \tag{B.3}$$

Any solution of Eq. (B.2) can be written as

$$X(t') = X_h(t') + \int_{t_i}^{t} \mathrm{d}t''\tilde{G}_{adv}(t',t'')\xi(t''), \tag{B.4}$$

where $X_h(t')$ is a solution of the homogeneous equation $(L\cdot X)(t')=0$ such that $X_h(t)=X$, $\dot{X}_h(t)=p/M$ and $\tilde{G}_{adv}(t',t'')$ is the advanced (i.e., $\tilde{G}_{adv}(t',t'')=0$ for $t'\geqslant t''$) Green function for the linear integro-differential operator associated to the kernel $L(t,t')$. The particular solution of the inhomogeneous Eq. (B.2) $\tilde{X}_i(t')=\int_{t_i}^{t}\mathrm{d}t''\tilde{G}_{adv}(t',t'')\xi(t'')$ has boundary conditions $\tilde{X}_i(t)=0$, $\partial\tilde{X}_i(t')/\partial t'|_{t'=t}=0$. Both $X_h(t')$ and $\tilde{G}_{adv}(t',t'')$ can be expressed in terms of the homogeneous solutions $u_1(t')$ and $u_2(t')$, which satisfy $u_1(t_i)=1$, $u_1(t)=0$ and $u_2(t_i)=0$, $u_2(t)=1$, respectively:

$$X_h(t') = X\left(u_2(t') - \frac{\dot{u}_2(t)}{\dot{u}_1(t)}u_1(t')\right) + \frac{(p/M)}{\dot{u}_1(t)}u_1(t'), \tag{B.5}$$

$$\tilde{G}_{adv}(t',t'') = -\frac{1}{M}\frac{u_1(t')u_2(t'') - u_2(t')u_1(t'')}{\dot{u}_1(t'')u_2(t'') - \dot{u}_2(t'')u_1(t'')}\theta(t''-t'). \tag{B.6}$$

We use the advanced propagator so that there is no dependence on the initial conditions at time $t'=t_i$ coming from the homogeneous solution but just on the final conditions at time $t'=t$, i.e., on those the Fokker–Planck equation is written in terms of. Using

expression (B.4) the first term within the expectation value appearing in Eq. (B.3) can be reexpressed as

$$\int_{t_i}^{t} \mathrm{d}tH(t,t')\left\langle\left\langle X(t')\delta(X(t)-X)\delta(M\dot{X}(t)-p)\right\rangle_{\xi}\right\rangle_{X_i,p_i}$$
$$= \int_{t_i}^{t} \mathrm{d}t' H(t,t')X_h(t')W_r(X,p,t)$$
$$+ \int_{t_i}^{t} \mathrm{d}t' \int_{t_i}^{t} \mathrm{d}t'' H(t,t')\tilde{G}_{adv}(t',t'')$$
$$\times \left\langle\left\langle \xi(t'')\delta(X(t)-X)\delta(M\dot{X}(t)-p)\right\rangle_{\xi}\right\rangle_{X_i,p_i} . \tag{B.7}$$

The first term on the right-hand side can in turn be written as

$$-(M\delta\Omega(t)X + 2A(t)p)W_r(X,p,t) , \tag{B.8}$$

where

$$\delta\Omega(t) = \frac{1}{M}\int_{t_i}^{t} \mathrm{d}t' H(t,t')[u_2(t') - (\dot{u}_2(t)/\dot{u}_1(t))u_1(t')] , \tag{B.9}$$

$$A(t) = \frac{1}{2}(M\dot{u}_1(t))^{-1}\int_{t_i}^{t} \mathrm{d}t' H(t,t')u_1(t') . \tag{B.10}$$

In order to find an expression for $\langle\xi(t')\delta(X(t)-X)\delta(M\dot{X}(t)-p)\rangle_{\xi}$ we use Novikov's formula for Gaussian stochastic processes [68], which corresponds essentially to use (3.2) and functionally integrate by parts with respect to $\xi(t)$,

$$\langle\xi(t')F(t;\xi]\rangle_{\xi} = \int_{t_i}^{t} \mathrm{d}t'' N(t',t'')\langle\delta F(t;\xi]/\delta\xi(t'')\rangle_{\xi} . \tag{B.11}$$

