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Haarlet analysis of Lipschitz regularity in metric measure spaces

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Abstract In this note we shall give a characterization of Lipschitz spaces on spaces of homogeneous type via Haar coefficients.

Keywords Lipschitz spaces, Haar basis, spaces of homogeneous type

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1 Introduction and statement of the results

We first briefly recall the basic properties of the general theory of spaces of homogeneous type. Assume that X is a set, a nonnegative symmetric function d on $X \times X$ is called a quasi-distance if there exists a constant K such that

$$d(x,y) \leqslant K[d(x,z)+d(z,y)], \tag{1.1}$$

for every $x, y, z \in X$, and $d(x,y)=0$ if and only if $x=y$.

We shall say that (X,d,μ) is a space of homogeneous type if d is a quasi-distance on X, μ is a positive Borel measure defined on a σ-algebra of subsets of X which contains the balls, and there exists a constant A such that

$$0 < \mu(B(x,2r)) \leqslant A\mu(B(x,r)) < \infty$$

holds for every $x \in X$ and every $r > 0$.

In [10] the authors prove that each quasi-metric space is metrizable and that d is equivalent to ρ^β, where ρ is a distance on X and $\beta \geqslant 1$. So that we shall assume along this paper that d is actually a distance on X, in other words that $K=1$ in (1.1).

In order to be able to apply Lebesgue differentiation theorem we shall also assume that continuous functions are dense in $L^1(X,\mu)$.

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Along this paper we shall deal with three at first different descriptions of α-regularity of real functions defined on X. The regularity parameter α shall always be positive.

In [10], Macías and Segovia introduce, in the setting of space of homogeneous type, the Campanato type functions given in [6] for the Euclidean case. They prove that under the condition of normality on the homogeneous space, these classes are exactly the standard Lipschitz spaces. Let us introduce these two classes of Lipschitz functions.

We define the Lipschitz spaces of order α, $\mathrm{Lip}(\alpha)$, as the space of functions f defined on X such that there exists a constant $C > 0$ such that

$$|f(x) - f(y)| \leqslant Cd(x,y)^\alpha$$

for every $x, y \in X$. We shall denote by $\|f\|_{\mathrm{Lip}(\alpha)}$ the infimun of all such constants C.

For $1 \leqslant q < \infty$ a real function f in $L^q_{\mathrm{loc}}(X,\mu)$ is said to belong to $\mathrm{Lip}(\alpha,q)$ if there exists a positive and finite constant C such that the inequality

$$\left(\frac{1}{\mu(B)}\int_B |f(x) - m_B(f)|^q d\mu(x)\right)^{1/q} \leqslant Cr(B)^\alpha \tag{1.2}$$

holds for every d-ball B in X, where $r(B)$ is the radius of B and $m_B(f) = \frac{1}{\mu(B)}\int_B f d\mu$. With $\|f\|_{\mathrm{Lip}(\alpha,q)}$ we denote the infimun of those constants C.

Let us point out that our definition of $\mathrm{Lip}(\alpha,q)$ coincides with the class $\mathrm{Lip}(\alpha,q)$ in [10] only when (X,d,μ) is normal in the sense that the measure of any d-ball is comparable to its radius. In fact, in [10] the authors define the element in the class $\mathrm{Lip}(\alpha,q)$ as those function in $L^q_{\mathrm{loc}}(X,\mu)$ such that the inequality (1.2) holds with $\mu(B)$ instead $r(B)$. It is easy to see that $\mathrm{Lip}(\alpha)$ implies $\mathrm{Lip}(\alpha,q)$. Actually, one of the main results in [10] is the converse. We shall show that both $\mathrm{Lip}(\alpha)$ and $\mathrm{Lip}(\alpha,q)$ have the same description in terms of Haar wavelets built on some wide classes of dyadic families. The following result is the first step in this direction.

Theorem 1.1. *Let (X,d,μ) be a space of homogeneous type and $f \in \mathrm{Lip}(\alpha,2)$. Then the inequality*

$$|\langle f,h\rangle| = \left|\int_X fh d\mu\right| \leqslant \|f\|_{\mathrm{Lip}(\alpha,2)}\ r(B)^\alpha\ \mu(B)^{1/2},$$

holds for every function h and every ball $B = B(z, r(B))$ such that

(i) $\int_X h d\mu = 0$;
(ii) $\int_X |h|^2 d\mu = 1$;
(iii) $\{x \in X : h(x) \neq 0\} \subseteq B$.

Proof. Let $f \in \mathrm{Lip}(\alpha,2)$ and h a function that satisfies (i) to (iii). From (i) and (iii) we have that

$$\int_X f(x)h(x)d\mu(x) = \int_B (f(x) - m_B(f))\, h(x)d\mu(x).$$

From the Hölder's inequality, (ii) and the fact that $f \in \mathrm{Lip}(\alpha,2)$ we obtain that

$$\begin{aligned}\left|\int_X f(x)h(x)d\mu(x)\right| &\leqslant \int_B |f(x) - m_B(f)|\,|h(x)|\,d\mu(x)\\ &\leqslant \left(\int_B |f(x) - m_B(f)|^2\,d\mu(x)\right)^{1/2}\\ &\leqslant \mu(B)^{1/2}\left(\frac{1}{\mu(B)}\int_B |f(x) - m_B(f)|^2\,d\mu(x)\right)^{1/2}\\ &\leqslant \|f\|_{\mathrm{Lip}(\alpha,2)}\mu(B)^{1/2}r(B)^\alpha.\end{aligned}$$

□

Notice that in the Euclidean context, any localized wavelet satisfies properties (i) to (iii). On an abstract metric space (X,d), for which no smoothness better than Lipschitz continuity makes sense, the first basic prototype of localized wavelet is the Haar wavelet. We are interested in a reciprocal of the previous theorem in terms of Haar type functions.

