Convolution of n-dimensional Tempered Ultradistributions and Field Theory *

C.G.Bollini and M.C.Rocca

Departamento de Física, Fac. de Ciencias Exactas,

Universidad Nacional de La Plata.

C.C. 67 (1900) La Plata. Argentina.

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Abstract

In this work, a general definition of convolution between two arbitrary Tempered Ultradistributions is given. When one of the Tem-

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pered Ultradistributions is rapidly decreasing this definition coincides with the definition of J. Sebastiao e Silva. In the four-dimensional case, when the Tempered Ultradistributions are even in the variables k^0 and ρ (see Section 5) we obtain an expression for the convolution, which is more suitable for practical applications. The product of two arbitrary even (in the variables x^0 and r) four dimensional distributions of exponential type is defined via the convolution of its corresponding Fourier Transforms. With this definition of convolution , we treat the problem of singular products of Green Functions in Quantum Field Theory. (For Renormalizable as well as for Nonrenormalizable Theories). Several examples of convolution of two Tempered Ultradistributions are given. In particular we calculate the convolution of two massless Wheeeler's propagators and the convolution of two complex mass Wheeler's propagators.

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1 Introduction

The question of the product of distributions with coincident point singularities is related in Field Theory, to the asymptotic behavior of loop integrals of propagators.

From a mathematical point of view, practically all definitions lead to limitations on the set of distributions that can be multiplied together to give another distribution of the same kind.

The properties of ultradistributions (ref.[1, 2]) are well adapted for their use in Field Theory. In this respect we have shown (ref.[3]) that it is possible to define in one dimensional space, the convolution of any pair of tempered ultradistributions, giving as a result another tempered ultradistribution. The next step is to consider the convolution of any pair of tempered ultradistribution in n-dimensional space. As we shall see, this follows from the formula obtained in ref.[3] for one dimensional space.

However, the resultant formula is rather complex to be used in practical applications and calculus. Then, for applications, it is convenient to consider the convolution of any two tempered ultradistributions which are even in the variables k^0 y ρ (see section 5).

Ultradistributions also have the advantage of being representable by means of analytic functions. So that, in general, they are easier to work with them and, as we shall see, have interesting properties. One of those properties is that Schwartz tempered distributions are canonical and continuously injected into tempered ultradistributions and as a consequence the Rigged Hilbert Space with tempered distributions is canonical and continuously included in the Rigged Hilbert Space with tempered ultradistributions.

This paper is organized as follow: in sections 2 and 3 we define the Distributions of Exponential Type and the Fourier transformed Tempered Ultradistributions. Each of them is part of a Guelfand's Triplet (or Rigged Hilbert Space [4]) together with their respective duals and a "middle term" Hilbert space. In section 4 we give a general expression for the convolution of any pair of n-dimensional tempered ultradistributions and some simple examples. In section 5 we obtain the expression for the convolution of any pair of even tempered ultradistributions. In section 6, we evaluate the convolution of two massless Wheeler's propagators. In section 7 we evaluate the convolution of two complex mass Wheeler's propagators. Finally, section 8 is reserved for a discussion of the principal results. For the benefit of the reader an Appendix is added containing some formulas utilized in the text.

2 Distributions of Exponential Type

For the sake of the reader we shall present a brief description of the principal properties of Tempered Ultradistributions.

Notations. The notations are almost textually taken from ref[2]. Let $\mathbb{R}^n$ (res. $\mathbb{C}^n$) be the real (resp. complex) n-dimensional space whose points are denoted by $x = (x_1, x_2, ..., x_n)$ (resp $z = (z_1, z_2, ..., z_n)$). We shall use the notations:

(i) $x + y = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n)$; $\alpha x = (\alpha x_1, \alpha x_2, ..., \alpha x_n)$

(ii)$x \geqq 0$ means $x_1 \geqq 0, x_2 \geqq 0, ..., x_n \geqq 0$

(iii)$x \cdot y = \sum\limits_{j=1}^{n} x_j y_j$

(iV)$| x |= \sum\limits_{j=1}^{n} | x_j |$

Let $\mathbb{N}^n$ be the set of n-tuples of natural numbers. If $p \in \mathbb{N}^n$, then $p = (p_1, p_2, ..., p_n)$, and p_j is a natural number, $1 \leqq j \leqq n$. $p + q$ denote $(p_1 + q_1, p_2 + q_2, ..., p_n + q_n)$ and $p \geqq q$ means $p_1 \geqq q_1, p_2 \geqq q_2, ..., p_n \geqq q_n$. x^p means $x_1^{p_1} x_2^{p_2} ... x_n^{p_n}$. We shall denote by $| p |= \sum\limits_{j=1}^{n} p_j$ and by D^p we denote the differential operator $\partial^{p_1+p_2+...+p_n}/\partial {x_1}^{p_1} \partial {x_2}^{p_2} ... \partial {x_n}^{p_n}$

For any natural k we define $x^k = x_1^k x_2^k ... x_n^k$ and $\partial^k/\partial x^k = \partial^{nk}/\partial x_1^k \partial x_2^k ... \partial x_n^k$

The space $\boldsymbol{\mathcal{H}}$ of test functions such that $e^{p|x|}|D^q \phi(x)|$ is bounded for any p and q is defined (ref.[2]) by means of the countably set of norms:

$$\|\hat{\phi}\|_p = \sup_{0\leq q\leq p,\, x} e^{p|x|} \left|D^q\hat{\phi}(x)\right| \quad , \quad p = 0, 1, 2, ... \tag{2.1}$$

According to reference[5] $\boldsymbol{\mathcal{H}}$ is a $\boldsymbol{\mathcal{K}}\{\mathbf{M_p}\}$ space with:

$$M_p(x) = e^{(p-1)|x|} \quad , \quad p = 1, 2, ... \tag{2.2}$$

$\boldsymbol{\mathcal{K}}\{\boldsymbol{e}^{(\mathbf{p-1})|\mathbf{x}|}\}$ satisfies condition $(\mathcal{N})$ of Guelfand (ref.[4]). It is a countable Hilbert and nuclear space:

$$\boldsymbol{\mathcal{K}}\{\boldsymbol{e}^{(\mathbf{p-1})|\mathbf{x}|}\} = \boldsymbol{\mathcal{H}} = \bigcap_{p=1}^{\infty} \boldsymbol{\mathcal{H}}_{\mathbf{p}} \tag{2.3}$$

where $\boldsymbol{\mathcal{H}}_{\mathbf{p}}$ is obtained by completing $\boldsymbol{\mathcal{H}}$ with the norm induced by the scalar product:

$$< \hat{\phi}, \hat{\psi} >_p = \int_{-\infty}^{\infty} e^{2(p-1)|x|} \sum_{q=0}^{p} D^q\overline{\hat{\phi}}(x) D^q\hat{\psi}(x)\ dx \quad ; \quad p = 1, 2, ... \tag{2.4}$$

where $dx = dx_1\ dx_2 ... dx_n$

If we take the usual scalar product:

$$< \hat{\phi}, \hat{\psi} >= \int_{-\infty}^{\infty} \overline{\hat{\phi}}(x)\hat{\psi}(x)\ dx \tag{2.5}$$

then $\boldsymbol{\mathcal{H}}$, completed with (2.5), is the Hilbert space $\mathbf{H}$ of square integrable functions.