We then obtain the following expression:

$$\langle\xi(t')\delta(X(t)-X)\delta(M\dot{X}(t)-p)\rangle_{\xi}$$
$$= -\int_{t_i}^{t} \mathrm{d}t'' N(t',t'')\left\langle\left(\frac{\delta X(t)}{\delta\xi(t'')}\frac{\partial}{\partial X} + M\frac{\delta\dot{X}(t)}{\delta\xi(t'')}\frac{\partial}{\partial p}\right)\right.$$
$$\left.\times\,\delta(X(t)-X)\delta(M\dot{X}(t)-p)\right\rangle_{\xi} , \tag{B.12}$$

where we used again the presence of the delta functions to substitute the functional derivatives $\delta/\delta X(t''')$ and $\delta/\delta\dot{X}(t''')$ by $-\delta(t'''-t)\partial/\partial X$ and $-\delta(t'''-t)M\partial/\partial p$, respectively, in the second equality. Functionally differentiating with respect to $\xi(t'')$

expression (A.6) for $X(t)$ and analogously for $\dot{X}(t)$ we get

$$\frac{\delta X(t')}{\delta \xi(t'')} = G_{ret}(t', t'') \,, \tag{B.13a}$$

$$\frac{\delta \dot{X}(t')}{\delta \xi(t'')} = \frac{\partial}{\partial t'} G_{ret}(t', t'') \,, \tag{B.13b}$$

which after substitution into (B.12) leads to

$$\left\langle \left\langle \xi(t')\delta(X(t) - X)\delta(M\dot{X}(t) - p) \right\rangle_{\xi} \right\rangle_{X_i, p_i}$$
$$= -\int_{t_i}^{t} \mathrm{d}t'' N(t', t'') \left(G_{ret}(t, t'') \frac{\partial}{\partial X} + M \frac{\partial G_{ret}(t, t'')}{\partial t} \frac{\partial}{\partial p} \right) W_r(X, p, t) \,. \tag{B.14}$$

The retarded Green function can also be expressed in terms of the solutions of the homogeneous equation $u_1(t)$ and $u_2(t)$, which were previously introduced, as

$$G_{ret}(t', t'') = \frac{1}{M} \frac{u_1(t')u_2(t'') - u_2(t')u_1(t'')}{\dot{u}_1(t'')u_2(t'') - \dot{u}_2(t'')u_1(t'')} \theta(t' - t'') \,. \tag{B.15}$$

Note that it is important to use now the expression in terms of the retarded propagator G_{ret} and the initial conditions X_i and p_i (at time $t' = t_i$), since the "final" conditions $X(t)$ and $M\dot{X}(t)$ depend on $\xi(t'')$ (for $t'' < t$). Putting all the terms together, i.e., (B.3), (B.7) and (B.14), we reach the final expression for (B.1)

$$\frac{\partial W_r}{\partial t} = \{H_R, W_r\}_{PB} + 2A(t) \frac{\partial (pW_r)}{\partial p} + B(t) \frac{\partial^2 W_r}{\partial X \partial p} + MC(t) \frac{\partial^2 W_r}{\partial p^2} \,, \tag{B.16}$$

where the Poisson bracket is defined following Eq. (2.11) (with $\Omega_R = \Omega_{ren} + \delta\Omega$), $\delta\Omega(t)$ and $A(t)$ are given by Eqs. (B.9) and (B.10), and

$$B(t) = \int_{t_i}^{t} \mathrm{d}t''' N(t, t''') G_{ret}(t, t''')$$
$$- \int_{t_i}^{t} \mathrm{d}t' H(t, t') \int_{t_i}^{t} \mathrm{d}t'' \tilde{G}_{adv}(t', t'') \int_{t_i}^{t} \mathrm{d}t''' N(t'', t''') G_{ret}(t, t''') \,, \tag{B.17}$$

$$C(t) = \int_{t_i}^{t} \mathrm{d}t''' N(t, t''') \frac{\partial G_{ret}(t, t''')}{\partial t} - \int_{t_i}^{t} \mathrm{d}t' H(t, t')$$
$$\times \int_{t_i}^{t} \mathrm{d}t'' \tilde{G}_{adv}(t', t'') \int_{t_i}^{t} \mathrm{d}t''' N(t'', t''') \frac{\partial G_{ret}(t, t''')}{\partial t} \,. \tag{B.18}$$

The last two expressions were obtained by combining the second term within the expectation value appearing in (B.3) and the second term on the right-hand side of Eq. (B.7). It should be taken into account that if we put back the $\hbar$'s, there appears one with every noise kernel in Eqs. (B.17) and (B.18).

