Convex potentials and optimal shift generated oblique duals in shift invariant spaces

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Abstract

We introduce an extension of the convex potentials for finite frames (e.g. the frame potential defined by Benedetto and Fickus) in the framework of Bessel sequences of integer translates of finite sequences in $L^2(\mathbb{R}^k)$. We show that under a natural normalization hypothesis, these convex potentials detect tight frames as their minimizers. We obtain a detailed spectral analysis of the frame operators of shift generated oblique duals of a fixed frame of translates. We use this result to obtain the spectral and geometrical structure of optimal shift generated oblique duals with norm restrictions, that simultaneously minimize every convex potential; we approach this problem by showing that the water-filling construction in probability spaces is optimal with respect to submajorization (within an appropriate set of functions) and by considering a non-commutative version of this construction for measurable fields of positive operators.

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1 Introduction

Let $\mathcal{W}$ be a closed subspace of a separable complex Hilbert space $\mathcal{H}$ and let $\mathbb{I}$ be a finite or countable infinite set. A (possibly finite) sequence $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}}$ in $\mathcal{W}$ is a frame for $\mathcal{W}$ if there exist positive constants $0 < a \leq b$ such that

$$a\,\|f\|^2 \leq \sum_{i\in\mathbb{I}} |\langle f, f_i\rangle|^2 \leq b\,\|f\|^2 \quad \text{for every} \quad f \in \mathcal{W}\,.$$

If $a = b$ then we say that $\mathcal{F}$ is a tight frame for $\mathcal{W}$.

A frame $\mathcal{F}$ for $\mathcal{W}$ allows for linear (typically redundant) and stable encoding-decoding schemes of vectors (signals) in $\mathcal{W}$. Indeed, if $\mathcal{V}$ is a closed subspace of $\mathcal{H}$ such that $\mathcal{V} \oplus \mathcal{W}^\perp = \mathcal{H}$ (e.g. $\mathcal{V} = \mathcal{W}$) then it is possible to find frames $\mathcal{G} = \{g_i\}_{i\in\mathbb{I}}$ for $\mathcal{V}$ such that

$$f = \sum_{i\in\mathbb{I}} \langle f, g_i\rangle\ f_i\ , \quad \text{for } f \in \mathcal{W}\,.$$

The representation above lies within the theory of oblique duality (see [17, 18, 22, 23]). In applied situations, it is usually desired to develop encoding-decoding schemes as above, with some additional features. In some cases, we search for schemes with prescribed properties (e.g., for which the sequence of norms $\{\|f_i\|^2\}_{i\in\mathbb{I}}$ as well as the spectral properties of the family $\mathcal{F}$ are given in advance) leading to what is known in the literature as frame design problem (see [3, 6, 12, 14, 34, 41]). In other cases, we search for numerically robust oblique dual pairs $(\mathcal{F}, \mathcal{G})$ as above, leading to what is known as optimal frame designs ([5, 18, 24, 35, 39, 46]).

In their seminal work [6], Benedetto and Fickus introduced a functional defined on finite sequences of (unit norm) vectors $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n}$ (where $\mathbb{I}_n = \{1, \ldots, n\}$), the so-called *frame potential*, given by

$$\mathrm{FP}\,(\mathcal{F}) = \sum_{i,\,j\,\in\mathbb{I}_n} |\langle f_i\,,\, f_j\rangle|^{\,2}\ . \tag{1}$$

In case $\dim \mathcal{H} = d \in \mathbb{N}$ then one of their major results shows that tight unit norm frames can be characterized as (local) minimizers of this functional, among unit norm frames. Since then, there has been interest in (local) minimizers of the frame potential within certain classes of frames, since such minimizers can be considered as natural substitutes of tight frames (see for example [13, 35]). Recently, there has been interest in the structure of minimizers of other potentials such as the so-called mean squared error (MSE) (see [24, 36, 41] and the references therein). Both the frame potential and the MSE are examples of the so-called convex potentials introduced in [35]. It turns out that minimizers of these convex potentials share the spectral and geometrical structure of minimizers of the frame potential. Now, it is a well known fact that in case $\mathcal{V} = \mathcal{W}$ then tight frames $\mathcal{F}$ for $\mathcal{W}$ - i.e. minimizers of convex potentials - give rise to optimal (numerically robust) dual pairs $(\mathcal{F}, \mathcal{G})$. Therefore, it seems apparent that in the general case $\mathcal{V} \oplus \mathcal{W}^{\perp}$ the construction of robust oblique dual pairs $(\mathcal{F}, \mathcal{G})$ is related with the construction of frames $\mathcal{F}$ which are minimizers of convex potentials (e.g. the frame potential).

It turns out that there is a single notion that ties all the previous problems together namely, the majorization preorder. Indeed, majorization is the key notion behind the frame design problems (see [3, 12, 14]) through natural extensions of the Schur-Horn theorem from matrix analysis (see [3, 9, 10, 30]). Moreover, the relation between majorization and tracial inequalities with respect to convex functions allows to apply this notion in the study of convex potentials ([5, 35, 36, 37, 39]). Unfortunately, the convex potentials considered in [35] (in particular, the frame potential) can only be defined for finite frames. Hence, in the infinite dimensional context we loose a tool which have proved useful as a measure of stability for frames in finite dimensional Hilbert spaces.

In this paper we show that there are natural analogues of the convex potentials (and in particular, of the frame potential) in the context of Bessel sequences of integer translates $E(\mathcal{F})$ of a finite family of vectors $\mathcal{F} = \{f_i\}_{i \in \mathbb{I}_n}$ that lie in a finitely generated shift invariant (FSI) subspace $\mathcal{V}$ of $L^2(\mathbb{R}^k)$. We take advantage of the detailed structure of $E(\mathcal{F})$ as a fibered family over the torus $\mathbb{T}^k$ (see [7, 8, 42]) and the theory of range functions for shift invariant (SI) subspaces and introduce the potential $P_{\varphi}^{\mathcal{V}}(E(\mathcal{F}))$ associated to a convex function φ and $\mathcal{V}$ as an integral over $\mathbb{T}^k$ of the corresponding potentials on the fibers (for related approaches to different problems in SI subspaces see [1, 27, 28]). In order to verify that our definition is a natural extension of the convex potentials for finite frames we show that under natural normalization conditions, a family of integer translates $E(\mathcal{F})$ that is a tight frame for a FSI subspace $\mathcal{V}$ is a minimizer of the convex potential associated to φ and $\mathcal{V}$; this is an example of the fact that the problems considered in here as well as our approach to deal with them are global in nature i.e., are not obtained fiber-wise.

The convex potentials in FSI subspaces raise several questions related with optimal design problems. In particular, given FSI subspaces $\mathcal{W}$, $\mathcal{V}$ such that $\mathcal{V} \oplus \mathcal{W}^{\perp} = L^2(\mathbb{R}^k)$ and a finite family $\mathcal{F} = \{f_i\}_{i \in \mathbb{I}_n}$ such that $E(\mathcal{F})$ is a frame for $\mathcal{W}$, we consider the problem of designing optimal oblique duals $E(\mathcal{G})$ which are translates of a family $\mathcal{G} = \{g_i\}_{i \in \mathbb{I}_n}$ in $\mathcal{V}$ and such that $\mathcal{G}$ satisfies the norm restrictions $\sum_{i \in \mathbb{I}_n} \|g_i\|^2 \geq w$, for $w > 0$. In order to deal with this problem we develop two new tools in the context of frames of translates. On the one hand, we obtain what we call the fine spectral structure of shift generated (SG) oblique dual frames of the fixed frame $E(\mathcal{F})$, which is a detailed description of the eigenvalues of the measurable field of positive operators defined on $\mathbb{T}^k$ corresponding to the frame operators of SG oblique duals of $E(\mathcal{F})$. As a consequence, we derive necessary and sufficient conditions for the existence of a tight SG oblique dual $E(\mathcal{G})$ of $E(\mathcal{F})$. On the other

hand, we consider the water-filling construction (both for functions in probability spaces as well as for measurable field of positive finite-rank operators) and show that this construction leads to optimal solutions of the oblique dual design problem; this is achieved by showing that the water-filling constructions are optimal with respect to majorization (considered in the general context of probability spaces) which is a result of independent interest. With these tools we completely solve the problem of designing optimal oblique dual frames with norm restriction mentioned before; it turns out that these optimal SG oblique duals are more stable than the so-called canonical oblique dual. We point out that the structure of the optimal solution is obtained in terms of a global analysis. As a byproduct we extend the so-called Fan-Pall interlacing theorem from matrix analysis to the context of measurable fields of positive matrices.

The paper is organized as follows. In section 2 we describe several preliminary notions and facts from frame theory, SI subspaces, frames of translates and majorization theory in probability spaces. In section 3 we introduce the convex potentials for frames of translates and show that are natural extensions of the convex potentials for finite frames. In section 4 we recall several facts from the theory of oblique duality in FSI subspaces and obtain the precise value of the aliasing norm corresponding to the consistent sampling in this setting. Then, we describe the fine spectral structure of oblique duals of a fixed SG frame. Since this result depends on an extension of the Fan-Pall interlacing theory, its proof is presented in an appendix (see section 6). In section 5 we study the problem of optimal design of oblique dual frames $E(\mathcal{G})$ - of a fixed finitely SG frame $E(\mathcal{F})$ - which satisfy certain norm restrictions. We first show that the water-filling construction for positive functions in probability spaces is optimal with respect to sub-majorization within a natural set of functions. We then construct optimal SG oblique dual with norm restrictions and explain the relation of our construction with a natural (non-commutative) water-filling construction for measurable field of positive finite-rank operators. The paper ends with an appendix section in which we develop the Fan-Pall inequalities for measurable fields of positive matrices as well as some consequences of this result.

2 Preliminaries

In this section we recall some basic facts related with frame theory, oblique duality and shift invariant (SI) subspaces of $L^2(\mathbb{R}^k)$. At the end of this section we describe majorization between functions in arbitrary probability spaces.

General Notation

Througout this work we shall use the following notation: the space of complex $d \times d$ matrices is denoted by $\mathcal{M}_d(\mathbb{C})$ and $\mathcal{M}_d(\mathbb{C})^+$ the set of positive semidefinite matrices. $\mathcal{G}l(d)$ is the group of invertible elements of $\mathcal{M}_d(\mathbb{C})$ and $\mathcal{G}l(d)^+ = \mathcal{M}_d(\mathbb{C})^+ \cap \mathcal{G}l(d)$. If $T \in \mathcal{M}_d(\mathbb{C})$, we denote by $\|T\|$ its spectral norm, by $\operatorname{rk} T = \dim R(T)$ the rank of T, and by $\operatorname{tr} T$ the trace of T.

Given $d \in \mathbb{N}$ we denote by $\mathbb{I}_d = \{1, \dots, d\} \subseteq \mathbb{N}$. For a vector $x \in \mathbb{R}^m$ we denote by $x^\downarrow$ (resp. $x^\uparrow$) the rearrangement of x in decreasing (resp. increasing) order. We denote by $(\mathbb{R}^d)^\downarrow = \{x \in \mathbb{R}^d : x = x^\downarrow\}$ the set of downwards ordered vectors, and $(\mathbb{R}^d)^\uparrow = \{x \in \mathbb{R}^d : \ x = x^\uparrow\}$.

Given $S \in \mathcal{M}_d(\mathbb{C})^+$, we write $\lambda(S) = \lambda^\downarrow(S) = (\lambda_1(S), \dots, \lambda_d(S)) \in (\mathbb{R}^d)^\downarrow$ the vector of eigenvalues of S - counting multiplicities - arranged in decreasing order. Similarly we denote

by $\lambda^{\uparrow}(S) \in (\mathbb{R}^d)^{\uparrow}$ the reverse ordered vector of eigenvalues of S.

If $W \subseteq \mathbb{C}^d$ is a subspace we denote by $P_W \in \mathcal{M}_d(\mathbb{C})^+$ the orthogonal projection onto W. Given $x\,,\, y \in \mathbb{C}^d$ we denote by $x \otimes y \in \mathcal{M}_d(\mathbb{C})$ the rank one matrix given by

$$x \otimes y\,(z) = \langle z\,,\, y\rangle\, x \quad \text{ for every } \quad z \in \mathbb{C}^d\,. \tag{2}$$

Note that, if $x \neq 0$, then the projection $P_x \stackrel{\text{def}}{=} P_{\text{span}\{x\}} = \|x\|^{-2}\, x \otimes x\,.$

2.1 Frames for subspaces and oblique duality

In what follows $\mathcal{H}$ denotes a separable complex Hilbert space and $\mathbb{I}$ denotes a finite or countable infinite set. Let $\mathcal{W}$ be a closed subspace of $\mathcal{H}$: recall that a sequence $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}}$ in $\mathcal{W}$ is a *frame* for $\mathcal{W}$ if there exist positive constants $0 < a \leq b$ such that

$$a\,\|f\|^2 \leq \sum_{i\in\mathbb{I}} |\langle f, f_i\rangle|^2 \leq b\,\|f\|^2 \quad \text{ for every } \quad f \in \mathcal{W}\,. \tag{3}$$

In general, if $\mathcal{F}$ satisfies the inequality to the right in Eq. (3) we say that $\mathcal{F}$ is a b-Bessel sequence.

Given a Bessel sequence $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}}$ we consider its *synthesis operator* $T_{\mathcal{F}} \in L(\ell^2(\mathbb{I}), \mathcal{H})$ given by $T_{\mathcal{F}}((a_i)_{i\in\mathbb{I}}) = \sum_{i\in\mathbb{I}} a_i\ f_i$ which, by hypothesis on $\mathcal{F}$, is a bounded linear transformation. We also consider $T^*_{\mathcal{F}} \in L(\mathcal{H}, \ell^2(\mathbb{I}))$ called the *analysis operator* of $\mathcal{F}$, given by $T^*_{\mathcal{F}}(f) = (\langle f, f_i\rangle)_{i\in\mathbb{I}}$ and the *frame operator* of $\mathcal{F}$ defined by $S_{\mathcal{F}} = T_{\mathcal{F}}\, T^*_{\mathcal{F}}$. It is straightforward to check that

$$\langle S_{\mathcal{F}} f, f\rangle = \sum_{i\in\mathbb{I}} |\langle f, f_i\rangle|^2 \quad \text{ for every } \quad f \in \mathcal{H}\ .$$

Hence, $S_{\mathcal{F}}$ is a positive semidefinite bounded operator; moreover, a Bessel sequence $\mathcal{F}$ in $\mathcal{W}$ is a frame for $\mathcal{W}$ if and only if $S_{\mathcal{F}}$ is an invertible operator when restricted to $\mathcal{W}$ or equivalently, if the range of $T_{\mathcal{F}}$ coincides with $\mathcal{W}$.

In order to describe oblique duality, we fix two closed subspaces $\mathcal{V}, \mathcal{W} \subseteq \mathcal{H}$ such that $\mathcal{W}^{\perp} \oplus \mathcal{V} = \mathcal{H}$, that is such that $\mathcal{W}^{\perp} + \mathcal{V} = \mathcal{H}$ and $\mathcal{W}^{\perp} \cap \mathcal{V} = \{0\}$. Hence, $\mathcal{W}^{\perp}$ is a common (algebraic) complement of $\mathcal{W}$ and $\mathcal{V}$. It is well known that in this case $P_{\mathcal{W}}|_{\mathcal{V}} : \mathcal{V} \to \mathcal{W}$ is a linear bounded isomorphism so, in particular, we see that $\dim \mathcal{V} = \dim \mathcal{W}$ as Hilbert spaces. Moreover, the conditions $\mathcal{W}^{\perp} \oplus \mathcal{V} = \mathcal{H}$ and $\mathcal{W} \oplus \mathcal{V}^{\perp} = \mathcal{H}$ are actually equivalent.

Fix a frame $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}}$ for $\mathcal{W}$. Following [22, 23] (see also [18]), given a Bessel sequence $\mathcal{G} = \{g_i\}_{i\in\mathbb{I}}$ in $\mathcal{V}$ we say that $\mathcal{G}$ is a (oblique) $\mathcal{V}$-dual of $\mathcal{F}$ if

$$f = \sum_{i\in\mathbb{I}} \langle f, g_i\rangle\ f_i \quad \text{ for every } \quad f \in \mathcal{W}\ .$$

It turns out (see [22, 23]) that if $\mathcal{G}$ is a $\mathcal{V}$-dual of $\mathcal{F}$ then $T_{\mathcal{F}}\, T^*_{\mathcal{G}} = P_{\mathcal{W}//\mathcal{V}^{\perp}}$, where $P_{\mathcal{W}//\mathcal{V}^{\perp}}$ denotes the oblique projection with range $\mathcal{W}$ and null space $\mathcal{V}^{\perp}$. In this case, by taking adjoints in the previous identity we also get that $T_{\mathcal{G}}\, T^*_{\mathcal{F}} = P^*_{\mathcal{W}//\mathcal{V}^{\perp}} = P_{\mathcal{V}//\mathcal{W}^{\perp}}$. Hence, $T_{\mathcal{G}}$ is onto $\mathcal{V}$ and then $\mathcal{G}$ is a frame for $\mathcal{V}$; moreover, we obtain the reconstruction formula

$$g = \sum_{i\in\mathbb{I}} \langle g, f_i\rangle\ g_i \quad \text{ for every } \quad g \in \mathcal{V}\ .$$

We consider the set of oblique $\mathcal{V}$-duals of $\mathcal{F}$ given by

$$\mathcal{D}_{\mathcal{V}}(\mathcal{F}) = \{\mathcal{G} = \{g_i\}_{i\in\mathbb{I}} \text{ is a } \mathcal{V}\text{-dual of } \mathcal{F}\}\ . \tag{4}$$

Remark 2.1. Let $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}}$ be a frame for $\mathcal{W}$. If we set $\mathcal{V} = \mathcal{W}$ then a Bessel sequence $\mathcal{G}$ in $\mathcal{W}$ is a $\mathcal{W}$-dual of $\mathcal{F}$ if it is a dual frame for $\mathcal{F}$ in $\mathcal{W}$ in the classical sense (see [16]) i.e. $T_{\mathcal{G}}\, T_{\mathcal{F}}^* = P_{\mathcal{W}}$. Hence

$$\mathcal{D}_{\mathcal{W}}(\mathcal{F}) = \mathcal{D}(\mathcal{F}) \stackrel{\text{def}}{=} \{\mathcal{G} = \{g_i\}_{i\in\mathbb{I}} \text{ is a dual frame for } \mathcal{F} \text{ in } \mathcal{W}\} \ .$$

Recall that there is a distinguished (classical) dual, called the canonical dual of $\mathcal{F}$, denoted $\mathcal{F}^{\#} = \{f_i^{\#}\}_{i\in\mathbb{I}}$, given by $f_i^{\#} = S_{\mathcal{F}}^{\dagger} f_i$ for $i \in \mathbb{I}$, where $S_{\mathcal{F}}^{\dagger}$ denotes the Moore-Penrose pseudo-inverse of the (closed range positive semidefinite operator) $S_{\mathcal{F}}$.

In the general context of oblique duality there also exists a distinguished $\mathcal{V}$-dual for $\mathcal{F}$, the so-called *canonical $\mathcal{V}$-dual*, which we denote by

$$\mathcal{F}_{\mathcal{V}}^{\#} = \{f_{\mathcal{V},i}^{\#}\}_{i\in\mathbb{I}} \quad \text{given by} \quad f_{\mathcal{V},i}^{\#} = P_{\mathcal{V}//\mathcal{W}^{\perp}} f_i^{\#} \quad \text{for every} \quad i \in \mathbb{I} \ ,$$

where $\mathcal{F}^{\#} = \{f_i^{\#}\}_{i\in\mathbb{I}}$ is the canonical dual of $\mathcal{F}$. It turns out that the encoding-decoding scheme based on the oblique dual pair $(\mathcal{F}, \mathcal{F}_{\mathcal{V}}^{\#})$ has several optimality properties (see [22, 23]). $\triangle$

2.2 Shift-invariant subspaces and frames of translates

In what follows we consider $L^2(\mathbb{R}^k)$ (with respect to Lebesgue measure) as a separable and complex Hilbert space. Recall that a closed subspace $\mathcal{V} \subseteq L^2(\mathbb{R}^k)$ is *shift-invariant* (SI) if $f \in \mathcal{V}$ implies $T_\ell f \in \mathcal{V}$ for any $\ell \in \mathbb{Z}^k$, where $T_y f(x) = f(x-y)$ is the translation by $y \in \mathbb{R}^k$. For example, if we take a subset $\mathcal{A} \subset L^2(\mathbb{R}^k)$ then,

$$\mathcal{S}(\mathcal{A}) = \overline{\text{span}}\ \{T_\ell f : f \in \mathcal{A},\ \ell \in \mathbb{Z}^k\} \tag{5}$$

is a shift-invariant subspace called the *SI subspace generated by* $\mathcal{A}$. Indeed, $\mathcal{S}(\mathcal{A})$ is the smallest SI subspace that contains $\mathcal{A}$. We say that a SI subspace $\mathcal{V}$ is *finitely generated* (FSI) if there exists a finite set $\mathcal{A} \subset L^2(\mathbb{R}^k)$ such that $\mathcal{V} = S(\mathcal{A})$. In this case, the *length* of $\mathcal{V}$ is the smallest cardinal $\#(\mathcal{A})$ such that $S(\mathcal{A}) = \mathcal{V}$.