In order to state the main result of this note we shall refer here to Sections 2 and 3 below, for the precise meaning of a dyadic family $\mathcal{D}$ in $\mathfrak{D}(\delta)$ on (X,d,μ), of a Haar system associated to $\mathcal{D}$ and for the notion of separating classes of dyadic families.

In the Euclidean case $\mathbb{R}^n$, the class of all arbitrary translations of the usual dyadic cubes $Q^j_{\vec{k}} = \prod_{i=1}^n I^j_{k_i}$, with $\vec{k}=(k_1,\ldots,k_n)\in\mathbb{Z}^n$, $j\in\mathbb{Z}$ and $I^j_{k_i}=[k_i 2^{-j},(k_i+1)2^{-j})$, has an important property which we shall call *separating*. In fact, it is easy to see that given two different points x and y in $\mathbb{R}^n$, then there exists $z\in\mathbb{R}^n$, $j\in\mathbb{Z}$ and $\vec{k}\in\mathbb{Z}^n$ such that, both x and y belong to two different subcubes $z+Q^{j+1}_{\vec{l}}$ and $z+Q^{j+1}_{\vec{m}}$ of $z+Q^j_{\vec{k}}$ and 2^{-j} is comparable to the Euclidean distance between x and y.

The precise definition of *separating classes* is given in Section 2 where we shall also prove their existence in our general setting of space of homogeneous type.

Let S be a given class of dyadic families $\mathcal{D}$ in $\mathfrak{D}(\delta)$. Let us write $\mathcal{H}$ to denote the set of all Haar functions h that belong to some Haar system $\mathcal{H}_{\mathcal{D}}$ associated to some $\mathcal{D}\in\mathcal{S}$. Let us define our third class of Lipschitz type functions. A function $f\in L^1_{\mathrm{loc}}(X,d,\mu)$ is said to belong to the Carlesson class $\mathcal{C}(\alpha,S)$ if there exists a positive constant C such that

$$|\langle f,h\rangle| \leqslant C\ \mathrm{diam}(Q_h)^\alpha\ \mu(Q_h)^{1/2}, \tag{1.3}$$

holds for every $h\in\mathcal{H}$, where Q_h is the smallest dyadic cube containing the set $\{x\in X: h(x)\neq 0\}$ and $\mathrm{diam}(E)=\sup\{d(x,y): x\in E, y\in E\}$.

Our main results are contained in the next two statements.

Theorem 1.2. *Let (X,d,μ) be a space of homogeneous type. There exist separating classes in $\mathfrak{D}(\delta)$ for some $\delta>0$.*

Theorem 1.3. *Let (X,d,μ) be a space of homogeneous type. Let $\mathcal{S}$ be a separating class of dyadic families on X. Then $\mathcal{C}(\alpha,S)\subseteq \mathrm{Lip}(\alpha)$ in the sense that in the Lebesgue class of each function in $\mathcal{C}(\alpha,\mathcal{S})$ there is a* Lip(α) *function.*

Collecting the results contained in Theorems 1.1–1.3 we have the next statement which contains Theorem 5 in [10].

Theorem 1.4. *Let (X,d,μ) be a space of homogeneous type.*

$$\mathrm{Lip}(\alpha)=\mathrm{Lip}(\alpha,q)=\mathcal{C}(\alpha,\mathcal{S})$$

for every $1\leqslant q<\infty$ and every separating family $\mathcal{S}$.

With almost the same proofs, the above results extend to moduli of continuity $\varphi(t)$ more general than t^α (see [2]). With the obvious definitions, regarding Lip(α) as Lip(φ) with $\varphi(t)=t^\alpha$, we now have

$$\mathrm{Lip}(\varphi)\subseteq\mathrm{Lip}(\varphi,q)\subseteq\mathcal{C}(\psi,\mathcal{S})\subseteq\mathrm{Lip}(\psi),$$

where $\psi(t)=\int_0^1\frac{\varphi(s)}{s}ds$, provided that $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ is a non-decreasing function and such that $\int_0^1\frac{\varphi(t)}{t}dt<\infty$. All these classes are the same when $\psi(t)\leqslant C\varphi(t)$ for some constant C.

We would like to point out that in order to prove that $f\in\mathrm{Lip}(\alpha)$ in Theorem 1.3 we actually only use (1.3) for a class of test function of Haar type much smaller than $\mathcal{H}$.

