THE QUANTUM DIVIDED POWER ALGEBRA OF A FINITE-DIMENSIONAL NICHOLS ALGEBRA OF DIAGONAL TYPE

NICOLÁS ANDRUSKIEWITSCH, IVÁN ANGIONO, FIORELA ROSSI BERTONE

Abstract. Let $\mathcal{B}_{\mathfrak{q}}$ be a finite-dimensional Nichols algebra of diagonal type corresponding to a matrix $\mathfrak{q}$. We consider the graded dual $\mathcal{L}_{\mathfrak{q}}$ of the distinguished pre-Nichols algebra $\widetilde{\mathcal{B}}_{\mathfrak{q}}$ from [A3] and the quantum divided power algebra $\mathcal{U}_{\mathfrak{q}}$, a suitable Drinfeld double of $\mathcal{L}_{\mathfrak{q}}\#\mathbf{k}\mathbb{Z}^{\theta}$. We provide basis and presentations by generators and relations of $\mathcal{L}_{\mathfrak{q}}$ and $\mathcal{U}_{\mathfrak{q}}$, and prove that they are noetherian and have finite Gelfand-Kirillov dimension.

1. Introduction

We fix an algebraically closed field $\mathbf{k}$ of characteristic zero. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra and $q \in \mathbf{k}$ a root of 1 (with some restrictions depending on $\mathfrak{g}$). In the theory of quantum groups, there are several Hopf algebras attached to $\mathfrak{g}$ and q:

◦ The Frobenius-Lusztig kernel (or small quantum group) $\mathfrak{u}_q(\mathfrak{g})$.
◦ The q-divided power algebra $\mathcal{U}_q(\mathfrak{g})$, see [L1, L2].
◦ The quantized enveloping algebra $U_q(\mathfrak{g})$, see [DK, DKP, DP].

These Hopf algebras have the following features:

◇ They admit triangular decompositions, e. g. $\mathfrak{u}_q(\mathfrak{g}) \simeq \mathfrak{u}_q^+(\mathfrak{g})\otimes\mathfrak{u}_q^0(\mathfrak{g})\otimes\mathfrak{u}_q^-(\mathfrak{g})$.
◇ The 0-part of this triangular decomposition is a Hopf subalgebra, actually a group algebra.
◇ The positive and negative parts are not Hopf subalgebras, but rather Hopf algebras in braided tensor categories, braided Hopf algebras for short.
◇ There are morphisms $\mathfrak{u}_q^+(\mathfrak{g}) \hookrightarrow \mathcal{U}_q^+(\mathfrak{g})$, $U_q^+(\mathfrak{g}) \twoheadrightarrow \mathfrak{u}_q^+(\mathfrak{g})$ of braided Hopf algebras, and ditto for the full Hopf algebras.
◇ The full Hopf algebras can be reconstructed from the positive part by standard procedures (bosonization, the Drinfeld double).
◇ The positive part $\mathfrak{u}_q^+(\mathfrak{g})$ has very special properties– it is a Nichols algebra.

Indeed, $\mathfrak{u}_q^+(\mathfrak{g})$ is completely determined by the matrix $\mathfrak{q} = (q^{d_i a_{ij}})$, where (a_{ij}) is the Cartan matrix of $\mathfrak{g}$ and $d_i \in \{1, 2, 3\}$ make $(d_i a_{ij})$ symmetric. In other words, $\mathfrak{u}_q^+(\mathfrak{g})$ is the Nichols algebra of diagonal type associated to $\mathfrak{q}$.

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The knowledge of the finite-dimensional Nichols algebras of diagonal type is crucial in the classification program of finite-dimensional Hopf algebras [AS]. Two remarkable results on these Nichols algebras are:

(a) The explicit classification [H2].
(b) The determination of their defining relations [A1, A2].

Let $\mathfrak{q} \in \mathbf{k}^{\theta\times\theta}$ with Nichols algebra $\mathcal{B}_{\mathfrak{q}}$ and assume that $\dim \mathcal{B}_{\mathfrak{q}} < \infty$. There are several reasons to consider the analogues of the braided Hopf algebras $U_q^+(\mathfrak{g})$ and $\mathcal{U}_q^+(\mathfrak{g})$, for $\mathcal{B}_{\mathfrak{q}}$, motivated by the classification of Hopf algebras with finite Gelfand-Kirillov dimension and by representation theory. The analogue $\widetilde{\mathcal{B}}_{\mathfrak{q}}$ of $U_q^+(\mathfrak{g})$ was introduced in [A2] and studied in [A3] under the name of distinguished pre-Nichols algebra. The definition of $\widetilde{\mathcal{B}}_{\mathfrak{q}}$ is by discarding some of the relations in [A2]. The purpose of this paper is to study the analogue $\mathcal{L}_{\mathfrak{q}}$ of $\mathcal{U}_q^+(\mathfrak{g})$; this is the graded dual of $\widetilde{\mathcal{B}}_{\mathfrak{q}}$ and although it could be called the distinguished post-Nichols algebra of $\mathfrak{q}$, we prefer to name it the Lusztig algebra as in [A+], where mentioned in passing.

The paper is organized as follows. Section 2 is devoted to preliminaries and Section 3 to Nichols algebras of diagonal type and distinguished pre-Nichols algebras. In Section 4 we discuss Lusztig algebras: we provide a basis and a presentation by generators and relations, and prove that they are noetherian and have finite Gelfand-Kirillov dimension. In Section 5 we introduce the quantum divided power algebra $\mathcal{U}_{\mathfrak{q}}$, that is a suitable Drinfeld double of $\mathcal{L}_{\mathfrak{q}}\#\mathbf{k}\mathbb{Z}^\theta$; we also provide a presentation by generators and relations, and prove that it is noetherian and has finite Gelfand-Kirillov dimension.

Remark 1.1. The quantum divided power algebras were introduced and studied in [GH, Hu]; they correspond to Nichols algebras of Cartan type $A_1 \times \cdots \times A_1$.

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2. Preliminaries and conventions

2.1. **Conventions.** If $\theta \in \mathbb{N}$, then we set $\mathbb{I}_\theta := \{1, 2, ..., \theta\}$; or simply $\mathbb{I}$ if no confusion arises. If Γ is a group, then $\widehat{\Gamma}$ is its group of characters, that is, one-dimensional representations.

Let $\mathbb{S}_n$ and $\mathbb{B}_n$ be the symmetric and braid groups in n letters, with standard generators $\tau_i = (i\, i+1)$, respectively σ_i, $i \in \mathbb{I}_{n-1}$. Let $s : \mathbb{S}_\theta \to \mathbb{B}_\theta$ be the (Matsumoto) section of the projection $\pi : \mathbb{B}_\theta \twoheadrightarrow \mathbb{S}_\theta$, $\pi(\sigma_i) = \tau_i$, $i \in \mathbb{I}_{n-1}$, given by $s(\omega) = \sigma_{i_1}\sigma_{i_2}...\sigma_{i_j}$, whenever $\omega = \tau_{i_1}\tau_{i_2}...\tau_{i_j} \in \mathbb{S}_\theta$ has length j.

We consider the $\mathbf{q}$-numbers in the polynomial ring $\mathbb{Z}[\mathbf{q}]$, $n \in \mathbb{N}$, $0 \le i \le n$,

$$(n)_{\mathbf{q}} = \sum_{j=0}^{n-1} \mathbf{q}^j, \qquad (n)^!_{\mathbf{q}} = \prod_{j=1}^{n} (j)_{\mathbf{q}}, \qquad \binom{n}{i}_{\mathbf{q}} = \frac{(n)^!_{\mathbf{q}}}{(n-i)^!_{\mathbf{q}}(i)^!_{\mathbf{q}}}.$$

If $q \in \mathbf{k}$, then $(n)_q$, $(n)_q^!$, $\binom{n}{i}_q$ are the respective evaluations at q.

We use the Heynemann-Sweedler notation for coalgebras and comodules; the counit of a coalgebra is denoted by ε, and the antipode of a Hopf algebra, by $\mathcal{S}$. All Hopf algebras in this paper have bijective antipode.

Let H be a Hopf algebra. A *Yetter-Drinfeld module* V over H is a H-module and a H-comodule satisfying the compatibility condition

$$\delta(h \cdot v) = h_{(1)} v_{(-1)} \mathcal{S}(h_{(3)}) \otimes h_{(2)} \cdot v_{(0)}, \qquad h \in H, v \in V.$$

Morphisms of Yetter-Drinfeld modules preserve the action and the coaction. Thus Yetter Drinfeld modules over H form a braided tensor category ${}^H_H\mathcal{YD}$, with braiding $c_{V,W}(v \otimes w) = v_{(-1)} \cdot w \otimes v_{(0)}$, $V, W \in {}^H_H\mathcal{YD}$, $v \in V$, $w \in W$. The full subcategory of finite-dimensional objects is rigid.

2.2. **Braided vector spaces and Nichols algebras.** A braided vector space is a pair (V, c) where V is a vector space and $c \in \operatorname{Aut}(V \otimes V)$ is a solution of the braid equation $(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c)$.

If V is a vector space, then we identify $V^* \otimes V^*$ with a subspace of $(V \otimes V)^*$ by $\langle f \otimes g, v \otimes w \rangle = \langle f, w \rangle \langle g, v \rangle$, for $v, w \in V$, $f, g \in V^*$.[1] If (V, c) is a finite-dimensional braided vector space, then (V^*, c^t) is its dual braided vector space, where $c^t : V^* \otimes V^* \to V^* \otimes V^*$ is $\langle c^t(f \otimes g), v \otimes w \rangle = \langle f \otimes g, c(v \otimes w) \rangle$.

We refer to [T] for the basic theory of braided Hopf algebras. If $R = \bigoplus_{n \geq 0} R^n$ is a graded braided Hopf algebra with $\dim R^n < \infty$ for all n, then its graded dual $R^d = \bigoplus_{n \geq 0} (R^n)^*$ is again a graded braided Hopf algebra. We use the variation of the Sweedler notation $\Delta(X) = X^{(1)} \otimes X^{(2)}$ for the coproducts in braided Hopf algebras.

The *Nichols algebra* of a braided vector space (V, c) is a graded braided Hopf algebra $\mathcal{B}(V) = \oplus_{n \geq 0} \mathcal{B}^n(V)$ with very rigid properties. There are several alternative definitions of Nichols algebras, see [AS]. We recall now two of these definitions.

Let $T(V) = \oplus_{n \geq 0} T^n(V)$ be the tensor algebra of V; it has a braiding c induced from V. Let $T(V) \underline{\otimes} T(V) = T(V) \otimes T(V)$ with the multiplication $(m \otimes m)(\mathrm{id} \otimes c \otimes \mathrm{id})$ and let $\Delta : T(V) \to T(V) \underline{\otimes} T(V)$ be the unique algebra map such that $\Delta(v) = v \otimes 1 + 1 \otimes v$, for all $v \in V$. Then $T(V)$ is a (graded) braided Hopf algebra with respect to Δ. Dually, consider the cotensor coalgebra $T^c(V)$ which is isomorphic to $T(V)$ as a vector space. It bears a multiplication making $T^c(V)$ a braided Hopf algebra with an analogous property, see e. g. [R, AG]. There exists only one morphism of braided Hopf algebras $\Theta : T(V) \to T^c(V)$ that it is the identity on V. The image of Θ is the Nichols algebra $\mathcal{B}(V)$ of V.

Here is the second description of $\mathcal{B}(V)$. Let $\mathfrak{S}$ be the partially ordered set of homogeneous Hopf ideals of $T(V)$ with trivial intersection with $\mathbf{k} \oplus V$. Then $\mathfrak{S}$ has a maximal element $\mathcal{J}(V)$ and $\mathcal{B}(V) = T(V)/\mathcal{J}(V)$ [AS].

[1] We prefer this identification instead of $\langle f \otimes g, v \otimes w \rangle = \langle f, v \rangle \langle g, w \rangle$ because it gives the right extension to tensor categories.

2.3. **Pre- and post-Nichols algebras.** For several purposes, it is useful to consider braided Hopf algebras $T(V)/I$, for various $I \in \mathfrak{S}$. These are called *pre-Nichols algebras* [M]. Indeed, $\mathfrak{Pre}(V) = \{T(V)/I : I \in \mathfrak{S}\}$ is a poset with ordering given by the surjections; so that it is isomorphic to $(\mathfrak{S}, \subseteq)$. The minimal element in $\mathfrak{Pre}(V)$ is $T(V)$, and the maximal is $\mathcal{B}(V)$. Dually, the poset $\mathfrak{Post}(V)$ consists of graded Hopf subalgebras $S = \bigoplus_{n\geq 0} S^n$ of $T^c(V)$ such that $S^1 = V$, ordered by the inclusion. Now the minimal element is $\mathcal{B}(V)$ and the maximal is $T^c(V)$. We shall call them *post-Nichols algebras*.