In order to describe the fine structure of a SI subspace we consider the following representation of $L^2(\mathbb{R}^k)$ (see [7, 8, 42] and [11] for extensions of these notions to the more general context of actions of locally compact abelian groups). Let $\mathbb{T} = [-1/2, 1/2)$ endowed with the Lebesgue measure and let $L^2(\mathbb{T}^k, \ell^2(\mathbb{Z}^k))$ be the Hilbert space of square integrable $\ell^2(\mathbb{Z}^k)$-valued functions that consists of all vector valued measurable functions $\phi : \mathbb{T}^k \to \ell^2(\mathbb{Z}^k)$ with the norm

$$\|\phi\|^2 = \int_{\mathbb{T}^k} \|\phi(x)\|_{\ell^2(\mathbb{Z}^k)}^2 \, dx < \infty.$$

Then, $\Gamma : L^2(\mathbb{R}^k) \to L^2(\mathbb{T}^k, \ell^2(\mathbb{Z}^k))$ defined for $f \in L^1(\mathbb{R}^k) \cap L^2(\mathbb{R}^k)$ by

$$\Gamma f : \mathbb{T}^k \to \ell^2(\mathbb{Z}^k) \ , \quad \Gamma f(x) = (\hat{f}(x+\ell))_{\ell\in\mathbb{Z}^k}, \tag{6}$$

extends uniquely to an isometric isomorphism between $L^2(\mathbb{R}^k)$ and $L^2(\mathbb{T}^k, \ell^2(\mathbb{Z}^k))$; here $\hat{f}$ denotes the Fourier transform of $f \in L^2(\mathbb{R}^k)$.

Let $\mathcal{V} \subseteq L^2(\mathbb{R}^k)$ be a SI subspace; for $x \in \mathbb{T}^k$ let $J_{\mathcal{V}}(x)$ be the closed subspace of $\ell^2(\mathbb{Z}^k)$ given by

$$J_{\mathcal{V}}(x) = \{\Gamma f(x) : \ f \in \mathcal{V}\}^{-\|\cdot\|} . \tag{7}$$

Then, the function $J_{\mathcal{V}} : \mathbb{T}^k \to \{$ closed subspaces of $\ell^2(\mathbb{Z}^k)\}$ is the *measurable range function* associated with $\mathcal{V}$. That is, if $P_{J_{\mathcal{V}}(x)}$ denotes the orthogonal projection onto $J_{\mathcal{V}}(x)$ for $x \in \mathbb{T}^k$ then for every ξ, $\eta \in \ell^2(\mathbb{Z}^k)$ the function $x \mapsto \langle P_{J_{\mathcal{V}}(x)}\, \xi\,,\, \eta\rangle$ is measurable and

$$\mathcal{V} = \{f \in L^2(\mathbb{R}^k) : \Gamma f(x) \in J_{\mathcal{V}}(x) \text{ for a.e. } x \in \mathbb{T}^k\}. \tag{8}$$

Indeed, by [8, Prop.1.5] Eq. (8) establishes a bijection between SI subspaces of $L^2(\mathbb{R}^k)$ and measurable range functions.

Given SI subspaces $\mathcal{V}$ and $\mathcal{W}$, recall that a bounded linear transformation $S \in L(\mathcal{V}, \mathcal{W})$ is *shift preserving* (SP) if $T_\ell\, S = S\, T_\ell$ for every $\ell \in \mathbb{Z}^k$. In this case, for $x \in \mathbb{T}^k$, let $\hat{S}_x \in L(J_{\mathcal{V}}(x), J_{\mathcal{W}}(x))$ be the linear transformation determined by

$$\hat{S}_x\big(\Gamma f(x)\,\big) = \Gamma(Sf)(x) \quad \text{ for } \ f \in \mathcal{V}\,. \tag{9}$$

Then, (see [8, Prop.4.5]) the function $\hat{S} : \mathbb{T}^k \to \coprod_{x\in\mathbb{T}^k} L(J_{\mathcal{V}}(x), J_{\mathcal{W}}(x))$ is weakly measurable, i.e. for every ξ, $\eta \in \ell^2(\mathbb{Z}^k)$ the function $x \mapsto \langle \hat{S}_x\, P_{J_{\mathcal{V}}(x)}\, \xi\,,\, \eta\rangle$ is measurable, and essentially bounded; indeed $\operatorname{ess\,sup}_{x\in\mathbb{T}^k} \|\hat{S}_x\| = \|S\|$. Conversely, if $s : \mathbb{T}^k \to \coprod_{x\in\mathbb{T}^k} L(J_{\mathcal{V}}(x), J_{\mathcal{W}}(x))$ is weakly measurable and essentially bounded then there exists a unique bounded linear transformation $S \in L(\mathcal{V}, \mathcal{W})$ that is SP and such that $\hat{S} = s$. For example, consider $P_{\mathcal{V}} \in L(L^2(\mathbb{R}^k))$ the orthogonal projection onto $\mathcal{V}$; then, $P_{\mathcal{V}}$ is SP so that $(P_{\mathcal{V}})\hat{}\,: \mathbb{T}^k \to L(\ell^2(\mathbb{Z}^k))$ is given by $(P_{\mathcal{V}})\hat{}_x = P_{J_{\mathcal{V}}(x)}$ i.e., the orthogonal projection onto $J_{\mathcal{V}}(x)$, for a.e. $x \in \mathbb{T}^k$.

The previous notions associated with SI subspaces and SP operators allow to develop a detailed study of frames of translates. Indeed, let $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}}$ be a (possibly finite) sequence in $L^2(\mathbb{R}^k)$. We denote by $E(\mathcal{F})$ the family of translates of $\mathcal{F}$, namely $E(\mathcal{F}) = \{T_\ell f_i\}_{(\ell,i)\in\mathbb{Z}^k\times\mathbb{I}}$. For $x \in \mathbb{T}^k$, let $\Gamma\mathcal{F}(x) = \{\Gamma f_i(x)\}_{i\in\mathbb{I}}$ which is a (possibly finite) sequence in $\ell^2(\mathbb{Z}^k)$. Then [8, 42] $E(\mathcal{F})$ is a b-Bessel sequence if and only if $\Gamma\mathcal{F}(x)$ is a b-Bessel sequence for a.e. $x \in \mathbb{T}^k$. In this case, we consider $T_{\Gamma\mathcal{F}(x)} : \ell^2(\mathbb{I}) \to \ell^2(\mathbb{Z}^k)$ and $S_{\Gamma\mathcal{F}(x)} : \ell^2(\mathbb{Z}^k) \to \ell^2(\mathbb{Z}^k)$ the synthesis and frame operators of $\Gamma\mathcal{F}(x)$, respectively, for $x \in \mathbb{T}^k$; it is straightforward to check that $S_{E(\mathcal{F})}$ is a SP operator.

If $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}}$ and $\mathcal{G} = \{g_i\}_{i\in\mathbb{I}}$ are such that $E(\mathcal{F})$ and $E(\mathcal{G})$ are Bessel sequences then (see [27, 42]) the following fundamental relation holds:

$$(T_{E(\mathcal{G})}\, T^*_{E(\mathcal{F})})\hat{}_x = T_{\Gamma\mathcal{G}(x)}\, T^*_{\Gamma\mathcal{F}(x)}\ , \quad \text{for a.e } x \in \mathbb{T}^k\,. \tag{10}$$

Eq. (10) has several consequences; indeed, if $\mathcal{W}$ is a SI subspace of $L^2(\mathbb{R}^k)$ and we assume further that $\mathcal{F}$, $\mathcal{G} \in \mathcal{W}^n$ then:

1. For every f, $g \in L^2(\mathbb{R}^k)$,

$$\langle S_{E(\mathcal{F})}\, f,\, g\rangle = \int_{\mathbb{T}^k} \langle S_{\Gamma\mathcal{F}(x)}\ \Gamma f(x),\, \Gamma g(x)\rangle_{\ell^2(\mathbb{Z}^k)}\ dx\ .$$

   This last fact implies that $(S_{E(\mathcal{F})})\hat{}_x = S_{\Gamma\mathcal{F}(x)}$ for a.e. $x \in \mathbb{T}^k$; moreover, it also holds that $E(\mathcal{F})$ is a frame for $\mathcal{W}$ with frame bounds $0 < a \le b$ if and only if $\Gamma\mathcal{F}(x)$ is a frame for $J_{\mathcal{W}}(x)$ with frame bounds $0 < a \le b$ for a.e. $x \in \mathbb{T}^k$ (see [8]).

2. Since $(P_{\mathcal{W}})\hat{}_x = P_{J_{\mathcal{W}}(x)}$ for a.e. $x \in \mathbb{T}^k$ then $E(\mathcal{G})$ is a (classical) dual for $E(\mathcal{F})$ in $\mathcal{W}$ if and only if $\Gamma\mathcal{G}(x)$ is a (classical) dual for $\Gamma\mathcal{F}(x)$ in $J_{\mathcal{W}}(x)$ for a.e. $x \in \mathbb{T}^k$ (see [8, 27, 28]).

2.3 Majorization in probability spaces

Majorization between vectors (see [4, 33]) has played a key role in frame theory. On the one hand, majorization allows to characterize the existence of frames with prescribed properties (see [3, 12, 14]). On the other hand, majorization is a preorder relation that implies a family of tracial inequalities; this last fact can be used to explain the structure of minimizers of the so-called Benedetto-Fickus frame potential ([6, 13]) as well as more general convex potentials for finite frames (see [34, 35, 36, 37, 39]). In the next section we extend the notion of convex potentials to the context of Bessel families of translates of finite sequences; therefore, we will need the following general notion of majorization between functions in probability spaces.

Throughout this section the triple $(X, \mathcal{X}, \mu)$ denotes a probability space i.e. $\mathcal{X}$ is a σ-algebra of sets in X and μ is a probability measure defined on $\mathcal{X}$. We shall denote by $L^\infty(X,\mu)^+ = \{f \in L^\infty(X,\mu) : f \geq 0\}$. For $f \in L^\infty(X,\mu)^+$, the *decreasing rearrangement* of f (see [33]), denoted $f^* : [0,1) \to \mathbb{R}^+$, is given by

$$f^*(s) \stackrel{\text{def}}{=} \sup\{t \in \mathbb{R}^+ : \ \mu\{x \in X : \ f(x) > t\} > s\} \quad \text{for every} \quad s \in [0,1)\,. \tag{11}$$

Remark 2.2. We mention some elementary facts related with the decreasing rearrangement of functions that we shall need in the sequel. Let $f \in L^\infty(X,\mu)^+$, then:

1. f^* is a right-continuous and non-increasing function.
2. f and f^* are equimeasurable i.e. for every Borel set $A \subset \mathbb{R}$ then $\mu(f^{-1}(A)) = |(f^*)^{-1}(A)|$, where $|B|$ denotes the Lebesgue measure of the Borel set $B \subset \mathbb{R}$. In turn, this implies that for every continuous $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ then: $\varphi \circ f \in L^\infty(X,\mu)$ iff $\varphi \circ f^* \in L^\infty([0,1])$ and in this case
$$\int_X \varphi \circ f \ d\mu = \int_0^1 \varphi \circ f^* \ dx\,.$$
3. If $g \in L^\infty(X,\mu)$ is such that $f \leq g$ then $0 \leq f^* \leq g^*$; moreover, in case $f^* = g^*$ then $f = g$.
4. If we consider the probability space $([0,1], \mathcal{B}, dt)$ - Borel sets in [0,1] with Lebesgue measure - then $f^* \in L^\infty([0,1], dt)$ is such that $(f^*)^* = f^*$.
5. If $c \in \mathbb{R}$ is such that $f + c \geq 0$ then $(f+c)^* = f^* + c$. $\triangle$

Definition 2.3. Let $f, g \in L^\infty(X,\mu)^+$ and let f^*, g^* denote their decreasing rearrangements. We say that f *submajorizes* g (in $(X, \mathcal{X}, \mu)$), denoted $g \prec_w f$, if

$$\int_0^s g^*(t) \ dt \ \leq \ \int_0^s f^*(t) \ dt \quad \text{for every} \quad 0 \leq s \leq 1\,.$$

If we further have that $\int_0^1 g^*(t) \ dt = \int_0^1 f^*(t) \ dt$ we say that f *majorizes* g and write $g \prec f$. $\triangle$

In order to check that majorization holds between functions in probability spaces, we can consider the so-called *doubly stochastic maps*. Recall that a linear operator D acting on $L^\infty(X,\mu)$ is a doubly-stochastic map if D is unital, positive and trace preserving i.e.

$$D(1_X) = 1_X\ , \quad D\big(L^\infty(X,\mu)^+\big) \subseteq L^\infty(X,\mu)^+ \quad \text{and} \quad \int_X D(f)(x)\ d\mu(x) = \int_X f(x)\ d\mu(x)$$

for every $f \in L^\infty(X,\mu)$. It is worth pointing out that D is necessarily a contractive map.

Our interest in majorization relies in its relation with integral inequalities in terms of convex functions. The following result summarizes this relation as well as the role of the doubly stochastic maps (see for example [15, 43]).

Theorem 2.4. Let $f,\, g \in L^\infty(X,\mu)^+$. Then the following conditions are equivalent:

1. $g \prec f$;
2. There is a doubly stochastic map D acting on $L^\infty(X,\mu)$ such that $D(f) = g$;
3. For every convex function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ we have that

$$\int_X \varphi(g(x))\; d\mu(x) \leq \int_X \varphi(f(x))\; d\mu(x)\;. \tag{12}$$

In case we only have $g \prec_w f$ then Eq. (12) holds if we assume further that φ is an increasing convex function. □

Example 2.5. The operator D given by $D(f) = (\int_X f\; d\mu)\cdot 1_X$ is a doubly stochastic map. Hence, we get the majorization relation $(\int_X f\; d\mu)\cdot 1_X \prec f$. Therefore, if $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is any convex function and $f \in L^\infty(X,\mu)^+$ then, by Theorem 2.4, we have that

$$\varphi(\int_X f\; d\mu) = \int_X \varphi((\int_X f\; d\mu)\cdot 1_X(x))\; d\mu(x) \leq \int_X \varphi(f(x))\; d\mu(x)\;, \tag{13}$$

which is an instance of the classical Jensen's inequality. Using the previous facts, notice that if $c \in \mathbb{R}$ is such that $0 \leq c \leq \int_X f\; d\mu$ then it is easy to see that $c\cdot 1_X \prec_w f$. △

The following result will play a key role in the study of the structure of minimizers of $\prec_w$ within (appropriate) sets of functions.

Proposition 2.6 ([15])**.** Let $f,\, g \in L^\infty(X,\mu)^+$ such that $g \prec_w f$. If there exists a non-decreasing and strictly convex function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\int_X \varphi(f(x))\; d\mu(x) = \int_X \varphi(g(x))\; d\mu(x) \quad \text{ then } \quad g^* = f^*\;. \qquad \square$$

With the notations of Example 2.5 notice that Proposition 2.6 implies (the known fact) that if φ is strictly convex and such that equality holds in Jensen's inequality Eq. (13) then $f^* = \int_X f\; d\mu$ and hence $f = \int_X f\; d\mu$.

3 Convex potentials for sequences of translates in FSI spaces

We begin by describing the convex potentials for finite sequences of vectors with respect to a finite dimensional subspace. We consider the sets

$$\mathrm{Conv}(\mathbb{R}^+) = \{\varphi : \mathbb{R}^+ \to \mathbb{R}^+\;,\; \varphi \text{ is a convex function } \}$$

and $\mathrm{Conv_s}(\mathbb{R}^+) = \{\varphi \in \mathrm{Conv}(\mathbb{R}^+)\ ,\ \varphi \text{ is strictly convex }\}$.

Now, given $\varphi \in \mathrm{Conv}(\mathbb{R}^+)$ and a finite dimensional subspace $\mathcal{W} \subset \mathcal{H}$, then the *convex potential* associated to $(\varphi, \mathcal{W})$, denoted by $P_\varphi^{\mathcal{W}}$, is defined as follows: for a finite sequence $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n} \in \mathcal{W}^n$ with frame operator $S_\mathcal{F} \in L(\mathcal{H})^+$,

$$P_\varphi^{\mathcal{W}}(\mathcal{F}) = \mathrm{tr}[\varphi(S_\mathcal{F})\, P_\mathcal{W}] \tag{14}$$

where $\varphi(S_\mathcal{F}) \in L(\mathcal{H})^+$ is obtained by functional calculus and $\mathrm{tr}(\cdot)$ denotes the usual (semi-finite) trace in $L(\mathcal{H})$. Notice that by construction, $P_\mathcal{W}\, S_\mathcal{F} = S_\mathcal{F}\, P_\mathcal{W} = S_\mathcal{F}$: then, it is clear that

$$P_\varphi^{\mathcal{W}}(\mathcal{F}) = \sum_{i\in\mathbb{I}_d} \varphi(\lambda_i(S_\mathcal{F}))\,, \tag{15}$$

where $d = \dim \mathcal{W}$ and $(\lambda_i(S_\mathcal{F}))_{i\in\mathbb{I}_d} \in (\mathbb{R}^+)^d$ denotes the vector of eigenvalues of the positive operator $S_\mathcal{F}|_\mathcal{W} \in L(\mathcal{W})^+$, counting multiplicities and arranged in non-increasing order (we use the convention $\mathbb{I}_0 = \emptyset$). In particular, if $\varphi \in \mathrm{Conv}(\mathbb{R}^+)$ is such that $\varphi(0) = 0$ we get that

$$P_\varphi^{\mathcal{W}}(\mathcal{F}) = \mathrm{tr}\,(\varphi(S_\mathcal{F})) = \mathrm{tr}\,(\varphi(G_\mathcal{F}))\,,$$

where the $n \times n$ matrix $G_\mathcal{F} = (\langle f_i,\, f_j\rangle)_{i,j\in\mathbb{I}_n}$ is the Gramian matrix of the finite sequence $\mathcal{F}$. That is, if $\varphi(0) = 0$, then $P_\varphi^{\mathcal{W}} = P_\varphi$ does not depend on $\mathcal{W}$. For example, in case $\varphi(x) = x^2$, then $P_\varphi^{\mathcal{W}}(\mathcal{F}) = P_\varphi(\mathcal{F})$ coincides with the frame potential: indeed, by Eq. (1) we have that

$$P_\varphi^{\mathcal{W}}(\mathcal{F}) = P_\varphi(\mathcal{F}) = \mathrm{tr}(S_\mathcal{F}^2) = \mathrm{tr}((T_\mathcal{F}^*\, T_\mathcal{F})^2) = \sum_{i,\,j\,\in\mathbb{I}_n} |\langle f_i\,,\, f_j\rangle|^{\,2} = \mathrm{FP}\,(\mathcal{F})\ . \tag{16}$$

For $\varphi \in \mathrm{Conv}(\mathbb{R}^+)$ and a finite dimensional subspace $\mathcal{W} \subset \mathcal{H}$, $P_\varphi^{\mathcal{W}}(\mathcal{F})$ is a measure of the spread of the eigenvalues of the frame operator of $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n} \in \mathcal{W}^n$. That is, (under suitable normalization hypothesis on $\mathcal{F}$) the smaller the value $P_\varphi^{\mathcal{W}}(\mathcal{F})$ is, the more concentrated the eigenvalues of $S_\mathcal{F}|_\mathcal{W} \in L(\mathcal{W})^+$ are. This is the main motivation for considering these convex potentials (see [35, 36, 39, 41]).

Next we extend the notion of convex potential to the context of finitely generated shift invariant systems in FSI subspaces.

Definition 3.1. Let $\mathcal{W}$ be a FSI subspace in $L^2(\mathbb{R}^k)$, let $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n} \in \mathcal{W}^n$ be such that $E(\mathcal{F})$ is a Bessel sequence and consider $\varphi \in \mathrm{Conv}(\mathbb{R}^+)$. Then the convex potential associated to $(\varphi, \mathcal{W})$ on $E(\mathcal{F})$, denoted $P_\varphi^{\mathcal{W}}(E(\mathcal{F}))$, is given by

$$P_\varphi^{\mathcal{W}}(E(\mathcal{F})) = \int_{\mathbb{T}^k} P_\varphi^{J_\mathcal{W}(x)}(\Gamma\mathcal{F}(x))\ dx \tag{17}$$

where $P_\varphi^{J_\mathcal{W}(x)}(\Gamma\mathcal{F}(x)) = \mathrm{tr}(\varphi(S_{\Gamma\mathcal{F}(x)})\,(P_\mathcal{W})_x^{\frown})$ is the convex potential associated with $(\varphi, J_\mathcal{W}(x))$ of the sequence $\Gamma\mathcal{F}(x) = \{\Gamma\, f_i(x)\}_{i\in\mathbb{I}_n}$ in $\ell^2(\mathbb{Z}^k)$, for every $x \in \mathbb{T}^k$. $\triangle$

Next we develop some notions and tools in order to show that the right hand side in Eq. (17) is well defined, namely that the function $\mathbb{T}^k \ni x \mapsto P_\varphi^{J_\mathcal{W}(x)}(\Gamma\mathcal{F}(x))$ is integrable.