Appendix C. Correlation functions and nonlocal influence action

Let us see how the fact that the influence action is nonlocal implies that the propagator for the reduced density matrix does not factorize in time and, thus, the system evolution is non-Markovian. In this appendix we will denote the integrand of the real part of the influence action by $\mathscr{H} \equiv \Delta(t)H(t,t')X(t')$ and the integrand of the imaginary part by $\mathscr{N} \equiv \Delta(t)N(t,t')\Delta(t')/2$.

When the influence action is local $\mathscr{H}(t,t') \equiv \tilde{H}(t)\delta(t-t')$, $\mathscr{N}(t,t') \equiv \tilde{N}(t)\delta(t-t')$ and we have

$$S_{IF}[x,x';t_f,t_i) = \int_{t_i}^{t_f} \mathrm{d}t \int_{t_i}^{t_f} \mathrm{d}t' \mathscr{H} + \mathrm{i} \int_{t_i}^{t_f} \mathrm{d}t \int_{t_i}^{t_f} \mathrm{d}t' \mathscr{N}$$
$$= \int_{t_i}^{t_f} \mathrm{d}t\tilde{H} + i \int_{t_i}^{t_f} \mathrm{d}t\tilde{N} \,, \tag{C.1}$$

where we introduced the notation $S_{IF}[x,x';t_f,t_i)$, which is a functional of $x(t)$ and $x'(t)$ and also depends on the variables t_i and t_f, to explicitly state the initial and final times defining the dependence domain considered for the functions $x(t)$ and $x'(t)$, which will play an important role in the subsequent discussion. Expression (C.1) can then be decomposed as follows

$$S_{IF}[x,x';t_f,t_i) = \left(\int_{t_1}^{t_f} \mathrm{d}t\tilde{H} + \mathrm{i} \int_{t_1}^{t_f} \mathrm{d}t\tilde{N} \right) + \left(\int_{t_i}^{t_1} \mathrm{d}t\tilde{H} + \mathrm{i} \int_{t_i}^{t_1} \mathrm{d}t\tilde{N} \right)$$
$$= S_{IF}[x,x';t_f,t_1) + S_{IF}[x,x';t_1,t_i) \,, \tag{C.2}$$

so that the influence functional factorizes

$$F_{IF}[x,x';t_f,t_i) = \mathrm{e}^{\mathrm{i}S_{IF}[x,x';t_f,t_i)} = F_{IF}[x,x';t_f,t_1)F_{IF}[x,x';t_1,t_i) \tag{C.3}$$

and so does the reduced density matrix propagator, as can be straightforwardly seen from its path integral representation

$$J(x_f,x'_f,t_f;x_i,x'_i,t_i) = \int_{x(t_i)=x_i}^{x(t_f)=x_f} \mathscr{D}x \int_{x'(t_i)=x'_i}^{x'(t_f)=x'_f} \mathscr{D}x' \mathrm{e}^{\mathrm{i}(S[x]-S[x']+S_{IF}[x,x';t_f,t_i))}$$
$$= \int \mathrm{d}x_1 \, \mathrm{d}x'_1 \left(\int_{x(t_1)=x_1}^{x(t_f)=x_f} \mathscr{D}x \int_{x'(t_1)=x'_1}^{x'(t_f)=x'_f} \mathscr{D}x' \mathrm{e}^{\mathrm{i}(S[x]-S[x']+S_{IF}[x,x';t_f,t_1))} \right)$$

$$\times \left(\int_{x(t_i)=x_i}^{x(t_1)=x_1} \mathscr{D}x \int_{x'(t_i)=x_i'}^{x'(t_1)=x_1'} \mathscr{D}x' \mathrm{e}^{\mathrm{i}(S[x]-S[x']+S_{IF}[x,x';t_1,t_i))} \right)$$