Remark 2.1. The map $\Phi : \mathfrak{Pre}(V) \to \mathfrak{Post}(V^*)$, $\Phi(R) = R^d$, is an anti-isomorphism of posets.

Proof. If $R = T(V)/I \in \mathfrak{Pre}(V)$, then $R^d = I^\perp$: hence, Φ is well-defined and it reverses the order. Also Φ is surjective, because for a given $S \in \mathfrak{Post}(V^*)$, $I = S^\perp$ is a graded Hopf ideal of $T(V)$ and $S = (T(V)/I)^d$. $\square$

3. Nichols algebras of diagonal type

A braided vector space (V, c) is of *diagonal type* if there exist a basis $x_1, \dots, x_\theta$ of V and a matrix $\mathfrak{q} = (q_{ij}) \in M_\theta(\mathbf{k}^\times)$ such that $c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$ for all $i, j \in \mathbb{I} = \mathbb{I}_\theta$. Let $H = \mathbf{k}G$ be a group algebra, $\chi_i \in \widehat{G}$ and $g_j \in Z(G)$ such that $\chi_j(g_i) = q_{ij}$, $i, j \in \mathbb{I}$. Then (V, c) is realized in ${}^H_H\mathcal{YD}$ by $h \cdot x_i = \chi_i(h) x_i$ and $\rho(x_i) = g_i \otimes x_i$ for all $i \in \mathbb{I}$, $h \in H$. We will only consider the case when $H = \mathbf{k}\mathbb{Z}^\theta$, $g_i = \alpha_i$ and $\chi_j \in \widehat{\mathbb{Z}^\theta}$ is given by $\chi_j(\alpha_i) = q_{ij}$, $i, j \in \mathbb{I}$. Here $\alpha_1, \dots, \alpha_\theta$ is the canonical basis of $\mathbb{Z}^\theta$.

Let $V^* \in {}^{\mathbf{k}\mathbb{Z}^\theta}_{\mathbf{k}\mathbb{Z}^\theta}\mathcal{YD}$; it is also a braided vector space of diagonal type, with matrix $\mathfrak{q}$. Indeed, if $y_1, \dots, y_\theta$ is the dual basis of $x_1, \dots, x_\theta$, then

$$\begin{aligned}\langle c^t(y_i \otimes y_j), x_h \otimes x_k\rangle &= \langle y_i \otimes y_j, c(x_h \otimes x_k)\rangle = q_{hk}\langle y_i \otimes y_j, x_k \otimes x_h\rangle \\ &= q_{hk}\delta_{jk}\delta_{ih} = q_{ij}\langle y_j \otimes y_i, x_h \otimes x_k\rangle.\end{aligned}$$

Since $T(V)$ and $\mathcal{B}_\mathfrak{q} = \mathcal{B}(V)$ are Hopf algebras in ${}^{\mathbf{k}\mathbb{Z}^\theta}_{\mathbf{k}\mathbb{Z}^\theta}\mathcal{YD}$, we may consider the bosonizations $T(V)\#\mathbf{k}\mathbb{Z}^\theta$ and $\mathcal{B}_\mathfrak{q}\#\mathbf{k}\mathbb{Z}^\theta$. We refer to [AS, §1.5] for the definition of the adjoint action of a Hopf algebra, respectively the braided adjoint ad_c action of a Hopf algebra in ${}^{\mathbf{k}\mathbb{Z}^\theta}_{\mathbf{k}\mathbb{Z}^\theta}\mathcal{YD}$. Then $\operatorname{ad}_c x \otimes \operatorname{id} = \operatorname{ad}(x\#1)$ if $x \in T(V)$ or $\mathcal{B}_\mathfrak{q}$, see [AS, (1-21)].

Now the matrix $\mathfrak{q}$ gives rise to a $\mathbb{Z}$-bilinear form $\Xi : \mathbb{Z}^\theta \times \mathbb{Z}^\theta \to \mathbf{k}^\times$ by $\Xi(\alpha_j, \alpha_k) = q_{jk}$ for all $j, k \in \mathbb{I}$. If $\alpha, \beta \in \mathbb{Z}^\theta$, we also set

$$q_{\alpha\beta} = \Xi(\alpha, \beta). \tag{1}$$

The algebra $T(V)$ is $\mathbb{Z}^\theta$-graded. If $x, y \in T(V)$ are homogeneous of degrees $\alpha, \beta \in \mathbb{Z}^\theta$ respectively, then their braided commutator is

$$[x, y]_c = xy - \text{multiplication} \circ c(x \otimes y) = xy - q_{\alpha\beta} yx. \tag{2}$$

Note that $\operatorname{ad}_c(x)(y) = [x, y]_c$ whenever x is primitive. We say that x $\mathfrak{q}$-commutes with a family $(y_i)_{i\in I}$ of homogeneous elements if $[x, y_i]_c = 0$, for all $i \in I$. Same considerations are valid in any braided graded Hopf algebra.

Define a matrix $(c^{\mathfrak{q}}_{ij})_{i,j\in\mathbb{I}}$ with entries in $\mathbb{Z}\cup\{-\infty\}$ by $c^{\mathfrak{q}}_{ii}=2$,

$$c^{\mathfrak{q}}_{ij} := -\min\left\{n\in\mathbb{N}_0 : (n+1)_{q_{ii}}(1-q_{ii}^{n}q_{ij}q_{ji})=0\right\}, \qquad i\neq j. \tag{3}$$

We assume from now on that $\dim\mathcal{B}_{\mathfrak{q}}<\infty$. Then $c^{\mathfrak{q}}_{ij}\in\mathbb{Z}$ for all $i,j\in\mathbb{I}$ [R, Section 3.2] and we may define the reflections $s^{\mathfrak{q}}_i\in GL(\mathbb{Z}^\theta)$, by $s^{\mathfrak{q}}_i(\alpha_j)=\alpha_j-c^{\mathfrak{q}}_{ij}\alpha_i$, $i,j\in\mathbb{I}$. Let $i\in\mathbb{I}$ and let $\rho_i(V)$ be the braided vector space of diagonal type with matrix $\rho_i(\mathfrak{q})$, where

$$\rho_i(\mathfrak{q})_{jk}=\Xi(s^{\mathfrak{q}}_i(\alpha_j),s^{\mathfrak{q}}_i(\alpha_k)), \qquad j,k\in\mathbb{I}. \tag{4}$$

The proofs of statements (a) and (b) in the Introduction have as a crucial ingredient the Weyl groupoid [H1] and the generalized root system [HY1]; the definitions involve the assignements $\mathfrak{q}\rightsquigarrow\rho_i(\mathfrak{q})$ described above. For our purposes, we just need to recall that

$$\Delta^{+}_{\mathfrak{q}} \text{ is the set of positive roots of } \mathcal{B}_{\mathfrak{q}}. \tag{5}$$

3.1. **Drinfeld doubles.** Let (V,c) be our fixed braided vector space of diagonal type with matrix $\mathfrak{q}$, realized in ${}^{\mathbf{k}\mathbb{Z}^\theta}_{\mathbf{k}\mathbb{Z}^\theta}\mathcal{YD}$ as above. In this Subsection, the hypothesis on the dimension of the Nichols algebra is not needed. We describe here the Drinfeld doubles of the bosonizations $T(V)\#\mathbf{k}\mathbb{Z}^\theta$, $\mathcal{B}_{\mathfrak{q}}\#\mathbf{k}\mathbb{Z}^\theta$ with respect to suitable bilinear forms. This construction goes back essentially to Drinfeld [Dr] and was adapted to different settings in various papers; here we follow [H3].

Definition 3.1. The Drinfeld double $\mathbf{U}_{\mathfrak{q}}$ of $T(V)\#\mathbf{k}\mathbb{Z}^\theta$ is the algebra generated by elements E_i, F_i, K_i, K_i^{-1}, L_i, L_i^{-1}, $i\in\mathbb{I}$, with defining relations

$$\begin{aligned}
XY&=YX, & X,Y&\in\{K_i^{\pm},L_i^{\pm}:i\in\mathbb{I}\},\\
K_iK_i^{-1}&=L_iL_i^{-1}=1, & E_iF_j-F_jE_i&=\delta_{i,j}(K_i-L_i).\\
K_iE_j&=q_{ij}E_jK_i, & L_iE_j&=q_{ji}^{-1}E_jL_i,\\
K_iF_j&=q_{ij}^{-1}F_jK_i, & L_iF_j&=q_{ji}F_jL_i.
\end{aligned}$$

Then $\mathbf{U}_{\mathfrak{q}}$ is a $\mathbb{Z}^\theta$-graded Hopf algebra, where the comultiplication and the grading are given, for $i\in\mathbb{I}$, by

$$\begin{aligned}
\Delta(K_i^{\pm1})&=K_i^{\pm1}\otimes K_i^{\pm1}, & \Delta(E_i)&=E_i\otimes 1+K_i\otimes E_i,\\
\Delta(L_i^{\pm1})&=L_i^{\pm1}\otimes L_i^{\pm1}, & \Delta(F_i)&=F_i\otimes L_i+1\otimes F_i.\\
\deg(K_i)&=\deg(L_i)=0, & \deg(E_i)&=\alpha_i=-\deg(F_i).
\end{aligned}$$

Let $\mathbf{U}^{+}_{\mathfrak{q}}$ (respectively, $\mathbf{U}^{-}_{\mathfrak{q}}$) be the subalgebra of $\mathbf{U}_{\mathfrak{q}}$ generated by E_i (respectively, F_i), $i\in\mathbb{I}$. Let $W=(V^*,\mathfrak{q}^t)$. [2] Moreover, $\mathbf{U}^{+}_{\mathfrak{q}}$ and $\mathbf{U}^{-}_{\mathfrak{q}}$ are Hopf algebras in ${}^{\mathbf{k}\mathbb{Z}^\theta}_{\mathbf{k}\mathbb{Z}^\theta}\mathcal{YD}$ via the actions and coactions

$$K_i\cdot E_j=q_{ij}E_j, \qquad \delta(E_i)=K_i\otimes E_i;$$

[2] Here and in Section 5 below, $\mathfrak{q}^t$ corresponds to V^* when realized as Yetter-Drinfeld module over the dual Hopf algebra.

$$L_i \cdot F_j = q_{ji} F_j, \qquad\qquad \delta(F_i) = L_i \otimes F_i.$$

Thus, there are isomorphisms $\psi^+ : T(V) \to \mathbf{U}_q^+$, $\psi^- : T(W) \to \mathbf{U}_q^-$ of Hopf algebras in ${}^{\mathbf{k}\mathbb{Z}^\theta}_{\mathbf{k}\mathbb{Z}^\theta}\mathcal{YD}$ given by $\psi^+(x_i) = E_i$ and $\psi^-(y_i) = F_i$.

Let

$$\mathfrak{u}_\mathfrak{q} = \mathbf{U}_\mathfrak{q}/(\psi^-(\mathcal{J}_{\mathfrak{q}^t}) + \psi^+(\mathcal{J}_\mathfrak{q}));$$

this is the Drinfeld double of $\mathcal{B}_\mathfrak{q}\#\mathbf{k}\mathbb{Z}^\theta$. We denote by E_i, F_i, K_i, L_i the elements of $\mathfrak{u}_\mathfrak{q}$ that are images of their homonymous in $\mathbf{U}_\mathfrak{q}$. Let $\mathfrak{u}^0$ (respectively, $\mathfrak{u}_\mathfrak{q}^+$, $\mathfrak{u}_\mathfrak{q}^-$) be the subalgebra of $\mathfrak{u}_\mathfrak{q}$ generated by K_i, L_i, (respectively, by E_i, by F_i), $i \in \mathbb{I}$. Then $\mathfrak{u}^0 \simeq \mathbf{k}\mathbb{Z}^{2\theta}$;

- there is a triangular decomposition $\mathfrak{u}_\mathfrak{q} \simeq \mathfrak{u}_\mathfrak{q}^+ \otimes \mathfrak{u}^0 \otimes \mathfrak{u}_\mathfrak{q}^-$;
- $\mathfrak{u}_\mathfrak{q}^+ \simeq \mathcal{B}_\mathfrak{q}$, $\mathfrak{u}_\mathfrak{q}^- \simeq \mathcal{B}_{\mathfrak{q}^t}$.

