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Powers of cycles, powers of paths, and distance graphs

Min Chih Lin [a], Dieter Rautenbach [b,*], Francisco Juan Soulignac [a], Jayme Luiz Szwarcfiter [c]

[a] Universidad de Buenos Aires, Facultad de Ciencias Exactas y Naturales, Departamento de Computación, Buenos Aires, Argentina
[b] Institut für Mathematik, Technische Universität Ilmenau, Postfach 100565, D-98684 Ilmenau, Germany
[c] Universidade Federal do Rio de Janeiro, Instituto de Matemática, NCE and COPPE, Caixa Postal 2324, 20001-970 Rio de Janeiro, RJ, Brazil

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Dedicated to Martin Charles Golumbic on the occasion of his 60th birthday

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ABSTRACT

In 1988, Golumbic and Hammer characterized the powers of cycles, relating them to circular arc graphs. We extend their results and propose several further structural characterizations for both powers of cycles and powers of paths. The characterizations lead to linear-time recognition algorithms of these classes of graphs. Furthermore, as a generalization of powers of cycles, powers of paths, and even of the well-known circulant graphs, we consider distance graphs. While the colorings of these graphs have been intensively studied, the recognition problem has been so far neglected. We propose polynomial-time recognition algorithms for these graphs under additional restrictions.

1. Introduction

In [15] Golumbic and Hammer proposed efficient algorithms for the maximum independent set problem restricted to circular arc graphs. As a simple reduction rule, they eliminate vertices whose closed neighbourhood contains the closed neighbourhood of another vertex. They prove that a circular arc graph which no longer allows such a reduction is isomorphic to the power of a cycle. In fact, their proof of this observation yields several equivalent characterizations of powers of cycles.

This nice connection between a well-known graph class and the powers of some very basic graph was our starting point for the present paper. We will first make all characterizations of powers of cycles implicit in [15] explicit and add some more. Then we prove a similar series of equivalent characterizations of powers of paths. Finally, we consider the so-called distance graphs, which generalize the powers of paths.

We need to review some notation and refer the reader to [3,14] for further details. We consider simple, finite, and undirected graphs G with vertex set $V(G)$ and edge set $E(G)$. The *order of* G is the cardinality of $V(G)$. For a vertex $u \in V(G)$, the *neighbourhood of* u *in* G is denoted by $N_G(u)$. The *degree of* u *in* G is $d_G(u) = |N_G(u)|$ and the *closed neighbourhood of* u *in* G is $N_G[u] = \{u\} \cup N_G(u)$. The *kth power* G^k of the graph G has the same vertex set as G and two distinct vertices u and v of G are adjacent in G^k if and only if their *distance* in G is at most k. If u and v are distinct vertices of G, then u and v are *twins*, if $N_G[u] = N_G[v]$ and v is a *dominator* of u, if $N_G[u] \subseteq N_G[v]$. A maximal sequence of at least two vertices $v_1, v_2, \ldots, v_l$ such that $N_G[v_i]$ properly contains $N_G[v_{i+1}]$ for $1 \le i \le l-1$ is a *dominator sequence*. A vertex u of G is *universal*, if $N_G[v] = V(G)$.

* Corresponding author. Fax: +49 3677 69 3272.
E-mail addresses: oscarlin@dc.uba.ar (M.C. Lin), dieter.rautenbach@gmx.de, dieter.rautenbach@tu-ilmenau.de (D. Rautenbach), fsoulign@dc.uba.ar (F.J. Soulignac), jayme@nce.ufrj.br (J.L. Szwarcfiter).

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The *path* of order n is denoted by P_n and the *cycle* of order n is denoted by C_n.

A graph is a *circular arc graph* if it is the intersection graph of open arcs on a circle. A *circular arc model* for a circular arc graph G is a collection $\mathcal{A} = \{a_v \mid v \in V(G)\}$ of open arcs a_v on a circle such that $uv \in E(G)$ if and only if a_u and a_v intersect. Fixing an orientation of the circle, the *extreme points* of the arcs can be distinguished into *starting points* and *ending points*. As noted in [14], we may assume that no two arcs have a common extreme point. If no arc in $\mathcal{A}$ contains another arc in $\mathcal{A}$, then $\mathcal{A}$ is a *proper circular arc model (PCA model)* and G is a *proper circular arc graph (PCA graph)*. If all arcs in $\mathcal{A}$ have the same lengths, then $\mathcal{A}$ is a *unit circular arc model (UCA model)* and G is a *unit circular arc graph (UCA graph)*.