Let $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n}$ be a finite sequence in $L^2(\mathbb{R}^k)$ such that $E(\mathcal{F})$ is a Bessel sequence. Recall that in this case $S_{E(\mathcal{F})}$ is a SP operator and that for $x \in \mathbb{T}^k$ then $(S_{E(\mathcal{F})})_x^{\frown} = S_{\Gamma\mathcal{F}(x)} \in L(\ell^2(\mathbb{Z}^k))^+$ is a positive and finite rank operator.

The next lemma is a reformulation of a result in [42] concerning the existence of measurable functions of eigenvalues and eigenvectors of measurable fields of positive semidefinite $n \times n$ matrices.

Lemma 3.2. *Let $\mathcal{W}$ be a FSI subspace in $L^2(\mathbb{R}^k)$ and let $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n} \in \mathcal{W}^n$ be such that $E(\mathcal{F})$ is a Bessel sequence. Then, there exist:*

1. *a measurable function $r : \mathbb{T}^k \to \mathbb{N}_{\geq 0}$ and measurable vector fields $v_j : \mathbb{T}^k \to \ell^2(\mathbb{Z}^k)$ for $j \in \mathbb{I}_n$ such that $r(x) \leq n$ and $\{v_j(x)\}_{j=1}^{r(x)}$ is an orthonormal system in $J_{\mathcal{W}}(x)$ for a.e. $x \in \mathbb{T}^k$;*

2. *bounded measurable functions $\lambda_j : \mathbb{T}^k \to \mathbb{R}^+$ for $j \in \mathbb{I}_n$, such that $\lambda_1 \geq \ldots \geq \lambda_n$, $\lambda_j(x) = 0$ if $j > r(x)$ and*

$$(S_{E(\mathcal{F})})_x = \sum_{j=1}^{r(x)} \lambda_j(x)\ v_j(x) \otimes v_j(x)\ , \quad \text{for a.e. } x \in \mathbb{T}^k\,. \tag{18}$$

If we assume further that $E(\mathcal{F})$ is a frame for $\mathcal{W}$ then $r(x) = \dim J_{\mathcal{W}}(x)$ and $\{v_j(x)\}_{j=1}^{r(x)}$ is an orthonormal basis (ONB) for $J_{\mathcal{W}}(x)$ for a.e. $x \in \mathbb{T}^k$.

Proof. Consider the measurable field of positive semidefinite matrices $G : \mathbb{T}^k \to \mathcal{M}_n(\mathbb{C})^+$ given by the Gramian $G(x) = (\langle \Gamma f_i(x), \Gamma f_j(x)\rangle)_{i,j\in\mathbb{I}_n}$, for $x \in \mathbb{T}^k$. Notice that $G(x)$ is the matrix representation of $T^*_{\Gamma\mathcal{F}(x)} T_{\Gamma\mathcal{F}(x)} \in L(\mathbb{C}^n)^+$ with respect to the canonical basis of $\mathbb{C}^n$ for $x \in \mathbb{T}^k$. In particular, if b denotes a Bessel (upper) bound of $E(\mathcal{F})$ then

$$\text{ess sup}_{x\in\mathbb{T}^k} \|G(x)\| = \text{ ess sup}_{x\in\mathbb{T}^k} \|T_{\Gamma\mathcal{F}(x)} T^*_{\Gamma\mathcal{F}(x)}\| = \|S_{E(\mathcal{F})}\| \leq b\ ,$$

by the remarks at the end of Section 2.2. We set $r(x) = \text{rk}(G(x)) = \text{rk}(S_{\Gamma\mathcal{F}(x)})$ for $x \in \mathbb{T}^k$; therefore $r(\cdot) : \mathbb{T}^k \to \mathbb{N}_{\geq 0}$ is a measurable function such that $r(x) \leq n$ for $x \in \mathbb{T}^k$. Hence, by considering a convenient finite partition of $\mathbb{T}^k$ into measurable sets we can assume, without loss of generality, that $r(x) = r \in \mathbb{N}$ for a.e. $x \in \mathbb{T}^k$.

Using results from [42], we see that there exist measurable functions $\lambda_j : \mathbb{T}^k \to \mathbb{R}^+$ and measurable vector fields $u_j : \mathbb{T}^k \to \mathbb{C}^n$, for $j \in \mathbb{I}_n$, such that: $\lambda_j(x) \geq \lambda_{j+1}(x)$ for $j \in \mathbb{I}_{n-1}$, $\{u_j(x)\}_{j\in\mathbb{I}_n}$ is an ONB of $\mathbb{C}^n$ and $G(x)u_j(x) = \lambda_j(x)\, u_j(x)$ for $j \in \mathbb{I}_n$ and a.e. $x \in \mathbb{T}^k$. In particular, the functions $\lambda_j : \mathbb{T}^k \to \mathbb{R}^+$ satisfy $0 \leq \lambda_j(x) \leq \|G(x)\| \leq b$ for a.e. $x \in \mathbb{T}^k$, $j \in \mathbb{I}_n$; these remarks prove item 2 above.

Take the polar decomposition $T_{\Gamma\mathcal{F}(x)} = U(x)\,|T_{\Gamma\mathcal{F}(x)}|$, where $U(x) : \mathbb{C}^n \to J_{\mathcal{W}}(x) \subset \ell^2(\mathbb{Z}^k)$ is (the unique) partial isometry with $\ker U(x) = \ker T_{\Gamma\mathcal{F}(x)}$ for a.e. $x \in \mathbb{T}^k$. Hence, in this case $U(x) = T_{\Gamma\mathcal{F}(x)}\,(G^{1/2}(x))^\dagger$ and therefore $U(\cdot) : \mathbb{T}^k \to L(\mathbb{C}^n, \ell^2(\mathbb{Z}^k))$ is a well defined measurable field of partial isometries. Then, $v_j : \mathbb{T}^k \to \ell^2(\mathbb{Z}^k)$ given by $v_j(x) = U(x)\, u_j(x) \in J_{\mathcal{W}}(x)$ for $j \in \mathbb{I}_n$ and $x \in \mathbb{T}^k$ are measurable vector fields such that $\{v_j(x)\}_{j\in\mathbb{I}_r}$ is an orthonormal system in $J_{\mathcal{W}}(x)$, for a.e. $x \in \mathbb{T}^k$; moreover, $(S_{E(\mathcal{F})})_x\, v_j(x) = \lambda_j(x)\, v_j(x)$ for $j \in \mathbb{I}_r$ and a.e. $x \in \mathbb{T}^k$. Since $\text{rk}(S_{E(\mathcal{F})})_x = r$ for a.e. $x \in \mathbb{T}^k$, then we see that Eq. (18) holds in this case.

Finally, notice that if $E(\mathcal{F})$ is a frame for $\mathcal{W}$ then we should have that $r = \text{rk}(S_{E(\mathcal{F})})_x = \dim J_{\mathcal{W}}(x)$ for a.e. $x \in \mathbb{T}^k$ which shows the last part of the statement. □

Remark 3.3. Let $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n}$ be a finite sequence in $L^2(\mathbb{R}^k)$ such that $E(\mathcal{F})$ is a Bessel sequence. In what follows we consider the function

$$\mathbb{T}^k \ni x \mapsto \lambda(S_{\Gamma\mathcal{F}(x)}) \stackrel{\text{def}}{=} (\lambda_j(S_{\Gamma\mathcal{F}(x)}))_{j\in\mathbb{N}} \in \left(\ell^1_+(\mathbb{Z}^k)\right)^\downarrow \text{ (decreasing sequences)}\ , \tag{19}$$

where $\lambda(S_{\Gamma\mathcal{F}(x)})$ is the sequence of eigenvalues of the positive semidefinite finite rank operator $(S_{E(\mathcal{F})})\hat{}_x = S_{\Gamma\mathcal{F}(x)} \in L(\ell^2(\mathbb{Z}^k))$, counting multiplicities and arranged in non-increasing order. Notice that the maps $\lambda_j(x)$ of Lemma 3.2 are uniquely determined (a.e.) by the equation $\lambda_j(x) = \lambda_j(S_{\Gamma\mathcal{F}(x)})$, for $j \in \mathbb{I}_n$. The last entries are $\lambda_j(S_{\Gamma\mathcal{F}(x)}) = 0$ for $j \geq n+1$ and $x \in \mathbb{T}^k$. Therefore, another way to state item 2 of Lemma 3.2 is to say that the map $\mathbb{T}^k \ni x \mapsto \lambda(S_{\Gamma\mathcal{F}(x)})$ of Eq. (19) is measurable. We refer to this function as the *fine spectral structure* of $E(\mathcal{F})$. $\triangle$

Remark 3.4. Consider the notations from Definition 3.1. We now show that the right hand side in Eq. (17) is well defined. Indeed, by Lemma 3.2 we get a spectral representation of $(S_{E(\mathcal{F})})\hat{}_{(\cdot)}$ as in Eq. (18) in terms of the bounded and measurable functions $\lambda_j(\cdot) : \mathbb{T}^k \to \mathbb{R}^+$, for $j \in \mathbb{I}_n$. If we consider the bounded and measurable function $d(x) = \dim J_{\mathcal{W}}(x) \geq r(x)$ for $x \in \mathbb{T}^k$ then, using Eq. (15) we see that

$$P_\varphi^{J_{\mathcal{W}}(x)}(\Gamma\mathcal{F}(x)) = \sum_{j \in \mathbb{I}_{r(x)}} \varphi(\lambda_j(x)) + (d(x) - r(x))\ \varphi(0) \quad \text{for a.e. } x \in \mathbb{T}^k\ .$$

Therefore, the non-negative function

$$\mathbb{T}^k \ni x \mapsto P_\varphi^{J_{\mathcal{W}}(x)}(\Gamma\mathcal{F}(x))$$

is bounded and measurable and therefore integrable on $\mathbb{T}^k$. This shows that the convex potential $P_\varphi^{\mathcal{W}}(E(\mathcal{F}))$ is a well defined non-negative real number. $\triangle$

Incidentally, Remark 3.4 above shows that if $\varphi(0) = 0$ then the convex potential $P_\varphi^{\mathcal{W}} = P_\varphi$ does not depend on the FSI subspace $\mathcal{W}$.

Example 3.5. Let $\mathcal{W}$ be a FSI subspace of $L^2(\mathbb{R}^k)$ and let $\mathcal{F} = \{f_i\}_{i \in \mathbb{I}_n} \in \mathcal{W}^n$. If we set $\varphi(x) = x^2$ for $x \in \mathbb{R}^+$ then, the corresponding potential on $E(\mathcal{F})$, that we shall denote $\mathrm{FP}\,(E(\mathcal{F}))$, is given by

$$\mathrm{FP}\,(E(\mathcal{F})) = \int_{\mathbb{T}^k} \mathrm{tr}(S^2_{\Gamma\mathcal{F}(x)})\ dx = \int_{\mathbb{T}^k} \sum_{i,\,j \in \mathbb{I}_n} |\langle \Gamma f_i(x), \Gamma f_j(x)\rangle|^2\ dx\ .$$

Hence, $\mathrm{FP}\,(E(\mathcal{F}))$ is a natural extension of the Benedetto-Fickus frame potential of Eq. (16). $\triangle$

Remark 3.6. Let $\mathcal{W}$ be a SI subspace of $L^2(\mathbb{R}^k)$ and let $A \in L(\ell^2(\mathbb{Z}^k))^+$ be a positive operator: in [21], E. Dutkay introduces the local trace function of A relative to $\mathcal{W}$, denoted $\tau_{\mathcal{W},\,A} : \mathbb{T}^k \to [0, \infty]$ as follows: for $x \in \mathbb{T}^k$,

$$\tau_{\mathcal{W},\,A}(x) = \mathrm{tr}(A\,(P_{\mathcal{W}})\hat{}_x)\ ,$$

where $\mathrm{tr}(\cdot)$ denotes the usual (semi-finite) trace in $L(\ell^2(\mathbb{Z}^k))$. We can extend the notion of local trace function as described above to the following setting: given $T \in L(L^2(\mathbb{R}^k))^+$ a positive and SP operator, we let the local trace function of T with respect to the SI subspace $\mathcal{W}$ be given by

$$\tau_{\mathcal{W},\,T}(x) = \mathrm{tr}(\hat{T}_x\,(P_{\mathcal{W}})\hat{}_x)\ , \quad x \in \mathbb{T}^k\,. \tag{20}$$

Notice that if $A \in L(\ell^2(\mathbb{Z}^k))^+$ and $T \in L(L^2(\mathbb{R}^k))^+$ is the unique positive and SP operator such that $\hat{T}_x = A$ for $x \in \mathbb{T}^k$ then

$$\tau_{\mathcal{W},\,A}(x) = \tau_{\mathcal{W},\,T}(x)\ , \quad x \in \mathbb{T}^k\,.$$

If we assume further that $\mathcal{W}$ is a FSI subspace, we consider $\varphi \in \mathrm{Conv}(\mathbb{R}^+)$ and take $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n} \in \mathcal{W}^n$ then

$$P_\varphi^{\mathcal{W}}(E(\mathcal{F})) = \int_{\mathbb{T}^k} \tau_{\mathcal{W},\varphi(S_{E(\mathcal{F})})}(x)\ dx\,,$$

where $\varphi(S_{E(\mathcal{F})}) \in L(L^2(\mathbb{R}^k))^+$ is obtained by the functional calculus. Indeed, notice that in this case $\varphi(S_{E(\mathcal{F})})$ is a SP operator such that

$$[\varphi(S_{E(\mathcal{F})})]\hat{}_x = \varphi(\,(S_{E(\mathcal{F})})\hat{}_x) = \varphi(S_{\Gamma\mathcal{F}(x)})\ , \quad \text{for a.e. } x \in \mathbb{T}^k\ . \qquad \triangle$$

Let $\mathcal{W}$ be a FSI subspace. In what follows we show that, under some natural restrictions, the convex potentials $P_\varphi^{\mathcal{W}}(E(\mathcal{F}))$ for finite sequences $\mathcal{F} \in \mathcal{W}^n$ detect tight frames for $\mathcal{W}$ as their minimizers (see Theorem 3.9 below). In turn, this last fact motivates the study of the structure of minimizers of convex potentials for finitely generated sequences in $L^2(\mathbb{R}^k)$ (under some restrictions) since these minimizers can be considered as natural substitutes of tight frames. In order to state the results on this matter, we introduce the following notions and notations.

Remark 3.7. Let $(X, \mathcal{X}, \mu_X), (Y, \mathcal{Y}, \mu_Y)$ be two measure spaces; we consider their direct sum, denoted $X \bigoplus Y$, given by the three-tuple $(X \oplus Y, \mathcal{X} \bigoplus \mathcal{Y}, \mu_X \oplus \mu_Y)$, where

1. $X \oplus Y = X \overset{d}{\cup} Y$ (the disjoint union of the sets); we further consider the canonical inclusions $\eta_X : X \to X \oplus Y$ and $\eta_Y : Y \to X \oplus Y$ of X and Y into their disjoint union; hence η_X and η_Y are injective functions such that $\eta_X(X) \cap \eta_Y(Y) = \emptyset$ and $\eta_X(X) \cup \eta_Y(Y) = X \oplus Y$.
2. $\mathcal{X} \bigoplus \mathcal{Y} = \{A \oplus B = \eta_X(A) \cup \eta_Y(B) : A \in \mathcal{X},\, B \in \mathcal{Y}\}$;
3. $\mu_X \oplus \mu_Y$ is the measure given by $\mu_X \oplus \mu_Y(A \oplus B) = \mu_X(A) + \mu_Y(B)$;

Notice that using the maps η_X and η_Y we can consider (as we sometimes do) $X,\, Y \subset X \oplus Y$. $\triangle$

Notations 3.8. In what follows we consider:

1. A FSI subspace of $L^2(\mathbb{R}^k)$ of length ℓ, denoted $\mathcal{W}$;
2. $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n} \in \mathcal{W}^n$ such that $E(\mathcal{F})$ is a Bessel sequence;
3. $d(x) = \dim J_{\mathcal{W}}(x) \leq \ell$, for $x \in \mathbb{T}^k$;
4. The Lebesgue measure on $\mathbb{R}^k$; denoted $|\cdot|$; $X_i = d^{-1}(i) \subseteq \mathbb{T}^k$ and $p_i = |X_i|$, $i \in \mathbb{I}_\ell$.
5. We denote by $C_{\mathcal{W}} = \sum_{i\in\mathbb{I}_\ell} i \cdot p_i$.
6. The spectrum of $\mathcal{W}$ is the measurable set $\mathrm{Spec}(\mathcal{W}) = \bigcup_{i\in\mathbb{I}_\ell} X_i = \{x \in \mathbb{T}^k : d(x) \neq 0\}$. $\triangle$

Theorem 3.9 (Structure of $P_\varphi^{\mathcal{W}}$ minimizers with norm restrictions)**.** *Consider the Notations 3.8 and assume that* $\sum_{i\in\mathbb{I}_n} \|f_i\|^2 = 1$. *If* $\varphi \in \mathrm{Conv}(\mathbb{R}^+)$, *then*

$$P_\varphi^{\mathcal{W}}(E(\mathcal{F})) \geq C_{\mathcal{W}}\ \varphi(C_{\mathcal{W}}^{-1})\,. \tag{21}$$

Moreover, if $\varphi \in \mathrm{Conv_s}(\mathbb{R}^+)$ *then equality holds in* (21) *iff* $E(\mathcal{F})$ *is a tight frame for* $\mathcal{W}$ *i.e.*

$$S_{E(\mathcal{F})} = C_{\mathcal{W}}^{-1}\ P_{\mathcal{W}}\,. \tag{22}$$

Proof. Let $(X_{ij}, \mathcal{X}_{ij}, |\cdot|_{ij})$ where $X_{ij} = X_i$, $\mathcal{X}_{ij} = \mathcal{X}_i$ the σ-algebra of Borel sets in X_i and $|\cdot|_{ij} = |\cdot|_i$ the Lebesgue measure in X_i, for $j \in \mathbb{I}_i$ and $i \in \mathbb{I}_\ell$. With the notations in Remark 3.7, we consider the measure space

$$(X, \mathcal{X}, \mu) = \bigoplus_{i \in \mathbb{I}_\ell} \bigoplus_{j \in \mathbb{I}_i} (X_{ij}, \mathcal{X}_{ij}, |\cdot|_{ij}) \ .$$

For $i \in \mathbb{I}_\ell$ and $j \in \mathbb{I}_i$ we further consider the canonical inclusions $\eta_{i,j} : X_{i,j} \to X$. Hence, for every $x \in X$ there exists unique $i \in \mathbb{I}_\ell$, $j \in \mathbb{I}_i$ and $\tilde{x} \in X_{i,j} = X_i$ such that $\eta_{i,j}(\tilde{x}) = x$. Notice that by construction, $\mu(X) = \sum_{i \in \mathbb{I}_\ell} i \cdot p_i = C_{\mathcal{W}}$.