3.2. **Lusztig isomorphisms and PBW bases.** G. Lusztig defined automorphisms of the quantized enveloping algebra $U_q(\mathfrak{g})$ of a simple Lie algebra $\mathfrak{g}$, see [L2]. These automorphisms satisfy the relations of the braid group covering the Weyl group of $\mathfrak{g}$; they are instrumental in the construction of Poincaré-Birkhoff-Witt (PBW) bases of $U_q(\mathfrak{g})$. These results were extended to the Drinfeld double of a finite-dimensional Nichols algebra of diagonal type in [H3], with the role of the Weyl group played here by the Weyl groupoid $\mathcal{W}_\mathfrak{q}$. The definition of the Lusztig isomorphisms in [H3] requires some hypotheses on the matrix $\mathfrak{q}$, that are always satisfied in the finite-dimensional case. So, let (V, c) and $\mathfrak{q}$ as above; recall that we assume that $\dim \mathcal{B}_\mathfrak{q} < \infty$. Fix $i \in \mathbb{I}$. We first recall the definition of the isomorphisms $\mathfrak{u}_\mathfrak{q} \to \mathfrak{u}_{\rho_i(\mathfrak{q})}$ [H3]. For $i \neq j \in \mathbb{I}$ and $n \in \mathbb{N}_0$, define the elements of $\mathfrak{u}_\mathfrak{q}$

$$E_{j,n} = (\operatorname{ad} E_i)^n E_j, \qquad\qquad F_{j,n} = (\operatorname{ad} F_i)^n F_j.$$

Let $\underline{E}_j$, $\underline{F}_j$, $\underline{K}_j$, $\underline{L}_j$ be the generators of $\mathfrak{u}_{\rho_i(\mathfrak{q})}$. Set

$$a_j(\mathfrak{q}) := (-c_{ij}^\mathfrak{q})^!_{q_{ii}} \prod_{s=0}^{-c_{ij}^\mathfrak{q}-1} (q_{ii}^s q_{ij} q_{ji} - 1), \qquad j \neq i. \tag{6}$$

Theorem 3.2. [H3, 6.11] *There are algebra isomorphisms $T_i : \mathfrak{u}_\mathfrak{q} \to \mathfrak{u}_{\rho_i(\mathfrak{q})}$ uniquely determined, for $h, j \in \mathbb{I}$, $j \neq i$, by*

$$T_i(K_h) = \underline{K}_i^{-c_{ih}^\mathfrak{q}} \underline{K}_h, \quad T_i(E_i) = \underline{F}_i \underline{L}_i^{-1}, \quad T_i(E_j) = \underline{E}_{j,-c_{ij}^\mathfrak{q}},$$
$$T_i(L_h) = \underline{L}_i^{-c_{ih}^\mathfrak{q}} \underline{L}_h, \quad T_i(F_i) = \underline{K}_i^{-1} \underline{E}_i, \quad T_i(F_j) = \frac{1}{a_j(\rho_i(\mathfrak{q}))} \underline{F}_{j,-c_{ij}^\mathfrak{q}}. \quad \square$$

Let $w \in \mathcal{W}_\mathfrak{q}$ be an element of maximal length and fix a reduced expression $w = \sigma_{i_1}^\mathfrak{q} \sigma_{i_2} \cdots \sigma_{i_M}$. If $k \in \mathbb{I}_M$ and $\mathbf{h} = (h_1, \dots, h_M) \in \mathbb{N}_0^M$, set

$$\beta_k = s_{i_1}^\mathfrak{q} \cdots s_{i_{k-1}}(\alpha_{i_k}), \tag{7}$$

$$E_{\beta_k} = T_{i_1} \cdots T_{i_{k-1}}(E_{i_k}) \in (\mathfrak{u}_\mathfrak{q}^+)_{\beta_k}, \tag{8}$$

$$\mathbf{E}^\mathbf{h} = E_{\beta_M}^{h_M} E_{\beta_{M-1}}^{h_{M-1}} \cdots E_{\beta_1}^{h_1}. \tag{9}$$

By [CH, Prop. 2.12], $\Delta_+^{\mathfrak{q}} = \{\beta_k | 1 \le k \le M\}$. Thus, we set

$$N_\beta = N_k = \operatorname{ord} q_{\beta\beta} \in \mathbb{N} \cup \{\infty\}, \qquad \text{if } \beta = \beta_k \in \Delta_+^{\mathfrak{q}}. \tag{10}$$

Theorem 3.3. [HY2, 4.5, 4.8, 4.9] *The following set is a basis of* $\mathfrak{u}_{\mathfrak{q}}^+$*:*

$$\{\mathbf{E}^{\mathbf{h}} \,|\, \mathbf{h} \in \mathbb{N}_0^M,\ 0 \le h_k < N_k,\ k \in \mathbb{I}_M\}. \quad \square$$

3.3. **Distinguished pre-Nichols algebra.** We now recall the definition of the distinguished pre-Nichols algebra from [A3]. Let $\mathfrak{q}$, V be as above. First, $i \in \mathbb{I}$ is a *Cartan vertex* of $\mathfrak{q}$ if

$$q_{ij}q_{ji} = q_{ii}^{c_{ij}^{\mathfrak{q}}}, \qquad \text{for all } j \neq i, \tag{11}$$

recall (3). Then the set of Cartan roots of $\mathfrak{q}$ is

$$\mathfrak{O}_{\mathfrak{q}} = \{s_{i_1}^{\mathfrak{q}} s_{i_2} \dots s_{i_k}(\alpha_i) \in \Delta_+^{\mathfrak{q}} : i \in \mathbb{I} \text{ is a Cartan vertex of } \rho_{i_k} \dots \rho_{i_2}\rho_{i_1}(\mathfrak{q})\}.$$

A set of defining relations of the Nichols algebra $\mathcal{B}_{\mathfrak{q}}$, i. e. generators of the ideal $\mathcal{J}_{\mathfrak{q}}$, was given in [A2, Theorem 3.1]. We now consider the ideal $\mathcal{I}_{\mathfrak{q}} \subset \mathcal{J}_{\mathfrak{q}}$ of $T(V)$ generated by all the relations in *loc. cit.*, but

- we exclude the power root vectors $E_\alpha^{N_\alpha}$, $\alpha \in \mathfrak{O}_{\mathfrak{q}}$,
- we add the quantum Serre relations $(\operatorname{ad}_c E_i)^{1-c_{ij}^{\mathfrak{q}}} E_j$ for those $i \neq j$ such that $q_{ii}^{c_{ij}^{\mathfrak{q}}} = q_{ij}q_{ji} = q_{ii}$.

Definition 3.4. [A3, 3.1] The *distinguished pre-Nichols algebra* of V is

$$\widetilde{\mathcal{B}}_{\mathfrak{q}} = T(V)/\mathcal{I}_{\mathfrak{q}}.$$

Let $\widetilde{\mathfrak{u}}_{\mathfrak{q}} = \mathbf{U}_{\mathfrak{q}}/(\psi^-(\mathcal{I}_{\mathfrak{q}^t}) + \psi^+(\mathcal{I}_{\mathfrak{q}}))$; this is the Drinfeld double of $\widetilde{\mathcal{B}}_{\mathfrak{q}}\#\mathbf{k}\mathbb{Z}^\theta$. It was shown in [A3] that there is a triangular decomposition $\widetilde{\mathfrak{u}}_{\mathfrak{q}} \simeq \widetilde{\mathfrak{u}}_{\mathfrak{q}}^+ \otimes \widetilde{\mathfrak{u}}^0 \otimes \widetilde{\mathfrak{u}}_{\mathfrak{q}}^-$ as above, with $\widetilde{\mathfrak{u}}^0 \simeq \mathfrak{u}^0 \simeq \mathbf{k}\mathbb{Z}^{2\theta}$.

If β_k is as in (7), $k \in \mathbb{I}_M$, then we set $\widetilde{N}_k = \begin{cases} N_k & \text{if } \beta_k \notin \mathfrak{O}_{\mathfrak{q}}, \\ \infty & \text{if } \beta_k \in \mathfrak{O}_{\mathfrak{q}}, \end{cases}$. For simplicity, we introduce

$$\mathtt{H} = \{\mathbf{h} \in \mathbb{N}_0^M : 0 \le h_k < \widetilde{N}_k, \text{ for all } k \in \mathbb{I}_M\} \tag{12}$$

Theorem 3.5.

(a) [A3, 3.4] *There exist algebra isomorphisms* $\widetilde{T}_i : \widetilde{\mathfrak{u}}_{\mathfrak{q}} \to \widetilde{\mathfrak{u}}_{\rho_i(\mathfrak{q})}$ *inducing the isomorphisms* $T_i : \mathfrak{u}_{\mathfrak{q}} \to \mathfrak{u}_{\rho_i(\mathfrak{q})}$.

(b) [A3, 3.6] *Let* $\widetilde{E}_{\beta_k}$, $\widetilde{\mathbf{E}}^{\mathbf{h}}$ *be the elements of* $\widetilde{\mathfrak{u}}_{\mathfrak{q}}$ *defined as in* (8), (9) *with* $\widetilde{T}_i$ *instead of* T_i. *Then* $\{\widetilde{\mathbf{E}}^{\mathbf{h}} \,|\, \mathbf{h} \in \mathtt{H}\}$ *is a basis of* $\widetilde{\mathfrak{u}}_{\mathfrak{q}}^+$. $\square$

As before, we have an isomorphism $\widetilde{\psi} : \widetilde{\mathcal{B}}_{\mathfrak{q}} \to \widetilde{\mathfrak{u}}_q^+$ of Hopf algebras in ${}^{\mathbf{k}\mathbb{Z}^\theta}_{\mathbf{k}\mathbb{Z}^\theta}\mathcal{YD}$, so we define

$$x_{\beta_k} = \widetilde{\psi}^{-1}(\widetilde{E}_{\beta_k}), \qquad k \in \mathbb{I}_M; \qquad \mathbf{x}^{\mathbf{h}} = \widetilde{\psi}^{-1}(\widetilde{\mathbf{E}}^{\mathbf{h}}), \qquad \mathbf{h} \in \mathtt{H}.$$

Note that $\widetilde{E}_{\beta_k}$ is a well-defined sequence of braided commutators in the elements E_i, $i \in \mathbb{I}$; then x_{β_k} is the same sequence of braided commutators in the x_i's. Also, $\mathbf{x}^{\mathbf{h}} = x_{\beta_M}^{h_M} x_{\beta_{M-1}}^{h_{M-1}} \cdots x_{\beta_1}^{h_1}$ and

$$\mathtt{B} = \{\mathbf{x}^{\mathbf{h}} \,|\, \mathbf{h} \in \mathtt{H}\}$$

is a basis of $\widetilde{\mathcal{B}}_{\mathfrak{q}}$. The Hilbert series of a graded vector space $V = \oplus_{n\in\mathbb{N}_0} V^n$ is $\mathcal{H}_V = \sum_{n\in\mathbb{N}_0} \dim V^n\, T^n \in \mathbb{Z}[[T]]$. It follows from Theorem 3.5 (b) that

$$\text{(13)} \qquad \operatorname{GKdim} \widetilde{\mathcal{B}}_{\mathfrak{q}} = |\mathfrak{O}_{\mathfrak{q}}|, \qquad \mathcal{H}_{\widetilde{\mathcal{B}}_{\mathfrak{q}}} = \prod_{\beta_k \in \mathfrak{O}_{\mathfrak{q}}} \frac{1}{1 - T^{\deg \beta}} \cdot \prod_{\beta_k \notin \mathfrak{O}_{\mathfrak{q}}} \frac{1 - T^{N_\beta \deg \beta}}{1 - T^{\deg \beta}}.$$

4. Lusztig algebras

Let $\mathfrak{q} = (q_{ij}) \in M_\theta(\mathbf{k}^\times)$, (V, c) the corresponding braided vector space of diagonal type and $(V^*, \mathfrak{q})$ the dual braided vector space. We still assume that $\mathcal{B}_{\mathfrak{q}}$ is finite-dimensional. As in [A+, 3.3.4], we define the *Lusztig algebra* $\mathcal{L}_{\mathfrak{q}}$ of (V, c) as the graded dual of the distinguished pre-Nichols algebra $\widetilde{\mathcal{B}}_{\mathfrak{q}}$ of $(V^*, \mathfrak{q})$; thus, $\mathcal{B}_{\mathfrak{q}} \subseteq \mathcal{L}_{\mathfrak{q}}$. In this Section we establish some basic properties of this algebra.