If we replace open arcs on a circle with open intervals in $\mathbb{R}$ in the above definitions, we obtain the notions of an *interval graph* and an *interval model*. The extreme points of the intervals can again be distinguished into starting points and ending points and we may assume that no two intervals in a model share an extreme point. If no interval in an interval model $\mathcal{A}$ contains another interval in $\mathcal{A}$, then $\mathcal{A}$ is a *proper interval model (PI model)* and G is a *proper interval graph (PI graph)*. If all intervals in $\mathcal{A}$ have the same lengths, then $\mathcal{A}$ is a *unit interval model (UI model)* and G is a *unit interval graph (UI graph)*. It is well known [2,13,26] that the classes of PI graphs and UI graphs coincide while UCA graphs form a proper subclass of PCA graphs. It is easy to see that every (proper, unit) interval model yields a (proper, unit) circular arc model for the same graph, i.e. (proper, unit) interval graphs are special (proper, unit) circular arc graphs.

A natural and important generalization of powers of cycles are *circulant graphs*: For $n \in \mathbb{N}$ and $D \subseteq \mathbb{N}$, the circulant graph C_n^D has vertex set $\{v_0, v_1, \ldots, v_{n-1}\}$ and $N_{C_n^D}(v_i) = \{v_{i+d} \mid |d| \in D\}$ for $0 \le i \le n-1$ where indices are identified modulo n. Clearly, we may assume $\max D \le \frac{n}{2}$ for every circulant graph C_n^D. Circulant graphs are the *Cayley graphs* of *cyclic groups* and due to their symmetry and connectivity properties, they have been proposed for various practical applications [1]. Isomorphism testing and recognition of circulant graphs had been long-standing open problems [23–25] and were completely solved only recently [12,22].

A similarly defined class of graphs are *distance graphs*: For $n \in \mathbb{N}$ and $D \subseteq \mathbb{N}$, the distance graph P_n^D has vertex set $\{v_0, v_1, \ldots, v_{n-1}\}$ and $N_{P_n^D}(v_i) = \{v_{i+d} \mid |d| \in D \text{ and } 0 \le i+d \le n-1\}$ for $0 \le i \le n-1$. Equivalently, $v_iv_j \in E(P_n^D)$ if and only if $|j-i| \in D$. Clearly, we may assume $\max D \le n-1$ for every distance graph P_n^D. Distance graphs lack the symmetry of circulant graphs and the algebraic methods used in [12,22] do not apply to them. At first sight they seem to generalize powers of paths in a similar way as circulant graphs generalize powers of cycles. Nevertheless, the circulant graph C_n^D with $\max D \le \frac{n}{2}$ is isomorphic to the distance graph $P_n^{D'}$ for $D' = D \cup \{n-d \mid d \in D\}$, i.e. every circulant graph is in fact also a distance graphs. Originally motivated by research due to Eggleton, Erdős, and Skilton [10,11] who considered coloring problems for infinite distance graphs, coloring problems for distance graphs and circulant graphs have been intensely studied [4,8,9,19,20,27,28]. While isomorphism testing and recognition of circulant graphs have been investigated for a long time, these problems seem to have been neglected for the more general distance graphs.

2. Powers of cycles and paths

Our first result collects several equivalent descriptions of powers of cycles. Theorem 1 in [15] actually only states that a circular arc graph without dominators is a power of a cycle. Nevertheless, the given arguments imply the following equivalences from our Theorem 1 below:

(i) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vii) $\Leftrightarrow$ (ix).

Theorem 1. *For a graph G of order n which is not complete, the following statements are equivalent.*

(i) *G is isomorphic to the kth power C_n^k of the cycle C_n of order n for some integer k.*
(ii) *G is a regular UCA graph with no twins.*
(iii) *G is a regular PCA graph with no twins.*
(iv) *G is a UCA graph without dominators.*
(v) *G is a PCA graph without dominators.*
(vi) *G is a UCA graph and in every UCA model of G the starting and ending points alternate.*
(vii) *G is a PCA graph and in every PCA model of G the starting and ending points alternate.*
(viii) *G is a UCA graph and in some UCA model of G the starting and ending points alternate.*
(ix) *G is a PCA graph and in some PCA model of G the starting and ending points alternate.*

Proof. The implications (ii) $\Rightarrow$ (iii), (viii) $\Rightarrow$ (ix), (vi) $\Rightarrow$ (viii), and (vii) $\Rightarrow$ (ix) are trivial.