The space of continuous linear functionals defined on $\boldsymbol{\mathcal{H}}$ is the space $\boldsymbol{\Lambda}_\infty$ of the distributions of the exponential type (ref.[2]).

The "nested space"

$$\boldsymbol{\mathfrak{H}} = (\boldsymbol{\mathcal{H}}, \mathbf{H}, \boldsymbol{\Lambda}_\infty) \tag{2.6}$$

is a Guelfand's triplet (or a Rigged Hilbert space [4]).

In addition we have: $\boldsymbol{\mathcal{H}} \subset \boldsymbol{\mathcal{S}} \subset \mathbf{H} \subset \boldsymbol{\mathcal{S}}' \subset \boldsymbol{\Lambda}_\infty$, where $\boldsymbol{\mathcal{S}}$ is the Schwartz space of rapidly decreasing test functions (ref[6]).

Any Guelfand's triplet $\boldsymbol{\mathfrak{G}} = (\boldsymbol{\Phi}, \mathbf{H}, \boldsymbol{\Phi}')$ has the fundamental property that a linear and symmetric operator on $\boldsymbol{\Phi}$, admitting an extension to a self-adjoint operator in $\mathbf{H}$, has a complete set of generalized eigen-functions in $\boldsymbol{\Phi}'$ with real eigenvalues.

3 Tempered Ultradistributions

The Fourier transform of a function $\hat{\phi} \in \boldsymbol{\mathcal{H}}$ is

$$\phi(z) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \overline{\hat{\phi}}(x)\; e^{iz\cdot x}\; dx \tag{3.1}$$

$\phi(z)$ is entire analytic and rapidly decreasing on straight lines parallel to the real axis. We shall call $\mathfrak{H}$ the set of all such functions.

$$\mathfrak{H} = \mathcal{F}\{\boldsymbol{\mathcal{H}}\} \tag{3.2}$$

It is a $\mathcal{Z}\{\mathbf{M_p}\}$ space (ref.[5]), countably normed and complete, with:

$$M_p(z) = (1 + |z|)^p \tag{3.3}$$

$\mathfrak{H}$ is also a nuclear space with norms:

$$\|\phi\|_{pn} = \sup_{z \in V_n} (1 + |z|)^p |\phi(z)| \tag{3.4}$$

where $V_k = \{z = (z_1, z_2, ..., z_n) \in \mathbb{C}^n : | \, \mathrm{Im} z_j \, | \leqq k, 1 \leqq j \leqq n\}$

We can define the usual scalar product:

$$< \phi(z), \psi(z) >= \int\limits_{-\infty}^{\infty} \phi(z)\psi_1(z) \; dz = \int\limits_{-\infty}^{\infty} \overline{\hat{\phi}}(x)\hat{\psi}(x) \; dx \tag{3.5}$$

where:

$$\psi_1(z) = \int\limits_{-\infty}^{\infty} \hat{\psi}(x) \; e^{-iz \cdot x} \; dx$$

and $dz = dz_1 \; dz_2 ... dz_n$

By completing $\mathfrak{H}$ with the norm induced by (3.5) we get the Hilbert space of square integrable functions.

The dual of $\mathfrak{H}$ is the space $\boldsymbol{\mathcal{U}}$ of tempered ultradistributions (ref.[2]). In other words, a tempered ultradistribution is a continuous linear functional defined on the space $\mathfrak{H}$ of entire functions rapidly decreasing on straight lines parallel to the real axis.

The set $\mathfrak{U} = (\mathfrak{H}, \mathrm{H}, \boldsymbol{\mathcal{U}})$ is also a Guelfand's triplet.

Moreover, we have: $\mathfrak{H} \subset \boldsymbol{\mathcal{S}} \subset \mathsf{H} \subset \boldsymbol{\mathcal{S}}' \subset \boldsymbol{\mathcal{U}}$.

$\boldsymbol{\mathcal{U}}$ can also be characterized in the following way (ref.[2]): let $\boldsymbol{\mathcal{A}}_\omega$ be the space of all functions $F(z)$ such that:

I- $F(z)$ is analytic for $\{z \in \mathbb{C}^n : |Im(z_1)| > p, |Im(z_2)| > p, ..., |Im(z_n)| > p\}$.

II- $F(z)/z^p$ is bounded continuous in $\{z \in \mathbb{C}^n : |Im(z_1)| \geqq p, |Im(z_2)| \geqq p, ..., |Im(z_n)| \geqq p\}$, where $p = 0, 1, 2, ...$ depends on $F(z)$.

Let Π be the set of all z-dependent pseudo-polynomials, $z \in \mathbb{C}^n$. Then $\boldsymbol{\mathcal{U}}$ is the quotient space:

III- $\boldsymbol{\mathcal{U}} = \boldsymbol{\mathcal{A}}_\omega / \Pi$

By a pseudo-polynomial we understand a function of z of the form $\sum_s z_j^s G(z_1, ..., z_{j-1}, z_{j+1}, ..., z_n)$ with $G(z_1, ..., z_{j-1}, z_{j+1}, ..., z_n) \in \boldsymbol{\mathcal{A}}_\omega$

Due to these properties it is possible to represent any ultradistribution as (ref.[2]):

$$F(\phi) = < F(z), \phi(z) > = \oint_\Gamma F(z)\phi(z)\, dz \tag{3.6}$$

$\Gamma = \Gamma_1 \cup \Gamma_2 \cup ...\Gamma_n$ where the path Γ_j runs parallel to the real axis from $-\infty$ to ∞ for $Im(z_j) > \zeta$, $\zeta > p$ and back from ∞ to $-\infty$ for $Im(z_j) < -\zeta$, $-\zeta < -p$. (Γ surrounds all the singularities of $F(z)$).

Formula (3.6) will be our fundamental representation for a tempered ultradistribution. Sometimes use will be made of "Dirac formula" for ultradistributions (ref.[1]):

$$F(z) = \frac{1}{(2\pi i)^n} \int_{-\infty}^{\infty} \frac{f(t)}{(t_1 - z_1)(t_2 - z_2)...(t_n - z_n)} \, dt \tag{3.7}$$

where the "density" $f(t)$ is such that

$$\oint_{\Gamma} F(z)\phi(z) \, dz = \int_{-\infty}^{\infty} f(t)\phi(t) \, dt \tag{3.8}$$

While $F(z)$ is analytic on Γ, the density $f(t)$ is in general singular, so that the r.h.s. of (3.8) should be interpreted in the sense of distribution theory.