4.1. **Presentation.** In the rest of the section we consider the bilinear form $\langle\, , \,\rangle : \widetilde{\mathcal{B}}_{\mathfrak{q}} \times \widetilde{\mathcal{B}}_{\mathfrak{q}}^* \to \mathbf{k}$ carried from the identification $V^* \otimes V^* \simeq (V \otimes V)^*$ in Section 2.2 which satisfies for all $x, x' \in \widetilde{\mathcal{B}}_{\mathfrak{q}}$, $y, y' \in \widetilde{\mathcal{B}}_{\mathfrak{q}}^*$

$$\langle y, xx' \rangle = \langle y^{(2)}, x \rangle \langle y^{(1)}, x' \rangle \qquad \text{and} \qquad \langle yy', x \rangle = \langle y, x^{(2)} \rangle \langle y', x^{(1)} \rangle.$$

If $\mathbf{h} \in \mathtt{H}$, then define $\mathbf{y}_{\mathbf{h}} \in \widetilde{\mathcal{B}}_{\mathfrak{q}}^*$ by $\langle \mathbf{y}_{\mathbf{h}}, \mathbf{x}^{\mathbf{j}} \rangle = \delta_{\mathbf{h},\mathbf{j}}$, $\mathbf{j} \in \mathtt{H}$. Then $\mathbf{y}_{\mathbf{h}} \in \mathcal{L}_{\mathfrak{q}}$ and $\{\mathbf{y}_{\mathbf{h}} \,|\, \mathbf{h} \in \mathtt{H}\}$ is a basis of $\mathcal{L}_{\mathfrak{q}}$.

Let $(\mathbf{h}_k)_{k \in \mathbb{I}_M}$ denote the canonical basis of $\mathbb{Z}^M$. If $k \in \mathbb{I}_M$ and $\beta = \beta_k \in \Delta_+^{\mathfrak{q}}$, then we denote the element $\mathbf{y}_{n\mathbf{h}_k}$ by $y_\beta^{(n)}$.

We recall some notation and results from [A3] and [AY]. For $i \in \mathbb{I}_M$, let

$$B^i = \langle \{x_{\beta_i}^{h_i} \cdots x_{\beta_1}^{h_1} | 0 \le h_j < N_j\} \rangle \subseteq \mathcal{B}_{\mathfrak{q}},$$
$$\mathbf{B}^i = \langle \{x_{\beta_M}^{h_M} \cdots x_{\beta_i}^{h_i} | 0 \le h_j < N_j\} \rangle \subseteq \mathcal{B}_{\mathfrak{q}},$$
$$\widetilde{B}^i = \langle \{x_{\beta_i}^{h_i} \cdots x_{\beta_1}^{h_1} | 0 \le h_j < \widetilde{N}_j\} \rangle \subseteq \widetilde{\mathcal{B}}_{\mathfrak{q}},$$
$$\widetilde{\mathbf{B}}^i = \langle \{x_{\beta_M}^{h_M} \cdots x_{\beta_i}^{h_i} | 0 \le h_j < \widetilde{N}_j\} \rangle \subseteq \widetilde{\mathcal{B}}_{\mathfrak{q}}.$$

We also denote by $\widetilde{L}^i$ and $\widetilde{\mathbf{L}}^i$ the analogous subspaces of $\mathcal{L}_{\mathfrak{q}}$:

$$\widetilde{L}^i = \langle \{y_{\beta_1}^{(h_1)} \cdots y_{\beta_i}^{(h_i)} | 0 \le h_j < \widetilde{N}_j\} \rangle \subseteq \mathcal{L}_{\mathfrak{q}},$$
$$\widetilde{\mathbf{L}}^i = \langle \{y_{\beta_i}^{(h_i)} \cdots y_{\beta_M}^{(h_M)} | 0 \le h_j < \widetilde{N}_j\} \rangle \subseteq \mathcal{L}_{\mathfrak{q}}.$$

Proposition 4.1. • [AY, 4.2, 4.11] *B^i (respectively $\mathbf{B}^i$) is a right (respectively left) coideal subalgebra of $\mathcal{B}_{\mathfrak{q}}$.*

• [A3, 4.1] *If $\beta \in \mathfrak{O}_{\mathfrak{q}}$, then $x_\beta^{N_\beta}$ $\mathfrak{q}$-commutes with every element of $\widetilde{\mathcal{B}}_{\mathfrak{q}}$.*

- [A3, 4.9] *If $\beta_i \in \mathfrak{O}_{\mathfrak{q}}$, then there exist $X(n_1,\dots,n_{i-1}) \in \widetilde{\mathcal{B}}_{\mathfrak{q}}$ such that*

$$\Delta(x_{\beta_i}^{N_{\beta_i}}) = x_{\beta_i}^{N_{\beta_i}} \otimes 1 + 1 \otimes x_{\beta_i}^{N_{\beta_i}} + \sum_{n_k \in \mathbb{N}_0} x_{\beta_{i-1}}^{n_{i-1}N_{\beta_{i-1}}} \dots x_{\beta_1}^{n_1 N_{\beta_1}} \otimes X(n_1,\dots,n_{i-1}). \qquad \square$$

Corollary 4.2. *$\widetilde{B}^i$ is a right coideal subalgebra of $\widetilde{\mathcal{B}}_{\mathfrak{q}}$.* $\square$

Let $Z_{\mathfrak{q}}^+$ be the subalgebra of $\widetilde{\mathcal{B}}_{\mathfrak{q}}$ generated by $x_\beta^{N_\beta}$, $\beta \in \mathfrak{O}_{\mathfrak{q}}$.

Theorem 4.3. [A3, 4.10, 4.13] *$Z_{\mathfrak{q}}^+$ is a braided normal Hopf subalgebra of $\widetilde{\mathcal{B}}_{\mathfrak{q}}$. Moreover $Z_{\mathfrak{q}}^+ = {}^{co\,\pi}\widetilde{\mathcal{B}}_{\mathfrak{q}}$, where π denotes the canonical projection of $\widetilde{\mathcal{B}}_{\mathfrak{q}}$ onto $\mathcal{B}_{\mathfrak{q}}$.* $\square$

Lemma 4.4. *Let x, x_1 and x_2 be elements in the PBW basis $\mathtt{B}$ of $\widetilde{\mathcal{B}}_{\mathfrak{q}}$. Write $\Delta(x)$ as a linear combination of $\{a \otimes b | a, b \in \mathtt{B}\}$. Assume that $x_1 \otimes x_2$ has a non-zero coefficient in $\Delta(x)$ (in this combination) and x_1x_2 (the concatenation of x_1 and x_2) is in $\mathtt{B}$. Then $x = x_1x_2$.*

Proof. Suppose that $x = x_{\beta_i}^{h_i} \cdots x_{\beta_1}^{h_1}$ with $h_i > 0$. Let

$$\begin{aligned} m(x) &= \min\{j \in \mathbb{N} : h_j \neq 0\}, \\ \mathcal{D}(x) &= \sum_{j=1}^{i}\sum_{t=1}^{h_j} \binom{h_j}{t}_{q_{\beta_j\beta_j}} x_{\beta_i}^{h_i} \cdots x_{\beta_j}^{t} \otimes x_{\beta_j}^{h_j - t} \cdots x_{\beta_1}^{h_1} + 1 \otimes x, \\ \widetilde{C}^i &= \langle \{x_{\beta_M}^{h_M} \cdots x_{\beta_1}^{h_1} \in \mathtt{B} | \, \exists j > i \text{ s.t. } h_j \neq 0\} \rangle. \end{aligned}$$

Observe that if $x_1 \otimes x_2$ appears in $\mathcal{D}(x)$, then $x = x_1x_2$. However, if $x_1 \otimes x_2 \in \sum_{u \in \widetilde{B}^i} u \otimes \widetilde{C}^{m(u)}$, then $x_1x_2 \notin \mathtt{B}$. Therefore the proof is completed by showing that

$$\Delta(x) \in \mathcal{D}(x) + \sum_{u \in \widetilde{B}^i} u \otimes \widetilde{C}^{m(u)}.$$

We proceed by induction on i. If $i = 1$, then $x = x_{\beta_1}^h$ and x_{β_1} is primitive, so $\Delta(x_{\beta_1}^h) = \sum_{0 \le k \le h} \binom{h}{k}_{q_{\beta_1\beta_1}} x_{\beta_1}^k \otimes x_{\beta_1}^{h-k} = \mathcal{D}(x_{\beta_1}^h)$. Let $i > 1$. Now we proceed by induction on h_i. Set $x' = x_{\beta_i}^{h_i - 1} x_{\beta_{i-1}}^{h_{i-1}} \cdots x_{\beta_1}^{h_1}$, so $x = x_{\beta_i}x'$. Notice that

$$\Delta(x_{\beta_i}) \in x_{\beta_i} \otimes 1 + 1 \otimes x_{\beta_i} + \widetilde{B}^{i-1} \otimes \widetilde{C}^i. \tag{14}$$

Indeed the analogous statement for $\mathcal{B}_{\mathfrak{q}}$ was proved in [AY, 4.3], but the same argument applies for $\widetilde{\mathcal{B}}_{\mathfrak{q}}$. By the inductive hypothesis and (14)

$$\begin{aligned} \Delta(x) &= \Delta(x_{\beta_i})\Delta(x') \\ &\in \big(x_{\beta_i} \otimes 1 + 1 \otimes x_{\beta_i} + \widetilde{B}^{i-1} \otimes \widetilde{C}^i\big)\Big(\mathcal{D}(x') + \sum_{u \in \widetilde{B}^i} u \otimes \widetilde{C}^{m(u)}\Big). \end{aligned}$$

Notice that $(x_{\beta_i}\otimes 1 + 1\otimes x_{\beta_i})\mathcal{D}(x') \in \mathcal{D}(x) + \sum_{u\in\widetilde{B}^i} u\otimes \widetilde{C}^{m(u)}$, since

$$(x_{\beta_i}\otimes 1 + 1\otimes x_{\beta_i})\Big(\sum_{t=1}^{h_i-1}\binom{h_i-1}{t}_{q_{\beta_i\beta_i}} x_{\beta_i}^t\otimes x_{\beta_i}^{h_i-1-t}\cdots x_{\beta_1}^{h_1} + 1\otimes x'\Big) =$$
$$x_{\beta_i}\otimes x' + \sum_{t=2}^{h_i}\binom{h_i-1}{t-1}_{q_{\beta_i\beta_i}} x_{\beta_i}^t\otimes x_{\beta_i}^{h_i-t}\cdots x_{\beta_1}^{h_1} +$$
$$\sum_{t=1}^{h_i-1} q_{\beta_i\beta_i}^t\binom{h_i-1}{t}_{q_{\beta_i\beta_i}} x_{\beta_i}^t\otimes x_{\beta_i}^{h_i-t}\cdots x_{\beta_1}^{h_1} + 1\otimes x_{\beta_i}x'$$

and for $h_i>1$, $1\le t<h_i$, we have $\binom{h_i-1}{t-1}_{q_{\beta_i\beta_i}} + q_{\beta_i\beta_i}^t\binom{h_i-1}{t}_{q_{\beta_i\beta_i}} = \binom{h_i}{t}_{q_{\beta_i\beta_i}}$. Also, $\widetilde{B}^{i-1}\subset\widetilde{B}^i$, $\widetilde{B}^i$ is a subalgebra and $\widetilde{C}^i z\subset\widetilde{C}^i$ for all $z\in\widetilde{\mathcal{B}}_{\mathfrak{q}}$, by [A3, 3.15], so

$$(\widetilde{B}^{i-1}\otimes\widetilde{C}^i)\mathcal{D}(x') \subset \widetilde{B}^{i-1}\widetilde{B}^i\otimes\widetilde{C}^i\widetilde{B}^i \subset \widetilde{B}^i\otimes\widetilde{C}^i.$$

As $x_{\beta_i}u\in\widetilde{B}^i$ for all $u\in\widetilde{B}^i$ and $m(u)=m(x_{\beta_i}u)$, then

$$x_{\beta_i}u\otimes\widetilde{C}^{m(u)} = x_{\beta_i}u\otimes\widetilde{C}^{m(x_{\beta_i}u)} \quad\text{and}\quad u\otimes x_{\beta_i}\widetilde{C}^{m(u)}\subset u\otimes\widetilde{C}^{m(u)}.$$

Finally, $\widetilde{B}^{i-1}u\otimes\widetilde{C}^i\widetilde{C}^{m(u)}\subset\widetilde{B}^i\otimes\widetilde{C}^i\subset\sum_{v\in\widetilde{B}^i} v\otimes\widetilde{C}^{m(v)}$ for all $u\in\widetilde{B}^i$. From these considerations the proof of the inductive step follows directly. $\square$

Corollary 4.5. *If $\beta\in\Delta_+^{\mathfrak{q}}$, then*

$$y_\beta^{(r)} = \frac{y_\beta^r}{(r)^!_{q_{\beta\beta}}}, \qquad r<N_\beta=\operatorname{ord} q_{\beta\beta}; \tag{15}$$

$$y_\beta^{(n)} = \frac{(y_\beta^{(N_\beta)})^s}{s!}y_\beta^{(r)}, \qquad \beta\in\mathfrak{O}_{\mathfrak{q}},\ n=sN_\beta+r,\ r<N_\beta. \tag{16}$$

Proof. Arguing inductively, we may suppose that $y_\beta^{r-1}=(r-1)^!_{q_{\beta\beta}}y_\beta^{(r-1)}$. If $x=\mathbf{x}^{\mathbf{h}}\in\widetilde{\mathcal{B}}_{\mathfrak{q}}$ such that

$$\langle y_\beta^r, x\rangle = \langle y_\beta^{r-1}, x^{(1)}\rangle\langle y_\beta, x^{(2)}\rangle \neq 0,$$

then by Lemma 4.4, $x=x_\beta^r$. Then

$$\langle y_\beta^r, x_\beta^r\rangle = \langle y_\beta^{r-1}, (x_\beta^r)^{(1)}\rangle\langle y_\beta, (x_\beta^r)^{(2)}\rangle = (r-1)^!_{q_{\beta\beta}}(r)_{q_{\beta\beta}} = (r)^!_{q_{\beta\beta}}.$$

The second equation follows immediately since $\langle y_\beta^{(N_\beta)}y_\beta^{(r)}, x_\beta^{N_\beta+r}\rangle = 1$. $\square$

The next lemma is crucial for the presentation of the algebra $\mathcal{L}_{\mathfrak{q}}$ by generators and relations.