(i) $\Rightarrow$ (ii): Clearly, C_n^k is regular and has not twins. If $x_0, x_1, \ldots, x_{n-1}$ are n equally spaced points on a circle $\mathcal{C}$, then the set $\mathcal{A}$ which contains the n open arcs of equal length with starting point x_i and ending point between x_{i+k} and x_{i+k+1} for $0 \le i \le n-1$ is a UCA model for C_n^k.

(ii) $\Rightarrow$ (iv) (and (iii) $\Rightarrow$ (v)): If $N_G[u] \subseteq N_G[v]$, then the regularity of G implies $N_G[u] = N_G[v]$ and the twin-freeness of G implies $u = v$. Hence G is a UCA (PCA) graph without dominators.

(iv) $\Rightarrow$ (vi) (and (v) $\Rightarrow$ (vii)): For contradiction, we assume that s_u and s_v are two consecutive extreme points of a UCA (PCA) model of G which are both starting points of the arcs a_u and a_v corresponding to the vertices u and v of G. This implies that every arc of the model which intersects a_u also intersects a_v and yields the contradiction $N_G[u] \subseteq N_G[v]$.

(ix) ⇒ (i): Let $s_0, t_0, s_1, t_1, \ldots, s_{n-1}, t_{n-1}$ be the cyclically consecutive extreme points of a PCA model $\mathcal{A}$ of G as in (ix). It suffices to prove the existence of some $k \in \mathbb{N}$ such that $\mathcal{A}$ consists of the open arcs with starting point s_i and ending point t_{i+k} for $0 \le i \le n-1$ and some $k \in \mathbb{N}$ where indices are identified modulo n.

For contradiction, we may assume that the arc starting with s_0 ends with t_k and that the arc starting with s_1 ends with t_{k+i} for some k with $i \neq 1$. Since the model is proper, we obtain $i \geq 2$. Let the arc ending with t_{k+1} start with s_j for some j. Again, since the model is proper, the arc from s_j to t_{k+1} is not contained in the arc from s_1 to t_{k+i} which implies $j < 1$. Similarly, the arc from s_j to t_{k+1} does not contain the arc from s_0 to t_k which implies $j > 0$. We obtain the contradiction that the integer j satisfies $0 < j < 1$.

In view of the following diagram of the implications this completes the proof.

$$\begin{array}{ccccccccccc} (i) & \Rightarrow & (ii) & \Rightarrow & (iv) & \Rightarrow & (vi) & \Rightarrow & (viii) & & \\ & & \Downarrow & & & & & & \Downarrow & & \\ & & (iii) & \Rightarrow & (v) & \Rightarrow & (vii) & \Rightarrow & (ix) & \Rightarrow & (i). \quad \square \end{array}$$

Our next result collects several equivalent descriptions of powers of paths. Before we can state it, we need some further definitions.

Let $\mathcal{A} = \{(s_i, t_i) \mid 1 \le i \le n\}$ be a proper interval model for a connected graph G with vertex set $\{v_1, v_2, \ldots, v_n\}$ such that

$$(s_i, t_i) \text{ corresponds to } v_i \quad \text{for } 1 \le i \le n \tag{1}$$

and

$$s_1 < s_2 < \cdots < s_n. \tag{2}$$

As noted by Roberts [26], the ordering $v_1, v_2, \ldots, v_n$ of the vertices of G is unique up to permutation of twins and up to reversion and is therefore called a *canonical ordering*.

If

$$p = \max\{i \mid 1 \le i \le n,\ v_i \in N_G[v_1]\} \quad \text{and} \quad q = \min\{i \mid 1 \le i \le n,\ v_i \in N_G[v_n]\}, \tag{3}$$

then

$$\{v_i \mid \min\{p, q\} \le i \le \max\{p, q\}\}$$

is the set of *middle* vertices. By Roberts' result [26], this set does not depend on the model $\mathcal{A}$. Note that the vertices v_i with $q \le i \le p$ are exactly the universal vertices of G.