2 Separating classes of dyadic families

The construction of dyadic type families of subsets in metric or quasi-metric spaces with some inner and outer metric control of the sizes of the dyadic sets is given in [7]. These families satisfy all the relevant properties of the usual dyadic cubes in $\mathbb{R}^n$ and are the basic tool to build wavelets on a metric space of homogeneous type (see [1] or [3]). The notion of dyadic families that we will consider here is contained in the following definition (see [5]).

Definition 2.1 (The class $\mathfrak{D}(\delta)$ of all dyadic families)**.** Let (X,d,μ) be a metric space of homogeneous type. We say that $\mathcal{D}=\cup_{j\in\mathbb{Z}}\mathcal{D}^j$ is a dyadic family on X with parameter $\delta\in(0,1)$, briefly that $\mathcal{D}$ belong $\mathfrak{D}(\delta)$, if each $\mathcal{D}^j$ is a family of Borel subsets Q of X, such that

(d.1) for every $j\in\mathbb{Z}$ the cubes in $\mathcal{D}^j$ are pairwise disjoint;

(d.2) for every $j\in\mathbb{Z}$ the family $\mathcal{D}^j$ almost covers X in the sense that $\mu(X\setminus\cup_{Q\in\mathcal{D}^j}Q)=0$;

(d.3) if $Q\in\mathcal{D}^j$ and $i<j$, then there exists a unique $\tilde{Q}\in\mathcal{D}^i$ such that $Q\subseteq\tilde{Q}$;

(d.4) if $Q\in\mathcal{D}^j$ and $\tilde{Q}\in\mathcal{D}^i$ with $i\leqslant j$, then either $Q\subseteq\tilde{Q}$ or $Q\cap\tilde{Q}=\emptyset$;

(d.5) there exist two constants a_1 and a_2 such that for each $Q\in\mathcal{D}^j$ there exists a point $x\in Q$ that satisfies $B(x,a_1\delta^j)\subseteq Q\subseteq B(x,a_2\delta^j)$.

The following properties that can easily be deduced from (d.1) to (d.5) are going to be frequently used in the sequel:

(d.6) there exists a positive integer N depending on a_i, $i=1,2$ in (d.5) and on the doubling constant A such that for every $j\in\mathbb{Z}$ and all $Q\in\mathcal{D}^j$ the inequalities $1\leqslant\#(\mathcal{L}(Q))\leqslant N$ hold, where $\mathcal{L}(Q)=\{Q'\in\mathcal{D}^{j+1}:Q'\subseteq Q\}$;

(d.7) the families $\tilde{\mathcal{D}}^j=\{Q\in\mathcal{D}^j:\#(\mathcal{L}(Q))>1\}$, $j\in\mathbb{Z}$ are pairwise disjoints;

(d.8) we have a well-defined function $\mathcal{J}:\tilde{\mathcal{D}}=\cup_{j\in\mathbb{Z}}\tilde{\mathcal{D}}^j\to\mathbb{Z}$ given by $Q\mapsto\mathcal{J}(Q)$ if $Q\in\tilde{\mathcal{D}}^{\mathcal{J}(Q)}$;

(d.9) there exists a positive constant c such that $\mu(Q)\leqslant c\mu(Q')$ for all $Q\in\tilde{\mathcal{D}}$ and every $Q'\in\mathcal{L}(Q)$.

Any Q in some dyadic family $\mathcal{D}$ shall be called cube or dyadic cube. Results like the one we propose in this paper are valid for continuous wavelet transforms on the Euclidean context (see [2, 8, 9, 11]). In our general setting we have no algebraic structure that allows us to speak of continuous translations. The following definition is designed to replace this limitation.

Definition 2.2 (Separating classes in $\mathfrak{D}(\delta)$)**.** We shall say that a class of dyadic families $\mathcal{S}\subseteq\mathfrak{D}(\delta)$ separates points of X if there exist two positive constant $c_1\leqslant c_2<\infty$ such that for every x and y in X with $x\neq y$ there exists $\mathcal{D}\in\mathcal{S}$ such that

(s1) there is a dyadic cube $Q\in\mathcal{D}$ such that x and y belong to Q;

(s2) there exist two different dyadic cubes Q_x and Q_y in $\mathcal{L}(Q)$ such that $x\in Q_x$ and $y\in Q_y$;

(s3) $c_1\delta^{\mathcal{J}(Q)}\leqslant d(x,y)\leqslant c_2\delta^{\mathcal{J}(Q)}$.