Another important property of the analytic representation is the fact that on Γ, $F(z)$ is bounded by a power of z (ref.[2]):

$$|F(z)| \leq C|z|^p \tag{3.9}$$

where C and p depend on F.

The representation (3.6) implies that the addition of a pseudo-polynomial $P(z)$ to $F(z)$ do not alter the ultradistribution:

$$\oint_{\Gamma} \{F(z) + P(z)\}\phi(z) \, dz = \oint_{\Gamma} F(z)\phi(z) \, dz + \oint_{\Gamma} P(z)\phi(z) \, dz$$

But:

$$\oint_{\Gamma} P(z)\phi(z) \, dz = 0$$

as $P(z)\phi(z)$ is entire analytic in some of the variables z_j (and rapidly decreasing),

$$\therefore \quad \oint\limits_{\Gamma} \{F(z)+P(z)\}\phi(z)\ dz = \oint\limits_{\Gamma} F(z)\phi(z)\ dz \qquad (3.10)$$

4 The Convolution

In ref.[3] we have defined and shown the existence of the convolution product between to arbitrary one dimensional tempered ultradistributions.

We now define:

$$H_\lambda(k) = \frac{i}{(2\pi)^n} \oint\limits_{\Gamma_1} \oint\limits_{\Gamma_2} \frac{k_1^\lambda F(k_1) k_2^\lambda G(k_2)}{k-k_1-k_2}\ dk_1\ dk_2 \qquad (4.1)$$

($k - k_1 - k_2 = \prod_{i=1}^{n}(k_i - k_{1i} - k_{2i})$). Let $\mathfrak{b}_i$ be a vertical band contained in the λ_i-plane $\mathfrak{p}_i$ Integral (4.1) is an analytic function of λ defined in a domain $\mathfrak{B}$ given by the cartesian product of vertical bands $\prod \mathfrak{b}_i$ contained in the cartesian product $\mathfrak{P} = \prod \mathfrak{p}_i$ of the n λ-planes. Moreover, it is bounded by a power of $|k|$. Then, according to the method of ref.[7], H_λ can be analytically continued to other parts of $\mathfrak{P}$. In particular near the origin we have the Laurent expansion:

$$H_\lambda(k) = \sum_n H^{(n)}(k)\lambda^n \qquad (4.2)$$

We now define the convolution product as the λ-independent term of (4.2):

$$H(k) = H^{(0)}(k) \tag{4.3}$$

The proof that $H^{(0)}(k)$ is a Tempered Ultradistribution is similar to the one given in ref.[3] for the one-dimensional case. For an immediate application of (4.1-4.3) we can evaluate the product of two arbitrary derivatives of a n-dimensional δ distribution. By calculating the convolution product of the Fourier transforms of $\delta^{(m)}(x)$ an $\delta^{(n)}(x)$, and then antitransforming, we can show that:

$$\delta^{(m)}(x) \cdot \delta^{(n)}(x) = 0 \tag{4.4}$$

extending the result obtained in ref.[3] for the one-dimensional case.

Likewise, we can obtain:

$$(x_{1+}^{\alpha_1}\ x_{2+}^{\alpha_2} ... x_{n+}^{\alpha_n}) \cdot (x_{1+}^{\beta_1}\ x_{2+}^{\beta_2} ... x_{n+}^{\beta_n}) = (x_{1+}^{\alpha_1+\beta_1}\ \ x_{2+}^{\alpha_2+\beta_2} ... x_{n+}^{\alpha_n+\beta_n}) \tag{4.5}$$

generalizing again the result of ref.[3].

As another example let us consider the product $(x^{-n_1}y^{-m_1}) \cdot (x^{-n_2}y^{-m_2})$

We have

$$\mathcal{F}\{(x^{-n_1}y^{-m_1}) \cdot (x^{-n_2}y^{-m_2})\} = \frac{(-i)^{n_1+n_2}}{(n_1+n_2-1)!} z_1^{n_1+n_2-1} \left[\frac{i}{4}\frac{z_1^{2\lambda_1}}{\lambda_1} + \frac{i}{2}\ln(z_1) + \right.$$

$$\left. \frac{\pi}{2} Sgn[\Im(z_1)]\right] \frac{(-i)^{m_1+m_2}}{(m_1+m_2-1)!} z_2^{m_1+m_2-1} \left[\frac{i}{4}\frac{z_2^{2\lambda_2}}{\lambda_2} + \frac{i}{2}\ln(z_2) + \frac{\pi}{2} Sgn[\Im(z_2)]\right] =$$

$$\frac{(-i)^{n_1+n_2}}{(n_1+n_2-1)!} z_1^{n_1+n_2-1} \left[\frac{i}{4\lambda_1}\left[1+2\lambda_1 \ln(z_1)\right] + \frac{i}{2}\ln(z_1) + \frac{\pi}{2}\mathrm{Sgn}[\Im(z_1)]\right] \times$$
$$\frac{(-i)^{m_1+m_2}}{(m_1+m_2-1)!} z_2^{m_1+m_2-1} \left[\frac{i}{4\lambda_2}\left[1+2\lambda_2 \ln(z_2)\right] + \frac{i}{2}\ln(z_2) + \frac{\pi}{2}\mathrm{Sgn}[\Im(z_2)]\right] \quad (4.6)$$

The $(\lambda_1;\lambda_2)$-independent term is:

$$\frac{(-i)^{n_1+n_2}\pi}{(n_1+n_2-1)!} z_1^{n_1+n_2-1} \left[\frac{1}{\pi i}\ln(z_1) - \frac{\pi}{2}\mathrm{Sgn}[\Im(z_1)]\right] \times$$
$$\frac{(-i)^{m_1+m_2}}{(m_1+m_2-1)!} z_2^{m_1+m_2-1} \left[\frac{1}{\pi i}\ln(z_2) - \frac{\pi}{2}\mathrm{Sgn}[\Im(z_2)]\right] \quad (4.7)$$

and it is recognized to be $\mathcal{F}\{x^{-n_1-n_2}y^{-m_1-m_2}\}$

5 The Convolution of even four-dimensional Tempered Ultradistributions

We pass now to consider the convolution of two even tempered ultradistributions.

The Fourier transform of a distribution of exponential type, even in the variables x^0 and $|\vec{x}|$ is by definition a even tempered ultradistribution in the variables k^0 and $\rho = (k_1^2+k_2^2+\cdots+k_n^2)^{1/2}$ Taking into account the equality:

$$\int\limits_{-\infty}^{+\infty} \hat{f}(x)\hat{\phi}(x)\ dx = \oint\limits_{\Gamma} F(k)\phi(k)\ dk = \int\limits_{-\infty}^{+\infty} f(k)\phi(k)\ dk \quad (5.1)$$

(where $F(k)$ and $f(k)$ are related by (3.7)) we conclude that $f(k)$ is even in k^0 and ρ.