Lemma 4.6. *Let $i\in\mathbb{I}_M$, $h_i<\widetilde{N}_{\beta_i}$ and $\mathbf{h}=(h_1,\dots,h_M)\in\mathbb{N}_0^M$, then*

$$\mathbf{y}_{\mathbf{h}} = y_{\beta_1}^{(h_1)}\cdots y_{\beta_M}^{(h_M)}. \tag{17}$$

Hence $\{y_{\beta_1}^{(h_1)}\cdots y_{\beta_M}^{(h_M)} \mid 0\le h_i<\widetilde{N}_{\beta_i}\}$ is a basis of $\mathcal{L}_{\mathfrak{q}}$.

Proof. The proof is by induction on $\mathrm{ht}(\mathbf{h}) := \sum_{i\in\mathbb{I}_M} h_i$. If $\mathrm{ht}(\mathbf{h}) = 1$ then $\mathbf{y_h} = y_\beta$ for some $\beta \in \Delta_+^{\mathfrak{q}}$ and the claim follows by definition.

Let $1 \le i_1 < \cdots < i_j \le M$, $n_k < \widetilde{N}_{\beta_{i_k}}$ and $n_1 = sN_{\beta_{i_1}} + r \neq 0$ where $r < N_{\beta_{i_1}}$. Let $y = y_{\beta_{i_1}}^{(n_1)} \dots y_{\beta_{i_j}}^{(n_j)} \in \mathcal{L}_{\mathfrak{q}}$. Since $\{\mathbf{y_h} \,|\, \mathbf{h} \in \mathtt{H}\}$ is a basis of $\mathcal{L}_{\mathfrak{q}}$, we can express y as the linear combination $y = \sum_{\mathbf{h}\in\mathtt{H}} c_{\mathbf{h}}\mathbf{y_h}$. Notice that $c_{\mathbf{h}} \neq 0$ if and only if $\langle y, x^{\mathbf{h}}\rangle \neq 0$.

If $r \neq 0$, then we write $y = \frac{1}{(r)_q} y_{\beta_{i_1}} y'$ where $y' = y_{\beta_{i_1}}^{(n_1-1)} \dots y_{\beta_{i_j}}^{(n_j)}$ and $q = q_{\beta_{i_1}\beta_{i_1}}$. Then $\langle y, x^{\mathbf{h}}\rangle = \frac{1}{(r)_q}\langle y_{\beta_{i_1}}, (x^{\mathbf{h}})^{(2)}\rangle\langle y', (x^{\mathbf{h}})^{(1)}\rangle$. By inductive hypothesis and Lemma 4.4, $c_{\mathbf{h}} \neq 0$ if and only if $\mathbf{h} = (0, \dots, n_1, \dots, n_k, 0, \dots)$. Moreover, the nonzero $c_{\mathbf{h}}$ is equal to 1 and the proof in this case is completed.

If $r = 0$, $n_1 = sN_{\beta_{i_1}}$, then we write $y = y_{\beta_{i_1}}^{(N_{\beta_{i_1}})} y'$. Arguing as above, (17) follows. Hence $\{y_{\beta_1}^{(h_1)} \cdots y_{\beta_M}^{(h_M)} | \, 0 \le h_i < \widetilde{N}_{\beta_i}\}$ is a basis of $\mathcal{L}_{\mathfrak{q}}$ because so is $\{\mathbf{y_h} : \mathbf{h} \in \mathtt{H}\}$ by definition. $\square$

We seek for a presentation of $\mathcal{L}_{\mathfrak{q}}$. Let us consider the algebra $\mathbb{L}$ presented by generators $\mathsf{y}_\beta^{(n)}$, $\beta \in \Delta_+^{\mathfrak{q}}$, $n \in \mathbb{N}$ with relations

$$\mathsf{y}_\beta^{(N_\beta)} = 0, \qquad \beta \in \Delta_+^{\mathfrak{q}} - \mathfrak{O}_{\mathfrak{q}}; \tag{18}$$

$$\mathsf{y}_\beta^{(h)}\mathsf{y}_\beta^{(j)} = \binom{h+j}{j}_{q_{\beta\beta}} \mathsf{y}_\beta^{(h+j)}, \qquad \beta \in \Delta_+^{\mathfrak{q}},\ h, j \in \mathbb{N}; \tag{19}$$

$$[\mathsf{y}_\beta^{(h)}, \mathsf{y}_\alpha^{(j)}]_c = \sum_{\mathsf{m}\in\mathsf{M}(\alpha,\beta,h,j)} \kappa_{\mathsf{m}}\, \mathsf{m}, \qquad \alpha < \beta \in \Delta_+^{\mathfrak{q}},\ 0 < h < N_\alpha,\ 0 < j < N_\beta; \tag{20}$$

$$[\mathsf{y}_\beta^{(N_\beta)}, \mathsf{y}_\alpha^{(N_\alpha)}]_c = \kappa_\gamma \mathsf{y}_\gamma^{(N_\gamma)} + \sum_{\substack{0<l<N_\beta,\, 0<i<N_\alpha \\ \mathsf{m}\in\mathsf{M}(\alpha,\beta,N_\alpha-i,N_\beta-l)}} \kappa_{\mathsf{m}}^{i,l}\, \mathsf{y}_\alpha^{(i)}\mathsf{m}\mathsf{y}_\beta^{(l)}, \qquad \alpha, \beta, \gamma \in \mathfrak{O}_{\mathfrak{q}},\ \alpha < \gamma < \beta; \tag{21}$$

$$[\mathsf{y}_\beta^{(j)}, \mathsf{y}_\alpha^{(N_\alpha)}]_c = \sum_{\substack{0<i<N_\alpha, \\ \mathsf{m}\in\mathsf{M}(\alpha,\beta,N_\alpha-i,j)}} \kappa_{\mathsf{m}}^{i,0}\, \mathsf{y}_\alpha^{(i)}\mathsf{m}, \qquad \alpha \in \mathfrak{O}_{\mathfrak{q}},\ \beta \in \Delta_+^{\mathfrak{q}},\ 0 < j < N_\beta. \tag{22}$$

Here we set

$$\mathsf{M}(\alpha, \beta, h, j) = \{\mathsf{m} = \mathsf{y}_{\beta_r}^{(h_r)} \cdots \mathsf{y}_{\beta_k}^{(h_k)} \in \widetilde{L}^\beta \cap \widetilde{\mathbf{L}}^\alpha : \deg \mathsf{m} = \deg \mathsf{y}_\alpha^{(h)} + \deg \mathsf{y}_\beta^{(j)}\};$$
$$\kappa_{\mathsf{m}}^{i,l} = \langle y_\beta^{(h)} y_\alpha^{(j)}, x_\beta^l x_{\beta_k}^{h_k} \cdots x_{\beta_r}^{h_r} x_\alpha^i\rangle;$$
$$\kappa_\gamma = \langle \mathsf{y}_\beta^{(N_\beta)}\mathsf{y}_\alpha^{(N_\alpha)}, x_\gamma^{N_\gamma}\rangle, \qquad \deg \mathsf{y}_\gamma^{(N_\gamma)} = \deg \mathsf{y}_\alpha^{(N_\alpha)} + \deg \mathsf{y}_\beta^{(N_\beta)}.$$

Theorem 4.7. *There is an algebra isomorphism $\Upsilon : \mathbb{L} \to \mathcal{L}_{\mathfrak{q}}$ given by*

$$\Upsilon(\mathsf{y}_\beta^{(n)}) = y_\beta^{(n)}, \qquad \beta \in \Delta_+^{\mathfrak{q}},\ n < \widetilde{N}_\beta.$$

Proof. We first prove that Υ is well-defined, i. e. that (18), ..., (22) are satisfied by the elements $y_\beta^{(n)} \in \mathcal{L}_\mathfrak{q}$. Relation (18) is trivial since $x_\beta^{N_\beta} = 0$ if $\beta \notin \mathfrak{O}_\mathfrak{q}$ and (19) is clear from (15).

For the other relations, given $\alpha < \beta$ and $h, j \in \mathbb{N}$, we write $y_\beta^{(h)} y_\alpha^{(j)} = \sum_{\mathbf{h}\in\mathbb{H}} c_\mathbf{h} \mathbf{y}_\mathbf{h}$. Then

$$c_\mathbf{h} = \langle y_\beta^{(h)} y_\alpha^{(j)}, \mathbf{x}^\mathbf{h}\rangle = \langle y_\alpha^{(j)}, (\mathbf{x}^\mathbf{h})^{(1)}\rangle\langle y_\beta^{(h)}, (\mathbf{x}^\mathbf{h})^{(2)}\rangle$$

is the coefficient of $x_\alpha^j \otimes x_\beta^h$ in the expression of $\Delta(\mathbf{x}^\mathbf{h})$ as linear combination of elements of the PBW basis in both sides of the tensor product.

If $j < N_\alpha$ and $h < N_\beta$, then $y_\alpha^{(j)}, y_\beta^{(h)} \in \mathcal{B}_\mathfrak{q}$. If $c_\mathbf{h} \neq 0$ then $\mathbf{x}^\mathbf{h}$ appears in the expression of $x_\alpha^j x_\beta^h$ in elements of the PBW basis, see [A1, Section 3]. Hence, by [HY2, 4.8] $\mathbf{x}^\mathbf{h} \in \mathbf{B}^\alpha \cap B^\beta$, and relation (20) is clear.

Let $\alpha, \beta \in \mathfrak{O}_\mathfrak{q}$, $j = N_\alpha$ and $h = N_\beta$. Suppose that there is $\mathbf{h} = (h_1, \dots, h_M)$ such that $c_\mathbf{h} \neq 0$ and $h_i \geq N_i$ for some $i \in \mathbb{I}_M$. As $x_{\beta_i}^{N_i}$ $\mathfrak{q}$-commutes with every element of $\widetilde{\mathcal{B}}_\mathfrak{q}$, we have $\mathbf{x}^\mathbf{h} = c\, x_{\beta_i}^{N_i} \mathbf{x}^{\mathbf{h}'}$, where $\mathbf{h}' = (h_1, \dots, h_i - N_i \dots, h_M)$ and $c = \Xi(h_M\beta_M + \dots + h_{i+1}\beta_{i+1}, N_i\beta_i) \in \mathbf{k}$. Then $\Delta(\mathbf{x}^\mathbf{h}) = c\,\Delta(x_{\beta_i}^{N_i})\Delta(\mathbf{x}^{\mathbf{h}'})$ and hence $\mathbf{x}^\mathbf{h} = x_{\beta_i}^{N_i}$ by Proposition 4.1. For the remaining $\mathbf{j}$ such that $c_\mathbf{j} \neq 0$ we have $j_i < N_i$ for all $i \in \mathbb{I}_M$. We write $x_\alpha^{N_\alpha} \otimes x_\beta^{N_\beta} = \xi(1 \otimes x_\beta^n)(x_\alpha^{N_\alpha - m} \otimes x_\beta^{N_\beta - n})(x_\alpha^m \otimes 1)$ where $\xi = \Xi^{-1}((N_\alpha - m)\alpha, n\beta)\Xi^{-1}(m\alpha, (N_\beta - n)\beta)$. Therefore, arguing as in the proof of (20) for $y_\beta^{(N_\beta - n)} y_\alpha^{(N_\alpha - m)}$, we obtain that $\mathbf{y_j} = \mathsf{y}_\alpha^{(m)} \mathsf{m} \mathsf{y}_\beta^{(n)}$, $\mathsf{m} \in \widetilde{L}^\beta \cap \widetilde{\mathbf{L}}^\alpha$. Here, either $m = N_\alpha$, $n = N_\beta$ so $\mathbf{y_j} = \Xi(N_\alpha\alpha, N_\beta\beta)\mathsf{y}_\alpha^{(N_\alpha)}\mathsf{y}_\beta^{(N_\beta)}$, or else $m < N_\alpha$ $n < N_\beta$. Hence relation (21) follows up to consider the correct degree for $\mathbf{y_h}$.