In order to prove Theorem 1.2 we shall use the basic construction given by Christ in [7] which we proceed to recall. We shall say that, for $\varepsilon>0$, $\mathcal{N}_\varepsilon$ is an ε-net in X if $\mathcal{N}_\varepsilon$ is a maximal ε-disperse subset of X. That is, $d(x,x')\geqslant\varepsilon$ for every $x,x'\in\mathcal{N}$ with $x\neq x'$ and if E is any other subset of X strictly containing $\mathcal{N}_\varepsilon$ then there exists $y,y'\in E$ with $y\neq y'$ such that $d(y,y')<\varepsilon$. It is easy to show that since (X,d) supports the doubling measure μ, then any ε-disperse subset $\mathcal{N}_\varepsilon$ of X is countable. Moreover $\mathcal{N}_\varepsilon$ is finite if and only if X is bounded. In the sequel we shall use the notation $\mathcal{N}_\varepsilon=\{x_k:k\in K(\varepsilon)\}$ to denote the elements of the ε-net $\mathcal{N}_\varepsilon$, where $K(\varepsilon)$ is an initial interval of positive integers, which could be the whole set $\mathbb{Z}^+$ of positive integers. For a fixed positive δ, the above considerations with $\varepsilon=\delta^j$, $j\in\mathbb{Z}$, gives rise to a sequence of δ^j-nets $\mathcal{N}_j=\{x_k^j:k\in K_j\}$, where $K_j=K(\delta^j)$. On the set $\mathcal{A}=\{(j,k):j\in\mathbb{Z}\text{ and }k\in K_j\}$, Christ introduce a tree structure that is closely related to the metric structure on X (see [7] for details). Let us remark here that for the construction given in [7] we do not need any nesting property of the sequence $\mathcal{N}_j$ of δ^j-nets. The partial order $\preceq$ on $\mathcal{A}$ defined in [7] which gives the tree structure on $\mathcal{A}$ satisfies the two following basic properties:

(a) if $d(x_k^j,x_l^{j-1})<\frac{\delta^{j-1}}{2}$, then $(j,k)\preceq(j-1,l)$;

(b) if $(j,k)\preceq(j-1,l)$, then $d(x_k^j,x_l^{j-1})<\delta^{j-1}$.

For a given sequence of δ^j-nets, $\mathcal{N}_j,\delta>0$, we shall say that such order belongs to the class $\mathcal{O}_\delta$, briefly, $\preceq\in\mathcal{O}_\delta$.

The Christ dyadic cube at the level j located at $k\in K_j$ is defined by

$$Q_k^j=\bigcup_{(i,l)\preceq(j,k)}B(x_l^i,a\delta^i). \tag{2.1}$$

The family of all these cubes Q_k^j satisfy Definition 2.1 for small values of the positive constants δ and a, where in (d.5) we can choose $x=x_k^j$.

The set Q_k^j shall be called the dyadic cube associated to $x_k^j \in \mathcal{N}_j$. The family $\mathcal{D}_\preceq$ of all those Q_k^j shall be called the Christ cubes associated to the family $\{\mathcal{N}_j : j \in \mathbb{Z}\}$ of δ^j-nets $\mathcal{N}_j, j \in \mathbb{Z}$ and the order $\preceq$. Notice that from (2.1) we have that

$$\text{if } (i,l) \preceq (j,k) \text{ then } Q_l^i \subseteq Q_k^j. \tag{2.2}$$

The following lemmas will be our key tools to prove Theorem 1.2.

Lemma 2.3. *Let $\mathcal{N}_j$, $j \in \mathbb{Z}$ be a sequence of δ^j-nets in X with $\delta < 1/2$ and let $\mathcal{D}_\preceq$ the Christ's cubes associated to family $\{\mathcal{N}_j : j \in \mathbb{Z}\}$ and to order $\preceq$ in $\mathcal{O}_\delta$. Assume that for some $j_0 \in \mathbb{Z}$, $k \in K_{j_0-1}$ and $k' \in K_{j_0+1}$ we have that $x_k^{j_0-1} = x_{k'}^{j_0+1} \in \mathcal{N}_{j_0-1} \cap \mathcal{N}_{j_0+1}$. Then there exists $s \in K_{j_0}$ such that*

$$(j_0+1, k') \preceq (j_0, s) \preceq (j_0-1, k).$$

Proof. Let $\preceq$ a order on $\mathcal{A} = \{(j,k) : j \in \mathbb{Z}, k \in K_j\}$ in $\mathcal{O}_\delta$. Let (j_0, s) the only element in $\mathcal{A}$ for which $(j_0+1, k') \preceq (j_0, s)$. To obtain the lemma all we need is to show that necessarily $(j_0, s) \preceq (j_0-1, k)$. In fact, from (b) we have

$$d(x_{k'}^{j_0+1}, x_s^{j_0}) < \delta^{j_0} < \frac{\delta^{j_0-1}}{2}.$$

Since the left-hand side is also $d(x_s^{j_0}, x_k^{j_0-1})$, (a) shows that $(j_0, s) \preceq (j_0-1, k)$. □

Since, as we have already observed, for the construction of Christ no nesting property of the nets $\mathcal{N}_j$ is need. This fact allows us to construct a particular sequence $\mathcal{N}$ of δ^j-nets associated to two given different points x and y in X. Set $\mathcal{N}$ to denote any sequence $(\mathcal{N}_j : j \in \mathbb{Z})$ of δ^j-nets with $x = x_1^j \in \mathcal{N}_j$ if j is odd and $y = x_1^j \in \mathcal{N}_j$ if j is even.

Lemma 2.4. *Let x and y be two points in X with $x \neq y$ and let $0 < \delta < 1/2$ be given. Let $\preceq$ be an order in $\mathcal{O}_\delta$ induced by the sequence $\mathcal{N}$ defined above. Then there exists $i \in \mathbb{Z}$ such that the dyadic family $\mathcal{D}_\preceq$ satisfies*

(1) $Q_1^{i+1} \subseteq Q_1^i$;

(2) *there exists $s \in K_{i+1}, s \neq 1$, such that $Q_1^{i+2} \subseteq Q_s^{i+1} \subseteq Q_1^i$;*

(3) $a\delta^{i+1} \leqslant d(x,y) \leqslant a_2\delta^i$, *where a is the constant in* (2.1) *and a_2 is the constant in* (d.5).