For most practical applications one has to deal with the convolution of two Lorentz invariant ultradistributions. They are particular cases of ultradistributions which are even in two relevant variables: one temporal and the other the spacial distance (The even ultradistributions).

Let as now consider $\hat{f} \in \mathbf{H}$ even. Then we can write:

$$\hat{f}(x_0, r) = \frac{i}{(2\pi)^3 r} \iint_{-\infty}^{+\infty} f(k_0, \rho) e^{-ik^0 x^0} e^{-i\rho r} \rho \, d\rho dk^0 \tag{5.2}$$

$$f(k_0, \rho) = -\frac{2\pi i}{\rho} \iint_{-\infty}^{+\infty} \hat{f}(x_0, r) e^{ik^0 x^0} e^{i\rho r} r \, dr dx^0 \tag{5.3}$$

Let as now take $\hat{g} \in \mathbf{H}$. Then according to (5.2):

$$\hat{f}(x)\hat{g}(x) = -\frac{1}{(2\pi)^6 r^2} \iiiint_{-\infty}^{+\infty} f(k_1^0, \rho_1) g(k_2^0, \rho_2) e^{-i(k_1^0 + k_2^0)x^0} e^{-i(\rho_1 + \rho_2)r} \times$$
$$\times \rho_1 \rho_2 \, d\rho_1 \, d\rho_2 \, dk_1^0 \, dk_2^0 \tag{5.4}$$

and Fourier transforming (5.4)

$$\mathcal{F}\{\hat{f}(x)\hat{g}(x)\}(k) = \frac{i}{(2\pi)^5 \rho} \int_{-\infty}^{+\infty} \cdots \int f(k_1^0, \rho_1) g(k_2^0, \rho_2) e^{i(k^0 - k_1^0 - k_2^0)x^0} e^{i(\rho - \rho_1 - \rho_2)r} \times$$
$$\times \rho_1 \rho_2 \, d\rho_1 \, d\rho_2 \, dk_1^0 \, dk_2^0 \, r^{-1} dr \, dx^0 \tag{5.5}$$

Evaluating the integral in the variable x^0 and calling $h(k^0, \rho) = \mathcal{F}\{\hat{f}(x)\hat{g}(x)\}(k)$ in (5.5) we obtain

$$h(k^0, \rho) = i \int_{-\infty}^{+\infty} \cdots \int f(k_1^0, \rho_1) g(k_2^0, \rho_2) \delta(k^0 - k_1^0 - k_2^0) \frac{e^{i(\rho - \rho_1 - \rho_2) r}}{\rho} \times$$

$$\times \rho_1 \rho_2 \, d\rho_1 \, d\rho_2 \, dk_1^0 \, dk_2^0 \, r^{-1} dr \tag{5.6}$$

We want now to extend $h(k^0, \rho)$ to the complex plane as a tempered ultradistribution. For this we can use for example, formula (3.7). First we consider the term

$$\frac{e^{i(\rho - \rho_1 - \rho_2) r}}{\rho} \tag{5.7}$$

The extension to the complex plane is:

$$\{\Theta(r) \, \Theta[\Im(\rho)] - \Theta(-r) \, \Theta[-\Im(\rho)]\} \, \frac{e^{i(\rho - \rho_1 - \rho_2) r}}{\rho} \tag{5.8}$$

where Θ is the Heaviside's step function and $\Im$ denotes "Imaginary part".

On the other hand the extension of

$$\delta(k^0 - k_1^0 - k_2^0) \tag{5.9}$$

is

$$-\frac{1}{2\pi i (k^0 - k_1^0 - k_2^0)} \tag{5.10}$$

Replacing [(5.8),(5.10)] in (5.6) and then integrating out the variable r we obtain:

$$H(k^0,\rho) = \frac{1}{2\pi\rho}\iiiint\limits_{-\infty}^{+\infty} \frac{f(k_1^0,\rho_1)g(k_2^0,\rho_2)}{k^0-k_1^0-k_2^0}\{\Theta[\Im(\rho)]\ln(\rho_1+\rho_2-\rho)+\Theta[-\Im(\rho)] \times$$

$$\ln(\rho-\rho_1-\rho_2)\}\,\rho_1\rho_2\,d\rho_1\,d\rho_2\,dk_1^0\,dk_2^0 \qquad (5.11)$$

where $H(k^0,\rho)$ is the extension of $f(k^0,\rho)$. Taking into account that $f(k_1^0,\rho_1)$ and $g(k_2^0,\rho_2)$ are even functions in the first and second variables (5.11) takes the form:

$$H(k^0,\rho) = \frac{1}{4\pi\rho}\iiiint\limits_{-\infty}^{+\infty} \frac{f(k_1^0,\rho_1)g(k_2^0,\rho_2)}{k^0-k_1^0-k_2^0}\ln[\rho^2-(\rho_1+\rho_2)^2] \times$$

$$\rho_1\rho_2\,d\rho_1\,d\rho_2\,dk_1^0\,dk_2^0 \qquad (5.12)$$

The expression (5.12) for $H(k^0,\rho)$ can be re-writted in the form

$$H(k^0,\rho) = \frac{1}{4\pi\rho}\oint\limits_{\Gamma_1^0}\oint\limits_{\Gamma_2^0}\oint\limits_{\Gamma_1}\oint\limits_{\Gamma_2} \frac{F(k_1^0,\rho_1)G(k_2^0,\rho_2)}{k^0-k_1^0-k_2^0}\ln[\rho^2-(\rho_1+\rho_2)^2] \times$$

$$\rho_1\rho_2\,d\rho_1\,d\rho_2\,dk_1^0\,dk_2^0 \qquad (5.13)$$

where $F(k_1^0,\rho_1)$ and $G(k_2^0,\rho_2)$ are respectively, the extensions of $f(k_1^0,\rho_1)$ and $g(k_2^0,\rho_2)$ and where we have taken: $|\Im(k^0)| > |\Im(k_1^0)| + |\Im(k_2^0)|$, $|\Im(\rho)| > |\Im(\rho_1)| + |\Im(\rho_2)|$. In addition Γ_1^0, Γ_2^0,Γ_1 and Γ_2 are respectively, paths (as we have described in section 3), in the variables k_1^0, k_2^0, ρ_1 and ρ_2, enclosing all the singularities of the integrand in (5.13). The difference between

$$\int \frac{2\rho}{\rho^2-(\rho_1+\rho_2)^2}\, d\rho \qquad \text{and} \qquad \ln[\rho^2-(\rho_1+\rho_2)^2]$$

is an entire analytic function. With this substitution in (5.13) we obtain

$$H(k^0,\rho) = \frac{1}{2\pi\rho}\int \rho\, d\rho \oint\limits_{\Gamma_1^0}\oint\limits_{\Gamma_2^0}\oint\limits_{\Gamma_1}\oint\limits_{\Gamma_2} \frac{F(k_1^0,\rho_1)G(k_2^0,\rho_2)}{k^0-k_1^0-k_2^0}\frac{1}{\rho^2-(\rho_1+\rho_2)^2}\times$$
$$\rho_1\rho_2\, d\rho_1\, d\rho_2\, dk_1^0\, dk_2^0 \tag{5.14}$$