For (22), $c_\mathbf{h} \neq 0$ implies $\mathbf{x}^\mathbf{h} \in \mathcal{B}_\mathfrak{q}$ by the same argument above, since $Z_\mathfrak{q}^+$ is a braided Hopf subalgebra by Theorem 4.3.

Hence, Υ is a morphism of algebras. By the presentation of $\mathbb{L}$ we can prove that $\{\mathsf{y}_{\beta_1}^{(h_1)} \dots \mathsf{y}_{\beta_M}^{(h_M)} : h_i < \widetilde{N}_i\}$ is a basis of $\mathbb{L}$. So, Υ maps a basis to a basis by Lemma 4.6 and then it is bijective. □

Example 4.8. Let $\theta = 3 \leq N$, $q \in \mathbf{k}^\times$, $\operatorname{ord} q = N$. We consider a diagonal braiding (of super type A) given by a matrix $\mathfrak{q} = (q_{ij})_{i,j\in\mathbb{I}_3}$ such that

$$q_{11} = q_{23}q_{32} = q, \qquad q_{12}q_{21} = q^{-1}, \qquad q_{22} = q_{33} = -1, \qquad q_{13}q_{31} = 1.$$

Let $\alpha_{jk} = \sum_{j\leq i\leq k} \alpha_i$; then $\Delta_\mathfrak{q}^+ = \{\alpha_{jk} : 1 \leq j \leq k \leq 3\}$, $\mathfrak{O}_\mathfrak{q}^+ = \{\alpha_1, \alpha_{23}, \alpha_{13}\}$.

The Lusztig algebra $\mathcal{L}_\mathfrak{q}$ is presented by generators $y_{jk}^{(n)}$, $1 \leq j \leq k \leq 3$, $n \in \mathbb{N}$ and relations:

$$y_{12}^{(2)} = y_2^{(2)} = y_3^{(2)} = 0,$$

$$
\begin{aligned}
y_{jk}^{(n)} y_{jk}^{(m)} &= \binom{n+m}{n}_{q_{jk}} y_{jk}^{(n+m)}, \quad n, m \in \mathbb{N},\\
[y_{12}, y_1]_c &= [y_{13}, y_1]_c = [y_3, y_1]_c = [y_{13}, y_{12}]_c = [y_2, y_{12}]_c = [y_{23}, y_{12}]_c = 0,\\
[y_2, y_{13}]_c &= [y_{23}, y_{13}]_c = [y_3, y_{13}]_c = [y_{23}, y_2]_c = [y_3, y_{23}]_c = 0,\\
[y_2, y_1]_c &= (1-q^{-1}) y_{12}, \qquad [y_3, y_{12}]_c = (1-q) y_{13},\\
[y_{23}, y_1]_c &= (1-q^{-1}) y_{13}, \qquad [y_3, y_2]_c = (1-q) y_{23},\\
[y_{23}^{(N)}, y_1]_c &= (1-q^{-1})(q_{21}q_{31})^{N-1} y_{13} y_{23}^{(N-1)},\\
[y_{23}, y_1^{(N)}]_c &= (1-q^{-1})(q_{21}q_{31})^{N-1} y_1^{(N-1)} y_{13},\\
[y_2, y_1^{(N)}]_c &= (1-q^{-1}) q_{21}^{N-1} y_1^{(N-1)} y_{12},\\
[y_{12}, y_1^{(N)}]_c &= [y_{13}, y_1^{(N)}]_c = [y_3, y_1^{(N)}]_c = 0,\\
[y_{13}^{(N)}, y_1]_c &= [y_{13}^{(N)}, y_{12}]_c = [y_2, y_{13}^{(N)}]_c = [y_{23}, y_{13}^{(N)}]_c = [y_3, y_{13}^{(N)}]_c = 0,\\
[y_{23}^{(N)}, y_{12}]_c &= [y_{23}^{(N)}, y_{13}]_c = [y_{23}^{(N)}, y_2]_c = [y_3, y_{23}^{(N)}]_c = 0,\\
[y_{13}^{(N)}, y_1^{(N)}]_c &= [y_{23}^{(N)}, y_{13}^{(N)}]_c = 0,\\
[y_{23}^{(N)}, y_1^{(N)}]_c &= (1-q^{-1})^N (q_{21}q_{31})^{N\frac{N-1}{2}} y_{13}^{(N)}\\
&\quad + \sum_{k=1}^{N-1} (1-q^{-1})^k (q_{21}q_{31})^{k\frac{2N-k-1}{2}} y_1^{(N-k)} y_{13}^{(k)} y_{23}^{(N-k)}.
\end{aligned}
$$

Indeed, to compute $y_{23}^{(N)} y_1^{(N)}$ in $\mathcal{L}_{\mathfrak{q}}$, we need to describe all $\mathbf{h} \in \mathtt{H}$, cf. (12), such that $x_1^N \otimes x_{23}^N$ appears in $\Delta(\mathbf{x}^{\mathbf{h}})$ with non-zero coefficient (also to be determined), where (for some numeration of $\Delta_{\mathfrak{q}}^+$)

$$\mathbf{x}^{\mathbf{h}} = x_3^{h_1} x_{23}^{h_2} x_2^{h_3} x_{123}^{h_4} x_{12}^{h_5} x_1^{h_6}.$$

One of these $\mathbf{x}^{\mathbf{h}}$ is $x_{23}^N x_1^N$, with coefficient $\mathfrak{q}_{N\alpha_1, N\alpha_2+N\alpha_3}$. Let $\mathbf{h}$ be as needed. We use the coproduct formulas in [A3, 5.1]. Clearly $h_1 = 0$. From $\Delta(x_{23}^{h_2})$, the only contribution is $(1 \otimes x_{23})^{h_2}$. Then we deduce easily that $h_3 = h_5 = 0$, and $h_6 = h_2 = N - h_4$. In this case, set $h_4 = k$ to simplify the notation, so

$$(1 \otimes x_{23})^{N-k} (x_1 \otimes x_{23})^k (x_1 \otimes 1)^{N-k} = (q_{21}q_{31})^{k\frac{2N-k-1}{2}} x_1^N \otimes x_{23}^N.$$

This gives the last relation, and the others are deduced analogously.

Corollary 4.9. *The algebra $\mathcal{L}_{\mathfrak{q}}$ is finitely generated.*

Proof. By (19), it is generated by $\{y_\beta : \beta \in \Delta_+^{\mathfrak{q}}\} \cup \{y_\alpha^{(N_\alpha)} : \alpha \in \mathfrak{O}_{\mathfrak{q}}\}$. □

Remark 4.10. Actually, the subalgebra $\mathcal{B}_{\mathfrak{q}} \subset \mathcal{L}_{\mathfrak{q}}$ is generated by its primitive elements $\{y_\alpha : \alpha \in \Pi_{\mathfrak{q}}\}$ where $\Pi_{\mathfrak{q}}$ denotes the set of simple roots $\alpha_1, \dots, \alpha_\theta$. Moreover, $y_\gamma^{(N_\gamma)} \in \mathbf{k}^\times [y_\beta^{(N_\beta)}, y_\alpha^{(N_\alpha)}]_c$ if and only if $x_\alpha^{N_\alpha} \otimes x_\beta^{N_\beta}$ appears with nonzero coefficient in $\Delta(x_\gamma^{N_\gamma})$. Hence,

$$\{y_\alpha : \alpha \in \Pi_{\mathfrak{q}}\} \cup \{y_\alpha^{(N_\alpha)} : \alpha \in \mathfrak{O}_{\mathfrak{q}},\, x_\alpha^{N_\alpha} \in \mathcal{P}(\widetilde{\mathcal{B}}_{\mathfrak{q}})\}$$

generates $\mathcal{L}_{\mathfrak{q}}$ as an algebra.

Proposition 4.11. $\widetilde{\mathbf{L}}^i$ *is a right coideal subalgebra of* $\mathcal{L}_{\mathfrak{q}}$.

Proof. From Theorem 4.7 we have that $y_{\beta_j}^{(n)} y_{\beta_i}^{(m)} \in \widetilde{\mathbf{L}}^i$ for $i < j$, thus $\widetilde{\mathbf{L}}^i$ is a subalgebra of $\mathcal{L}_{\mathfrak{q}}$. On the other hand, we know that $\langle y_\beta^{(n)}, xx'\rangle = \langle (y_\beta^{(n)})^{(2)}, x\rangle\langle (y_\beta^{(n)})^{(1)}, x'\rangle$. Therefore $y_{\mathbf{j}} \otimes y_{\mathbf{h}}$ appears with nonzero coefficient in $\Delta(y_\beta^{(n)})$ if and only if x_β^n appears with nonzero coefficient in the expression of $x^{\mathbf{h}}x^{\mathbf{j}}$ in the PBW basis. The last condition implies that $x^{\mathbf{h}} \in \widetilde{B}^\beta$ and $x^{\mathbf{j}} \in \widetilde{\mathbf{B}}^\beta$. Hence,

$$\Delta(y_\beta^{(n)}) \in \sum_{i=0}^{n} y_\beta^{(i)} \otimes y_\beta^{(n-i)} + \widetilde{\mathbf{L}}^\beta \otimes \widetilde{L}^\beta.$$

Hence $\Delta(y_{\beta_i}^{(n_i)} \dots y_{\beta_M}^{(n_M)}) = \Delta(y_{\beta_i}^{(n_i)})\Delta(y_{\beta_{i+1}}^{(n_{i+1})} \dots y_{\beta_M}^{(n_M)}) \in \widetilde{\mathbf{L}}^i \otimes \mathcal{L}_{\mathfrak{q}}$ and the proof is complete. □

4.2. **Noetherianity and Gelfand-Kirillov dimension.** We argue as in the pre-Nichols case [A3, Section 3.4], cf. [DP]. Let us consider the lexicographic order in $\mathbb{N}_0^M$, so that $\mathbf{h}_M < \dots < \mathbf{h}_1$, where $(\mathbf{h}_j)_{j\in\mathbb{I}_M}$ denotes the canonical basis of $\mathbb{Z}^M$.

Lemma 4.12. *Let* $\mathcal{L}_{\mathfrak{q}}(\mathbf{h})$ *be the subspace of* $\mathcal{L}_{\mathfrak{q}}$ *generated by* $\mathbf{y_j}$*, with* $\mathbf{j} \le \mathbf{h}$*. Then* $\mathcal{L}_{\mathfrak{q}}(\mathbf{h})$ *is an* $\mathbb{N}_0^M$*-algebra filtration of* $\mathcal{L}_{\mathfrak{q}}$.