Proof. Let x and y be two points in X with $x \neq y$. Let $j_0 \in \mathbb{Z}$ be such that

$$\delta^{j_0+1} \leqslant 2d(x,y) < \delta^{j_0}. \tag{2.3}$$

Let $\preceq$ in $\mathcal{O}_\delta$ be an order associated to $\mathcal{N}$.

Let

$$i = \begin{cases} j_0+1, & \text{if } Q_1^{j_0+2} \subseteq Q_1^{j_0+1}, \\ j_0, & \text{if } Q_1^{j_0+2} \nsubseteq Q_1^{j_0+1}. \end{cases}$$

Let us check that this value of i satisfies (1), (2) and (3). Let us start by checking (1). If $i = j_0+1$, this is the case when $Q_1^{j_0+2} \subseteq Q_1^{j_0+1}$, there is nothing to prove since $Q_1^{i+1} = Q_1^{j_0+2}$ and $Q_1^i = Q_1^{j_0+1}$. On the other hand, if $i = j_0$, from (2.3) and (a) we have that $(j_0+1, 1) \preceq (j_0, 1)$, hence from (2.2) we get that $Q_1^{i+1} = Q_1^{j_0+1} \subseteq Q_1^{j_0} = Q_1^i$.

Since $x_1^i = x_1^{i+2} \in \mathcal{N}_i \cap \mathcal{N}_{i+2}$, applying Lemma 2.3 we get that there exists $s \in K_{i+1}$ such that

$$(i+2, 1) \preceq (i+1, s) \preceq (i, 1). \tag{2.4}$$

Then, from (2.2) we have that $Q_1^{i+2} \subseteq Q_s^{i+1} \subseteq Q_1^i$. So that in order to check (2) we only need to show that $s \neq 1$. If $i = j_0$ then $Q_1^{i+2} \nsubseteq Q_1^{i+1}$, so that $s \neq 1$ in this case. On the other hand, notice that since $\delta < 1/2$ we have that $\delta^{j_0+2} < \delta^{j_0+1}/2 \leqslant d(x,y) = d(x_1^{j_0+3}, x_1^{j_0+2})$. Then, from (b) we get that $(j_0+3, 1) \npreceq (j_0+2, 1)$, that is, $(i+2, 1) \npreceq (i+1, 1)$ when $i = j_0+1$. Now from (2.4) we get that $s \neq 1$ also in the case $i = j_0+1$.

Finally we shall see that (3) holds. Notice that from (1), (d.5) and (2.1) we have that $x_1^{i+1} \in B(x_1^i, a_2\delta^i)$. Thus, $d(x,y) = d(x_1^{i+1}, x_1^i) < a_2\delta^i$. On one hand, we have that $x_1^{i+1} \in Q_1^{i+1}$ and $x_1^i = x_1^{i+2} \in Q_s^{i+1}$ where $s \neq 1$ is the integer in (2). On the other hand, from (d.1), we have that $Q_1^{i+1} \cap Q_s^{i+1} = \emptyset$. Therefore, $d(x,y) = d(x_1^{i+1}, x_1^i) \geqslant a\delta^{i+1}$. □

Proof of Theorem 1.2. We shall prove that for every couple of points x and y in X with $x \neq y$ there exists a dyadic family $\mathcal{D}_{(x,y)} = \mathcal{D}_{\preceq}$ for some order $\preceq$ associated to $\mathcal{N}$ that such (s1) to (s3) hold. Thus, taking $\mathcal{S} = \{\mathcal{D}_{(x,y)} : x, y \in X, x \neq y\}$ we shall obtain the result.

Let x and y be are two points in X with $x \neq y$ and let i and s be the integers provided by Lemma 2.4 associated to the family $\mathcal{N}$ of δ^j-nets. Assume that $i = 2u$ for some $u \in \mathbb{Z}$. Consider the dyadic cubes $Q_y = Q_1^{i+1}, Q_x = Q_s^{i+1}$ and $Q = Q_1^i$. From (1) and (2) in Lemma 2.4, the definition of dyadic cubes given by (2.1) and (d.1), we obtain (s1) and (s2). Finally from (3) in Lemma 2.4 we get that $d(x,y)$ is comparable with $\delta^{\mathcal{J}(Q)}$ which is (s3). The case of i is odd we can proceed in similar way. □

3 Proof of Theorems 1.3 and 1.4

Let us first define what we mean by a Haar system associated to a dyadic family.