Now we can use the method of ref.[3] to define the convolution for the case in which $F(k_1^0,\rho_1)$ and $G(k_2^0,\rho_2)$ are tempered ultradistributions. We define:

$$H_{\lambda_0\lambda}(k^0,\rho) = \frac{1}{2\pi\rho}\int \rho\, d\rho \oint\limits_{\Gamma_1^0}\oint\limits_{\Gamma_2^0}\oint\limits_{\Gamma_1}\oint\limits_{\Gamma_2} \frac{{k_1^0}^{\lambda_0}\rho_1^{\lambda+1}F(k_1^0,\rho_1){k_2^0}^{\lambda_0}\rho_2^{\lambda+1}G(k_2^0,\rho_2)}{k^0-k_1^0-k_2^0}\times$$
$$\frac{1}{\rho^2-(\rho_1+\rho_2)^2}\, d\rho_1\, d\rho_2\, dk_1^0\, dk_2^0 \tag{5.15}$$

Integral (5.15) is an analytic function of (λ_0,λ) bounded by a power of $|k|$ and defined in a domain $\mathfrak{B}$ given by the cartesian product of a vertical band $\mathfrak{b}_0$ contained in the λ_0-plane and vertical band $\mathfrak{b}$ contained in the λ-plane. We can again extend this domain using the method given in ref.[7] and perform the Laurent expansion :

$$H_{\lambda_0\lambda}(k^0,\rho) = \sum_{mn} H^{(m,n)}(k^0,\rho)\lambda_0^m\lambda^n \tag{5.16}$$

We define the convolution product as the (λ_0, λ)- independent term of (5.16).

$$H(k) = H(k^0, \rho) = H^{(0,0)}(k^0, \rho) \tag{5.17}$$

The proof that $H(k)$ is an ultradistribution is similar to the one given in ref.[3] for the one-dimensional case.

To simplify the evaluation of (5.15) we define:

$$L_{\lambda_0 \lambda}(k^0, \rho) = \oint\limits_{\Gamma_1^0} \oint\limits_{\Gamma_2^0} \oint\limits_{\Gamma_1} \oint\limits_{\Gamma_2} \frac{{k_1^0}^{\lambda_0} \rho_1^{\lambda+1} F(k_1^0, \rho_1) {k_2^0}^{\lambda_0} \rho_2^{\lambda+1} G(k_2^0, \rho_2)}{k^0 - k_1^0 - k_2^0} \times$$

$$\frac{1}{\rho^2 - (\rho_1 + \rho_2)^2} \, d\rho_1 \, d\rho_2 \, dk_1^0 \, dk_2^0 \tag{5.18}$$

so that

$$H_{\lambda_0 \lambda}(k^0, \rho) = \frac{1}{2\pi\rho} \int L_{\lambda_0 \lambda}(k^0, \rho) \, \rho \, d\rho \tag{5.19}$$

Now we go to show that the cut on the real axis of (5.17) $h_{\lambda_0 \lambda}(k^0, \rho)$ is a even function of k^0 and ρ. For this purpose we consider

$$H_{\lambda_0 \lambda}(k^0, \rho) = \frac{1}{4\pi\rho} \oint\limits_{\Gamma_1^0} \oint\limits_{\Gamma_2^0} \oint\limits_{\Gamma_1} \oint\limits_{\Gamma_2} \frac{{k_1^0}^{\lambda_0} \rho_1^{\lambda+1} F(k_1^0, \rho_1) {k_2^0}^{\lambda_0} \rho_2^{\lambda+1} G(k_2^0, \rho_2)}{k^0 - k_1^0 - k_2^0} \times$$

$$\ln[\rho^2 - (\rho_1 + \rho_2)^2] \, d\rho_1 \, d\rho_2 \, dk_1^0 \, dk_2^0 \tag{5.20}$$

(5.20) is explicitly odd in ρ. For the variable k^0 we take on account that $e^{i\pi\lambda_0\{Sgn[\Im(k_1^0)] + Sgn[\Im(k_2^0)]\}} = 1$ and as a consequence (5.20) is odd in k^0 too. We consider now the parity in variable ρ.

$$\oint\limits_{\Gamma_0}\oint\limits_{\Gamma} H_{\lambda_0\lambda}(k^0,-\rho)\phi(k^0,\rho)\;dk^0\;d\rho = -\iint\limits_{-\infty}^{+\infty} h_{\lambda_0\lambda}(k^0,-\rho)\phi(k^0,\rho)\;dk^0\;d\rho =$$

$$-\oint\limits_{\Gamma_0}\oint\limits_{\Gamma} H_{\lambda_0\lambda}(k^0,\rho)\phi(k^0,\rho)\;dk^0\;d\rho = -\iint\limits_{-\infty}^{+\infty} h_{\lambda_0\lambda}(k^0,\rho)\phi(k^0,\rho)\;dk^0\;d\rho \quad (5.21)$$

Thus we have

$$h_{\lambda_0\lambda}(k^0,-\rho) = h_{\lambda_0\lambda}(k^0,\rho) \quad (5.22)$$

The proof for the variable k^0 is similar.

6 The Convolution of two massless Wheeler's Propagators

The massless Wheeler's propagator w_0 is given by:

$$w_0(k) = \frac{i}{k_0^2-\rho^2} \quad (6.1)$$

It can be extended to the complex plane as a tempered ultradistribution in the variables k^0 and ρ:

$$W_0(k) = -i\frac{Sgn\Im(k^0)}{8k^0}\left[\frac{Sgn\Im(\rho)-Sgn\Im(k^0)}{\rho-k^0} - \frac{Sgn\Im(\rho)+Sgn\Im(k^0)}{\rho+k^0}\right] \quad (6.2)$$

where $Sgn(x)$ is the function sign of the variable x.