Let $\lambda_{E(\mathcal{F})} : X \to \mathbb{R}^+$ be the measurable function of eigenvalues of $E(\mathcal{F})$ defined as follows: for $x \in X$, let $(i\,,\, j) \in \mathbb{I}_\ell \times \mathbb{I}_i$ and $\tilde{x} \in X_{i,j} = X_i$ be (uniquely determined) such that $\eta_{i,j}(\tilde{x}) = x$; in this case we set

$$\lambda_{E(\mathcal{F})}(x) = \lambda_j(\ (S_{E(\mathcal{F})})\hat{}_{\tilde{x}}) = \lambda_j(S_{\Gamma\mathcal{F}(\tilde{x})}) \ ,$$

where $\lambda(S_{\Gamma\mathcal{F}(\tilde{x})}) \in \left(\ell^1_+(\mathbb{Z}^k)\right)^\downarrow$ is the fine spectral structure of $E(\mathcal{F})$ defined in Eq. (19). We claim that if $\varphi \in \mathrm{Conv}(\mathbb{R}^+)$, then

$$P_\varphi^{\mathcal{W}}(E(\mathcal{F})) = \int_X \varphi(\lambda_{E(\mathcal{F})}(x)) \ d\mu(x) \tag{23}$$

Indeed, for Eq. (17)

$$P_\varphi^{\mathcal{W}}(E(\mathcal{F})) = \int_{\mathbb{T}^k} P_\varphi^{J_{\mathcal{W}}(x)}(\Gamma\mathcal{F}(x)) \ dx = \int_{\mathrm{Spec}(\mathcal{W})} P_\varphi^{J_{\mathcal{W}}(x)}(\Gamma\mathcal{F}(x)) \ dx \ ,$$

where $P_\varphi^{J_{\mathcal{W}}(x)}(\Gamma\mathcal{F}(x))$ is the convex potential associated with $(\varphi, J_{\mathcal{W}}(x))$ of the finite sequence $\Gamma\mathcal{F}(x) = \{\Gamma\, f_i(x)\}_{i \in \mathbb{I}_n}$ in $\ell^2(\mathbb{Z}^k)$ as defined in Eq. (15) (notice that $P_\varphi^{J_{\mathcal{W}}(x)}(\Gamma\mathcal{F}(x)) = 0$ for $x \in \mathbb{T}^k \setminus \mathrm{Spec}(\mathcal{W})$). Therefore, if $x \in X_i$ for some $i \in \mathbb{I}_\ell$ then

$$P_\varphi^{J_{\mathcal{W}}(x)}(\Gamma\mathcal{F}(x)) = \sum_{j=1}^{i} \varphi(\lambda_j(S_{\Gamma\mathcal{F}(x)})) \, .$$

For $i \in \mathbb{I}_\ell$ we have that

$$\int_{X_i} P_\varphi^{J_{\mathcal{W}}(x)}(\Gamma\mathcal{F}(x)) \, dx = \int_{X_i} \sum_{j=1}^{i} \varphi(\lambda_j(S_{\Gamma\mathcal{F}(x)})) \, dx = \int_{\oplus_{j=1}^i X_{ij}} \varphi(\lambda_{E(\mathcal{F})}(x)) \, d\mu(x) \ .$$

Therefore, since $\mathrm{Spec}(\mathcal{W}) = \bigcup\limits_{i \in \mathbb{I}_\ell} X_i$ and $X = \oplus_{i \in \mathbb{I}_\ell} \oplus_{j \in \mathbb{I}_i} X_{i,j}$,

$$P_\varphi^{\mathcal{W}}(E(\mathcal{F})) = \sum_{i \in \mathbb{I}_\ell} \int_{X_i} P_\varphi^{J_{\mathcal{W}}(x)}(\Gamma\mathcal{F}(x)) \, dx = \sum_{i \in \mathbb{I}_\ell} \int_{\oplus_{j=1}^i X_{ij}} \varphi(\lambda_{E(\mathcal{F})}(x)) \, d\mu(x) = \int_X \varphi(\lambda_{E(\mathcal{F})}(x)) \, d\mu(x) \, ,$$

which proves Eq. (23). In particular, if we take $\varphi(x) = x$ in Eq. (23) we get that

$$\int_X \lambda_{E(\mathcal{F})}(x) \, d\mu(x) = \int_{\mathbb{T}^k} \mathrm{tr}(S_{\Gamma\mathcal{F}(x)}) \, dx = \int_{\mathbb{T}^k} \sum_{i \in \mathbb{I}_n} \|\Gamma\, f_i(x)\|^2 \, dx = \sum_{i \in \mathbb{I}_n} \|f_i\|^2 = 1 \ .$$

Consider the probability measure $\tilde{\mu} = C_{\mathcal{W}}^{-1}\,\mu$. Then, as in Example 2.5, we have that

$$\int_X \lambda_{E(\mathcal{F})}(x)\,d\tilde{\mu}(x) = C_{\mathcal{W}}^{-1} \implies C_{\mathcal{W}}^{-1}\cdot 1_X \prec \lambda_{E(\mathcal{F})} \quad (\text{ in } (X,\mathcal{X},\tilde{\mu}))\,. \tag{24}$$

If we let $\varphi \in \mathrm{Conv}(\mathbb{R}^+)$ then, using the previous facts and Theorem 2.4, we get that

$$\begin{aligned}\varphi\left(C_{\mathcal{W}}^{-1}\right) = \textstyle\int_X \varphi\left(C_{\mathcal{W}}^{-1}\cdot 1_X\right)\,d\tilde{\mu}(x) &\overset{2.4}{\leq} \textstyle\int_X \varphi(\lambda_{E(\mathcal{F})}(x))\,d\tilde{\mu}(x)\\ &= C_{\mathcal{W}}^{-1}\textstyle\int_X \varphi(\lambda_{E(\mathcal{F})}(x))\,d\mu(x) \overset{(23)}{=} C_{\mathcal{W}}^{-1}\,P_\varphi^{\mathcal{W}}(E(\mathcal{F}))\,,\end{aligned}$$

which proves Eq. (21). If $\varphi \in \mathrm{Conv_s}(\mathbb{R}^+)$ and also $P_\varphi^{\mathcal{W}}(E(\mathcal{F})) = \varphi(C_{\mathcal{W}}^{-1})\,C_{\mathcal{W}}$, using Eq. (23) and the majorization relation in Eq. (24), we get that

$$\int_X \varphi(\lambda_{E(\mathcal{F})}(x))\,d\tilde{\mu}(x) = \int_X \varphi(C_{\mathcal{W}}^{-1})\,d\tilde{\mu}(x)\,.$$

Hence, by Proposition 2.6,

$$(\lambda_{E(\mathcal{F})})^* = C_{\mathcal{W}}^{-1}\,1_{[0,1]} \implies \lambda_i(\,(S_{E(\mathcal{F})})\hat{}_x) = C_{\mathcal{W}}^{-1} \quad \text{for } i\in\mathbb{I}_{d(x)} \text{ and a.e. } x\in\mathbb{T}^k\,.$$

Therefore, $S_{E(\mathcal{F})} = C_{\mathcal{W}}^{-1}\,P_{\mathcal{W}}$ i.e. $E(\mathcal{F})$ is a tight frame for $\mathcal{W}$. Conversely, notice that if $S_{E(\mathcal{F})} = C_{\mathcal{W}}^{-1}\,P_{\mathcal{W}}$ then lower bound in Eq. (21) is attained. □

Remark 3.10. Notice that Theorem 3.9 above indicates that minimizers of the convex potentials have a nice *global* structure. That is, the structure of the minimizers of the convex potentials is not obtained by glueing minimizers of the corresponding (local) convex potential in each fiber. △

4 Fine spectral structure of shift generated oblique duals in FSI subspaces

Throughout this section $\mathcal{V}$, $\mathcal{W}\subseteq L^2(\mathbb{R}^k)$ denote FSI subspaces such that $\mathcal{V}\oplus\mathcal{W}^\perp = L^2(\mathbb{R}^k)$ and $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n}\in\mathcal{W}^n$ denotes a finite sequence such that $E(\mathcal{F})$ is a frame for $\mathcal{W}$.

Next we recall some characterizations of the condition $\mathcal{S}\oplus\mathcal{T}^\perp = L^2(\mathbb{R}^k)$ for SI subspaces and a characterization of shift generated (SG) oblique duals of $E(\mathcal{F})$; these results together with [5] allow us to obtain the exact value of the aliasing norm corresponding to the consistent sampling induced by the FSI subspaces $\mathcal{V}$ and $\mathcal{W}$. In Section 4.2 we obtain a detailed description of the fine spectral structure (i.e. eigenvalues) of the frame operators of SG oblique $\mathcal{V}$-duals of the (fixed) frame $E(\mathcal{F})$ for $\mathcal{W}$. We will apply these results in Section 5, where we compute SG oblique dual frames with norm restrictions that simultaneously minimize the convex potentials $P_\varphi^{\mathcal{V}}$ for all $\varphi\in\mathrm{Conv}(\mathbb{R}^+)$.

4.1 SG oblique duals and aliasing in FSI subspaces

Following [27] (see also [17, 26, 28]) we consider the set of SG $\mathcal{V}$-duals of $E(\mathcal{F})$:

$$\mathcal{D}_{\mathcal{V}}^{SG}(\mathcal{F}) = \mathcal{D}_{\mathcal{V}}^{SG}(E(\mathcal{F})) = \{E(\mathcal{G})\in D_{\mathcal{V}}(E(\mathcal{F}))\,:\,\mathcal{G}=\{g_i\}_{i\in\mathbb{I}_n}\in\mathcal{V}^n\}\,. \tag{25}$$

In case $\mathcal{V} = \mathcal{W}$ then we write $\mathcal{D}^{SG}(\mathcal{F}) = \mathcal{D}^{SG}_{\mathcal{W}}(E(\mathcal{F}))$ (which is the class of SG duals of type I, in the terminology of [27]). Given $E(\mathcal{G}) \in \mathcal{D}^{SG}_{\mathcal{V}}(\mathcal{F})$ we obtain the following (structured) reconstruction formulas: for every $f \in \mathcal{W}$ and $g \in \mathcal{V}$,

$$f = \sum_{(\ell,i)\in\mathbb{Z}^k\times\mathbb{I}_n} \langle f,\, T_\ell\, g_i\rangle\; T_\ell\, f_i \qquad \text{and} \qquad g = \sum_{(\ell,i)\in\mathbb{Z}^k\times\mathbb{I}_n} \langle g,\, T_\ell\, f_i\rangle\; T_\ell\, g_i\,.$$

Next we describe some results related with the general assumption for studying oblique duality, namely $\mathcal{V} \oplus \mathcal{W}^\perp = L^2(\mathbb{R}^k)$, for the FSI subspaces $\mathcal{V}$ and $\mathcal{W}$, as well as SG oblique duality. The next two results can be derived using combinations of results and techniques in [2, 31, 32].

Lemma 4.1. *With the previous notations and assumptions, let $J_{\mathcal{V}}$ and $J_{\mathcal{W}}$ denote the range functions of the SI subspaces $\mathcal{V}$ and $\mathcal{W}$, respectively. Then,*

1. *$\mathcal{W}^\perp$ is a SI subspace with range function $J_{\mathcal{W}^\perp}(x) = [J_{\mathcal{W}}(x)]^\perp$ for a.e. $x \in \mathbb{T}^k$;*
2. *If $Q = P_{\mathcal{V}//\mathcal{W}^\perp}$ then Q is a shift preserving operator;*
3. *$J_{\mathcal{V}}(x) \oplus J_{\mathcal{W}}(x)^\perp = \ell^2(\mathbb{Z}^k)$ and $\hat{Q}_x = P_{J_{\mathcal{V}}(x)//J_{\mathcal{W}}(x)^\perp}$ for a.e. $x \in \mathbb{T}^k$.*
4. *$E(\mathcal{G}) \in \mathcal{D}^{SG}_{\mathcal{V}}(\mathcal{F}) \iff \Gamma\mathcal{G}(x)$ is $J_{\mathcal{V}}(x)$ − dual of $\Gamma\mathcal{F}(x)$, for a.e $x \in \mathbb{T}^k$.* □

Remark 4.2. Let $\mathcal{S}$ and $\mathcal{T}$ be closed subspaces of $L^2(\mathbb{R}^k)$. In order to characterize when the (algebraic) sum of these subspaces is a closed subspace we recall the Dixmier angle between $\mathcal{S}$ and $\mathcal{T}$, denoted by $[\mathcal{S},\, \mathcal{T}^\perp]_D \in [0, \pi]$, given by

$$\cos[\mathcal{S}, \mathcal{T}]_D = \sup\{|\langle v,\, w\rangle|, v \in \mathcal{S}_1, w \in \mathcal{T}_1\}\,, \tag{26}$$

where $\mathcal{S}_1 = \{f \in \mathcal{S} : \|f\| = 1\}$ (and similar for $\mathcal{T}_1$). It is well known (see [20]) that $[\mathcal{S}, \mathcal{T}]_D > 0$ if and only if $\mathcal{S} \cap \mathcal{T} = \{0\}$ and $\mathcal{S} + \mathcal{T}$ is a closed subspace of $L^2(\mathbb{R}^k)$.

Assume further that $\mathcal{S} \oplus \mathcal{T} = L^2(\mathbb{R}^k)$ and let $Q = P_{\mathcal{S}//\mathcal{T}}$ be the corresponding oblique projection. Then (see [20])

$$\|Q\| = \frac{1}{\sin[\mathcal{S}, \mathcal{T}]_D}\;. \qquad \triangle$$

Proposition 4.3. *Let $\mathcal{S}, \mathcal{T} \subseteq L^2(\mathbb{R}^k)$ be SI subspaces of $L^2(\mathbb{R}^k)$. The following statements are equivalent:*

1. *$\mathcal{S} \oplus \mathcal{T}^\perp = L^2(\mathbb{R}^k)$;*
2. *$J_{\mathcal{S}}(x) \oplus J_{\mathcal{T}}(x)^\perp = \ell^2(\mathbb{Z}^k)$ for a.e. $x \in \mathbb{T}^k$ and $\operatorname{ess\,sup}_{x\in\mathbb{T}^k} \|P_{J_{\mathcal{S}}(x)//J_{\mathcal{T}}(x)^\perp}\| < \infty$;*
3. *$J_{\mathcal{S}}(x)^\perp \cap J_{\mathcal{T}}(x) = \{0\}$ and $\operatorname{ess\,inf}_{x\in\mathbb{T}^k} [J_{\mathcal{S}}(x), J_{\mathcal{T}}(x)^\perp]_D > 0$.*

In this case we have that $[\mathcal{S}, \mathcal{T}^\perp]_D = \operatorname{ess\,inf}_{x\in\mathbb{T}^k} [J_{\mathcal{S}}(x), J_{\mathcal{T}}(x)^\perp]_D$. □

As an application of the previous results we compute the exact value of the aliasing norm (see [22, 29]) in terms of the relative geometry of the FSI subspaces $\mathcal{V}$ and $\mathcal{W}$. Indeed, recall that the aliasing norm corresponding to the consistent sampling

$$f \mapsto \tilde{f} = P_{\mathcal{W}//\mathcal{V}^\perp} f\;, \quad \text{for } f \in L^2(\mathbb{R}^k)$$

denoted $A(\mathcal{V},\, \mathcal{W})$, is given by

$$A(\mathcal{V},\, \mathcal{W}) = \sup_{e\in\mathcal{W}^\perp} \frac{\|P_{\mathcal{W}//\mathcal{V}^\perp} e\|}{\|e\|} = \|P_{\mathcal{W}//\mathcal{V}^\perp}\, P_{\mathcal{W}^\perp}\|\,. \tag{27}$$

The aliasing norm is a measure of the incidence of $\mathcal{W}^\perp$ in the consistent sampling induced by $P_{\mathcal{W}//\mathcal{V}^\perp}$ and it plays a role in applications of oblique duality.

Definition 4.4. Let $\mathcal{S},\, \mathcal{T} \subset L^2(\mathbb{R}^k)$ be closed subspaces. We define the aperture between $\mathcal{S}$ and $\mathcal{T}$, denoted $[\mathcal{S},\mathcal{T}]^a \in [0,\pi/2)$, as the angle given by

$$\cos([\mathcal{S},\mathcal{T}]^a) = \inf_{f\in\mathcal{T},\,\|f\|=1} \|P_{\mathcal{S}} f\|\,. \qquad \triangle$$

Remark 4.5. With the notations of Definition 4.4, we point out that the aperture $[\mathcal{S},\mathcal{T}]^a$ coincides with the notion of angle between the subspaces $\mathcal{S}$ and $\mathcal{T}$ as defined in [47] (and $\cos([\mathcal{S},\mathcal{T}]^a)$ is also known as the infimum cosine angle from [31]). It is known that the following relation holds (see [31, 32]):

$$\cos([\mathcal{S},\mathcal{T}]_D)^2 = 1-\cos([\mathcal{S},\mathcal{T}]^a)^2 \implies [\mathcal{S},\mathcal{T}]^a = \pi/2 - [\mathcal{S},\mathcal{T}^\perp]_D\,.$$

Hence, using the relations above and Proposition 4.3 (see also [31]) we get that if $\mathcal{S},\, \mathcal{T} \subset L^2(\mathbb{R}^k)$ are SI subspaces such that $\mathcal{S}^\perp \oplus \mathcal{T} = L^2(\mathbb{R}^k)$ then

$$[\mathcal{S},\mathcal{T}]^a = \operatorname{ess\,sup}_{x\in\mathbb{T}^k} [J_{\mathcal{S}}(x), J_{\mathcal{T}}(x)]^a < \pi/2\,. \qquad \triangle$$

Consider again the notations of Definition 4.4 and assume further that $L^2(\mathbb{R}^k) = \mathcal{S}\oplus\mathcal{T}^\perp$. Then, using Remarks 4.2 and 4.5 we see that

$$\|P_{\mathcal{S}//\mathcal{T}^\perp}\| = \frac{1}{\sin[\mathcal{S},\mathcal{T}^\perp]_D} = \frac{1}{\cos[\mathcal{S},\mathcal{T}]^a}\,.$$

From this we obtain the following upper bound for the aliasing norm (see [48])

$$A(\mathcal{S},\, \mathcal{T}^\perp) \le \|P_{\mathcal{S}//\mathcal{T}^\perp}\| = \frac{1}{\cos[\mathcal{S},\mathcal{T}]^a}\,.$$

Notice that this known bound is not sharp; indeed, if we take $\mathcal{S}=\mathcal{T}$ then $A(\mathcal{S},\, \mathcal{T}^\perp) = 0$ but $\cos[\mathcal{S},\mathcal{T}]^a = 1$.

Next we compute the exact value of the aliasing norm.

Proposition 4.6. *With the previous notations and assumptions, the aliasing norm $A(\mathcal{V},\mathcal{W})$ corresponding to the FSI oblique pair $(\mathcal{V},\mathcal{W})$ is given by*

$$A(\mathcal{V},\, \mathcal{W}) = \tan([\mathcal{V},\, \mathcal{W}]^a)\,.$$

Proof. Notice that by assumption $J_{\mathcal{V}}(x)$ and $J_{\mathcal{W}}(x)$ are finite dimensional subspaces of $\ell^2(\mathbb{Z}^k)$ and, by Proposition 4.3, we see that $J_{\mathcal{V}}(x)^\perp \oplus J_{\mathcal{W}}(x) = \ell^2(\mathbb{Z}^k)$, for a.e. $x\in\mathbb{T}^k$. Hence, we can apply the results from [5], and conclude that

$$A(J_{\mathcal{V}}(x),\, J_{\mathcal{W}}(x)) = \|P_{J_{\mathcal{W}}(x)//J_{\mathcal{V}}(x)^\perp}\; P_{J_{\mathcal{W}}(x)^\perp}\| = \tan([J_{\mathcal{V}}(x), J_{\mathcal{W}}(x)]^a)\;, \qquad \text{for a.e. } x\in\mathbb{T}^k\,.$$

Therefore, using Remark 4.5, we get that

$$A(\mathcal{V},\, \mathcal{W}) = \|P_{\mathcal{W}//\mathcal{V}^\perp}\, P_{\mathcal{W}^\perp}\| = \operatorname{ess\,sup}_{x\in\mathbb{T}^k} \tan([J_{\mathcal{V}}(x), J_{\mathcal{W}}(x)]^a) = \tan([\mathcal{V},\, \mathcal{W}]^a)\,. \qquad \square$$

Conjecture 4.7. We conjecture that Proposition 4.6 holds for the consistent sampling corresponding to an oblique decomposition $\mathcal{S}\oplus\mathcal{T}^\perp = \mathcal{H}$ in an arbitrary Hilbert space $\mathcal{H}$. By the results from [5] the conjecture holds for finite dimensional $\mathcal{S}$ and $\mathcal{T}$. By Proposition 4.6 this conjecture holds for some infinite dimensional subspaces $\mathcal{S}$ and $\mathcal{T}$ as well. $\triangle$

4.2 Fine spectral structure of SG oblique duals

Let $E(\mathcal{G}) \in \mathcal{D}_{\mathcal{V}}^{SG}(\mathcal{F})$ and let $S_{E(\mathcal{G})}$ denote the frame operator of $E(\mathcal{G})$. Recall that in this case $S_{E(\mathcal{G})}$ is a shift preserving (SP) operator such that $(S_{E(\mathcal{G})})\hat{}_x = S_{\Gamma\mathcal{G}(x)}$ for a.e. $x \in \mathbb{T}^k$ and the fine spectral structure of $E(\mathcal{G})$, as defined in Eq (19), is the function $\mathbb{T}^k \ni x \mapsto \lambda(S_{\Gamma\mathcal{G}(x)}) = (\lambda_j(S_{\Gamma\mathcal{G}(x)}))_{j\in\mathbb{N}}$, the sequence of eigenvalues of the positive finite rank operator $S_{\Gamma\mathcal{G}(x)}$, counting multiplicities and arranged in non-increasing order for a.e. $x \in \mathbb{T}^k$.

In the next result we consider the measurable function $d : \mathbb{T}^k \to \{0, \dots, n\}$ given by $d(x) = \dim J_{\mathcal{W}}(x) = \dim J_{\mathcal{V}}(x)$ for a.e. $x \in \mathbb{T}^k$.