Definition 3.1. Let $\mathcal{D}$ be a dyadic family on (X, d, μ) such that $\mathcal{D} \in \mathfrak{D}(\delta)$. A system $\mathcal{H}_{\mathcal{D}}$ of simple Borel measurable real functions h on X is a Haar system associated to $\mathcal{D}$ if it satisfies

(h.1) for each $h \in \mathcal{H}_{\mathcal{D}}$ there exists a unique $j \in \mathbb{Z}$ and a cube $Q = Q_h \in \tilde{\mathcal{D}}^j$ such that $\{x \in X : h(x) \neq 0\} \subseteq Q$, and this property does not hold for any cube in $\mathcal{D}^{j+1}$;

(h.2) for every $Q \in \tilde{\mathcal{D}}$ there exist exactly $M_Q = \#(\mathcal{L}(Q)) - 1 \geqslant 1$ functions $h \in \mathcal{H}_{\mathcal{D}}$ such that (h.1) holds; we shall write $\mathcal{H}_Q$ to denote the set of all these functions h;

(h.3) for each $h \in \mathcal{H}_{\mathcal{D}}$ we have that $\int_X h d\mu = 0$;

(h.4) for each $Q \in \tilde{\mathcal{D}}$ let V_Q denote the vector space of all functions on Q which are constant on each $Q' \in \mathcal{L}(Q)$. Then the system $\{\frac{\chi_Q}{(\mu(Q))^{1/2}}\} \cup \mathcal{H}_Q$ is an orthonormal basis for V_Q.

It is easy to show, following the proof in [1] (see also [3]), that given $\mathcal{D}$ in $\mathfrak{D}(\delta)$ it is allways possible to construct Haar systems supported on the elements Q of $\tilde{\mathcal{D}}$. This means that there exist systems $\mathcal{H}_{\mathcal{D}}$ of functions h on X satisfying (h.1) to (h.4) for all $\mathcal{D}$ in $\mathfrak{D}(\delta)$. Observe also that from (d.9) we get that there exists a positive constant C such that

$$\|h\|_\infty \leqslant C\mu(Q_h)^{-1/2}, \tag{3.1}$$

for all $h \in \mathcal{H}_{\mathcal{D}}$. Here, as ussual, $\|f\|_\infty$ is the L^∞-norm of the function f which is defined as the μ-essential least upper bound of f.

We define $\mathcal{H}$ as the set of all Haar functions h that belong to some Haar system $\mathcal{H}_{\mathcal{D}}$ associated to some dyadic family $\mathcal{D} \in \mathcal{S}$ for some separating class $\mathcal{S}$. As already mentioned in the introduction, the proof of Theorem 1.3 requires the condition (1.3) only for a much smaller class of test functions than $\mathcal{H}$. In fact, given $\mathcal{D}$ in $\mathfrak{D}(\delta)$, $Q \in \tilde{\mathcal{D}}$ and $Q' \in \mathcal{L}(Q)$, the function

$$h_{Q,Q'} = \left(\frac{\mu(Q \setminus Q')}{\mu(Q)\mu(Q')}\right)^{1/2} \chi_{Q'} - \left(\frac{\mu(Q')}{\mu(Q)\mu(Q \setminus Q')}\right)^{1/2} \chi_{Q \setminus Q'} \tag{3.2}$$

is said to belong to $\mathcal{T}_{\mathcal{D}}$. Let us point out that when the standard Grahm-Schmidt orthonormalization algorithm is applied to the indicator functions of Q and $\#\mathcal{L}(Q) - 1$ cubes Q'' in $\mathcal{L}(Q)$, the first element in $\mathcal{H}_Q$ (see (h.4)) belongs to $\mathcal{T}_{\mathcal{D}}$. Hence, for each separating class $\mathcal{S}$ we have that

$$\bigcup_{\mathcal{D} \in \mathcal{S}} \mathcal{T}_{\mathcal{D}} \subseteq \mathcal{H}.$$

The following statement collects the main tools that we shall use in the proof of Theorem 1.3.

Proposition 3.2. *Let f be a locally integrable function on X and let $\mathcal{D}$ be a dyadic family in $\mathfrak{D}(\delta)$. Then*

(1) *for all $Q \in \mathcal{D}$ and every $Q' \in \mathcal{L}(Q)$ we have*

$$|m_Q(f) - m_{Q'}(f)| = \frac{\mu(Q \setminus Q')}{\mu(Q)} |m_{Q'}(f) - m_{Q \setminus Q'}(f)|;$$

(2) *for all $Q \in \tilde{\mathcal{D}}$ and every $h_{Q,Q'} \in \mathcal{T}_{\mathcal{D}}$, we have*

$$|\langle f, h_{Q,Q'} \rangle| = \left(\frac{\mu(Q')\mu(Q \setminus Q')}{\mu(Q)} \right)^{1/2} |m_{Q'}(f) - m_{Q \setminus Q'}(f)|.$$

Proof. Notice that if $Q = Q'$ then the equality in (1) is trivial. On the other hand, since for $Q \in \tilde{\mathcal{D}}$ we have $(\frac{1}{\mu(Q')} - \frac{1}{\mu(Q)})\mu(Q') = \frac{\mu(Q \setminus Q')}{\mu(Q)}$, we get

$$\begin{aligned} |m_{Q'}(f) - m_Q(f)| &= \left| \frac{1}{\mu(Q')} \int_{Q'} f(z) d\mu(z) - \frac{1}{\mu(Q)} \int_{Q' \cup (Q \setminus Q')} f(z) d\mu(z) \right| \\ &= \left| \left(\frac{1}{\mu(Q')} - \frac{1}{\mu(Q)} \right) \mu(Q') m_{Q'}(f) - \frac{\mu(Q \setminus Q')}{\mu(Q)} m_{Q \setminus Q'}(f) \right| \\ &= \frac{\mu(Q \setminus Q')}{\mu(Q)} |m_{Q'}(f) - m_{Q \setminus Q'}(f)|, \end{aligned}$$

and (1) hold.