We can now evaluate the convolution of two massless Wheeler's propagators. Then according to (5.18) and (6.2) we can write:

$$L_{\lambda_0\lambda}(k^0,\rho) = -\oint_{\Gamma_1^0}\oint_{\Gamma_2^0}\oint_{\Gamma_1}\oint_{\Gamma_2} \frac{Sgn\Im(k_1^0)}{8k_1^0}\left[\frac{Sgn\Im(\rho_1)-Sgn\Im(k_1^0)}{\rho_1-k_1^0} - \frac{Sgn\Im(\rho_1)+Sgn\Im(k_1^0)}{\rho_1+k_1^0}\right]$$

$$\frac{Sgn\Im(k_2^0)}{8k_2^0}\left[\frac{Sgn\Im(\rho_2)-Sgn\Im(k_2^0)}{\rho_2-k_2^0} - \frac{Sgn\Im(\rho_2)+Sgn\Im(k_2^0)}{\rho_2+k_2^0}\right]\times$$

$$\frac{{k_1^0}^{\lambda_0}\rho_1^{\lambda+1}{k_2^0}^{\lambda_0}\rho_2^{\lambda+1}}{(k^0-k_1^0-k_2^0)[\rho^2-(\rho_1+\rho_2)^2]}\, d\rho_1\, d\rho_2\, dk_1^0\, dk_2^0 \tag{6.3}$$

equation (6.3) can be written as:

$$L_{\lambda_0\lambda}(k^0,\rho) = -\oint_{\Gamma_1^0}\oint_{\Gamma_2^0}\iint_{-\infty}^{+\infty}\left\{\frac{Sgn\Im(k_1^0)}{8\rho_1}\left[\frac{1}{k_1^0-\rho_1}-\frac{1}{k_1^0+\rho_1}\right]\times\right.$$

$$\left[(\rho_1+i0)^{\lambda+1}+(\rho_1-i0)^{\lambda+1}\right]+\frac{1}{8k_1^0}\left[\frac{1}{k_1^0+\rho_1}-\frac{1}{k_1^0-\rho_1}\right]\times$$

$$\left.\left[(\rho_1+i0)^{\lambda+1}-(\rho_1-i0)^{\lambda+1}\right]\right\}\left\{\frac{Sgn\Im(k_2^0)}{8\rho_2}\left[\frac{1}{k_2^0-\rho_2}-\frac{1}{k_2^0+\rho_2}\right]\times\right.$$

$$\left[(\rho_2+i0)^{\lambda+1}+(\rho_2-i0)^{\lambda+1}\right]+\frac{1}{8k_2^0}\left[\frac{1}{k_2^0+\rho_2}-\frac{1}{k_2^0-\rho_2}\right]\times$$

$$\left.\left[(\rho_2+i0)^{\lambda+1}-(\rho_2-i0)^{\lambda+1}\right]\right\}\frac{{k_1^0}^{\lambda_0}{k_2^0}^{\lambda_0}\, d\rho_1\, d\rho_2\, dk_1^0\, dk_2^0}{(k^0-k_1^0-k_2^0)[\rho^2-(\rho_1+\rho_2)^2]} \tag{6.4}$$

Integrating (6.4) in the variable k_1^0 we obtain

$$L_\lambda(k^0,\rho) = -\oint_{\Gamma_2^0}\iint_{-\infty}^{+\infty}\left\{\frac{i\pi}{4\rho_1}Sgn\Im(k^0)\left[\frac{1}{k_2^0-(k^0-\rho_1)}-\frac{1}{k_2^0-(k^0+\rho_1}\right]\times\right.$$

$$\left[(\rho_1+i0)^{\lambda+1}+(\rho_1-i0)^{\lambda+1}\right]+\frac{i\pi}{4\rho_1}\left[\frac{2}{k_2^0-k^0}-\frac{1}{k_2^0-(k^0-\rho_1)}-\frac{1}{k_2^0-(k^0-\rho_1)}\right]\times$$

$$\left[(\rho_1+i0)^{\lambda+1}-(\rho_1-i0)^{\lambda+1}\right]\Big\}\ \left\{\frac{Sgn\Im(k_2^0)}{8\rho_2}\left[\frac{1}{k_2^0-\rho_2}-\frac{1}{k_2^0+\rho_2}\right]\times\right.$$

$$\left[(\rho_2+i0)^{\lambda+1}+(\rho_2-i0)^{\lambda+1}\right]+\frac{1}{8k_2^0}\left[\frac{1}{k_2^0+\rho_2}-\frac{1}{k_2^0-\rho_2}\right]\times$$

$$\left.\left[(\rho_2+i0)^{\lambda+1}-(\rho_2-i0)^{\lambda+1}\right]\right\}\frac{d\rho_1\ d\rho_2\ dk_2^0}{\rho^2-(\rho_1+\rho_2)^2} \tag{6.5}$$

where we have selected $\lambda_0=0$ due to the fact the integral is convergent for $\lambda_0=0$.

There have a sole term in (6.5) whose integral is not null. It is:

$$L_\lambda(k^0,\rho)=-\oint_{\Gamma_2^0}\iint_{-\infty}^{+\infty}\frac{i\pi}{4\rho_1}Sgn\Im(k^0)\left[\frac{1}{k_2^0-(k^0-\rho_1)}-\frac{1}{k_2^0-(k^0+\rho_1)}\right]\times$$

$$\left[(\rho_1+i0)^{\lambda+1}+(\rho_1-i0)^{\lambda+1}\right]\frac{Sgn\Im(k_2^0)}{8\rho_2}\left[\frac{1}{k_2^0-\rho_2}-\frac{1}{k_2^0+\rho_2}\right]\times$$

$$\left[(\rho_2+i0)^{\lambda+1}+(\rho_2-i0)^{\lambda+1}\right]\frac{d\rho_1\ d\rho_2\ dk_2^0}{\rho^2-(\rho_1+\rho_2)^2} \tag{6.6}$$

Evaluation of (6.6) gives:

$$L_\lambda(k^0,\rho)=\frac{\pi^2k^0}{2}\iint_{-\infty}^{+\infty}\left[(\rho_1+i0)^{\lambda+1}+(\rho_1-i0)^{\lambda+1}\right]\left[(\rho_2+i0)^{\lambda+1}+(\rho_2-i0)^{\lambda+1}\right]$$

$$\frac{d\rho_1\ d\rho_2}{\left[(k_0^2+\rho_1{}^2-\rho_2{}^2)^2-4k_0^2\rho_1{}^2\right]\left[\rho^2-(\rho_1+\rho_2)^2\right]} \tag{6.7}$$