Lemma 4.8. *Let $\mathcal{G} = \{g_i\}_{i\in\mathbb{I}_n} \in \mathcal{V}^n$ be such that $E(\mathcal{G})$ is a frame for $\mathcal{V}$. Let $B \in L(L^2(\mathbb{R}^k))^+$ be a shift preserving operator such that $R(B) \subseteq \mathcal{V}$. Then, there exists $\mathcal{Z} = \{z_i\}_{i\in\mathbb{I}_n} \in \mathcal{V}^n$ such that $B = S_{E(\mathcal{Z})}$ and $T_{E(\mathcal{G})}\, T^*_{E(\mathcal{Z})} = 0$ if and only if $\operatorname{rk}(\hat{B}_x) \le n - d(x)$ for a.e. $x \in \mathbb{T}^k$.*

Proof. First notice that by considering a convenient finite partition of $\mathbb{T}^k$ into measurable sets we can assume, without loss of generality, that $d(x) = d \in \mathbb{N}$ for a.e. $x \in \mathbb{T}^k$. By Lemma 3.2 there exist measurable vector fields $v_j : \mathbb{T}^k \to \ell^2(\mathbb{Z}^k)$ for $j \in \mathbb{I}_n$ such that if $\lambda(S_{\Gamma\mathcal{G}(\cdot)}) = (\lambda_j(\cdot))_{j\in\mathbb{N}}$ denotes the fine spectral structure of $E(\mathcal{G})$ then

$$(S_{E(\mathcal{G})})\hat{}_x = \sum_{j\in\mathbb{I}_d} \lambda_j(x)\; v_j(x) \otimes v_j(x)\;, \qquad \text{for a.e. } x \in \mathbb{T}^k\,. \tag{28}$$

Moreover, in this case $\{v_j(x)\}_{j\in\mathbb{I}_d}$ is an ONB of $J_{\mathcal{V}}(x)$ for a.e. $x \in \mathbb{T}^k$. Assume that $B \in L(L^2(\mathbb{R}^k))^+$ is a shift preserving operator such that $R(B) \subseteq \mathcal{V}$ and such that $\operatorname{rk}(\hat{B}_x) \le n - d$ for a.e. $x \in \mathbb{T}^k$. Since $\hat{B}_x \in L(\ell^2(\mathbb{Z}^k))^+$ is such that $R(\hat{B}_x) \subseteq J_{\mathcal{V}}(x)$ for a.e. $x \in \mathbb{T}^k$ then, using the measurable vector fields $\{v_j\}_{j\in\mathbb{I}_d}$ as above (indeed, the measurable field of matrix representations $[\hat{B}_x] \in \mathcal{M}_d(\mathbb{C})^+$ of $\hat{B}_x$ with respect to $\{v_j(x)\}_{j\in\mathbb{I}_d}$) we get measurable fields $w_j : \mathbb{T}^k \to \ell^2(\mathbb{Z}^k)$ for $j \in \mathbb{I}_d$, such that $\{w_j(x)\}_{j\in\mathbb{I}_d}$ is an ONB of $J_{\mathcal{V}}(x)$ and $\hat{B}_x\, w_j(x) = \lambda_j(\hat{B}_x)\, w_j(x)$ for $j \in \mathbb{I}_d$ and a.e. $x \in \mathbb{T}^k$. In particular, we see that

$$\hat{B}_x^{1/2} = \sum_{j=1}^{\min\{d,n-d\}} \lambda_j(\hat{B}_x)^{1/2}\, w_j(x) \otimes w_j(x)\;, \quad \text{for a.e. } x \in \mathbb{T}^k\;.$$

Let $V : \mathbb{T}^k \to L(\mathbb{C}^n, \ell^2(\mathbb{Z}^k))$ be the measurable field of partial isometries given by

$$V(x)\, u_j(x) = \begin{cases} w_{j-d} & \text{if} \quad d+1 \le j \le d + \min\{d, n-d\}\,, \\ 0 & \text{otherwise.} \end{cases}$$

Hence $V(x)\, V^*(x)$ is the orthogonal projection onto $\operatorname{span}\{w_j(x) : \; j \in \mathbb{I}_{\min\{d,d-n\}}\}$ and $R(V^*(x)\,) \subseteq \ker V(x)^\perp \subseteq \operatorname{span}\{u_j(x) : d+1 \le j \le n\} \implies T_{\Gamma\mathcal{G}(x)} V^*(x) = 0$ for a.e. $x \in \mathbb{T}^k$.

For $i \in \mathbb{I}_n$ we set $z_i \in \mathcal{V}$ determined uniquely by $\Gamma z_i(x) = \hat{B}_x^{1/2}\, V(x) e_i$ for a.e. $x \in \mathbb{T}^k$, where $\{e_i\}_{i\in\mathbb{I}_n}$ denotes the canonical ONB of $\mathbb{C}^n$. If we set $\mathcal{Z} = \{z_i\}_{i\in\mathbb{I}_n}$ then $T_{\Gamma\mathcal{Z}(x)} = \hat{B}_x^{1/2}\, V(x)$ for a.e. $x \in \mathbb{T}^k$; hence, using Eq. (10), we see that

$$(T_{E(\mathcal{G})}\, T^*_{E(\mathcal{Z})})\hat{}_x = T_{\Gamma\mathcal{G}(x)} T^*_{\Gamma\mathcal{Z}(x)} = T_{\Gamma\mathcal{G}(x)} V^*(x) \hat{B}_x^{1/2} = 0\;, \qquad \text{for a.e. } x \in \mathbb{T}^k\,.$$

On the other hand, notice that

$$(S_{E(\mathcal{Z})})\hat{}_x = S_{\Gamma\mathcal{Z}(x)} = \hat{B}_x^{1/2}\, V(x)\, V^*(x)\, \hat{B}_x^{1/2} = \hat{B}_x\;, \qquad \text{for a.e. } x \in \mathbb{T}^k\;.$$

Conversely, assume that there exists $\mathcal{Z} = \{z_i\}_{i\in\mathbb{I}_n} \in \mathcal{V}^n$ such that $B = S_{E(\mathcal{Z})}$ and $T_{E(\mathcal{G})}\, T^*_{E(\mathcal{Z})} = 0$. Then, by Eq. (10), we get that $0 = T_{\Gamma\mathcal{G}(x)}\, T^*_{\Gamma\mathcal{Z}(x)}$ and hence $\mathrm{rk}(T^*_{\Gamma\mathcal{Z}(x)}) \leq n - \mathrm{rk}(T_{\Gamma\mathcal{G}(x)}) = n - d(x)$ for a.e. $x \in \mathbb{T}^k$. Therefore,

$$\mathrm{rk}\,(S_{E(\mathcal{Z})}\hat{)}_x = \mathrm{rk}(S_{\Gamma\mathcal{Z}(x)}) = \mathrm{rk}(T_{\Gamma\mathcal{Z}(x)}) = \mathrm{rk}(T^*_{\Gamma\mathcal{Z}(x)}) \leq n - d(x) \quad \text{for a.e. } x \in \mathbb{T}^k\ . \qquad \square$$

Definition 4.9. Let $\mathcal{G} = \{g_i\}_{i\in\mathbb{I}_n}$ be such that $E(\mathcal{G})$ is a frame for $\mathcal{V}$ with frame operator $A = S_{E(\mathcal{G})}$. Recall that $d(x) = \dim J_{\mathcal{V}}(x)$ for $x \in \mathbb{T}^k$. Then, we consider

$$U_{\mathcal{V}}(E(\mathcal{G})\,) = \Big\{ A+B : B \in L(L^2(\mathbb{R}^k))^+ \text{ is SP, } R(B) \subset \mathcal{V},\ \mathrm{rk}(\hat{B}_x) \leq n-d(x)\,, \text{ for a.e. } x \in \mathbb{T}^k \Big\}. \qquad \triangle$$

Proposition 4.10. *Let $E(\mathcal{F})^{\#}_{\mathcal{V}} = \{T_\ell\, f^{\#}_{\mathcal{V},\,i}\}_{(\ell,i)\in\mathbb{Z}^k\times\mathbb{I}_n}$ denote the canonical $\mathcal{V}$-dual of $\mathcal{F}$. Then,*

$$\{S_{E(\mathcal{G})} :\ E(\mathcal{G}) \in \mathcal{D}^{SG}_{\mathcal{V}}(\mathcal{F})\} = U_{\mathcal{V}}(E(\mathcal{F})^{\#}_{\mathcal{V}})\ . \tag{29}$$

Proof. Let $\mathcal{G} = \{g_i\}_{i\in\mathbb{I}_n} \in \mathcal{V}^n$ be such that $E(\mathcal{G}) \in \mathcal{D}^{SG}_{\mathcal{V}}(\mathcal{F})$. Let $\mathcal{Z} = \{z_i\}_{i\in\mathbb{I}_n} \in \mathcal{V}^n$ be given by $z_i = g_i - f^{\#}_{\mathcal{V},\,i}$ for $i \in \mathbb{I}_n$. Then $E(\mathcal{Z}) = \{T_\ell\, z_i\}_{(\ell,\,i)\in\mathbb{Z}^k\times\mathbb{I}_n}$ is a Bessel sequence in $\mathcal{V}$ such that $T_{E(\mathcal{G})} = T_{E(\mathcal{F})^{\#}_{\mathcal{V}}} + T_{E(\mathcal{Z})}$. In this case we have that $T_{E(\mathcal{Z})}\, T^*_{E(\mathcal{F})} = 0$ and therefore $T_{E(\mathcal{Z})}\, T^*_{E(\mathcal{F})^{\#}_{\mathcal{V}}} = 0$, since $R(T^*_{E(\mathcal{F})}) = R(T^*_{E(\mathcal{F})^{\#}_{\mathcal{V}}})$. Thus,

$$S_{E(\mathcal{G})} = (T_{E(\mathcal{F})^{\#}_{\mathcal{V}}} + T_{E(\mathcal{Z})})\,(T_{E(\mathcal{F})^{\#}_{\mathcal{V}}} + T_{E(\mathcal{Z})})^* = S_{E(\mathcal{F})^{\#}_{\mathcal{V}}} + S_{E(\mathcal{Z})}\ .$$

We conclude that $B = S_{E(\mathcal{Z})} \in L(L^2(\mathbb{R}^k))^+$ is SP, $R(S_{E(\mathcal{Z})}) \subset \mathcal{V}$ and, by Lemma 4.8, that $\mathrm{rk}\,(S_{E(\mathcal{Z})}\hat{)}_x \leq n - d(x)$ for a.e. $x \in \mathbb{T}^k$.

Conversely, if $S \in U_{\mathcal{V}}(E(\mathcal{F})^{\#}_{\mathcal{V}})$ then $S = S_{E(\mathcal{F})^{\#}_{\mathcal{V}}} + B$, where $B \in L(L^2(\mathbb{R}^k))^+$ is SP, $R(B) \subset \mathcal{V}$ and $\mathrm{rk}(\hat{B}_x) \leq n - d(x)$ for a.e. $x \in X$. By Lemma 4.8 we see that there exists $\mathcal{Z} = \{z_i\}_{i\in\mathbb{I}_n}$ such that $T_{E(\mathcal{Z})}\, T^*_{E(\mathcal{F})} = 0$ and $B = S_{E(\mathcal{Z})}$. If we let $\mathcal{G} = \{g_i\}_{i\in\mathbb{I}_n}$ be given by $g_i = f^{\#}_{\mathcal{V},\,i} + z_i$ for $i \in \mathbb{I}_n$, then $E(\mathcal{G})$ is a Bessel sequence in $\mathcal{V}$ such that $T_{E(\mathcal{G})} = T_{E(\mathcal{F})^{\#}_{\mathcal{V}}} + T_{E(\mathcal{Z})}$. Using that $T_{E(\mathcal{Z})}\, T^*_{E(\mathcal{F})^{\#}_{\mathcal{V}}} = 0$ we conclude, as before, that

$$S_{E(\mathcal{G})} = S_{E(\mathcal{F})^{\#}_{\mathcal{V}}} + S_{E(\mathcal{Z})} = S_{E(\mathcal{F})^{\#}_{\mathcal{V}}} + B = S \qquad \square$$

Proposition 4.10 shows that the set of frame operators of SG $\mathcal{V}$-duals of a fixed frame $\mathcal{F}$ can be described in terms of the additive model $U_{\mathcal{V}}(E(\mathcal{F})^{\#}_{\mathcal{V}})$ introduced in Definition 4.9. It turns out that the fine spectral structure of the elements of $U_{\mathcal{V}}(E(\mathcal{F})^{\#}_{\mathcal{V}})$ can be described using a natural extension of the Fan-Pall interlacing theorem for measurable fields of positive matrices. We develop both results in the Appendix section (see Theorems 6.3 and 6.4). As a consequence we obtain the following

Theorem 4.11 (Fine spectral structure of $\mathcal{V}$-duals)**.** *Let $E(\mathcal{F})^{\#}_{\mathcal{V}}$ be the canonical $\mathcal{V}$-dual frame of $E(\mathcal{F})$. Denote by $A = S_{E(\mathcal{F})^{\#}_{\mathcal{V}}}$ and by $\lambda^{\#}_{\mathcal{V},\,i}(x) = \lambda_i(\hat{A}_x)$, $i \in \mathbb{N}$, $x \in \mathbb{T}^k$. Let m be the measurable function given by $m(x) = 2d(x) - n$, for $x \in \mathbb{T}^k$. Given a measurable function $\mu : \mathbb{T}^k \to (\ell^1(\mathbb{N})^+)^{\downarrow}$ (decreasing sequences) described as $\mu = (\mu_i)_{i\in\mathbb{N}}$, the following are equivalent:*

1. *There exists $E(\mathcal{G}) \in \mathcal{D}^{SG}_{\mathcal{V}}(\mathcal{F})$ such that $\mu(x) = \lambda(\,(S_{E(\mathcal{G})}\hat{)}_x) = \lambda(S_{\Gamma\mathcal{G}(x)})$ for every $x \in \mathbb{T}^k$.*

2. $\mu(x) = 0$ *for every* $x \notin Spec(\mathcal{V})$. *If* $x \in Spec(\mathcal{V})$ *then* $\mu_j(x) = 0$ *for* $j \geq d(x) + 1$ *and*

   (a) *in case that* $m(x) \leq 0$, *then* $\mu_i(x) \geqslant \lambda^{\#}_{\mathcal{V},i}(x)$ *for* $i \in \mathbb{I}_{d(x)}$;

   (b) *in case that* $m(x) \geq 1$, *then* $\mu_i(x) \geqslant \lambda^{\#}_{\mathcal{V},i}(x)$ *for* $i \in \mathbb{I}_{d(x)}$ *and*

$$\mu_{n-d(x)+i}(x) = \mu_{d(x)-m(x)+i}(x) \leq \lambda^{\#}_{\mathcal{V},i}(x) \quad \text{for} \quad i \in \mathbb{I}_{m(x)} \ .$$

Proof. It follows from Proposition 4.10 and Theorem 6.4. □

As a consequence of the description of the fine spectral structure of $\mathcal{V}$-duals of $E(\mathcal{F})$ we characterize the existence of *tight* $\mathcal{V}$-duals of $E(\mathcal{F})$ that are *shift generated* (compare with [25]).

Corollary 4.12. *With the notations of Theorem 4.11 then there exists a c-tight* $\mathcal{V}$*-dual* $E(\mathcal{G}) \in \mathcal{D}^{SG}_{\mathcal{V}}(\mathcal{F})$ *if and only if*

1. $A = S_{E(\mathcal{F})^{\#}_{\mathcal{V}}} \leq c \cdot P_{\mathcal{V}}$;

2. $\mathrm{rk}((c \cdot P_{\mathcal{V}} - S_{E(\mathcal{F})^{\#}_{\mathcal{V}}})(x)) \leq \min\{d(x),\, n - d(x)\}$ *for a.e.* $x \in \ Spec(\mathcal{V})$.

Proof. Theorem 4.11 imply that there exists a $\mathcal{V}$-dual $E(\mathcal{G}) \in \mathcal{D}^{SG}_{\mathcal{V}}(\mathcal{F})$ such that $S_{E(\mathcal{G})} = c \cdot P_{\mathcal{V}}$ if and only if $c \geq \lambda_i(\hat{A}_x)$ for $i \in \mathbb{I}_{d(x)}$ and $\lambda_i(\hat{A}_x) = c$ for $i \in \mathbb{I}_{m(x)}$ whenever $m(x) = 2d(x) - n \geq 1$, for a.e. $x \in \mathrm{Spec}(\mathcal{V})$. These last two conditions are equivalent to the fact that $c \cdot P_{\mathcal{V}} \geq S_{E(\mathcal{F})^{\#}_{\mathcal{V}}}$ and

$$\mathrm{rk}((c \cdot P_{\mathcal{V}} - A)(x)) \leq d(x) - m(x) = n - d(x) \quad \text{whenever} \quad m(x) \geq 1 \ .$$

Also notice that in case $m(x) \leq 0$ then $n - d(x) \geq d(x) = \dim J_{\mathcal{V}}(x)$. The proof follows from these remarks. □

Remark 4.13. Consider the notations of Theorem 4.11. As a consequence of Corollary 4.12, we get the following dichotomy related with the existence of tight oblique $\mathcal{V}$-duals of $E(\mathcal{F})$:

1. If $n \geq 2\, d(x)$ for a.e. $x \in \mathbb{T}^k$ then for every $c \geq \|S_{E(\mathcal{F})^{\#}_{\mathcal{V}}}\|$ there exists $E(\mathcal{G}) \in \mathcal{D}^{SG}_{\mathcal{V}}(\mathcal{F})$ that is a c-tight frame for $\mathcal{V}$.

2. If there exists $N \subseteq \mathbb{T}^k$ with positive Lebesgue measure such that $n < 2\, d(x)$ for a.e. $x \in N$ and there exists a c-tight frame $E(\mathcal{G}) \in \mathcal{D}^{SG}_{\mathcal{V}}(\mathcal{F})$ then $c = \|S_{E(\mathcal{F})^{\#}_{\mathcal{V}}}\|$. △

5 Applications: optimal SG-duals with norm restrictions

As before, we consider two FSI subspaces $\mathcal{V}$ and $\mathcal{W}$ such that $\mathcal{W}^{\perp} \oplus \mathcal{V} = L^2(\mathbb{R}^k)$ and $\mathcal{F} = \{f_i\}_{i \in \mathbb{I}_n} \in \mathcal{W}^n$ such that $E(\mathcal{F})$ is a frame for $\mathcal{W}$.

As a consequence of the description of the fine spectral structure of elements in $\mathcal{D}^{SG}_{\mathcal{V}}(\mathcal{F})$, we see that the canonical $\mathcal{V}$-dual is optimal with respect to several criteria. Nevertheless, in applied situations, the canonical dual might not be the best choice: for example, we can be interested in duals of $E(\mathcal{F})$ such that the spectrum of their frame operators are as

concentrated as possible. Ideally, we would search for tight dual frames for $E(\mathcal{F})$, although Corollary 4.12 shows that there are restrictions for the existence of such duals.

In order to search for alternate $\mathcal{V}$-duals that are spectrally more stable, we proceed as follows: for $w \geq \sum_{i\in\mathbb{I}_n} \|f^{\#}_{\mathcal{V},i}\|^2$, where $E(\mathcal{F})^{\#}_{\mathcal{V}} = \{T_\ell\, f^{\#}_{\mathcal{V},i}\}_{(\ell,i)\in\mathbb{Z}^k\times\mathbb{I}_n}$, we consider

$$\mathcal{D}^{SG}_{\mathcal{V},w}(\mathcal{F}) = \mathcal{D}^{SG}_{\mathcal{V},w}(E(\mathcal{F})) \stackrel{\text{def}}{=} \left\{E(\mathcal{G}) \in \mathcal{D}^{SG}_{\mathcal{V}}(\mathcal{F}) :\ \mathcal{G} = \{g_i\}_{i\in\mathbb{I}_n} \text{ and } \sum_{i\in\mathbb{I}_n} \|g_i\|^2 \geq w\right\}.$$

Notice that if $w > \sum_{i\in\mathbb{I}_n} \|f^{\#}_{\mathcal{V},i}\|^2$ then $E(\mathcal{F})^{\#}_{\mathcal{V}} \notin \mathcal{D}^{SG}_{\mathcal{V},w}(\mathcal{F})$ and therefore, it is natural to ask whether there is an optimal dual fulfilling the previous requirements. Using the identity

$$\sum_{i\in\mathbb{I}_n} \|g_i\|^2 = \int_{\mathbb{T}^k} \sum_{i\in\mathbb{I}_n} \|\Gamma g_i(x)\|^2\, dx = \int_{\mathbb{T}^k} \operatorname{tr}(\,(S_{E(\mathcal{G})})^{\frown}_x)\, dx = \int_{\mathbb{T}^k} \sum_{i\in\mathbb{N}} \mu_i(x)\, dx \tag{30}$$

where $\lambda(\,(S_{E(\mathcal{G})})^{\frown}_x) = (\mu_i(x))_{i\in\mathbb{N}}$ for a.e. $x \in \mathbb{T}^k$, we see that Theorem 4.11 gives a complete solution to a frame design problem in the sense that it allows to get a complete description of the eigenvalue lists of the frame operators of elements in $\mathcal{D}^{SG}_{\mathcal{V},w}(\mathcal{F})$. It is then natural to seek for those oblique SG-duals $E(\mathcal{G}) \in \mathcal{D}^{SG}_{\mathcal{V},w}(\mathcal{F})$ that minimize the convex potentials $P^{\mathcal{V}}_{\varphi}$, for $\varphi \in \operatorname{Conv}(\mathbb{R}^+)$; in order to deal with this problem we first examine a construction known as water-filling in terms of submajorization, in the general context of measure spaces (see Theorem 5.5). We then apply these results together with the properties of submajorization and results from matrix analysis to conclude that there are structural optimal duals with norm restrictions. These optimal solutions are obtained in terms of a non-commutative water-filling construction.

5.1 Water-filling in measure spaces

The water-filling construction goes back to the work of Shanon [45], as the solution of an optimal spectral allocation problem (see [19]). The water-filling strategy has also been the main tool in the design of channels with optimal capacity (see [46] and the more recent works on iterative water-filling techniques [40, 44]).

As a first step towards an extension of this construction, we examine its scalar counter-part in the general context of measure spaces. In the next section we show that the water-filling technique produces optimal solutions in the general (non-commutative) context of measurable fields of positive semidefinite matrix valued functions.