Proof. It is enough to prove that $\mathbf{y_h}\mathbf{y_j} \in \mathcal{L}_{\mathfrak{q}}(\mathbf{h}+\mathbf{j})$ for all $\mathbf{h},\mathbf{j} \in \mathtt{H}$. First we consider the case when $\mathbf{h} = n\mathbf{h}_k$, $\mathbf{j} = m\mathbf{h}_l$, $k,l \in \mathbb{I}_M$, $n,m \in \mathbb{N}$. We claim that $y_{\beta_k}^{(n)} y_{\beta_l}^{(m)} \in \mathcal{L}_{\mathfrak{q}}(n\mathbf{h}_k + m\mathbf{h}_l)$. This follows by definition when $k \le l$. If $l < k$, then $[y_{\beta_k}^{(n)}, y_{\beta_l}^{(m)}]_c \in \sum_{j<m} y_{\beta_l}^{(j)}.\widetilde{\mathbf{L}}^{l+1}$ by Theorem 4.7, thus

$$y_{\beta_k}^{(n)} y_{\beta_l}^{(m)} \in \mathcal{L}_{\mathfrak{q}}(n\mathbf{h}_k + m\mathbf{h}_l) \qquad \text{since} \qquad \sum_{j=l+1}^{M} a_j\mathbf{h}_j < n\mathbf{h}_k + m\mathbf{h}_l.$$

The Lemma follows by reordering the factors of $\mathbf{y_h}\mathbf{y_j}$, for any $\mathbf{h},\mathbf{j} \in \mathbb{N}_0^M$. □

We now consider the corresponding graded algebra

$$\operatorname{gr} \mathcal{L}_{\mathfrak{q}} = \oplus_{\mathbf{h}\in\mathbb{N}_0^M} \operatorname{gr}^{\mathbf{h}} \mathcal{L}_{\mathfrak{q}}, \qquad \text{where} \qquad \operatorname{gr}^{\mathbf{h}} \mathcal{L}_{\mathfrak{q}} = \mathcal{L}_{\mathfrak{q}}(\mathbf{h})/\sum_{\mathbf{j}<\mathbf{h}} \mathcal{L}_{\mathfrak{q}}(\mathbf{j}).$$

Lemma 4.13. *The algebra* $\operatorname{gr}\mathcal{L}_{\mathfrak{q}}$ *is presented by generators* $\mathrm{y}_k^{(n)}$*,* $k \in \mathbb{I}_M$*,* $n \in \mathbb{N}$*, and relations*

$$\mathrm{y}_k^{(N_k)} = 0, \quad \beta_k \notin \mathfrak{O}_{\mathfrak{q}},$$

$$\mathtt{y}_k^{(n)}\mathtt{y}_k^{(m)} = \binom{n+m}{m}_{q_{\beta_k\beta_k}} \mathtt{y}_k^{(n+m)},$$
$$[\mathtt{y}_k^{(n)}, \mathtt{y}_l^{(m)}]_c = 0, \quad l < k.$$

Proof. Let $\mathcal{G}$ be the algebra presented by the generators and relations above and $\pi : \mathcal{G} \to \operatorname{gr}\mathcal{L}_{\mathfrak{q}}$ given by $\mathtt{y}_k^{(n)} \mapsto y_{\beta_k}^{(n)}$. By Theorem 4.7, the relations above hold in $\operatorname{gr}\mathcal{L}_{\mathfrak{q}}$. By a direct computation, $\mathcal{G}$ has a basis

$$\{\mathtt{y}_1^{(h_1)} \dots \mathtt{y}_M^{(h_M)} : h_i < \widetilde{N}_i\}.$$

On the other hand, $\mathbf{y_h} \in \mathcal{L}_{\mathfrak{q}}(\mathbf{h}) - \sum_{\mathbf{j}<\mathbf{h}} \mathcal{L}_{\mathfrak{q}}(\mathbf{j})$. Hence the projection of the PBW basis of $\mathcal{L}_{\mathfrak{q}}$ is a basis of $\operatorname{gr}\mathcal{L}_{\mathfrak{q}}$ and π is an isomorphism. $\square$

Proposition 4.14. *The algebra $\mathcal{L}_{\mathfrak{q}}$ is Noetherian.*

Proof. Let $\mathcal{Z}^+$ be the subalgebra of $\operatorname{gr}\mathcal{L}_{\mathfrak{q}}$ generated by $\{y_\beta^{(N_\beta)} : \beta \in \mathfrak{O}_{\mathfrak{q}}\}$. Then $\mathcal{Z}^+$ is a quantum affine space and $\operatorname{gr}\mathcal{L}_{\mathfrak{q}}$ is a finitely generated free $\mathcal{Z}^+$-module. Hence $\operatorname{gr}\mathcal{L}_{\mathfrak{q}}$ is Noetherian and so is $\mathcal{L}_{\mathfrak{q}}$. $\square$

We compute either from Lemma 4.6 or else from Lemma 4.13 the Gelfand-Kirillov dimension of $\mathcal{L}_{\mathfrak{q}}$.

Proposition 4.15. $\operatorname{GKdim}\mathcal{L}_{\mathfrak{q}} = |\mathfrak{O}_{\mathfrak{q}}|$. $\square$

5. Quantum divided power algebras

5.1. **Definition.** Let $\mathfrak{q}$, (V, c) be as above with $\dim \mathcal{B}_{\mathfrak{q}} < \infty$. Let $W = V^*$, with matrix $\mathfrak{q}^t$, see footnote 2, and let $\{z_\beta^{(n)} : \beta \in \Delta_+^{\mathfrak{q}}, n \in \mathbb{N}\}$ be the generators of $\mathcal{L}_{\mathfrak{q}^t}$. Here we consider $W \in {}^{\mathbf{k}\mathbb{Z}^\theta}_{\mathbf{k}\mathbb{Z}^\theta}\mathcal{YD}$ via the equivalence of categories between ${}^{(\mathbf{k}\mathbb{Z}^\theta)^*}_{(\mathbf{k}\mathbb{Z}^\theta)^*}\mathcal{YD}$ and ${}^{\mathbf{k}\mathbb{Z}^\theta}_{\mathbf{k}\mathbb{Z}^\theta}\mathcal{YD}$. Then we have a natural evaluation map such that $\langle w \otimes w', v \otimes v' \rangle = \langle w \otimes v' \rangle \langle w' \otimes v \rangle$. In this section we define the *quantum divided power algebra* $\mathcal{U}_{\mathfrak{q}}$ of (V, c) and we establish some of its basic properties.

Let Γ and Λ be two copies of $\mathbb{Z}^\theta$, generated by $(K_i)_{i\in\mathbb{I}}$ and $(L_i)_{i\in\mathbb{I}}$ respectively; so that $(K_i^{\pm1})_{i\in\mathbb{I}}$ and $(L_i^{\pm1})_{i\in\mathbb{I}}$ are the generators of $\mathbf{k}\Gamma$ and $\mathbf{k}\Lambda$, respectively. Set $K_\alpha = K_1^{a_1} \dots K_\theta^{a_\theta}$ and $L_\alpha = L_1^{a_1} \dots L_\theta^{a_\theta}$ for $\alpha = (a_1, \dots, a_\theta) \in \mathbb{Z}^\theta$. Then $\mathcal{L}_{\mathfrak{q}} \in {}^{\mathbf{k}\Gamma}_{\mathbf{k}\Gamma}\mathcal{YD}$, $\mathcal{L}_{\mathfrak{q}^t} \in {}^{\mathbf{k}\Lambda}_{\mathbf{k}\Lambda}\mathcal{YD}$ with structure determined by the formulae

$$K_\alpha^{\pm1} \cdot y_\beta^{(n)} = q_{\alpha\beta}^{\pm n} y_\beta^{(n)}, \qquad \rho(y_\beta^{(n)}) = K_\beta^n \otimes y_\beta^{(n)};$$
$$L_\alpha^{\pm1} \cdot z_\beta^{(n)} = q_{\beta\alpha}^{\pm n} z_\beta^{(n)}, \qquad \rho(z_\beta^{(n)}) = L_\beta^n \otimes y_\beta^{(n)}.$$

Therefore, we can consider the bosonizations $\mathcal{L}_{\mathfrak{q}}\#\mathbf{k}\Gamma$ and $\mathcal{L}_{\mathfrak{q}^t}\#\mathbf{k}\Lambda$.

We define next the quantum double of $\mathcal{L}_{\mathfrak{q}}\#\mathbf{k}\Gamma$ and $\mathcal{L}_{\mathfrak{q}^t}\#\mathbf{k}\Lambda$ following [J, 3.2.2]. For this we need a Hopf pairing between them.

Lemma 5.1. *There is a unique bilinear form* $(\,|\,): T^c(V)\times(T^c(W))^{\mathrm{cop}}\to\mathbf{k}$ *such that* $(1|1)=1$,

$$\begin{aligned}
&(y_i|z_j)=\delta_{ij}, && i,j\in\mathbb{I};\\
&(y|zz')=(y^{(1)}|z)(y^{(2)}|z'), && y\in T^c(V),\ z,z'\in T^c(W);\\
&(yy'|z)=(y|z^{(1)})(y'|z^{(2)}), && y,y'\in T^c(V),\ z\in T^c(W);\\
&(y|z)=0, && |y|\neq|z|,\ y\in T^c(V),\ z\in T^c(W).
\end{aligned}$$

Proof. Let $\mathbf{T}^n=\sum_{\sigma\in\mathbb{S}_n}s(\sigma):(T^c)^n(W)\to T^n(W)$, where $s:\mathbb{S}_n\to\mathbb{B}_n$ is the Matsumoto section, see [AG, §3.2]. Let $\langle\,,\,\rangle: T^c(V)\otimes T(W)^{\mathrm{op}}\to\mathbf{k}$ be the evaluation map. We define $(1|1)=1$,

$$\begin{aligned}
&(y|z)=\langle y,\mathbf{T}^n(z)\rangle, && y\in(T^c)^n(V), z\in(T^c)^n(W)\\
&(y|z)=0, && y\in(T^c)^n(V), z\in(T^c)^m(W), n\neq m.
\end{aligned}$$

Note that $\mathbf{T}^{i+j}=\mathbf{T}_{i,j}(\mathbf{T}^i\otimes\mathbf{T}^j)$ with $\mathbf{T}_{i,j}=\sum s(\sigma^{-1})$ where the sum is over all (i,j)-shuffles σ. Then, for $y\in(T^c)^n(V)$, $z\in(T^c)^{n-i}(W)$, $z'\in(T^c)^i(W)$,

$$\begin{aligned}
(y|zz')&=\langle y,\mathbf{T}^n(z'z)\rangle=\langle y,\mathbf{T}_{i,n-i}(\mathbf{T}^i\otimes\mathbf{T}^{n-i})(z'z)\rangle\\
&=\langle y,\mathbf{T}_{i,n-i}(\mathbf{T}^i(z')\otimes\mathbf{T}^{n-i}(z))\rangle=\langle y^{(1)},\mathbf{T}^{n-i}(z)\rangle\langle y^{(2)},\mathbf{T}^i(z')\rangle\\
&=(y^{(1)}|z)(y^{(2)}|z')
\end{aligned}$$

The other conditions are clear. □

This bilinear form restricts to $\mathcal{L}_{\mathfrak{q}}\times(\mathcal{L}_{\mathfrak{q}^t})^{\mathrm{cop}}$ and then it can be extended to a bilinear form between their bosonizations. Then we may define a skew-Hopf pairing between $\mathcal{L}_{\mathfrak{q}}\#\mathbf{k}\Gamma$ and $\mathcal{L}_{\mathfrak{q}^t}\#\mathbf{k}\Lambda$, or equivalently:

Corollary 5.2. *There is a unique Hopf pairing*

$$(\,|\,):\mathcal{L}_{\mathfrak{q}}\#\mathbf{k}\Gamma\times(\mathcal{L}_{\mathfrak{q}^t}\#\mathbf{k}\Lambda)^{\mathrm{cop}}\to\mathbf{k}$$

such that for all $Y,Y'\in\mathcal{L}_{\mathfrak{q}}\#\mathbf{k}\Gamma$, $Z,Z'\in(\mathcal{L}_{\mathfrak{q}^t}\#\mathbf{k}\Lambda)^{\mathrm{cop}}$, $y_\alpha^{(n)}\in\mathcal{L}_{\mathfrak{q}}$, $K_\alpha\in\mathbf{k}\mathbb{Z}^\theta$, $z_\beta^{(m)}\in\mathcal{L}_{\mathfrak{q}^t}$ *and* $L_\beta\in\mathbf{k}\mathbb{Z}^\theta$

$$\begin{aligned}
&(Y|ZZ')=(Y_{(1)}|Z)(Y_{(2)}|Z'),\quad (YY'|z)=(Y|Z_{(1)})(Y'|Z_{(2)}),\\
&(y_\alpha^{(n)}|z_\beta^{(m)})=\delta_{n\alpha,m\beta},\quad (y_\alpha^{(n)}|L_\beta)=0,\quad (K_\alpha|z_\beta^{(m)})=0,\quad (K_\alpha|L_\beta)=q_{\alpha\beta}.
\end{aligned}$$

Moreover, this pairing satisfies the equation $(yK|zL)=(y|z)(K|L)$. □

Let $\mathcal{U}_{\mathfrak{q}}$ be the Drinfeld double of $\mathcal{L}_{\mathfrak{q}}\#\mathbf{k}\Gamma$ and $(\mathcal{L}_{\mathfrak{q}^t}\#\mathbf{k}\Lambda)^{\mathrm{cop}}$ with respect to the Hopf pairing in Corollary 5.2. In other words:

Definition 5.3. Let $\mathcal{U}_{\mathfrak{q}}$ be the unique Hopf algebra such that

(1) $\mathcal{U}_{\mathfrak{q}}=(\mathcal{L}_{\mathfrak{q}}\#\mathbf{k}\Gamma)\otimes(\mathcal{L}_{\mathfrak{q}^t}\#\mathbf{k}\Lambda)$ as vector spaces,
(2) the maps $Y\mapsto Y\otimes 1$ and $Z\mapsto 1\otimes Z$ are Hopf algebra morphisms,

(3) the product is given by

$$(Y \otimes Z)(Y' \otimes Z') = (Y'_{(1)}|\mathcal{S}(Z_{(1)}))YY'_{(2)} \otimes Z_{(2)}Z'(Y'_{(3)}|Z_{(3)})$$

for all $Y, Y' \in \mathcal{L}_{\mathfrak{q}}\#\mathbf{k}\Gamma$ and $Z, Z' \in (\mathcal{L}_{\mathfrak{q}^t}\#\mathbf{k}\Lambda)^{\mathrm{cop}}$.