Theorem 2. *For a connected graph G of order n which is not complete, the following statements are equivalent.*

(i) *G is isomorphic to the kth power P_n^k of the path P_n of order n for some integer k.*
(ii) *G is a UI graph in which all twins are universal and whose middle vertices have the same degree.*
(iii) *G is a UI graph in which all twins are universal. Furthermore, if $v_1, v_2, \ldots, v_n$ is a canonical ordering and p and q are as in (3), then the only dominator sequences of G are*

$$v_r, v_{\min\{p,q\}-1}, v_{\min\{p,q\}-2}, \ldots, v_1$$

with $\min\{p, q\} \le r \le p$ *and*

$$v_s, v_{\max\{p,q\}+1}, v_{\max\{p,q\}+2}, \ldots, v_n$$

with $q \le s \le \max\{p, q\}$.

(iv) *G is a UI graph and for all PI models $\mathcal{A} = \{(s_i, t_i) \mid 1 \le i \le n\}$ with (1) and (2), and p and q as in (3), the starting and ending points between s_p and t_q alternate.*
(v) *G is a UI graph and for some UI model $\mathcal{A} = \{(s_i, t_i) \mid 1 \le i \le n\}$ with (1) and (2), and p and q as in (3), the starting and ending points between s_p and t_q alternate.*

Proof. (i) ⇒ (ii): Clearly, in P_n^k all twins are universal and $\left\{\left(i, i + k + \frac{1}{2}\right) \mid 1 \le i \le n\right\}$ is a UI model for P_n^k which yields the desired degree property.

(ii) ⇒ (iii): Since G is connected and not complete, we obtain $1 < \min\{p, q\} \le \max\{p, q\} < n$. By (1) to (3), this implies

$$N_G[v_1] \subseteq N_G[v_2] \subseteq \cdots \subseteq N_G[v_{\min\{p,q\}-1}] \subseteq N_G[v_r]$$

for $\min\{p, q\} \le r \le p$, and

$$N_G[v_n] \subseteq N_G[v_{n-1}] \subseteq \cdots \subseteq N_G[v_{\max\{p,q\}+1}] \subseteq N_G[v_s]$$

for $q \le s \le \max\{p, q\}$. Since $v_1, \ldots, v_{\min\{p,q\}-1}$ are not adjacent to v_n and $v_{\max\{p,q\}+1}, \ldots, v_n$ are not adjacent to v_1, all these vertices are not universal and (ii) implies that all the above inclusions are proper. This yields the dominator sequences described in (iii). It remains to prove that there are no further dominator sequences.

Since, by (ii), all middle vertices have the same degree, every dominator sequence contains at most one middle vertex.

If $i < \min\{p, q\}$ and $p < j$, then $v_1 \in N_G[v_i] \setminus N_G[v_j]$. Furthermore, either $q \leq j$ which implies $v_n \in N_G[v_j] \setminus N_G[v_i]$, or $q > j$ which implies that v_j is a middle vertex and, by (ii), $d_G(v_j) = d_G(v_p) > d_G(v_i)$. Hence, in both cases, v_i and v_j do not both appear in one dominator sequence.

Similarly, if $i > \max\{p, q\}$ and $j < q$, then v_i and v_j do not both appear in one dominator sequence. Altogether, this implies that there are no further dominator sequences as those described in (iii).

(iii) $\Rightarrow$ (iv): Let $\mathcal{A}$ be a PI model for G. For contradiction, we may assume, by symmetry, that there are two consecutive extreme points which are starting point s_i and s_{i+1} between s_p and t_q. By (3), the extreme point following s_p is t_1 which implies $i > p$. Since $\mathcal{A}$ is a proper interval model, we obtain $N_G[v_i] \subseteq N_G[v_{i+1}]$. Since $v_1 \notin N_G[v_i] \cup N_G[v_{i+1}]$, the vertices v_i and v_{i+1} are not universal. Hence, by (iii), v_{i+1} and v_i are no twins and they appear in this order in some dominator sequence of G. By (iii), this implies the contradiction $i + 1 \leq p$.

(iv) $\Rightarrow$ (v): trivial.

(v) $\Rightarrow$ (i): Let $\mathcal{A}$ be a UI model as described in (v). By (3), (v), and the fact that $s_i < s_j$ implies $t_i < t_j$, the order of the extreme points is as follows

$$s_1 < s_2 < \cdots < s_p < t_1 < s_{p+1} < t_2 < s_{p+2} < t_3 < \cdots < s_n < t_q < t_{q+1} < \cdots < t_n$$

which implies that $p = n - q - 1$ and that G is isomorphic to C_n^{p-1}. □

In view of the corresponding recognition algorithms for circular arc graphs [7,18,21] and interval graphs [5,6,16,17], Theorems 1 and 2 imply that powers of cycles and paths can be recognized in linear time.