We can evaluate now the integral in the variable ρ_2 in (6.7). The result is:

$$L_\lambda(k^0,\rho)=\frac{\pi^3}{16\rho}\frac{(1+\cos\pi\lambda)^2}{\sin\frac{\pi(\lambda+1)}{2}}\int_0^\infty d\rho_1\ \ \rho_1^\lambda\times$$

$$\left\{ \frac{e^{-\frac{i\pi}{2}(\lambda+1)\mathrm{Sgn}\Im(k^0)}(k^0+\rho_1)^{\lambda+1} - e^{-\frac{i\pi}{2}(\lambda+1)\mathrm{Sgn}\Im(\rho)}(\rho+\rho_1)^{\lambda+1}}{(\rho-k^0)\left(\frac{\rho+k^0}{2}+\rho_1\right)} - \right.$$

$$\frac{e^{-\frac{i\pi}{2}(\lambda+1)\mathrm{Sgn}\Im(k^0)}(k^0+\rho_1)^{\lambda+1} - e^{\frac{i\pi}{2}(\lambda+1)\mathrm{Sgn}\Im(\rho)}(\rho_1-\rho)^{\lambda+1}}{(\rho+k^0)\left(\frac{\rho-k^0}{2}-\rho_1\right)} -$$

$$\frac{e^{\frac{i\pi}{2}(\lambda+1)\mathrm{Sgn}\Im(k^0)}(\rho_1-k^0)^{\lambda+1} - e^{-\frac{i\pi}{2}(\lambda+1)\mathrm{Sgn}\Im(\rho)}(\rho_1+\rho)^{\lambda+1}}{(\rho+k^0)\left(\frac{\rho-k^0}{2}+\rho_1\right)} +$$

$$\left. \frac{e^{\frac{i\pi}{2}(\lambda+1)\mathrm{Sgn}\Im(k^0)}(\rho_1-k^0)^{\lambda+1} - e^{\frac{i\pi}{2}(\lambda+1)\mathrm{Sgn}\Im(\rho)}(\rho_1-\rho)^{\lambda+1}}{(\rho-k^0)\left(\frac{\rho+k^0}{2}-\rho_1\right)} \right\} \tag{6.8}$$

The evaluation of (6.8) is tedious task. Fortunately lim $\lambda \to 0$ can be taken without problem in the finals steps of the calculation. The result is:

$$L(k^0,\rho) = \frac{\pi^3}{4\rho}\left[\frac{\pi}{2}\mathrm{Sgn}\Im(k^0)\mathrm{Sgn}\Im(k^0+\rho) + \frac{\pi}{2}\mathrm{Sgn}\Im(\rho)\mathrm{Sgn}\Im(k^0+\rho) \ + \right.$$

$$\left. \frac{\pi}{2}\mathrm{Sgn}\Im(k^0)\mathrm{Sgn}\Im(\rho-k^0) - \mathrm{Sgn}\Im(\rho-k^0)\right] \tag{6.9}$$

Eq. (6.9) can be written:

$$L(k^0,\rho) = \frac{\pi^4}{8\rho}\left[\left(\mathrm{Sgn}\Im(k^0)+\mathrm{Sgn}\Im(\rho)\right)\mathrm{Sgn}\Im(\rho+k^0) \ + \right.$$

$$\left. \left(\mathrm{Sgn}\Im(k^0)-\mathrm{Sgn}\Im(\rho)\right)\mathrm{Sgn}\Im(\rho-k^0)\right] \ =$$

$$\frac{\pi^4}{4\rho}\,\mathrm{Sgn}\Im(k^0)\mathrm{Sgn}\Im(\rho) \tag{6.10}$$

Taking into account that:

$$H(k^0,\rho) = \frac{1}{2\pi\rho}\int L(k^0,\rho)\rho\ d\rho$$

we obtain:

$$H(k^0,\rho) = \frac{\pi^3}{8} Sgn\Im(k^0) Sgn\Im(\rho) = [W_0 * W_0](k^0,\rho) \tag{6.11}$$

(The symbol $*$ indicates the convolution product).

Thus the cut of $H(k^0,\rho)$ along the real axis, i.e., the distribution $h(k^0,\rho)$ is:

$$h(k^0,\rho) = \frac{\pi^3}{2} = [w_0 * w_0](k^0,\rho) \tag{6.12}$$

7 The Convolution of two complex mass Wheeler's Propagators

The complex mass Wheeler's propagator is:

$$w_\mu(x) = -\frac{i\pi}{2}\frac{\mu^{n/2-1}}{(2\pi)^{n/2}} Q_-^{1/2(1-n/2)} J_{1-n/2}(\mu Q_-^{1/2}) \tag{7.1}$$

and it Fourier transform has the expression:

$$W_\mu(k^0,\rho) = -\frac{iSgn[\Im(k^0)]}{8\sqrt{k_0^2-\mu^2}}\left[\frac{Sgn[\Im(\rho)] - Sgn[\Im(\sqrt{k_0^2-\mu^2})]}{\rho-\sqrt{k_0^2-\mu^2}} - \right.$$
$$\left.\frac{Sgn[\Im(\rho)] + Sgn[\Im(\sqrt{k_0^2-\mu^2})]}{\rho+\sqrt{k_0^2-\mu^2}}\right] \tag{7.2}$$

Using (7.2) we have now:

$$L(k^0,\rho) = -\oint\limits_{\Gamma_1^0}\oint\limits_{\Gamma_2^0}\oint\limits_{\Gamma_1}\oint\limits_{\Gamma_2} \frac{Sgn[\Im(k_1^0)]}{8\sqrt{k_1^{02}-\mu_1^2}}\left[\frac{Sgn[\Im(\rho_1)] - Sgn[\Im(\sqrt{k_1^{02}-\mu_1^2})]}{\rho_1-\sqrt{k_1^{02}-\mu_1^2}} - \right.$$

$$\frac{Sgn[\Im(\rho_1)] + Sgn[\Im(\sqrt{k_1^{02} - \mu_1^2})]}{\rho + 1 + \sqrt{k_1^{02} - \mu_1^2}}\Bigg] \frac{Sgn[\Im(k_2^0)]}{8\sqrt{k_2^{02} - \mu_2^2}} \Bigg[\frac{Sgn[\Im(\rho_2)] - Sgn[\Im(\sqrt{k_2^{02} - \mu_2^2})]}{\rho_2 - \sqrt{k_2^{02} - \mu_2^2}} -$$

$$\frac{Sgn[\Im(\rho_2)] + Sgn[\Im(\sqrt{k_2^{02} - \mu_2^2})]}{\rho_2 + \sqrt{k_2^{02} - \mu_2^2}}\Bigg] \frac{\rho_1\rho_2 d\rho_1 \, d\rho_2 \, dk_1^0 \, dk_2^0}{(k^0 - k_1^0 - k_2^0)[\rho^2 - (\rho_1 + \rho_2)^2]} \tag{7.3}$$

where we have selected $\lambda_0 = \lambda = 0$ due to that (7.3) is convergent in this point (Observe the reader that it is due to the definition of $L(k^0, \rho)$). Now (7.3) is equal to:

$$L(k^0, \rho) = -\frac{1}{4} \oint_{\Gamma_1^0} \oint_{\Gamma_2^0} \iint_{-\infty}^{+\infty} \frac{Sgn[\Im(k_1^0)]}{\rho_1^2 + \mu_1^2 - k_1^{02}} \; \frac{Sgn[\Im(k_2^0)]}{\rho_2^2 + \mu_2^2 - k_2^{02}} \times$$