$$= \int \mathrm{d}x_1\, \mathrm{d}x_1' J(x_f,x_f',t_f;x_1,x_1',t_1) J(x_1,x_1',t_1;x_i,x_i',t_i)\,, \tag{C.4}$$

where use was made of both the fact that the system action is local and (C.3) applied to definition (2.7) for the reduced density matrix propagator. This property allows one to obtain the quantum correlation functions for the system from the propagators of the reduced density matrix, which are solutions of the master equation. To illustrate this fact, consider as an example the quantum correlation function $\langle \hat{x}(t_2)\hat{x}(t_1)\rangle$ with $t_2 > t_1$, defined by

$$\mathrm{Tr}[\hat{x}(t_2)\hat{x}(t_1)\hat{\rho}(t_i)] = \int \mathrm{d}x_f \int^{x(t_f)=x_f} \mathscr{D}x \int^{x'(t_f)=x_f} \mathscr{D}x' x(t_2)x(t_1)$$

$$\times \mathrm{e}^{\mathrm{i}(S[x]-S[x']+S_{IF}[x,x';t_f,t_i))} \rho_r(x_i,x_i',t_i)$$

$$= \int \mathrm{d}x_f \int \mathrm{d}x_i\, \mathrm{d}x_i' \int \mathrm{d}x_2\, \mathrm{d}x_2' x_2 \int \mathrm{d}x_1\, \mathrm{d}x_1' x_1 J(x_f,x_f,t_f;x_2,x_2',t_2)$$

$$\times J(x_2,x_2',t_2;x_1,x_1',t_1) J(x_1,x_1',t_1;x_i,x_i',t_i) \rho_r(x_i,x_i',t_i)\,. \tag{C.5}$$

Here the path integrals in the intermediate steps were decomposed in a way completely analogous to that used in (C.4). Hence, the information on the correlation functions can be essentially obtained from the master equation when the influence action is local.

On the other hand, when the influence action is nonlocal

$$S_{IF}[x,x';t_f,t_i) = \int_{t_i}^{t_f} \mathrm{d}t \int_{t_i}^{t_f} \mathrm{d}t' \mathscr{H} + \mathrm{i} \int_{t_i}^{t_f} \mathrm{d}t \int_{t_i}^{t_f} \mathrm{d}t' \mathscr{N}$$

$$= \left(\int_{t_i}^{t_1} \mathrm{d}t \int_{t_i}^{t_1} \mathrm{d}t' \mathscr{H} + \int_{t_i}^{t_1} \mathrm{d}t \int_{t_1}^{t_f} \mathrm{d}t' \mathscr{H} \right.$$

$$\left. + \int_{t_1}^{t_f} \mathrm{d}t \int_{t_i}^{t_1} \mathrm{d}t' \mathscr{H} + \int_{t_1}^{t_f} \mathrm{d}t \int_{t_1}^{t_f} \mathrm{d}t' \mathscr{H} \right)$$

$$+ \mathrm{i} \left(\int_{t_i}^{t_1} \mathrm{d}t \int_{t_i}^{t_1} \mathrm{d}t' \mathscr{N} + \int_{t_i}^{t_1} \mathrm{d}t \int_{t_1}^{t_f} \mathrm{d}t' \mathscr{N} \right.$$

$$\left. + \int_{t_1}^{t_f} \mathrm{d}t \int_{t_i}^{t_1} \mathrm{d}t' \mathscr{N} + \int_{t_1}^{t_f} \mathrm{d}t \int_{t_1}^{t_f} \mathrm{d}t' \mathscr{N} \right)\,. \tag{C.6}$$

The cross terms like $\int_{t_i}^{t_1} \mathrm{d}t \int_{t_1}^{t_f} \mathrm{d}t' \mathscr{N}$ do not allow the influence action to be separated into terms that depend either on the "history" of the system just for times smaller than t_1 or just for times greater than t_1 (as happened in Eq. (C.2)). This fact makes it impossible to factorize the influence functional as was done in Eq. (C.3) and

consequently implies that neither the reduced density matrix propagators factorize in the sense of Eq. (C.4) nor the quantum correlation functions can be obtained from the reduced density matrix propagators as was done in Eq. (C.5). It is, thus, clear how the nonlocality of the influence action leads to a non-Markovian evolution for the system and the impossibility to obtain the correlation functions from the propagators for the reduced density matrix.

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