3. Distance graphs

For a set $D = \{d_1, d_2, \ldots, d_k\} \subseteq \mathbb{N}$ and $n \in \mathbb{N}$, we consider the distance graph P_n^D with vertex set $\{v_0, v_1, \ldots, v_{n-1}\}$ where

$$v_i v_j \text{ is an edge} \quad \text{if and only if} \quad |i - j| \in D. \tag{4}$$

The next lemma collects some simple observations about P_n^D.

Lemma 3. *Let* $1 \leq d_1 < d_2 < \cdots < d_k \leq n - 1$.

(i) P_n^D *has* $\sum_{i=1}^{k}(n - d_i)$ *edges.*
(ii) *If* P_n^D *is connected, then the greatest common divisor* $gcd(D)$ *of the elements in* D *equals* 1.
(iii) *If* $gcd(D) = 1$ *and* $d_k \leq n - gcd\left(\left\{d_i \mid 1 \leq i \leq k, d_i \leq \frac{n-1}{2}\right\}\right)$, *then* P_n^D *is connected.*

Proof. (i) Since there are exactly $n - d_i$ edges of the form $v_i v_{i+d_i}$, the statement follows.

(ii) If P_n^D is connected, then there is a path from v_0 to v_1. This implies that 1 is an integral linear combination of the elements of D and hence $gcd(D) = 1$.

(iii) Let $d = gcd\left(\left\{d_i \mid 1 \leq i \leq k, d_i \leq \frac{n-1}{2}\right\}\right)$. Let

$$d = \sum_{\mu=1}^{r} a_\mu - \sum_{\nu=1}^{s} b_\nu \tag{5}$$

be such that $a_\mu, b_\nu \in \left\{d_i \mid 1 \leq i \leq k, d_i \leq \frac{n-1}{2}\right\}$ for $1 \leq \mu \leq r$ and $1 \leq \nu \leq s$, and $r + s$ is minimum. (The existence of such a representation of d follows from the Euclidean algorithm.) Clearly, $r \geq 1$. Furthermore, if $r + s \geq 2$, then $r, s \geq 1$.

Claim. *For every* $0 \leq i \leq n - 1 - d$, *there is a path in* P_n^D *from* v_i *to* v_{i+d}.

Proof of the Claim. We will argue by induction on $r + s$. If $r + s = 1$, then $d \in D$ and $v_i v_{i+d}$ is an edge of P_n^D. If $r + s > 1$ and $i \leq \frac{n-1}{2}$, then $v_i v_{i+a_1}$ is an edge of P_n^D and, by induction, there is a path from v_{i+a_1} to $v_{i+d} = v_{(i+a_1)+(d-a_1)}$. If $r + s > 1$ and $i \geq \frac{n-1}{2}$, then $v_i v_{i-b_1}$ is an edge of P_n^D and, by induction, there is a path from v_{i-b_1} to $v_{i+d} = v_{(i-b_1)+(d+b_1)}$. This completes the proof of the claim. □

By the claim, for every $0 \leq i \leq n-1$, there are paths between v_i and vertices v_j and $v_{j'}$ with $j, j' \equiv i \bmod d$, $0 \leq j \leq d-1$ and $n - d \leq j' \leq n - 1$. Since $d_k \leq n - d$, this implies that for every $0 \leq i \leq n - 1$ and every d' with $|d'| \in D$, there is a path from v_i to a vertex v_j with $j \equiv (i + d') \bmod d$.

Since $gcd(D) = 1$, 1 is an integral linear combination of the elements of D. This implies, by an inductive argument, that for every $0 \leq i \leq n - 2$, there is a path from v_i to a vertex v_j with $j \equiv (i + 1) \bmod d$. Applying the claim again, we obtain that there is a path from v_i to v_{i+1} which completes the proof. □

As we have already observed in the introduction, circulant graphs are special distance graphs. Since isomorphism testing and recognition of circulant graphs were major achievements, these problems will be very hard for distance graphs. In order to represent a circulant graph as a distance graphs P_n^D, the set D typically contains elements which are larger than $\frac{n-1}{2}$. In the sequel we will restrict our attention to the case

$$1 = d_1 < d_2 < \cdots < d_k \leq \frac{n-1}{2}. \tag{6}$$

This assumption essentially results in distance graphs which seem closer to powers of paths than to circulant graphs. The assumption $d_1 = 1$ ensures that the path $v_0v_1v_2 \cdots v_{n-1}$ is always contained in P_n^D. The assumption $d_k \leq \frac{n-1}{2}$ ensures that for $0 \leq i \leq k-1$, the set of vertices of P_n^D of degree $k+i$ is exactly