By the construction of $\mathcal{U}_{\mathfrak{q}}$, there is a triangular decomposition, via the multiplication, $\mathcal{U}_{\mathfrak{q}} \simeq \mathcal{U}_{\mathfrak{q}}^+ \otimes \mathcal{U}^0 \otimes \mathcal{U}_{\mathfrak{q}}^-$ where

$$\mathcal{U}_{\mathfrak{q}}^+ \simeq \mathcal{L}_{\mathfrak{q}}, \qquad \mathcal{U}_{\mathfrak{q}}^- \simeq \mathcal{L}_{\mathfrak{q}^t}, \qquad \mathcal{U}^0 \simeq \mathbf{k}(\mathbb{Z}^\theta \times \mathbb{Z}^\theta).$$

We give a presentation of the algebra $\mathcal{U}_{\mathfrak{q}}$ by generators and relations. The tensor product signs in elements of $\mathcal{U}_{\mathfrak{q}}$ will be omitted.

Proposition 5.4. *The algebra $\mathcal{U}_{\mathfrak{q}}$ is generated by the elements $y_\beta^{(n)}$, $z_\beta^{(n)}$, $K_\beta^{\pm 1}$, $L_\beta^{\pm 1}$ for $\beta \in \Delta_+^{\mathfrak{q}}$, $n \in \mathbb{N}$; and relations* (18), …, (22) *between the $y_\beta^{(n)}$'s, similar relations for the $z_\beta^{(n)}$'s plus the relations*

$$K_\beta K_\beta^{-1} = L_\beta^{-1} L_\beta = 1, \qquad K_\beta^{\pm 1} L_\alpha^{\pm 1} = L_\alpha^{\pm 1} K_\beta^{\pm 1} \tag{23}$$

$$K_\alpha y_\beta^{(n)} = q_{\alpha\beta}^n y_\beta^{(n)} K_\alpha, \qquad L_\alpha y_\beta^{(n)} = q_{\beta\alpha}^{-n} y_\beta^{(n)} L_\alpha, \tag{24}$$

$$K_\alpha z_\beta^{(n)} = q_{\alpha\beta}^{-n} z_\beta^{(n)} K_\alpha, \qquad L_\alpha z_\beta^{(n)} = q_{\beta\alpha}^{n} z_\beta^{(n)} L_\alpha, \tag{25}$$

$$zy = (y^{(1)}|\mathcal{S}(z^{(3)}))\,(K_2K_3|L_3^{-1})\,(y^{(3)}|z^{(1)})\,y^{(2)}K_3 z^{(2)}L_3, \tag{26}$$

for all $\alpha, \beta \in \Delta_+^{\mathfrak{q}}$, $n, m \in \mathbb{N}$. Here in (26) *$y = y_\beta^{(n)} \in \mathcal{L}_{\mathfrak{q}}$, $z = z_\alpha^{(m)} \in \mathcal{L}_{\mathfrak{q}^t}$, and denote $K_i = (y^{(i)})_{(-1)}$ and $L_i = (z^{(i)})_{(-1)}$ for the coactions of $\mathbf{k}\Gamma$ and $\mathbf{k}\Lambda$ respectively.* $\square$

Note that if $y = y_{\alpha_i}$, $z = z_{\alpha_j}$ with $\alpha_i, \alpha_j \in \Pi_{\mathfrak{q}}$, then y, z are primitives and relation (26) is $zy - yz = \delta_{ij}(K_i - L_i)$.

5.2. **Basic properties.** Proceeding as in [DP, A3], we will prove that the algebra $\mathcal{U}_{\mathfrak{q}}$ is Noetherian. For each $\mathbf{h}, \mathbf{j} \in \mathtt{H}$, $K \in \Gamma$, $L \in \Lambda$, set

$$d_1(\mathbf{y_h}KL\mathbf{z_j}) = \sum_{i \in \mathbb{I}_M} (h_i + j_i)\,\mathrm{ht}(\beta_i),$$

$$d(\mathbf{y_h}KL\mathbf{z_j}) = \Big(d_1(\mathbf{y_h}KL\mathbf{z_j}),\, h_1, \dots, h_M,\, j_1, \dots, j_M\Big) \in \mathbb{N}_0^{2M+1}.$$

Consider the lexicographic order in $\mathbb{N}_0^{2M+1}$. If $\mathbf{u} \in \mathbb{N}_0^{2M+1}$, then we set

$$\mathcal{U}_{\mathfrak{q}}(\mathbf{u}) = \text{span of } \{\mathbf{y_h}KL\mathbf{z_j} : \mathbf{h}, \mathbf{j} \in \mathtt{H},\, K \in \Gamma,\, L \in \Lambda,\, d(\mathbf{y_h}KL\mathbf{z_j}) \le \mathbf{u}\}.$$

Lemma 5.5. $(\mathcal{U}_{\mathfrak{q}}(\mathbf{u}))_{\mathbf{u} \in \mathbb{N}_0^{2M+1}}$ *is an $\mathbb{N}_0^{2M+1}$-algebra filtration of $\mathcal{U}_{\mathfrak{q}}$.*

Proof. It is enough to prove that $(\mathbf{y_h}KL\mathbf{z_j})(\mathbf{y_{h'}}K'L'\mathbf{z_{j'}}) \in \mathcal{U}_{\mathfrak{q}}(\mathbf{u} + \mathbf{u}')$ for all $\mathbf{h}, \mathbf{j}, \mathbf{h}', \mathbf{j}' \in \mathtt{H}$, $K, K' \in \Gamma$ and $L, L' \in \Lambda$ where $d(\mathbf{y_h}KL\mathbf{z_j}) = \mathbf{u}$ and $d(\mathbf{y_{h'}}K'L'\mathbf{z_{j'}}) = \mathbf{u}'$.

First we claim that

$$d_1(z_\beta^{(n)} y_\alpha^{(m)} - y_\alpha^{(m)} z_\beta^{(n)}) < m\,\mathrm{ht}(\alpha) + n\,\mathrm{ht}(\beta). \tag{27}$$

Indeed, since the coproduct in $\mathcal{L}_{\mathfrak{q}}$ (resp. $\mathcal{L}_{\mathfrak{q}^t}$) is graded, we have that $d_1((y_\alpha^{(m)})^{(2)}) < m\,\mathrm{ht}(\alpha)$ if $(y_\alpha^{(m)})^{(1)} \neq 1$ (resp. $d_1((z_\beta^{(n)})^{(2)}) < n\,\mathrm{ht}(\beta)$ if $(z_\beta^{(n)})^{(1)} \neq 1$). Hence, for $K \in \Gamma$ and $L \in \Lambda$ we have

$$d_1((y_\alpha^{(m)})^{(2)} KL (z_\beta^{(n)})^{(2)}) \leq m\,\mathrm{ht}(\alpha) + n\,\mathrm{ht}(\beta)$$

and by Proposition 5.4 the claim follows.

Since K, L $\mathfrak{q}$-commutes with all elements of $\mathcal{L}_{\mathfrak{q}}$ and $\mathcal{L}_{\mathfrak{q}^t}$ for all $K \in \Gamma$ and $L \in \Lambda$. We proceed as in Lemma 4.12 and we reduce the proof to the product between $z_{\beta_i}^{(n)}$ and $y_{\beta_j}^{(m)}$. It follows directly by (27) that

$$z_{\beta_i}^{(n)} y_{\beta_j}^{(m)} \in \mathcal{U}_{\mathfrak{q}}(m\,\mathrm{ht}(\beta_j) + n\,\mathrm{ht}(\beta_i), \delta_j, \delta_i). \quad \square$$

We consider the associated graded algebra $\mathrm{gr}\,\mathcal{U}_{\mathfrak{q}} = \oplus_{\mathbf{v}\in\mathbb{N}_0^{2M+1}} \mathcal{U}_{\mathfrak{q}}{}^{\mathbf{v}}$ where $\mathcal{U}_{\mathfrak{q}}{}^{\mathbf{v}} = \mathcal{U}_{\mathfrak{q}}(\mathbf{v}) / \sum_{\mathbf{u}<\mathbf{v}} \mathcal{U}_{\mathfrak{q}}(\mathbf{u})$.

Corollary 5.6. *The algebra* $\mathrm{gr}\,\mathcal{U}_{\mathfrak{q}}$ *is presented by generators* $\mathtt{y}_j^{(n)}$, $\mathtt{z}_j^{(n)}$, $K_j^{\pm 1}$, $L_j^{\pm 1}$, $j \in \mathbb{I}_M$, $n \in \mathbb{N}$ *and relations*

$$\begin{aligned}
&RS = SR, && R, S \in \{K_j^{\pm 1}, L_j^{\pm 1} : j \in \mathbb{I}_M\}\\
&K_\beta K_\beta^{-1} = L_\beta L_\beta^{-1} = 1 && \mathtt{y}_k^{(n)} \mathtt{z}_l^{(m)} = \mathtt{z}_l^{(m)} \mathtt{y}_k^{(n)}\\
&\mathtt{y}_k^{(N_k)} = 0, \quad \beta_k \notin \mathfrak{O}_{\mathfrak{q}}, && \mathtt{z}_k^{(N_k)} = 0, \quad \beta_k \notin \mathfrak{O}_{\mathfrak{q}},\\
&\mathtt{y}_k^{(n)} \mathtt{y}_k^{(m)} = \binom{n+m}{m}_{q_{\beta_k\beta_k}} \mathtt{y}_k^{(n+m)}, && \mathtt{z}_k^{(n)} \mathtt{z}_k^{(m)} = \binom{n+m}{m}_{q_{\beta_k\beta_k}} \mathtt{z}_k^{(n+m)},\\
&[\mathtt{y}_k^{(n)}, \mathtt{y}_l^{(m)}]_c = 0, \quad l < k, && [\mathtt{z}_k^{(n)}, \mathtt{z}_l^{(m)}]_c = 0, \quad l < k,\\
&K_\alpha y_\beta^{(n)} = q_{\alpha\beta}^n y_\beta^{(n)} K_\alpha, && K_\alpha z_\beta^{(n)} = q_{\alpha\beta}^{-n} z_\beta^{(n)} K_\alpha,\\
&L_\alpha y_\beta^{(n)} = q_{\beta\alpha}^{-n} y_\beta^{(n)} L_\alpha, && L_\alpha z_\beta^{(n)} = q_{\beta\alpha}^{n} z_\beta^{(n)} L_\alpha.
\end{aligned}$$

Proof. The proof of this statement is similar to the proof of Lemma 4.13 if we check that $y_k^{(n)} z_l^{(m)} = z_l^{(m)} y_k^{(n)}$ for all $y_k^{(n)} \in \mathcal{L}_{\mathfrak{q}}$ and $z_l^{(m)} \in \mathcal{L}_{\mathfrak{q}^t}$; but this follows by (27). $\square$

Proposition 5.7. *The algebra* $\mathcal{U}_{\mathfrak{q}}$ *is Noetherian and* $\mathrm{GKdim}\,\mathcal{U}_{\mathfrak{q}} = 2|\mathfrak{O}_{\mathfrak{q}}| + 2\theta$.

Proof. Let $\mathcal{Z}$ be the subalgebra of $\mathrm{gr}\,\mathcal{U}_{\mathfrak{q}}$ generated by $\{K_i, L_i : i \in \mathbb{I}\}$ and $\{y_\beta^{(N_\beta)}, z_\beta^{(N_\beta)} : \beta \in \mathfrak{O}_{\mathfrak{q}}\}$. Then $\mathcal{Z}$ is the localization of a quantum affine space and $\mathrm{gr}\,\mathcal{U}_{\mathfrak{q}}$ is a free $\mathcal{Z}$-module of rank $\prod_{i\in\mathbb{I}_M} N_i$. Therefore $\mathrm{gr}\,\mathcal{U}_{\mathfrak{q}}$ is Noetherian and so is $\mathcal{U}_{\mathfrak{q}}$. Moreover, by [KL, Prop. 6.6],

$$\mathrm{GKdim}\,\mathcal{U}_{\mathfrak{q}} = \mathrm{GKdim}\,\mathrm{gr}\,\mathcal{U}_{\mathfrak{q}} = \mathrm{GKdim}\,\mathcal{Z} = 2|\mathfrak{O}_{\mathfrak{q}}| + 2\theta. \quad \square$$

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FaMAF-CIEM (CONICET), Universidad Nacional de Córdoba, Medina Allende s/n, Ciudad Universitaria, 5000 Córdoba, República Argentina.

E-mail address: (andrus|angiono|rossib)@mate.uncor.edu