The proof of (2) is a consequence of (3.2). □

Proof of Theorem 1.3. The proof of this theorem follows the lines of the case in the real line given in [2]. The key tools are the Lebesgue differentiation theorem and Proposition 3.2. Let $\mathcal{S}$ be a separating class in $\mathfrak{D}(\delta)$ and let f be a function in $\mathcal{C}(\alpha, S)$. We shall prove that f equals, with the possible exception of a set of measure zero, a function in $\mathrm{Lip}(\alpha)$. Let x and y be two points in X such that $x \neq y$ and both are Lebesgue points of f. Since $\mathcal{S}$ is a separating class in $\mathfrak{D}(\delta)$, there exist $\mathcal{D} \in \mathcal{S}$, $Q \in \mathcal{D}$ and Q_x, Q_y in $\mathcal{L}(Q)$ such that (s1) to (s3) in Definition 2.2 hold.

Now, we consider the two sequences $(Q_n : n \in \mathbb{Z}_0^+)$ and $(R_n : n \in \mathbb{Z}_0^+)$ of dyadic cubes defined as follows. Here $\mathbb{Z}_0^+$ denotes the set of all nonnegative integers. For $n = 0$ we define $Q_0 = Q$. For $n = 1$ we define $Q_1 = Q_x$. In general, for each integer $n \geqslant 2$, we take $Q_n \in \mathcal{L}(Q_{n-1})$ such that $x \in \overline{Q_n}$ where $\overline{Q_n}$ is the closure of Q_n. Notice that from (d.2) the dyadic cube Q_n is well defined for each n. Analogously we define $R_0 = Q$, $R_1 = Q_y$, and in general, for each integer $n \geqslant 2$, $R_n \in \mathcal{L}(R_{n-1})$ such that $y \in \overline{R_n}$. It is important to observe that, from the construction of these sequences and since $Q_1 \cap R_1 = \emptyset$, then $Q_n \cap R_n = \emptyset$ for all positive integer n.

Associated to the sequences $(Q_n : n \in \mathbb{Z}_0^+)$ and $(R_n : n \in \mathbb{Z}_0^+)$ we consider the two following sequences $(h_n : n \in \mathbb{Z}_0^+)$ and $(\tilde{h}_n : n \in \mathbb{Z}_0^+)$ of test functions of Haar type in $\mathcal{T}_{\mathcal{D}}$. For each $n \in \mathbb{Z}_0^+$ we take $h_n = h_{Q_n, Q_{n+1}}$ and $\tilde{h}_n = h_{R_n, R_{n+1}}$ as in (3.2). Thus,

$$\begin{aligned} |f(x) - f(y)| &\leqslant |f(x) - m_{Q_k}(f) - f(y) + m_{R_k}(f)| + |m_{Q_0}(f) - m_{R_0}(f)| \\ &\quad + \sum_{i=1}^{k} |m_{Q_i}(f) - m_{Q_{i-1}}(f)| + \sum_{i=0}^{k-1} |m_{R_i}(f) - m_{R_{i+1}}(f)| \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}. \end{aligned}$$

Since $Q_0 = R_0 = Q$, we have that $\mathrm{II} = 0$. For I, since both x and y are differentiation points for f, we can choose k large enough in order to get that $\mathrm{I} \leqslant d(x, y)^\alpha$. Here we have used Lebesgue differentiation through dyadic sets which can be applied from (d.5) (see also [4]).

Since III and IV are similar we shall only deal with III. If $Q_{i-1} = Q_i$ then we have that $|m_{Q_i}(f) - m_{Q_{i-1}}(f)| = 0$. Set $I_1 = \{i \in \mathbb{Z}^+ : Q_{i-1} \in \tilde{\mathcal{D}}\}$. Then,

$$\mathrm{III} = \sum_{i \in I_1} |m_{Q_i}(f) - m_{Q_{i-1}}(f)|.$$

From Proposition 3.2 and (d.9) we have that

$$\begin{aligned}|m_{Q_i}(f) - m_{Q_{i-1}}(f)| &= \frac{\mu(Q_{i-1}\setminus Q_i)}{\mu(Q_{i-1})}|m_{Q_i}(f) - m_{Q_{i-1}\setminus Q_i}(f)| \\ &= \left(\frac{\mu(Q_{i-1}\setminus Q_i)}{\mu(Q_i)\mu(Q_{i-1})}\right)^{1/2}\left|\int_X f(z)h_{i-1}(z)d\mu(z)\right| \\ &\leqslant C\mu(Q_i)^{-1/2}\left|\int_X f(z)h_{i-1}(z)d\mu(z)\right|,\end{aligned}$$

for all $i \in I_1$. Then, from (1.3) and (d.9) we obtain that

$$|m_{Q_i}(f) - m_{Q_{i-1}}(f)| \leqslant C\mu(Q_i)^{-1/2}\operatorname{diam}(Q_{i-1})^\alpha \mu(Q_{i-1})^{1/2} \leqslant C \operatorname{diam}(Q_{i-1})^\alpha,$$

for all $i \in I_1$.