$$\{v_j \mid d_i \leq j \leq d_{i+1} - 1\} \cup \{v_j \mid n - d_{i+1} \leq j \leq n - d_i - 1\} \tag{7}$$

where $d_0 := 0$. Furthermore, the vertices v_j with $d_k \leq j \leq n-1-d_k$ are all of degree $2k$. Hence P_n^D has $2(d_{i+1} - d_i)$ vertices of degree $k+i$ for $0 \leq i \leq k-1$ and the set D is uniquely determined by the degree sequence of the graph P_n^D.
We consider the recognition problem for distance graphs. Equivalently, for a distance graph P_n^D given up to isomorphism, we consider the problem to reconstruct the set D and an ordering $v_0, v_1, \ldots, v_{n-1}$ of its vertices which satisfies (4). As we have already observed, the set D is uniquely determined by the degree sequence of P_n^D.

We call some r with $1 \leq r \leq \frac{n-1}{2}$ an *index of ambiguity of* P_n^D, if there is an index $r < s \leq n-1-r$ such that

$$N_{P_n^D}(v_r) \cap \{v_j \mid 0 \leq j \leq r-1\} = N_{P_n^D}(v_s) \cap \{v_j \mid 0 \leq j \leq r-1\}, \tag{8}$$

$$N_{P_n^D}(v_r) \cap \{v_j \mid n-r \leq j \leq n-1\} = N_{P_n^D}(v_s) \cap \{v_j \mid n-r \leq j \leq n-1\}, \tag{9}$$

$$d_{P_n^D}(v_r) = d_{P_n^D}(v_s), \tag{10}$$

$$N_{P_n^D}[v_r] \neq N_{P_n^D}[v_s]. \tag{11}$$

We call v_s a *cuckoo twin* of v_r. The role of these notions is captured by the following result.

Theorem 4. *Let D satisfy (6). If P_n^D has no index of ambiguity, then D and an ordering $v_0, v_1, \ldots, v_{n-1}$ of its vertices which satisfies (4) can be obtained from P_n^D in time $O(n^2)$.*

Proof. Since $d_1 = 1$, P_n^D has exactly two vertices of degree k. Since $v_i \mapsto v_{n-1-i}$ is an automorphism of P_n^D, we can select any of the two vertices as v_0 and the other as v_{n-1}.

In view of an inductive approach, we assume that we have already identified the vertices in

$$U = \{v_j \mid 0 \leq j \leq r-1\} \cup \{v_j \mid n-r \leq j \leq n-1\}$$

for some $r \geq 1$. Now, since P_n^D has no index of ambiguity, v_r and v_{n-1-r} are uniquely determined by D, their degrees, and their neighbours within U. (Note that a vertex v of the same degree as v_r and with the same neighbours within U, does not satisfy (11). Hence v and v_r are twins and we can select an arbitrary such vertex as v_r.)

Clearly, this approach can be implemented in quadratic time. □

The next lemma captures some properties of indices of ambiguity.

Lemma 5. *Let D satisfy (6). If r is an index of ambiguity of P_n^D, the index s is such that v_s is a cuckoo twin of v_r, and $d_{P_n^D}(v_r) = k+i$ for some $0 \leq i \leq k$, then the following statements hold.*

(i) $1 \leq i \leq k-1$.
(ii) $d_i \leq r \leq d_{i+1} - 1$ *and* $n - d_{i+1} \leq s \leq n-1-d_i$.
(iii) $s - r = d_k - d_i = d_{k-1} - d_{i-1} = \cdots = d_{k-i+1} - d_1$. *Furthermore, v_r has a unique cuckoo twin v_s, and v_s is no cuckoo twin of a vertex $v_{r'}$ with $r' \neq r$.*
(iv) $N_{P_n^D}(v_r) \cap \{v_j \mid s+1 \leq j \leq n-1\} = N_{P_n^D}(v_s) \cap \{v_j \mid s+1 \leq j \leq n-1\}$.
(v) $n - d_i - 1 - 2(d_{i+1} - d_i - 1) \leq d_k \leq n - d_i - 1$.

Proof. Since $r \geq 1$ and $1 \in D$, we obtain $i \geq 1$. If $i = k$, then $r - d_k \geq 0$ and $v_{r-d_k} \in N_{P_n^D}(v_r) \setminus N_{P_n^D}(v_s)$ which contradicts (8). Hence $i \leq k-1$ and (i) follows.