Throughout this section the triple $(X, \mathcal{X}, \mu)$ denotes a probability space. Recall that we denote by $L^\infty(X,\mu)^+ = \{f \in L^\infty(X,\mu) : f \geq 0\}$.

Definition 5.1 (Water-filling at level c)**.** Let $f \in L^\infty(X,\mu)^+$. Given $c \geq \operatorname{ess\,inf} f \geq 0$ we consider $f_c \in L^\infty(X,\mu)^+$ given by $f_c = \max\{f, c\} = f + (c-f)^+$, where g^+ denotes the positive part of a real function g. △

In order to study the submajorization properties of the function f_c obtained by the water-filling construction as above, we consider the following result in which we obtain a simple relation between the decreasing rearrangements of f and f_c.

Lemma 5.2. *Let $f \in L^\infty(X,\mu)^+$ and let $c \geq \operatorname{ess\,inf} f \geq 0$. Consider the number*

$$s_0 = \mu\{x \in X :\ f(x) > c\}\,. \quad \textit{Then} \quad f_c^*(s) = \begin{cases} f^*(s) & \textit{if} \quad 0 \leq s < s_0\,; \\ c & \textit{if} \quad s_0 \leq s \leq 1\,. \end{cases} \tag{31}$$

Proof. Notice that by Eq. (11), for $0 \leq s < s_0$ we have that

$$\begin{aligned} f^*(s) &= \sup\{t \in \mathbb{R}^+ :\ \mu\{x \in X :\ f(x) > t\} > s\} \\ &= \sup\{t \in \mathbb{R}^+ :\ \mu\{x \in X :\ f(x) > t\} > s \ \text{ and } \ t \geq c\} \\ &= \sup\{t \in \mathbb{R}^+ :\ \mu\{x \in X :\ f_c(x) > t\} > s \ \text{ and } \ t \geq c\} \\ &= \sup\{t \in \mathbb{R}^+ :\ \mu\{x \in X :\ f_c(x) > t\} > s\} = f_c^*(s)\ . \end{aligned}$$

It is straightforward to see that if $s_0 \leq s \leq 1$ then $f_c^*(s) = c$. □

In order to prove Theorem 5.5 below, we shall need an explicit statement of some re-parametrized versions of the basics results of section 2.3:

Lemma 5.3. *Let $a\,,\ b \in \mathbb{R}$ be such that $a < b$ and let $k \in L^\infty([\,a\,,\,b\,],\,\nu)^+$ be a non-increasing right continuous function, where $\nu = (b-a)^{-1}\,dt$ is the normalized Lebesgue measure on $[a, b]$. Then*

1. *The decreasing rearrangement $k^*(t) = k\big(\,(b-a)\,t + a\,\big)$ for every $t \in [0\,,\,1)$.*
2. *Fix a constant $c \in \mathbb{R}$. Then*

$$(b-a)\,c \leq \int_a^b k(t)\,dt \implies (s-a)\,c \leq \int_a^s k(t)\,dt \quad \text{for every} \quad s \in [a\,,\,b]\ .$$

Proof. Straightforward. □

With the notations of Lemma 5.3 above, notice that item 2. is a restatement (using the re-parametrization from item 1) of the submajorization inequalities corresponding to $c \prec_w k$ in $([a, b],\,\nu)$ whenever $c \leq \int_{[a,b]} k\ d\nu$ (see Example 2.5).

Remark 5.4. Let $f \in L^\infty(X, \mu)^+$ and consider $\phi : [\operatorname{ess\,inf} f, \infty) \to \mathbb{R}^+$ given by

$$\phi(c) = \int_X f_c\ d\mu = \int_X f(x) + (c - f(x))^+\ d\mu(x)\,.$$

Then, it is easy to see that ϕ has the following properties:

1. $\phi(\operatorname{ess\,inf} f) = \int_X f\ d\mu$ and $\lim_{c\to+\infty} \phi(c) = +\infty$;
2. ϕ is continuous and strictly increasing.

Hence, for every $w \geq \int_X f\ d\mu$ there exists a unique $c(w) = c \geq \operatorname{ess\,inf} f$ such that

$$\phi(c(w)) = w \quad \text{i.e.} \quad \int_X f_{c(w)}\ d\mu = w\ . \tag{32}$$

△

Theorem 5.5 ($\prec_w$-optimality of water-filling)**.** *Let $f \in L^\infty(X, \mu)^+$, take $w \geq \int_X f\ d\mu$ and consider the constant $c(w) = c$ as in Remark 5.4. Then, for every $h \in L^\infty(X, \mu)^+$,*

$$f \leq h \quad \text{and} \quad \int_X h\ d\mu \geq w \implies f_c \prec_w h\ .$$

Proof. Assume that $f \leq h$ and $\int_X h \, d\mu \geq w$. If we let $s_0 = \mu\{x \in X : \ f(x) > c\}$ then, by Lemma 5.2, we have that Eq. (31) holds. Thus, using Remark 2.2, we see that if $0 \leq s < s_0$ then

$$\int_0^s f_c^*(t) \, dt = \int_0^s f^*(t) \, dt \leq \int_0^s h^*(t) \, dt \, . \tag{33}$$

Fix now $s_0 \leq s \leq 1$ and consider

$$\alpha = \int_0^{s_0} h^*(t) \ dt - \int_0^{s_0} f_c^*(t) \ dt \geq 0 \quad \text{and} \quad k = h^* + \frac{1}{1-s_0} \, \alpha \in L^\infty([0,1], \ dt)^+ \, .$$

Notice that k is a non-increasing right continuous map. In this case we get that

$$\begin{aligned} \int_{s_0}^1 k(t) \, dt &= \int_{s_0}^1 h^*(t) \ dt + \alpha \\ &= \int_{s_0}^1 h^*(t) \, dt + \left(\int_0^{s_0} h^*(t) \, dt - \int_0^{s_0} f_c^*(t) \ dt \right) \\ &\geq w - (w - (1-s_0)\, c) = (1-s_0)\, c \, . \end{aligned}$$

Then, Lemma 5.3 (applied to the map $k|_{[s_0\,,\,1]}$) implies that

$$(s-s_0)\, c \leq \int_{s_0}^s k(t) \, dt = \int_{s_0}^s h^*(t) \, dt + \frac{s-s_0}{1-s_0} \, \alpha \quad \text{for every} \quad s \in [s_0 \, , \, 1] \, .$$

Hence, using the inequality above and Lemma 5.2, we conclude that for $s_0 \leq s < 1$

$$\begin{aligned} \int_0^s f_c^*(t) &= \int_0^{s_0} h^*(t) \ dt - \alpha + (s-s_0)\, c \\ &\leq \int_0^s h^*(t) \ dt + \left(\frac{s-s_0}{1-s_0} - 1 \right) \, \alpha \leq \int_0^s h^*(t) \ dt \, . \end{aligned}$$

This last fact together with Eq. (33) show that $f_c \prec_w h$. □

Theorem 5.5 above implies a family of integral inequalities in terms of convex functions involving the water-filling of a function f at level c. We will need these facts in order to show the optimality properties of the non-commutative version of waterfilling.

Corollary 5.6. *With the notations of Theorem 5.5, if $\varphi \in \mathrm{Conv}(\mathbb{R}^+)$ is non-decreasing then*

$$\int_X \varphi \circ h \, d\mu \geq \int_X \varphi \circ f_c \, d\mu \, . \tag{34}$$

If there is a non-decreasing $\varphi \in \mathrm{Conv}_{\mathrm{s}}(\mathbb{R}^+)$ such that equality holds in Eq. (34) then $h = f_c$.

Proof. The first claim is a consequence of the submajorization relation in Theorem 5.5 and Theorem 2.4. If we further assume that $\varphi \in \mathrm{Conv}_{\mathrm{s}}(\mathbb{R}^+)$ is such that equality holds in Eq. (34) then, by Proposition 2.6, we see that $f_c^* = h^*$. Let $B = \{x \in X : \ f(x) > c\}$ so that $s_0 = \mu(B)$. Then, it is straightforward to show that

$$(f \cdot 1_B)^*(s) = \begin{cases} f^*(s) & \text{if} \quad s \in [0, s_0) \, ; \\ 0 & \text{if} \quad s \in [s_0, 1) \, . \end{cases}$$

Notice that, in particular, $(f \cdot 1_B)^* = 1_{[0,s_0)} \cdot f_c^*$. On the other hand we have $(h \cdot 1_B)^* = 1_{[0,\, s_0)} \cdot (h \cdot 1_B)^*$. Hence, since $h \geq h \cdot 1_B \geq f \cdot 1_B$, by Remark 2.2 we have that

$$h^* \geq (h \cdot 1_B)^* = 1_{[0,s_0)} \cdot (h \cdot 1_B)^* \geq (f \cdot 1_B)^* = 1_{[0,s_0)} \cdot f_c^* \quad \Longrightarrow \quad (h \cdot 1_B)^* = (f \cdot 1_B)^* .$$

Therefore, again by Remark 2.2, we get that $h \cdot 1_B = f \cdot 1_B = f_c \cdot 1_B > c \cdot 1_B$, where the last facts follow from Definition 5.1. Finally, notice that

$$\mu(h^{-1}(\{c\})) = |(h^*)^{-1}(\{c\})| = |(f_c^*)^{-1}(\{c\})| = 1 - \mu(B) \ ,$$

which shows that $h \cdot 1_{X \setminus B} = c \cdot 1_{X \setminus B}$ and hence $h = f_c$. □

5.2 Optimal SG-duals with norm restrictions: NC water-filling

In what follows we show the existence of *structural* optimal SG oblique duals of a fixed frame $E(\mathcal{F})$ with norm restrictions, as described at the beginning of Section 5. That is, we explicitly construct a dual frame $E(\mathcal{G}^{\rm op}) \in \mathcal{D}^{SG}_{\mathcal{V},w}(\mathcal{F})$ such that for every $E(\mathcal{G}) \in \mathcal{D}^{SG}_{\mathcal{V},w}(\mathcal{F})$ then

$$P^{\mathcal{V}}_{\varphi}(E(\mathcal{G}^{\rm op})) \leq P^{\mathcal{V}}_{\varphi}(E(\mathcal{G})) \, ,$$

for every convex potential $P^{\mathcal{V}}_{\varphi}$ associated to a non-decreasing $\varphi \in \mathrm{Conv}(\mathbb{R}^+)$. Moreover, the arguments involved in this construction show that (structural) optimal SG oblique duals with norm restrictions share several spectral properties. We end the section with a non-commutative counter-part of the water-filling construction for functions, that allows to describe the spectral structure of optimal SG oblique duals in $\mathcal{D}^{SG}_{\mathcal{V},w}(\mathcal{F})$ in some detail.

Theorem 5.7 (Optimal duals in $\mathcal{D}^{SG}_{\mathcal{V},w}(\mathcal{F})$)**.** *Let $\mathcal{V}$ and $\mathcal{W}$ be FSI subspaces of $L^2(\mathbb{R}^k)$ such that $\mathcal{W}^\perp \oplus \mathcal{V} = L^2(\mathbb{R}^k)$. Let $\mathcal{F} = \{f_i\}_{i \in \mathbb{I}_n}$ be such that $E(\mathcal{F})$ is a frame for $\mathcal{W}$ and $w > \sum_{i \in \mathbb{I}_n} \|f^{\#}_{\mathcal{V},i}\|^2$, where $E(\mathcal{F})^{\#}_{\mathcal{V}} = \{T_\ell \, f^{\#}_{\mathcal{V},i}\}_{(\ell,i) \in \mathbb{Z}^k \times \mathbb{I}_n}$. Then, there exists $\mathcal{G}^{\rm op} = \{g_i^{\rm op}\}_{i \in \mathbb{I}_n} \in \mathcal{V}^n$ such that:*

1. *$E(\mathcal{G}^{\rm op}) \in \mathcal{D}^{SG}_{\mathcal{V},w}(\mathcal{F})$ and $\sum_{i \in \mathbb{I}_n} \|g_i^{\rm op}\|^2 = w$.*
2. *For every $\mathcal{G} = \{g_i\}_{i \in \mathbb{I}_n}$ such that $E(\mathcal{G}) \in \mathcal{D}^{SG}_{\mathcal{V},w}(\mathcal{F})$ and every non-decreasing $\varphi \in \mathrm{Conv}(\mathbb{R}^+)$ we have that*
$$P^{\mathcal{V}}_{\varphi}(E(\mathcal{G}^{\rm op})) \leq P^{\mathcal{V}}_{\varphi}(E(\mathcal{G})) \ .$$

Proof. Let $d(x) = \dim J_{\mathcal{V}}(x) = \dim J_{\mathcal{W}}(x)$ for $x \in \mathbb{T}^k$. For each $i \in \mathbb{I}_n$, let $X_i = d^{-1}(i) \subseteq \mathbb{T}^k$, $p_i = |X_i|$ (the Lebesgue measure of X_i) and $r_i = \min\{n - i, \, i\}$. Since $E(\mathcal{F})$ is a frame for $\mathcal{W}$ then $\mathrm{Spec}(\mathcal{V}) = \mathrm{Spec}(\mathcal{W}) = \cup_{i \in \mathbb{I}_n} X_i$. Also, for $i \in \mathbb{I}_n$ and $j \in \mathbb{I}_{r_i}$ we consider the measure space $(X_{ij}, \mathcal{X}_{ij}, |\cdot|_{ij})$, where $X_{ij} = X_i$, $\mathcal{X}_{ij} = \mathcal{X}_i$ denotes the σ-algebra of Borel sets in X_i and $|\cdot|_{ij} = |\cdot|_i$ denotes the Lebesgue measure in X_i. Then, using Remark 3.7, we construct the measure space

$$(Y, \mathcal{Y}, \nu) = \bigoplus_{i \in \mathbb{I}_n} \bigoplus_{j \in \mathbb{I}_{r_i}} (X_{ij}, \mathcal{Y}_{ij}, |\cdot|_{ij}).$$

In particular, $\nu(Y) = \sum_{i \in \mathbb{I}_n} r_i \cdot p_i$. We further consider the canonical inclusion maps $\eta_{i,j} : X_{i,j} \to Y$ for $i \in \mathbb{I}_n$ and $j \in \mathbb{I}_{r_i}$.

Let $\mathcal{G} = \{g_i\}_{i \in \mathbb{I}_n}$ be such that $E(\mathcal{G}) \in \mathcal{D}^{SG}_{\mathcal{V}}(\mathcal{F})$. We shall denote by $A = S_{E(\mathcal{F})^{\#}_{\mathcal{V}}}$ and $S = S_{E(\mathcal{G})} \in L(L^2(\mathbb{R}^k))^+$. By Proposition 4.10, $S = S_{E(\mathcal{F})^{\#}_{\mathcal{V}}} + B = A + B$, for some

$B \in L(L^2(\mathbb{R}^k))^+$ which is SP, $R(B) \subset \mathcal{V}$ and $\mathrm{rk}(\hat{B}_x) \leq n - d(x)$ for a.e. $x \in \mathbb{T}^k$. Let $i \in \mathbb{I}_n$ and $x \in X_i$; using Lidskii's additive inequality (see [4]) we get that

$$\big(\, \lambda_{i-j+1}(\hat{A}_x) + \lambda_j(\hat{B}_x)\,\big)_{j\in\mathbb{I}_i} \prec \big(\, \lambda_j(\,\hat{S}_x)\,\big)_{j\in\mathbb{I}_i}\,, \tag{35}$$

while $\lambda_j(\,\hat{S}_x) = 0$ for $j \geq i+1$. Notice that $R(\hat{B}_x) \subset J_{\mathcal{V}}(x)$ and $\mathrm{rk}(\hat{B}_x) \leq n - i$. Therefore $\mathrm{rk}(\hat{B}_x) \leq \min\{n-i,\, i\} = r_i$. Then, for $x \in X_i$ we have that

$$\lambda_{i-j+1}(\,\hat{A}_x) + \lambda_j(\hat{B}_x) = \begin{cases} \lambda_{i-j+1}(\hat{A}_x) + \lambda_j(\hat{B}_x) & \text{if} \quad 1 \leq j \leq r_i\,; \\ \lambda_{i-j+1}(\,\hat{A}_x) & \text{if} \quad r_i + 1 \leq j \leq i\,. \end{cases} \tag{36}$$

Now, Eq. (35) together with Eq. (36) imply that, for any $\varphi \in \mathrm{Conv}(\mathbb{R}^+)$: if $x \in X_i$ then

$$\sum_{j=1}^{r_i} \varphi(\lambda_{i-j+1}(\,\hat{A}_x) + \lambda_j(\hat{B}_x)) + \sum_{j=r_i+1}^{i} \varphi(\lambda_{i-j+1}(\,\hat{A}_x)\,) \leq \sum_{j\in\mathbb{I}_i} \varphi(\,\lambda_j(\hat{S}_x)\,)\ . \tag{37}$$

With the previous notations, we now consider the measurable function $h : Y \to \mathbb{R}^+$ defined as follows: for $x \in Y$, let $(i\,,\, j) \in \mathbb{I}_n \times \mathbb{I}_{r_i}$ and $\tilde{x} \in X_{i,j} = X_i$ be (uniquely determined) such that $\eta_{i,j}(\tilde{x}) = x$; in this case we set $h(x) = \lambda_{i-j+1}(\,\hat{A}_{\tilde{x}}) + \lambda_j(\hat{B}_{\tilde{x}})$. If we let $w_0 = \sum_{i\in\mathbb{I}_n} \|f^{\#}_{\mathcal{V},i}\|^2$ and we assume that $E(\mathcal{G}) \in \mathcal{D}^{SG}_{\mathcal{V},w}(\mathcal{F})$ then, using Eq. (30) we see that

$$\int_{\mathbb{T}^k} \mathrm{tr}(\hat{B}_x)\ dx = \int_{\mathbb{T}^k} \mathrm{tr}(\,\hat{S}_x - \hat{A}_x)\ dx \geq w - w_0 \geq 0\ .$$

Consider now the measurable function $f : Y \to \mathbb{R}^+$ given as before by $f(x) = \lambda_{i-j+1}(\,\hat{A}_{\tilde{x}})$ for $\tilde{x} \in X_{ij} = X_i$, with $(i\,,\, j) \in \mathbb{I}_n \times \mathbb{I}_{r_i}$ such that $\eta_{i,j}(\tilde{x}) = x$. Arguing as in the proof of Theorem 3.9 we get that

$$\int_Y f\ d\nu = \sum_{i\in\mathbb{I}_n} \int_{X_i} \sum_{j\in\mathbb{I}_{r_i}} \lambda_{i-j+1}(\,\hat{A}_x)\ dx\,. \tag{38}$$

Moreover, the previous facts show that if $E(\mathcal{G}) \in \mathcal{D}^{SG}_{\mathcal{V},w}(\mathcal{F})$ we have that $h \geq f$ and

$$\int_Y h(x)\ d\nu(x) = \sum_{i\in\mathbb{I}_n} \sum_{j\in\mathbb{I}_{r_i}} \int_{X_i} (\lambda_{i-j+1}(\,\hat{A}_x) + \lambda_j(\hat{B}_x))\ dx \geq (w - w_0) + \int_Y f(x)\ d\nu(x) \ \stackrel{\text{def}}{=}\ w'\ .$$

Let $c = c(w')$ be as in Remark 5.4 and consider f_c as in Definition 5.1, both with respect to the probability space $(Y, \mathcal{Y}, \tilde{\nu})$, where $\tilde{\nu} = \nu(Y)^{-1}\,\nu$. By Corollary 5.6 and the previous remarks we see that if $\varphi \in \mathrm{Conv}(\mathbb{R}^+)$ is non-decreasing then

$$\int_Y \varphi \circ f_c\ d\nu \leq \int_Y \varphi \circ h\ d\nu\,. \tag{39}$$

For $j \in \mathbb{I}_n$ we consider the measurable functions $\xi_j : \mathrm{Spec}(\mathcal{V}) \to \mathbb{R}^+$ defined as follows: for $i \in \mathbb{I}_n$ and $x \in X_i$,

$$\xi_j(x) = \begin{cases} f_c(\eta_{ij}(x)) = \max\{c\,,\, \lambda_{i-j+1}(\,\hat{A}_x)\} & \text{if} \quad 1 \leq j \leq r_i \\ \lambda_{i-j+1}(\,\hat{A}_x) & \text{if} \quad r_i + 1 \leq j \leq i \\ 0 & \text{if} \quad i+1 \leq j \leq n \end{cases}\ , \tag{40}$$

Notice that by construction, if $\varphi \in \mathrm{Conv}(\mathbb{R}^+)$ then

$$\int_Y \varphi \circ f_c \, d\nu = \sum_{i\in\mathbb{I}_n} \int_{X_i} \sum_{j\in\mathbb{I}_{r_i}} \varphi(\xi_j(x)) \, dx \, . \tag{41}$$

Using the definition of f and the properties of f_c from Remark 5.4, we see that if $x \in X_i$, then there exist $\lambda_1^{\mathrm{op}}(x) \geq \ldots \geq \lambda_{r_i}^{\mathrm{op}}(x) \geq 0$ such that

$$\xi_j(x) = \lambda_{i-j+1}(\hat{A}_x) + \lambda_j^{\mathrm{op}}(x) \quad \text{ for every } \quad j \leq r_i \, . \tag{42}$$