Notice that, since $Q \in \mathcal{D}^{\mathcal{J}(Q)}$, we have that Q_n belong to $\mathcal{D}^{\mathcal{J}(Q)+n}$ for each $n \in \mathbb{Z}_0^+$. Thus, from (d.5) we have that $\operatorname{diam}(Q_n)$ is bounded by $a_2\delta^{\mathcal{J}(Q)+n}$, for each $n \in \mathbb{Z}_0^+$. Now, since $0 < \delta < 1$ and $\alpha > 0$ we have that

$$\begin{aligned}\mathrm{III} &\leqslant C\sum_{i\in I_1}\operatorname{diam}(Q_{i-1})^\alpha \\ &\leqslant (a_2)^\alpha C\sum_{i\in I_1}(\delta^{\mathcal{J}(Q)+i-1})^\alpha \\ &\leqslant (a_2)^\alpha C(\delta^{\mathcal{J}(Q)})^\alpha\sum_{i\in\mathbb{Z}^+}(\delta^{i-1})^\alpha \\ &\leqslant (a_2)^\alpha C(\delta^{\mathcal{J}(Q)})^\alpha.\end{aligned}$$

On the other hand, since the dyadic family $\mathcal{D}$ belongs to the separating class $\mathcal{S}$ in $\mathfrak{D}(\delta)$, from (s3) we have that $(\delta^{\mathcal{J}(Q)})^\alpha \leqslant Cd(x,y)^\alpha$. Similarly, using $\tilde{h}_n$, we obtain that

$$\mathrm{IV} \leqslant Cd(x,y)^\alpha.$$

Hence

$$|f(x) - f(y)| \leqslant Cd(x,y)^\alpha,$$

for almost every x and y in X. Thus, redefining the function f on a set of null measure, we have that $f \in \operatorname{Lip}(\alpha)$. □

Proof of Theorem 1.4. First notice that for every ball B in X and all x in B we have that the inequality

$$\left(\int_B |f(z) - m_B(f)|^q\, d\mu(z)\right)^{1/q} \leqslant 2\left(\int_B |f(z) - f(x)|^q\, d\mu(z)\right)^{1/q}, \tag{3.3}$$

holds for each $1 \leqslant q < \infty$.

Thus, if $f \in \operatorname{Lip}(\alpha)$ we have that

$$\left(\int_B |f(z) - m_B(f)|^q d\mu(z)\right)^{1/q} \leqslant 2\left(\int_B d(z,x)^\alpha d\mu(z)\right)^{1/q} \leqslant 2\operatorname{diam}(B)^\alpha\ \mu(B)^{1/q},$$

for all ball B in X and therefore $f \in \operatorname{Lip}(\alpha, q)$, $1 \leqslant q < \infty$. Now, we shall prove that $\operatorname{Lip}(\alpha, q), 1 \leqslant q < \infty$ implies $\mathcal{C}(\alpha, S)$ for all separating class $\mathcal{S}$ in $\mathfrak{D}(\delta)$. It is easy to see, from Hölder inequality, that $\operatorname{Lip}(\alpha, q), 1 \leqslant q < \infty$ implies $\operatorname{Lip}(\alpha, 1)$. Therefore we only have to prove that $\operatorname{Lip}(\alpha, 1)$ implies $\mathcal{C}(\alpha, S)$. Notice that if $f \in \operatorname{Lip}(\alpha, 1)$ then the inequality

$$\frac{1}{\mu(Q)}\int_Q |f(x) - m_Q(f)|d\mu(x) \leqslant C\operatorname{diam}(Q)^\alpha \tag{3.4}$$

holds for every dyadic cube Q in $\mathcal{D}$ and all dyadic family $\mathcal{D}$ in $\mathfrak{D}(\delta)$. Let $\mathcal{S}$ be a separating class in $\mathfrak{D}(\delta)$ and $h \in \mathcal{T}_{\mathcal{D}}$, from (h.1), (h.3), (3.1) and the inequality (3.4) we have that

$$\begin{aligned}
|\langle f, h\rangle| &= \left|\int_{Q_h} (f(z) - m_{Q_h}(f))\, h(z) d\mu(z)\right| \\
&\leqslant \int_{Q_h} |f(z) - m_{Q_h}(f)|\, |h(z)|\, d\mu(z) \\
&\leqslant C\mu(Q_h)^{-1/2} \int_{Q_h} |f(z) - m_{Q_h}(f)|\, d\mu(z) \\
&\leqslant C \operatorname{diam}(Q_h)^{\alpha}\, \mu(Q_h)^{1/2},
\end{aligned}$$

which implies that $f \in \mathcal{C}(\alpha, S)$. Finally, from Theorem 1.3 we have the inclusion $\mathcal{C}(\alpha, S) \subseteq \mathrm{Lip}(\alpha)$ and we are done. □

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