Since $r \leq \frac{n-1}{2}$, (7) implies $d_i \leq r \leq d_{i+1} - 1$. Furthermore, (6) implies that v_r has exactly k neighbours v_j with $j > r$ and exactly i neighbours v_j with $j < r$. If $s \leq \frac{n-1}{2}$, then (6) and (8) imply that v_s has k neighbours v_j with $j > s$, i neighbours v_j with $0 \leq j \leq r-1$ and at least one further neighbour v_{s-1}. This implies $d_{P_n^D}(v_s) \geq k+i+1$ which contradicts (10). Hence $s > \frac{n-1}{2}$, v_s has exactly k neighbours v_j with $j < s$ and exactly i neighbours v_j with $j > s$. By (7), $n - d_{i+1} \leq s \leq n-1-d_i$ and (ii) follows.

Since $N_{P_n^D}(v_r) \cap \{v_j \mid 0 \le j \le r-1\} = \{v_{r-d_1}, v_{r-d_2}, \dots, v_{r-d_i}\}$, (8) implies

$$s - r = d_k - d_i = d_{k-1} - d_{i-1} = \cdots = d_{k-i+1} - d_1.$$

Since $s = r + d_k - d_i$, the cuckoo twin v_s of v_r is uniquely determined. If v_s is also the cuckoo twin of a vertex $v_{r'}$ for an index of ambiguity r' different from r, then the degree of v_s implies that $v_{r'}$ has exactly i neighbours v_j with $j < r'$ which coincide with the i neighbours $v_{r-d_1}, v_{r-d_2}, \dots, v_{r-d_i}$ of v_s. This clearly implies the contradiction $r' = r$ and (iii) follows.

Furthermore, $N_{P_n^D}(v_s) \cap \{v_j \mid s+1 \le j \le n-1\} = \{v_{s+d_1}, v_{s+d_2}, \dots, v_{s+d_i}\}$, and, by (iii), (iv) follows. (ii) and (iii) imply $n - 2d_{i+1} + 1 \le d_k - d_i \le n - 2d_i - 1$, and (v) follows. □

In view of Theorem 4, situations with no or with only few indices of ambiguity are of interest. The following two corollaries make this more precise.

Corollary 6. *Let D satisfy* (6).

(i) *If* $d_k < \frac{n+2}{3}$, *then* P_n^D *has no index of ambiguity.*
(ii) *If* $\delta = \max\{d_{i+1} - d_i \mid 1 \le i \le k-1\}$, *then* P_n^D *has at most* $\delta - 1$ *indices of ambiguity.*

Proof. Let r be an index of ambiguity and let $d_{P_n^D}(v_r) = k + i$. Let v_s be the cuckoo twin of v_r.

(i) By Lemma 5(v), $n \le d_k + 2d_{i+1} - d_i - 1 \le 3d_k - 2$ which implies the contradiction $d_k \ge \frac{n+2}{3}$.

(ii) By the definition of δ, the vertex v_r has a neighbour v_j with $j \le \delta - 1$. By Lemma 5(ii), $\frac{n+1}{2} \le n - d_{i+1} \le s \le j + d_k \le \delta - 1 + \frac{n-1}{2}$. By Lemma 5(iii), there are at most $\delta - 1$ cuckoo twins and hence also at most $\delta - 1$ indices of ambiguity. □

Corollary 7. *Let D satisfy* (6). *If* P_n^D *has l indices of ambiguity, then D and an ordering* $v_0, v_1, \dots, v_{n-1}$ *of its vertices which satisfies* (4) *can be obtained from* P_n^D *in time* $O(4^l n^2)$.

Proof. Note that if r is an index of ambiguity, then also $n - 1 - r$ satisfies similar conditions as r. If P_n^D has l indices of ambiguity, then a similar strategy as used in the proof of Theorem 4 can be applied: Every time an index r of ambiguity is reached, one has to branch into 4 possibilities according to the two choices for each of v_r and v_{n-1-r}. Since the branching depth is at most l, the resulting time complexity of this modified approach is $O(4^l n^2)$. □

By Lemma 5(iii), indices of ambiguity typically lead to repeated differences among the elements of D. Therefore, we will consider the following choice for D with most repeated differences:

$$D = \{1 + (i-1)p \mid 1 \le i \le k\} \tag{12}$$

for some $p, k \in \mathbb{N}$. Note that for $p = 1$, $P_n^D = P_n^k$. Furthermore, for $p = 2$, P_n^D is a proper interval bigraph [16].