Let $\mu = (\mu_j)_{j\in\mathbb{N}} : \mathbb{T}^k \to \ell^1(\mathbb{N})^+$ such that: $\mu_j(x) = 0$ for $j \in \mathbb{N}$ whenever $x \in \mathbb{T}^k \setminus \mathrm{Spec}(\mathcal{V})$, while for $i \in \mathbb{I}_n$ and $x \in X_i$ then $\mu_j(x) = 0$ for $j \geq i+1$ and

$$(\mu_j(x))_{j\in\mathbb{I}_i} = [(\xi_j(x))_{j\in\mathbb{I}_i}]^{\downarrow} \, . \tag{43}$$

Putting the previous remarks together we see that $\mu = (\mu_j)_{j\in\mathbb{N}}$ satisfies the conditions of item 2 in Theorem 4.11. Thus, there exists $\mathcal{G}^{\mathrm{op}} = \{g_i^{\mathrm{op}}\}_{i\in\mathbb{I}_n}$ such that $E(\mathcal{G}^{\mathrm{op}}) \in \mathcal{D}_{\mathcal{V}}^{SG}(\mathcal{F})$ and $\lambda(\, (S_{E(\mathcal{G}^{\mathrm{op}})})_x) = (\mu_j(x))_{j\in\mathbb{N}}$ for a.e. $x \in \mathbb{T}^k$. In this case, if we consider Eq. (30), use Eqs. (42), (43) and we take $\varphi(x) = x$ in Eq. (41) we have that

$$\begin{aligned}
\sum_{i\in\mathbb{I}_n} \|g_i^{\mathrm{op}}\|^2 &= \int_{\mathbb{T}^k} \sum_{j\in\mathbb{N}} \mu_j(x) \, dx = \sum_{i\in\mathbb{I}_n} \int_{X_i} \left(\sum_{j\in\mathbb{I}_{r_i}} \xi_j(x) + \sum_{j=r_i+1}^{i} \lambda_{i-j+1}(\hat{A}_x) \right) \, dx \\
&= \int_Y f_c \, d\nu + \sum_{i\in\mathbb{I}_n} \int_{X_i} \sum_{j=r_i+1}^{i} \lambda_{i-j+1}(\, \hat{A}_x) \, dx \\
&= (w - w_0) + \int_{\mathbb{T}^k} \mathrm{tr}(\, \hat{A}_x) \, dx = w \, ,
\end{aligned}$$

where we have also used the relation in Eq. (38) above. In particular, $\mathcal{G}^{\mathrm{op}}$ satisfies item 1. in the statement. Now, if $E(\mathcal{G}) \in \mathcal{D}_{\mathcal{V},w}^{SG}(\mathcal{F})$, using Eqs. (37), (39), (40) and (43) then,

$$\begin{aligned}
P_\varphi^{\mathcal{V}}(E(\mathcal{G})) &\geq \int_Y \varphi \circ h \, d\nu + \sum_{i\in\mathbb{I}_n} \int_{X_i} \sum_{j=r_i+1}^{i} \varphi \circ \lambda_{i-j+1}(\hat{A}_x) \, dx \\
&\geq \int_Y \varphi \circ f_c \, d\nu + \sum_{i\in\mathbb{I}_n} \int_{X_i} \sum_{j=r_i+1}^{i} \varphi \circ \lambda_{i-j+1}(\hat{A}_x) \, dx = P_\varphi^{\mathcal{V}}(E(\mathcal{G}^{\mathrm{op}}))
\end{aligned}$$

where we have also used Eq. (41) and the fact that $\lambda((S_{E(\mathcal{G}^{\mathrm{op}})})_x) = \mu(x)$ for a.e. $x \in \mathbb{T}^k$. □

Corollary 5.8 (Essential uniqueness of optimal $\mathcal{V}$-duals with norm restrictions)**.** *With the notations of Theorem 5.7 ant its proof, assume that $\mathcal{G} = \{g_i\}_{i\in\mathbb{I}_n}$ is such that $E(\mathcal{G}) \in \mathcal{D}_{\mathcal{V},w}^{SG}(\mathcal{F})$ and that there exists a non-decreasing $\varphi \in \mathrm{Conv}_{\mathrm{s}}(\mathbb{R}^+)$ such that*

$$P_\varphi^{\mathcal{V}}(E(\mathcal{G}^{\mathrm{op}})) = P_\varphi^{\mathcal{V}}(E(\mathcal{G})) \, . \tag{44}$$

Let $B \in L(L^2(\mathbb{R}^k))^+$ be SP, with $R(B) \subset \mathcal{V}$ and such that $S_{E(\mathcal{G})} = S_{E(\mathcal{F})_{\mathcal{V}}^{\#}} + B = A + B$. Then,

1. $\sum_{i\in\mathbb{I}_n} \|g_i\|^2 = w$;

2. *There exist $c > 0$ and measurable vector fields $v_i : \mathbb{T}^k \to \ell^2(\mathbb{Z}^k)$ for $i \in \mathbb{I}_n$ such that $\{v_i(x)\}_{i\in\mathbb{I}_{d(x)}}$ is an ONB of $J_{\mathcal{V}}(x)$ for a.e. $x \in Spec(\mathcal{V})$,*

$$(S_{E(\mathcal{F})^{\#}_{\mathcal{V}}})^{\frown}_x = \hat{A}_x = \sum_{i\in\mathbb{I}_{d(x)}} \lambda_i(\hat{A}_x)\ v_i(x) \otimes v_i(x)\ , \quad \text{for a.e. } x \in Spec(\mathcal{V})$$

and such that for a.e. $x \in Spec(\mathcal{V})$ we have that

$$\hat{B}_x = \sum_{i=r(x)+1}^{d(x)} \big(c - \lambda_i(\hat{A}_x)\,\big)^+\ v_i(x) \otimes v_i(x)\ ,$$

where $r(x) = \max\{2d(x) - n,\, 0\}$, for $x \in \mathbb{T}^k$.

The constant c $(= c(w'))$ > 0 does not depend on $\mathcal{G}$. Moreover, in this case $P^{\mathcal{V}}_{\psi}(E(\mathcal{G})) = P^{\mathcal{V}}_{\psi}(E(\mathcal{G}^{\rm op}))$ for every non-decreasing $\psi \in \mathrm{Conv}(\mathbb{R}^+)$.

Proof. We use the notions and notations from the proof of Theorem 5.7. Arguing as in the last part of the proof of Theorem 5.7 we see that Eq. (44) implies that

$$\int_Y \varphi \circ h\ d\tilde{\nu} = \int_Y \varphi \circ f_c\ d\tilde{\nu}\,,$$

where $\tilde{\nu} = \nu(Y)^{-1}\,\nu$ is the probability measure obtained by normalization of ν. By Corollary 5.6 we get that $h = f_c$, where $c = c(w')$ is as in the proof of Theorem 5.7. Therefore, for $i \in \mathbb{I}_n$ and $x \in X_i$, then

$$\lambda_{i-j+1}(\hat{A}_x) + \lambda_j(\hat{B}_x) = \begin{cases} \max\{\lambda_{i-j+1}(\hat{A}_x),\, c\} & \text{if} \quad 1 \le j \le r_i\,; \\ \lambda_{i-j+1}(\hat{A}_x) & \text{if} \quad r_i + 1 \le j \le i\,. \end{cases} \tag{45}$$

Moreover, by Eq. (37) and the properties of h we have that

$$\begin{aligned} P^{\mathcal{V}}_{\varphi}(E(\mathcal{G})) &= \sum_{i\in\mathbb{I}_n} \int_{X_i} \sum_{j\in\mathbb{I}_i} \varphi(\lambda_j(\,(S_{E(\mathcal{G})})^{\frown}_x)\,) \\ &\ge \int_Y \varphi \circ h\ d\nu + \sum_{i\in\mathbb{I}_n} \int_{X_i} \sum_{j=r_i+1}^{i} \varphi \circ \lambda_{i-j+1}(\hat{A}_x)\ dx = P^{\mathcal{V}}_{\varphi}(E(\mathcal{G}))\ . \end{aligned}$$

Therefore, we should have equality Eq. (37) for a.e. $x \in X_i$ and $i \in \mathbb{I}_n$. Since φ is strictly convex, then the majorization relation in Eq. (35) together with the case of equality in Lidskii's inequality (see the Appendix section in [38]) imply that for $i \in \mathbb{I}_n$ and a.e. $x \in X_i$ there exists an ONB $\{z_j(x)\}_{j\in\mathbb{I}_i}$ of $J_{\mathcal{V}}(x)$ (but not necessarily of measurable vector fields as functions of x) such that

$$(S_{E(\mathcal{F})^{\#}_{\mathcal{V}}})^{\frown}_x = \sum_{j\in\mathbb{I}_i} \lambda_j(\hat{A}_x)\ z_j(x) \otimes z_j(x) \ \text{ and } \ \hat{B}_x = \sum_{j\in\mathbb{I}_i} \lambda_{i-j+1}(\hat{B}_x)\ z_j(x) \otimes z_j(x)\,. \tag{46}$$

Let $P \in L(L^2(\mathbb{R}^k))^+$ denote the orthogonal projection onto $\mathcal{R} = \overline{R(B)}$, so that P is SP and $\hat{P}_x = P_{R(\hat{B}_x)}$ for every $x \in \mathbb{T}^k$. Let $p : \mathbb{T}^k \to \mathbb{N}$ be the measurable function given

by $p(x) = \mathrm{tr}(\hat{P}_x)$ for $x \in \mathbb{T}^k$. Then, by inspection of Eqs. (45) and (46) we see that $P\, S_{E(\mathcal{F})^{\#}_{\mathcal{V}}} = S_{E(\mathcal{F})^{\#}_{\mathcal{V}}}\, P$,

$$\hat{P}_x\, \hat{A}_x\, \hat{P}_x + \hat{B}_x = c \cdot \hat{P}_x \ , \qquad \text{for a.e. } x \in \mathbb{T}^k$$

and, for $i \in \mathbb{I}_n$ and $x \in X_i$,

$$(I - P)\hat{}_x\, \hat{A}_x\, (I - P)\hat{}_x = \sum_{j=1}^{i-p(x)} \lambda_j(\hat{A}_x)\ z_j(x) \otimes z_j(x)\,.$$

Since $a + (c-a)^+ = \max\{a,\, c\}$ for $a,\, c \geq 0$, these last facts imply the existence of measurable vector fields $v_i : \mathbb{T}^k \to \ell^2(\mathbb{Z}^k)$ for $i \in \mathbb{I}_n$ with the desired properties; indeed, the previous identities show that we just have to consider measurable fields of eigenvectors of the operators $P\, S_{E(\mathcal{F})^{\#}_{\mathcal{V}}}\, P$ and $(I - P)\, S_{E(\mathcal{F})^{\#}_{\mathcal{V}}}\, (I - P)$, whose existence follow from Lemma 3.2.

Finally, if $\mathcal{G}^{\mathrm{op}}$ is as in Theorem 5.7, a careful inspection of the proof of that theorem shows that

$$\lambda(\ (S_{E(\mathcal{G})})\hat{}_x) = \lambda(\ (S_{E(\mathcal{G}^{\mathrm{op}})})\hat{}_x)\ , \qquad \text{for a.e. } x \in \mathrm{Spec}(\mathcal{V})\,,$$

which implies the optimality properties of $E(\mathcal{G})$ for a non-decreasing $\psi \in \mathrm{Conv}(\mathbb{R}^+)$. □

Notice that with the notations of Corollary 5.8 above, we see that for a.e. $x \in \mathbb{T}^k$ then

$$(S_{E(\mathcal{G}^{\mathrm{op}})})\hat{}_x = \sum_{i=1}^{r(x)} \lambda_i(\hat{A}_x)\ v_i(x) \otimes v_i(x) + \sum_{i=r(x)+1}^{d(x)} \max\{\lambda_i(\hat{A}_x),\, c\}\ v_i(x) \otimes v_i(x)\,,$$

where we have used that $a + (c-a)^+ = \max\{a,\, c\}$ for $a,\, c \geq 0$. In particular, notice that

$$\lambda_{d(x)}(S_{E(\mathcal{G}^{\mathrm{op}})})\hat{}_x \geq \max\{c\,,\ \lambda_{d(x)}(\hat{A}_x)\}$$

which implies that the condition number of $(S_{E(\mathcal{G}^{\mathrm{op}})})\hat{}_x$ is smaller than or equal to the condition number of $\hat{A}_x = (S_{E(\mathcal{F})^{\#}_{\mathcal{V}}})\hat{}_x$ - both acting on $J_{\mathcal{V}}(x)$ - for a.e. $x \in \mathbb{T}^k$. That is, the optimal oblique dual $E(\mathcal{G}^{\mathrm{op}})$ improves the (spectral) stability of the canonical oblique dual $E(\mathcal{F})^{\#}_{\mathcal{V}} = E(\mathcal{F}^{\#}_{\mathcal{V}})$.

The representation of $(S_{E(\mathcal{G}^{\mathrm{op}})})\hat{}_x$ above motivates the following construction, which also characterizes all elements of $\mathcal{D}^{SG}_{\mathcal{V},w}(\mathcal{F})$ which are minimal in the sense of Theorem 5.7.

Definition 5.9 (Non-commutative water-filling at level c in $U_{\mathcal{V}}(E(\mathcal{G}))$). Let $\mathcal{G} = \{g_i\}_{i\in\mathbb{I}_n}$ be such that $E(\mathcal{G})$ is a frame for $\mathcal{V}$ with frame operator $A = S_{E(\mathcal{G})}$. By Lemma 3.2 we can consider measurable vector fields $v_j : \mathbb{T}^k \to \ell^2(\mathbb{Z}^k)$ for $j \in \mathbb{I}_n$ such that

$$\hat{A}_x = \sum_{j\in\mathbb{I}_{d(x)}} \lambda_j(\hat{A}_x)\ v_j(x) \otimes v_j(x) \tag{47}$$

is a spectral representation of $\hat{A}_x$, where $\{v_j(x)\}_{j\in\mathbb{I}_{d(x)}}$ is an ONB of $J_{\mathcal{V}}(x)$ (here $d(x) = \dim J_{\mathcal{V}}(x) \leq n$), for a.e. $x \in \mathbb{T}^k$.

Given $c \geq 0$ then we define the (non-commutative) water-filling of A at level c with respect to the representation in Eq. (47), denoted $A_c \in U_{\mathcal{V}}(E(\mathcal{G})\,)$, as the unique positive SP operator such that operator $R(A_c) \subset \mathcal{V}$ and

$$\hat{A}_c(x) \stackrel{\text{def}}{=} (A_c)_x^{\frown} = \sum_{i \in \mathbb{I}_{r(x)}} \lambda_i(\hat{A}_x)\; v_i(x) \otimes v_i(x) + \sum_{i=r(x)+1}^{d(x)} \max\{\lambda_i(\hat{A}_x),\, c\}\; v_i(x) \otimes v_i(x) \quad (48)$$

where $r(x) = \max\{2d(x) - n, 0\}$ (recall $\mathbb{I}_0 = \emptyset$) for a.e. $x \in \mathbb{T}^k$. $\triangle$

Remark 5.10. With the notations of Definition 5.9:

1. We point out that $\hat{A}_c$ as described in Eq. (48) is a well defined measurable field of positive semidefinite operators that is essentially bounded.

2. Notice that in the spectral representation of $\hat{A}_c(x)$ given in Eq. (48), the eigenvalues are not necessarily arranged in non-increasing order.

3. Finally notice that $A_c \in U_{\mathcal{V}}(E(\mathcal{G})\,)$, since $\hat{A}_c(x) - \hat{A}_x$ is a positive operator with rank at most $d(x) - r(x) \leq n - d(x)$, for a.e. $x \in \mathbb{T}^k$. $\triangle$

We end this section with the following comments: with the notions and notations of Theorem 5.7, let $A = S_{E(\mathcal{F})_{\mathcal{V}}^{\#}}$ and consider measurable vector fields $v_i : \mathbb{T}^k \to \ell^2(\mathbb{Z}^k)$, for $i \in \mathbb{I}_n$, such that

$$\hat{A}_x = \sum_{i \in \mathbb{I}_{d(x)}} \lambda_i(\hat{A}_x)\; v_i(x) \otimes v_i(x) \quad (49)$$

is a spectral representation of $\hat{A}_x$ with respect to an eigen-basis $\{v_i(x)\}_{i \in \mathbb{I}_{d(x)}}$, where $d(x) = \dim(J_{\mathcal{V}}(x))$, for a.e. $x \in \mathbb{T}^k$. Let $c > 0$ be such that, if A_c is the water-filling of A at level c with respect to the representation in Eq. (49) then,

$$\int_{\mathbb{T}^k} \operatorname{tr}(\hat{A}_c(x))\; dx = w\;.$$

By construction $A_c \in U_{\mathcal{V}}(E(\mathcal{F})_{\mathcal{V}}^{\#})$ and therefore, by Proposition 4.10, there exists $\mathcal{G}_0 \in \mathcal{D}_{\mathcal{V},w}^{SG}(\mathcal{F})$ such that $S_{\mathcal{G}_0} = A_c$. As we have already noticed, $\mathcal{G}^{\text{op}}$ from Theorem 5.7 is constructed in this way; hence, in this case we have that for every non-decreasing $\varphi \in \text{Conv}(\mathbb{R}^+)$,

$$P_\varphi^{\mathcal{V}}(E(\mathcal{G}_0)) \leq P_\varphi^{\mathcal{V}}(E(\mathcal{G}))\;, \qquad \text{for every } \mathcal{G} \in \mathcal{D}_{\mathcal{V},w}^{SG}(\mathcal{F})\,.$$

Moreover, by Corollary 5.8, any structural optimal frame $\mathcal{G} \in \mathcal{D}_{\mathcal{V},w}^{SG}(\mathcal{F})$ (i.e. such that $\mathcal{G}$ is a $P_\varphi^{\mathcal{V}}$-minimizer in $\mathcal{D}_{\mathcal{V},w}^{SG}(\mathcal{F})$ for every $\varphi \in \text{Conv}(\mathbb{R}^+)$) is obtained in this way. That is, the structural optimal SG $\mathcal{V}$-dual frames for $E(\mathcal{F})$ with norm restrictions are exactly those $\mathcal{G} \in \mathcal{D}_{\mathcal{V},w}^{SG}(\mathcal{F})$ for which their frame operators are obtained in terms of the non-commutative water-filling construction from Definition 5.9.

6 Appendix: spectral structure of $U_{\mathcal{V}}(E(\mathcal{G})\,)$

In what follows we consider a measure space $(X, \mathcal{X}, \rho)$, such that $X \subset \mathbb{T}^k$ is a Borel set, $\mathcal{X}$ denotes the σ-algebra of Borel sets in X and ρ is the Lebesgue measure restricted to $\mathcal{X}$.

Proposition 6.1. *Let $G : X \to \mathcal{M}_n(\mathbb{C})^+$ be a bounded measurable field of positive semidefinite matrices with associated measurable eigenvalues $\lambda_j : X \to \mathbb{R}^+$ for $j \in \mathbb{I}_n$ such that $\lambda_1 \geq \ldots \geq \lambda_n$. Assume that the measurable functions $\beta_j : X \to \mathbb{R}^+$ for $j \in \mathbb{I}_{n-1}$ satisfy the interlacing conditions*

$$\lambda_j(x) \geq \beta_j(x) \geq \lambda_{j+1}(x) \qquad x\text{-a.e. for} \quad j \in \mathbb{I}_{n-1} \ . \tag{50}$$

Then there exists a measurable map $W : X \to \mathcal{M}_{n,\,n-1}(\mathbb{C})$ such that $W^*(x)\,W(x) = I_{n-1}$ and

$$\lambda(W^*(x)\,G(x)\,W(x)\,) = (\beta_1(x), \ldots, \beta_{n-1}(x)\,) \ , \quad \text{for a.e.} \quad x \in X \ . \tag{51}$$

Proof. We argue by induction on n (the size of G). Notice that $\beta_1 \geq \ldots \geq \beta_{n-1}$ by Eq. (50). Using the results of [42], we can consider measurable vector fields $u_j : X \to \mathbb{C}^n$ for $j \in \mathbb{I}_n$ such that $\{u_j(x)\}_{j\in\mathbb{I}_n}$ is an ONB of eigenvectors of $G(x)$ for a.e. $x \in X$.