$$\frac{\rho_1\rho_2}{(k^0 - k_1^0 - k_2^0)[\rho^2 - (\rho_1 + \rho_2)^2]} d\rho_1 \, d\rho_2 \, dk_1^0 \, dk_2^0 \tag{7.4}$$

and can be re-written as:

$$L(k^0, \rho) = -\frac{1}{16} \oint_{\Gamma_1^0} \oint_{\Gamma_2^0} \iint_{-\infty}^{+\infty} \frac{Sgn[\Im(k_1^0)]}{\sqrt{\rho_1^2 + \mu_1^2}} \left[\frac{1}{k_1^0 - \sqrt{\rho_1^2 + \mu_1^2}} - \frac{1}{k_1^0 + \sqrt{\rho_1^2 + \mu_1^2}}\right] \times$$

$$\frac{Sgn[\Im(k_2^0)]}{\sqrt{\rho_2^2 + \mu_2^2}} \left[\frac{1}{k_2^0 - \sqrt{\rho_2^2 + \mu_2^2}} - \frac{1}{k_2^0 + \sqrt{\rho_2^2 + \mu_2^2}}\right] \frac{1}{(k^0 - k_1^0 - k_2^0)} \times$$

$$\frac{\rho_1\rho_2}{\rho^2 - (\rho_1 + \rho_2)^2} d\rho_1 \, d\rho_2 \, dk_1^0 \, dk_2^0 \tag{7.5}$$

Taking into account that:

$$\oint_{\Gamma_1^0} \oint_{\Gamma_2^0} \frac{Sgn[\Im(k_1^0)]Sgn[\Im(k_2^0)}{k^0 - k_1^0 - k_2^0} \left[\frac{1}{k_1^0 - \sqrt{\rho_1^2 + \mu_1^2}} - \frac{1}{k_1^0 + \sqrt{\rho_1^2 + \mu_1^2}}\right] \times$$

$$\left[\frac{1}{k_2^0 - \sqrt{\rho_2^2 + \mu_2^2}} - \frac{1}{k_2^0 + \sqrt{\rho_2^2 + \mu_2^2}}\right] dk_1^0 \, dk_2^0 =$$

$$-\frac{32\pi^2 k^0 \sqrt{\rho_1^2+\mu_1^2}\sqrt{\rho_2^2+\mu_2^2}}{[k_0^2+(\rho_2^2+\mu_2^2)-(\rho_1^2+\mu_2^2)]^2-4k_0^2(\rho_2^2+\mu_2^2)} \tag{7.6}$$

Replacing this result in (7.5) we obtain

$$L(k^0,\rho) = 2\pi^2 k^0 \iint\limits_{-\infty}^{+\infty} \frac{1}{[k_0^2+(\rho_2^2+\mu_2^2)-(\rho_1^2+\mu_2^2)]^2-4k_0^2(\rho_2^2+\mu_2^2)} \times$$

$$\frac{\rho_1\rho_2}{\rho^2-(\rho_1+\rho_2)^2}\, d\rho_1\, d\rho_2 \tag{7.7}$$

Taking into account that

$$\int \frac{\rho\, d\rho}{\rho^2-(\rho_1+\rho_2)^2} = \Theta[\Im(\rho)]\ln(\rho_1+\rho_2-\rho)+\Theta[-\Im(\rho)]\ln(\rho-\rho_1-\rho_2) \tag{7.8}$$

and using the result (7.7) we obtain

$$H(k^0,\rho) = \frac{\pi k^0}{\rho} \iint\limits_{-\infty}^{+\infty} \frac{1}{[k_0^2+(\rho_2^2+\mu_2^2)-(\rho_1^2+\mu_2^2)]^2-4k_0^2(\rho_2^2+\mu_2^2)} \times$$

$$\Theta[\Im(\rho)]\ln(\rho_1+\rho_2-\rho)+\Theta[-\Im(\rho)]\ln(\rho-\rho_1-\rho_2)\, d\rho_1\, d\rho_2 \tag{7.9}$$

The equation (7.9) can be written in the real ρ-axis as:

$$H(k^0,\rho) = \frac{i\pi^2 k^0}{\rho} \iint\limits_{-\infty}^{+\infty} \frac{Sgn(\rho_1+\rho_2-\rho)\,\rho_1\rho_2\, d\rho_1\, d\rho_2}{[k_0^2+(\rho_2^2+\mu_2^2)-(\rho_1^2+\mu_2^2)]^2-4k_0^2(\rho_2^2+\mu_2^2)} \tag{7.10}$$

After the evaluation of double integral of (7.10) we obtain:

$$H(k^0,\rho) = \frac{\pi^3 Sgn[\Im(k^0)]}{4(k_0^2-\rho^2)}\sqrt{(k_0^2-\rho^2+\mu_2^2-\mu_1^2)^2-4(k_0^2-\rho^2)\mu_2^2} =$$

$$[W_{\mu_1} * W_{\mu_2}](k^0,\rho) \tag{7.11}$$

8 Discussion

In a earlier paper [3] we have shown the existence of the convolution of two one-dimensional tempered ultradistributions. In this paper we have extended these procedure to n-dimensional space. In four-dimensional space we have obtained a expression for the convolution of two tempered ultradistributions even in the variables k^0 and ρ.

When we use the perturbative development in Quantum Field Theory, we have to deal with products of distributions in configuration space, or else, with convolutions in the Fourier transformed p-space. Unfortunately, products or convolutions (of distributions) are in general ill-defined quantities. However, in physical applications one introduces some "regularization" scheme, which allows us to give sense to divergent integrals. Among these procedures we would like to mention the dimensional regularization method (ref. [8, 9]). Essentially, the method consists in the separation of the volume element ($d^\nu p$) into an angular factor ($d\Omega$) and a radial factor ($p^{\nu-1}dp$). First the angular integration is carried out and then the number of dimensions ν is taken as a free parameter. It can be adjusted to give a convergent integral, which is an analytic function of ν.

Our formula (4.1) is similar to the expression one obtains with dimensional regularization. However, the parameters λ are completely independents of any dimensional interpretation.

All ultradistributions provide integrands (in (4.1)) that are analytic functions along the integration paths. The parameters λ permit us to control the possible tempered asymptotic behavior (cf. eq. (3.9)). The existence of a region of analyticity for each λ, and a subsequent continuation to the point of interest (ref. [3]), defines the convolution product.

For tempered ultradistributions (even in the variables k^0 and ρ) we have obtained formula (5.15) for which are valid similar considerations to those given for (4.1) The properties described below show that tempered ultradistributions provide an appropriate framework for applications to physics. Furthermore, they can "absorb" arbitrary pseudo-polynomials, thanks to eq. (3.10). A property that is interesting for renormalization theory. For this reason we decided to begin this paper and also for the benefit of the reader we began this paper with a summary of the main characteristics of n-dimensional tempered ultradistributions and their Fourier transformed distributions of the exponential type.

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