Theorem 8. *If D satisfies* (12) *for some* $p, k \in \mathbb{N}$ *and* $1 + (k-1)p \le \frac{n-1}{2}$, *then D and an ordering* $v_0, v_1, \dots, v_{n-1}$ *of its vertices which satisfies* (4) *can be obtained from* P_n^D *in time* $O(n^2)$.

Proof. We only need to argue how to resolve the indices of ambiguity. Therefore, let r be an index of ambiguity and let v_s be the cuckoo twin of v_r.

By (8), (11), and Lemma 5 (iv), there is some j with $r < j < s$ such that $v_j \in N_{P_n^D}(v_s) \setminus N_{P_n^D}(v_r)$. This implies $p \ge 3$ and $s - j = 1 + (i_1 - 1)p$ for some $1 \le i_1 \le k$. Clearly, we may assume that $n \ge 4$. In this case v_1 is the unique neighbour of v_0 of degree $k+1$ which implies $r \ge 2$. Since v_r is adjacent to v_{r-1} and v_s is a cuckoo twin of v_r, v_s is adjacent to v_{r-1}, i.e. $s - (r-1) = 1 + (i_2 - 1)p$ for some $1 \le i_2 \le k$ with $i_2 > i_1$. Now $j - (r-2) = -(s-j) + (s-(r-1)) + 1 = -(1 + (i_1 - 1)p) + (1 + (i_2 - 1)p) + 1 = 1 + ((i_2 - i_1) - 1)p$ and $1 \le i_2 - i_1 \le k$ which implies that v_{r-2} is adjacent to v_j. Since $p \ge 3$, v_{r-2} is non-adjacent to the vertices in the non-empty set $N_{P_n^D}(v_r) \setminus N_{P_n^D}(v_s)$. This allows to distinguish between v_r and its cuckoo twin v_s and completes the proof. □

4. Induced subgraphs

The graph classes to which we have related the powers of cycles, the powers of paths, and the distance graphs are hereditary. Therefore, it makes sense to consider the induced subgraphs of these graphs.

Theorem 9. (i) *A graph is an induced subgraph of a power of a cycle if and only it is a UCA graph.*
(ii) *A graph is an induced subgraph of a power of a path if and only it is a UI graph.*
(iii) *Every graph is an induced subgraph of a distance graph.*

Proof. Since the proofs of (i) and (ii) are very similar, we will only give details for the proofs of (ii) and (iii).

(ii) By Theorem 2, powers of paths are UI graphs and, hence, so are their induced subgraphs. For the converse, we assume that G is a UI graph and that $\mathcal{A} = \{(s_v, t_v) \mid v \in V(G)\}$ is a UI model for G. As we have noted in the introduction, we may assume that all $2|V(G)|$ extreme points are distinct. Therefore, there is some $n \in \mathbb{N}$ such that strictly between every two consecutive extreme points of $\mathcal{A}$ there are at least two points from the set $\mathbb{Z}/n = \left\{\frac{i}{n} \mid i \in \mathbb{Z}\right\}$.

If I_1 and I_2 are two open intervals of the same length, then $|I_1 \cap \mathbb{Z}/n|$ and $|I_2 \cap \mathbb{Z}/n|$ differ by at most one. Therefore, suitably replacing every extreme point x with one of the two smallest elements of $\mathbb{Z}/n$ which are larger than x, we obtain a UI model $\mathcal{A}'$ for G which uses only extreme points from $\mathbb{Z}/n$. Suitably adding further intervals of the same length with starting points in $\mathbb{Z}/n$ yields in a UI model for a power of a path.

(iii) In view of a simple inductive argument, we may assume that $G-v$ is an induced subgraph of P_n^D for some n and D with $\max D \le n-1$. If $N_G(v) = \{v_{i_1}, v_{i_2}, \ldots, v_{i_l}\}$, then G is an induced subgraph of $P_{2n}^{D'}$ with $D' = D \cup \{n+i_1, n+i_2, \ldots, n+i_l\}$. □

5. Conclusion

We have presented several characterizations of powers of cycles and powers of paths relating them to well-known graphs classes. Furthermore, we studied the recognition problem for distance graphs which generalize powers of paths. The main problem left open in this paper is the recognition of distance graphs without further simplifying assumptions.

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