Assume first that $\beta_{n-1} = \lambda_n$. Set $G'(x) = V(x)^*\,G(x)\,V(x)$ where $V(x)$ is the $n \times (n-1)$ matrix whose columns are the vectors $u_1(x), \ldots, u_{n-1}(x)$, for $x \in X$. Then, G' is a bounded measurable field of (diagonal) positive semidefinite matrices of size $n-1$ with measurable eigenvalues $\lambda_j : X \to \mathbb{R}^+$ for $j \in \mathbb{I}_{n-1}$. If we assume that we can find a measurable function $Z : X \to \mathcal{M}_{n-1,\,n-2}(\mathbb{C})$ such that $Z^*(x)Z(x) = I_{n-2}$ and $\lambda(Z^*(x)G'(x)\,Z(x)) = (\beta_1(x), \ldots, \beta_{n-2}(x))$ for a.e. $x \in X$, we let

$$W(x) = \begin{pmatrix} Z(x) & 0_{n-1} \\ 0_{n-2}^t & 1 \end{pmatrix} \ , \quad \text{for } x \in X \, .$$

Then, it is easy to see that $W : X \to \mathcal{M}_{n,n-1}(\mathbb{C})$ has the desired properties. By iterating the previous argument and considering a convenient partition of X into measurable sets, we can assume without loss of generality that

$$\lambda_j(x) > \beta_j(x) > \lambda_{j+1}(x) \ , \quad \text{for a.e. } x \in X \ , \quad j \in \mathbb{I}_{n-1} \, .$$

In this case we set

$$\gamma_j(x) = \frac{\displaystyle\prod_{i\in\mathbb{I}_{n-1}} (\lambda_j(x) - \beta_i(x))}{\displaystyle\prod_{k\neq j} (\lambda_j(x) - \lambda_k(x))} \ , \quad \text{for } x \in X \ , \quad j \in \mathbb{I}_n \, .$$

The previous assumptions (strict interlacing inequalities) imply that $\gamma_j(x) > 0$ is defined for a.e. $x \in X$; moreover, the functions $\gamma_j : X \to \mathbb{R}^+$ are measurable for $j \in \mathbb{I}_n$.

Set $\xi_j = \gamma_j^{1/2} : X \to \mathbb{R}^+$ for $j \in \mathbb{I}_n$, let $v = \sum_{j\in\mathbb{I}_n} \xi_j\, u_j : X \to \mathbb{C}^n$ and let $P : X \to \mathcal{M}_n(\mathbb{C})^+$ given by $P(x) = I - P_{v(x)}$ (the orthogonal projection onto $\{v(x)\}^\perp$, notice that $v(x) \neq 0$ a.e.). Let $p_x(t) \in \mathbb{R}[t]$ denote the characteristic polynomial of $P(x)\,G(x)\,P(x)$. Then, a well known argument in terms of alternate tensor products (see [4]) shows that

$$p_x(t) = t \sum_{j\in\mathbb{I}_n} \gamma_j(x) \prod_{k\neq j} (t - \lambda_k(x)) \implies p_x(\lambda_j(x)) = \lambda_j(x) \prod_{i\in\mathbb{I}_{n-1}} (\lambda_j(x) - \beta_i(x))$$

for a.e. $x \in X$, $j \in \mathbb{I}_n$ and $p_x(0) = 0$. Therefore,

$$p_x(t) = t \prod_{j\in\mathbb{I}_{n-1}} (t - \beta_j) \quad \text{and} \quad \sum_{j\in\mathbb{I}_n} \xi_j^2(x) = 1 \ , \quad \text{for a.e. } x \in X \ ,$$

by comparing the leading coefficients of the two representations of the polynomial. This last normalization condition shows, in particular, that $P(x) = I - v(x) \otimes v(x)$ for $x \in X$ a.e. and hence P is a measurable function.

Finally, let $\{w_j : X \to \mathbb{C}^n\}_{j \in \mathbb{I}_n}$ be a measurable ONB of eigenvectors functions for P such that $P(x)\, w_n(x) = 0$ for a.e. $x \in X$. Set $W : X \to \mathcal{M}_{n,\, n-1}(\mathbb{C})$ such that $W(x)$ is the $n \times n-1$ matrix whose columns are the vectors $w_1(x), \ldots, w_{n-1}(x)$; then W is a measurable function with the desired properties. □

Lemma 6.2. *Let $\lambda_j : X \to \mathbb{R}^+$ for $j \in \mathbb{I}_n$ be measurable functions such that $\lambda_1 \geq \ldots \geq \lambda_n$. Let $d \in \mathbb{I}_{n-1}$ and let $\beta_j : X \to \mathbb{R}^+$ for $j \in \mathbb{I}_d$ be measurable functions such that $\beta_1 \geq \ldots \geq \beta_d$ and such that they satisfy the interlacing inequalities*

$$\lambda_j(x) \geq \beta_j(x) \geq \lambda_{n-d+j}(x) \ , \quad \text{for .a.e. } x \in X \ , \quad j \in \mathbb{I}_d \ . \tag{52}$$

Then, there exist measurable functions $\gamma_{i\,,\,j} : X \to \mathbb{R}^+$ for $0 \leq i \leq n-d$ and $j \in \mathbb{I}_{n-i}$ such that:

1. *$\gamma_{0\,,\,j} = \lambda_j$ for $j \in \mathbb{I}_n$ and $\gamma_{n-d\,,\,j} = \beta_j$ for $j \in \mathbb{I}_d$;*
2. *For $0 \leq i \leq n-d$ then $\gamma_{i\,,\,j}(x) \geq \gamma_{i\,,\,j+1}(x)$ for $j \in \mathbb{I}_{n-i-1}$, for a.e. $x \in X$;*
3. *For $0 \leq i \leq n-d-1$ then $\gamma_{i\,,\,j}(x) \geq \gamma_{i+1\,,\,j}(x) \geq \gamma_{i\,,\,j+1}(x)$ for $j \in \mathbb{I}_{n-i-1}$, for a.e. $x \in X$.*

Proof. We argue by (decreasing) induction in terms of d. Notice that the statement is trivially true if $d = n-1$. Assume that the result is true for $d+1$ interlacing measurable functions for some $d \in \mathbb{I}_{n-2}$. Given the measurable functions β_j for $j \in \mathbb{I}_d$ as above, we shall construct measurable functions $\alpha_j : X \to \mathbb{R}^+$ for $j \in \mathbb{I}_{d+1}$ such that

$$\lambda_j \geq \alpha_j \geq \lambda_{n-(d+1)+j} \quad \text{for} \quad j \in \mathbb{I}_{d+1} \quad \text{and} \quad \alpha_j \geq \beta_j \geq \alpha_{j+1} \quad \text{for} \quad j \in \mathbb{I}_d \,, \tag{53}$$

and hence $\alpha_1 \geq \ldots \geq \alpha_{d+1}$. Notice that the lemma would be a consequence of this construction and the inductive hypothesis (where the maps α_j play the role of $\gamma_{n-d+1\,,\,j}$ for $j \in \mathbb{I}_{d+1}$).

First notice that by the interlacing inequalities in Eq. (52) we have that

$$\min\{\lambda_{r+1}\,,\, \beta_r\} \geq \max\{\beta_{r+1}\,,\, \lambda_{n-d+r}\} \ , \quad \text{for } r \in \mathbb{I}_{d-1} \ \text{ and } \ \text{for a.e. } x \in X \,. \tag{54}$$

We define $\alpha_j : X \to \mathbb{R}^+$, for $j \in \mathbb{I}_{d+1}$, as follows:

$$\alpha_j := \begin{cases} \max\{\beta_j\,,\, \lambda_{n-(d+1)+j}\} & \text{if} \quad 1 \leq j \leq d\,; \\ \min\{\beta_d\,,\, \lambda_{d+1}\} & \text{if} \quad j = d+1\,. \end{cases} \tag{55}$$

By construction the functions α_j are measurable, and it is easy to check (by using Eq. (54)) that they satisfy Eq. (53). □

The following result is the Fan-Pall interlacing inequalities theorem for measurable fields of positive operators.

Theorem 6.3. *Let $G : X \to \mathcal{M}_n(\mathbb{C})^+$ be a bounded measurable field of positive semidefinite matrices with associated measurable eigenvalues $\lambda_j : X \to \mathbb{R}^+$ for $j \in \mathbb{I}_n$ such that $\lambda_1 \geq \ldots \geq \lambda_n$. Let $d \in \mathbb{I}_{n-1}$ and let $\beta_j : X \to \mathbb{R}^+$ for $j \in \mathbb{I}_d$ be measurable functions such that $\beta_1 \geq \ldots \geq \beta_d$. Then the following conditionas are equivalent:*

1. *$\lambda_j(x) \geq \beta_j(x) \geq \lambda_{n-d+j}(x)$, for a.e. $x \in X$, $j \in \mathbb{I}_d$.*
2. *There exists a projection valued measurable function $P : X \to \mathcal{M}_n(\mathbb{C})^+$ such that*

$$\operatorname{rk} P(x) = d \quad \text{and} \quad \lambda(P(x)\, G(x)\, P(x)) = (\beta_1(x), \ldots, \beta_d(x), 0_{n-d}) \ , \quad \text{for a.e. } x \in X \, .$$

Proof. Assume first that the functions $\{\beta_j\}_{j\in\mathbb{I}_d}$ satisfy the interlacing inequalities in item 1. Let $\gamma_{i\,,\,j} : X \to \mathbb{R}^+$ for $0 \leq i \leq n-d$ and $j \in \mathbb{I}_{n-i}$ be measurable functions as in By Lemma 6.2. By Proposition 6.1 there exists a measurable function $W_1 : X \to \mathcal{M}_{n\,,\,n-1}(\mathbb{C})$ such that

$$W_1(x)^* W_1(x) = I_{n-1} \quad \text{and} \quad \lambda(W_1(x)^* G\, W_1(x)) = (\gamma_{1\,,\,1}(x), \ldots, \gamma_{1\,,\,n-1}(x)) \, , \quad \text{for a.e. } x \in X \, .$$

Arguing as before, using Proposition 6.1 we can construct for $2 \leq i \leq n-d$ measurable functions $W_i : X \to \mathcal{M}_{n-i+1\,,\,n-i}(\mathbb{C})$ such that $W_i(x)^* W_i(x) = I_{n-i}$ for a.e. $x \in X$ and

$$\lambda(W_i(x)^* \cdots W_1(x)^* G(x)\, W_1(x) \cdots W_i(x)) = (\gamma_{i\,,\,1}(x), \ldots, \gamma_{i\,,\,n-i}(x)) \ , \quad \text{for a.e. } x \in X \ .$$

Let $W = W_1 \cdots W_{n-d} : X \to \mathcal{M}_{n\,,\,d}(\mathbb{C})$ which is measurable by construction and notice that

$$W^*(x) W(x) = I_d \quad \text{and} \quad \lambda(W(x)^* G(x)\, W(x)) = (\beta_1(x), \ldots, \beta_d(x)) \ , \quad \text{for a.e. } x \in X \ .$$

Hence, if we set $P = WW^* : X \to \mathcal{M}_n(\mathbb{C})$ then P is a measurable field of projections with the desired properties.

Conversely, assume that there exists a projection valued measurable function $P : X \to \mathcal{M}_n(\mathbb{C})^+$ satisfying item 2. Then item 1 is a straightforward consequence of the so-called Cauchy interlacing inequalities from matrix analysis (see for example [4]). □

Let $\mathcal{G} = \{g_i\}_{i\in\mathbb{I}_n}$ be such that $E(\mathcal{G})$ is a frame for the SI subspace $\mathcal{V}$, with frame operator $A = S_{E(\mathcal{G})}$. Recall that (see Definition 4.9)

$$U_{\mathcal{V}}(E(\mathcal{G})\,) = \{A+B : B \in L(L^2(\mathbb{R}^k))^+ \text{ is SP, } R(B) \subset \mathcal{V}, \operatorname{rk}(\hat{B}_x) \leq n-d(x) \text{ , for a.e. } x \in \mathbb{T}^k\} \, .$$

Using the Fan-Pall inequalities for measurable fields of matrices we can now describe the fine spectral structure of the elements in $U_{\mathcal{V}}(E(\mathcal{G})\,)$

Theorem 6.4. *Let $\mathcal{V}$ be a SI subspace in $L^2(\mathbb{R}^k)$ with Spec$(\mathcal{V}) \subseteq X$ and let $\mathcal{G} = \{g_i\}_{i\in\mathbb{I}_n}$ be such that $E(\mathcal{G})$ is a frame for $\mathcal{V}$ with frame operator $A = S_{E(\mathcal{G})}$. Let $d : X \to \mathbb{N}$ be the measurable function given by $d(x) = \dim J_{\mathcal{V}}(x)$, for $x \in$ Spec$(\mathcal{V})$, and let $m(\cdot) = 2d(\cdot) - n$. Given a measurable function $\mu : X \to (\ell^1_+(\mathbb{N})^+)^{\downarrow}$ (decreasing sequences) the following are equivalent:*

1. *There exists $C \in U_{\mathcal{V}}(E(\mathcal{G})\,)$ such that $\mu(x) = \lambda(\hat{A}_x)$ for a.e. $x \in X$;*
2. *$\mu(x) = 0$ for every $x \notin$ Spec$(\mathcal{V})$. If $x \in$ Spec$(\mathcal{V})$ then $\mu_j(x) = 0$ for $j \geq d(x)+1$ and*
   (a) *in case that $m(x) \leq 0$, then $\mu_i(x) \geqslant \lambda_i(\hat{A}_x)$ for $i \in \mathbb{I}_{d(x)}$;*

(b) in case that $m(x) \geq 1$, *then* $\mu_i(x) \geqslant \lambda_i(\hat{A}_x)$ *for* $i \in \mathbb{I}_{d(x)}$ *and*

$$\mu_{n-d(x)+i}(x) = \mu_{d(x)-m(x)+i}(x) \leq \lambda_i(\hat{A}_x) \quad \textit{for} \quad i \in \mathbb{I}_{m(x)} \ .$$

Proof. First notice that by considering a convenient finite partition of X into measurable sets we can assume, without loss of generality, that $d(x) = d \in \mathbb{N}$ for a.e. $x \in \mathrm{Spec}(\mathcal{V})$.

Let $C \in U_{\mathcal{V}}(E(\mathcal{G}))$, and assume that $\mu = \lambda(\hat{C})$. By hypothesis, there exists $B \in L(L^2(\mathbb{R}^k))^+$ SP, with $R(B) \subset \mathcal{V}$, $\mathrm{rk}(\hat{B}_x) \leq n - d(x)$ for a.e. $x \in \mathbb{T}^k$, such that $C = A + B$. By Lemma 4.8 there exists $\mathcal{Z} = \{z_i\}_{i\in\mathbb{I}_n} \in \mathcal{V}^n$ such that $T_{E(\mathcal{G})}\, T^*_{E(\mathcal{Z})} = 0$ and $B = S_{E(\mathcal{Z})}$. If we let $\mathcal{G} + \mathcal{Z} = \{g_i + z_i\}_{i\in\mathbb{I}_n}$ then $T_{E(\mathcal{G}+\mathcal{Z})} = T_{E(\mathcal{G})} + T_{E(\mathcal{Z})}$ and

$$S_{E(\mathcal{G}+\mathcal{Z})} = T_{E(\mathcal{G}+\mathcal{Z})}\, T^*_{E(\mathcal{G}+\mathcal{Z})} = S_{E(\mathcal{G})} + S_{E(\mathcal{Z})} = A + S_{E(\mathcal{Z})} = C$$

with $R(\hat{C}_x) = J_{\mathcal{V}}(x)$ and $\dim J_{\mathcal{V}}(x) = d \leq n$. Then,

$$\lambda_j\big(\,(T^*_{E(\mathcal{G}+\mathcal{Z})}\ T_{E(\mathcal{G}+\mathcal{Z})})\hat{}_x\big) = \mu_j(x) \quad \text{ for } j \in \mathbb{I}_d \text{ and a.e. } x \in \ \mathrm{Spec}(\mathcal{V}) \ .$$

Moreover, if we let $P:\ \mathrm{Spec}(\mathcal{V}) \to \mathcal{M}_n(\mathbb{C})^+$ be the projection valued measurable function such that $P(x)$ is the orthogonal projection onto span $\{\Gamma\mathcal{G}(x)\} = R(T^*_{\Gamma\mathcal{G}(x)})$ then, using again that $T_{E(\mathcal{G})}\, T^*_{E(\mathcal{Z})} = 0$ we see that

$$P(x)\ [\,(T^*_{E(\mathcal{G}+\mathcal{Z})}\ T_{E(\mathcal{G}+\mathcal{Z})})\hat{}_x\,]\ P(x) = (T^*_{E(\mathcal{G})}\ T_{E(\mathcal{G})})\hat{}_x \ , \quad \text{ for a.e. } x \in \ \mathrm{Spec}(\mathcal{V}) \, .$$

Since $\mathrm{rk}(P(x)) = d \leq n$ and

$$\lambda_j((T^*_{E(\mathcal{G})}\ T_{E(\mathcal{G})})\hat{}_x) = \lambda_j((T_{E(\mathcal{G})}\ T^*_{E(\mathcal{G})})\hat{}_x) = \lambda_j(\hat{A}_x) \ , \quad \text{ for } j \in \mathbb{I}_d \text{ and a.e. } x \in \ \mathrm{Spec}(\mathcal{V})$$

then, using Theorem 6.3 we conclude that that the Fan-Pall inequalities hold between

$$(\mu_1(x), \ldots, \mu_d(x), 0_{n-d}) \quad \text{and} \quad (\lambda_1(\hat{A}_x)\,,\ \ldots\ ,\ \lambda_d(\hat{A}_x)\,,\ 0_{n-d}) \ .$$

A careful inspection of these inequalities for the previous vectors shows that the inequalities in items 2.a and 2.b. above hold (according to the relation between n and d).

Conversely, let $\mu : \mathrm{Spec}(\mathcal{V}) \to \ell^1(\mathbb{N})^+$ satisfy the conditions in item 2 and let $D_\mu(\cdot) : \mathrm{Spec}(\mathcal{V}) \to \mathcal{M}_n(\mathbb{C})^+$ be the measurable field of positive semidefinite matrices such that $D_\mu(x)$ is the diagonal matrix with main diagonal $(\mu_1(x), \ldots, \mu_d(x), 0_{n-d})$ for $x \in X$. Then, by Theorem 6.3 there exists a projection valued measurable function $P : \mathrm{Spec}(\mathcal{V}) \to \mathcal{M}_n(\mathbb{C})^+$ such that

$$\mathrm{tr}(P(x)) = d \quad \text{ and } \quad \lambda(P(x)\ D_\mu(x)\ P(x)) = (\lambda_1(\hat{A}_x)\,,\ \ldots\ ,\ \lambda_d(\hat{A}_x)\,,\ 0_{n-d}) \in (\mathbb{R}^+)^n \ .$$

In this case we see that

$$\lambda(D_\mu^{1/2}\, P(x)\, D_\mu^{1/2}) = (\lambda_1(\hat{A}_x)\,,\ \ldots\ ,\ \lambda_d(\hat{A}_x)\,,\ 0_{n-d}) \in (\mathbb{R}^+)^n \, .$$

Let $D_\lambda(x)$ be the diagonal matrix with main diagonal $(\lambda_1(\hat{A}_x)\,,\ \ldots\ ,\ \lambda_d(\hat{A}_x)\,,\ 0_{n-d})$. By taking an appropriate measurable field of unitary matrices $U(x) : \mathrm{Spec}(\mathcal{V}) \to \mathcal{M}_n(\mathbb{C})$ we conclude that

$$D_\lambda(x) = U(x)^*[D_\mu^{1/2}\, P(x)\, D_\mu^{1/2}]\, U(x) \ , \quad \text{ for a.e. } x \in \ \mathrm{Spec}(\mathcal{V}) \, . \tag{56}$$

Arguing as in the proof of Lemma 4.8 we see that there exist measurable fields of vectors $v_j : \mathrm{Spec}(\mathcal{V}) \to \ell^2(\mathbb{Z}^k)$, for $j \in \mathbb{I}_d$, such that $\hat{A}_x\, v_j(x) = \lambda_j(\hat{A}_x)\; v_j(x)$ and $\mathcal{B}(x) = \{v_j(x)\}_{i\in\mathbb{I}_d}$ is an ONB of $J_{\mathcal{V}}(x)$ for a.e. $x \in \mathrm{Spec}(\mathcal{V})$. We finally consider $B \in L(L^2(\mathbb{R}^k))^+$ S.P. with $R(B) \subset \mathcal{V}$, uniquely determined by the condition:

$$[\hat{B}_x]_{\mathcal{B}(x)} = U(x)^*[D_\mu^{1/2}\,(I - P(x))\,D_\mu^{1/2}]\,U(x)\;, \qquad \text{for a.e. } x \in \mathrm{Spec}(\mathcal{V}) \tag{57}$$

where $[\hat{B}_x]_{\mathcal{B}(x)}$ stands for the matrix representation of $\hat{B}_x$ with respect to the ONB $\mathcal{B}(x)$ of $J_{\mathcal{V}}(x)$; in particular, using that $\mathrm{rk}(I-P(x)) = n-d$, we conclude that $\mathrm{rk}(\hat{B}_x) \leq n-d$ for a.e. $x \in \mathrm{Spec}(\mathcal{V})$. On the other hand, by construction of $\mathcal{B}(x)$, we have that $[\hat{A}_x]_{\mathcal{B}(x)} = D_\lambda(x)$: thus, using Eqs (56) and (57) we have that

$$[\hat{A}_x + \hat{B}_x]_{\mathcal{B}(x)} = [\hat{A}_x]_{\mathcal{B}(x)} + [\hat{B}_x]_{\mathcal{B}(x)} = U(x)^*\; D_\mu\; U(x)\;, \qquad \text{for a.e. } x \in \mathrm{Spec}(\mathcal{V})\,.$$

This last fact implies $C = A + B \in U_{\mathcal{V}}(E(\mathcal{G})\,)$ satisfies that $\lambda_j(\hat{C}_x) = \mu_j(x)$ for $j \in \mathbb{N}$ and a.e. $x \in \mathrm{Spec}(\mathcal{V})$. □

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