NEGATIVE RICCI CURVATURE ON SOME NON-SOLVABLE LIE GROUPS II

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Abstract. We construct many examples of Lie groups G with compact Levi factor, admitting a left-invariant metric with negative Ricci curvature. We start with a Lie algebra which is a semidirect product $\mathfrak{g} = (\mathfrak{h} \oplus \mathfrak{a}) \ltimes \mathfrak{n}$ and we obtain examples where $\mathfrak{h} = \mathfrak{su}(n)$ and $\mathfrak{so}(n)$, $\mathfrak{a}$ is one dimensional and $\mathfrak{n}$ is a representation of $\mathfrak{h}$ in the space of homogeneous polynomials. In the case $\mathfrak{h} = \mathfrak{su}(2)$ we get a more general construction where $\mathfrak{n}$ can be any nilpotent Lie algebra where $\mathfrak{su}(2)$ acts. We also prove a general result in the case when $\mathfrak{h}$ is a semisimple Lie algebra of non-compact type.

1. Introduction

In this paper we are interested in homogeneous negative Ricci curvature, as a continuation of the work started in [15]. It is proved there that if V is a non-trivial real representation of $\mathfrak{su}(2)$ extended to $\mathfrak{u}(2)$ by letting the center act as multiples of the identity, then the Lie algebra $\mathfrak{u}(2) \ltimes V$ admits an inner product with negative Ricci curvature. Before that, the only Lie groups in the literature that were known to admit a left-invariant metric with negative Ricci curvature were either semisimple (see [2], [3]) or solvable (see [1], [12], [13]). We refer to [12] or [15] for a more detailed summary of the known results on negative Ricci curvature in the homogenous case.

Another question that arises naturally in this context is whether the existence of a left-invariant metric with negative Ricci curvature impose topological obstructions on a Lie group. First recall that if K is a maximal compact subgroup of a Lie group G, then all the nontrivial topology of G is in K, in the sense that as a differentiable manifold, G is the product of K by a euclidean space. Therefore, from the semisimple examples in [3], it follows that it is possible to get the topologies of almost all the compact simple Lie groups with the following exceptions:

$$\mathrm{SU}(2), \quad \mathrm{SU}(3), \quad \mathrm{SO}(5), \quad \mathrm{SO}(7), \quad \mathrm{Sp}(3), \quad \mathrm{Sp}(4), \quad \mathrm{Sp}(5), \quad G_2.$$

See the remark after [3, Theorem 2.1]. Recall that In [15] we obtained the topology of SU(2).

In this work, we extend the results in [15] in many ways finding families of examples of Lie groups admitting a metric with negative Ricci curvature. We construct Lie algebras as semidirect products $\mathfrak{g} = (\mathfrak{a} \oplus \mathfrak{k}) \ltimes \mathfrak{n}$ where $\mathfrak{k}$ is a compact semisimple Lie algebra, $\mathfrak{a}$ is abelian and $\mathfrak{n}$ is a nilpotent Lie algebra. Note that if G is the simply connected Lie group with Lie algebra $\mathfrak{g}$ then the topology of G is in the subgroup K with Lie algebra $\mathfrak{k}$. We obtain examples where $\mathfrak{k}$ is $\mathfrak{su}(n)$, $n \geq 3$ and $\mathfrak{so}(m)$ for $m \geq 3$ and therefore we get, in particular, the topologies of SU(3), SO(5) and SO(7) that does not follow from the semisimple examples.

First, we consider the representations of $\mathfrak{su}(m)$ on the space of complex-valued polynomials on $\mathbb{C}^m$.

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Theorem 1.1. *Let $V = \mathcal{P}_{m,n}(\mathbb{C})$, be the usual real representation of $\mathfrak{su}(m)$ on the space of complex homogeneous polynomials of degree n in m variables extended to $\mathfrak{u}(m)$ by letting the center act as multiples of the identity. Hence the Lie algebra $\mathfrak{u}(m) \ltimes V$ admits a inner product with negative Ricci curvature for all $n, m \geq 2$.*

Using the same methods, we also consider the case of $\mathfrak{so}(m)$ with the standard representation on the space of polynomials.

Theorem 1.2. *Let V be the standard real representation of $\mathfrak{so}(m)$ on the space of complex-valued homogeneous polynomials of degree n on $\mathbb{R}^m$. If $\mathfrak{g} = (\mathbb{R}Z \oplus \mathfrak{so}(m)) \ltimes V$ is the Lie algebra such that $[Z, \mathfrak{so}(m)] = 0$ and Z acts as the identity on V, then $\mathfrak{g}$ admits an inner product with negative Ricci curvatures for any $n, m \geq 3$.*

On the other hand, we consider any algebra with Levi factor $\mathfrak{su}(2)$ and show that under some condition, $\mathfrak{g}$ admits an inner product with negative Ricci curvature.

Theorem 1.3. *Let $\mathfrak{g} = (\mathbb{R}Z \oplus \mathfrak{su}(2)) \ltimes \mathfrak{n}$ be a Lie algebra where $\mathfrak{n}$ is any nilpotent Lie algebra and $[Z, \mathfrak{su}(2)] = 0$. If $[\mathfrak{su}(2), \mathfrak{n}] \neq 0$ and $\operatorname{ad} Z$ is a positive multiple of the identity on each $\mathfrak{su}(2)$-irreducible subspace of $\mathfrak{n}$, then $\mathfrak{g}$ admits an inner product with negative Ricci curvature.*

Another case where we can apply our method is when one starts with the non-compact dual of $\mathfrak{su}(m)$, $\mathfrak{sl}(m, \mathbb{R})$. In this way, by the Weyl's unitary trick, for each representation of $\mathfrak{su}(m)$ one gets a representation of $\mathfrak{sl}(m, \mathbb{R})$. We then show that $(\mathbb{R}Z \oplus \mathfrak{sl}(m, \mathbb{R})) \ltimes V$ admits an inner product with negative Ricci curvature for $m \geq 2$, where V is the representation on the complex homogeneous polynomials in m variables viewed as real. Although this comes from a continuous argument, in each case one can actually have explicitly the inner product.

As a generalization of this we consider Lie algebras $(\mathfrak{a} \oplus \mathfrak{h}) \ltimes \mathfrak{n}$ where $\mathfrak{n}$ is nilpotent, $\mathfrak{a}$ is abelian and $\mathfrak{h}$ is semisimple of non-compact type, and obtain the following existence result.

Theorem 1.4. *Let $\mathfrak{g} = (\mathfrak{a} \oplus \mathfrak{h}) \ltimes \mathfrak{n}$ be a Lie algebra where $\mathfrak{h}$ is a semisimple Lie algebra with no compact factors, $\mathfrak{n}$ is nilpotent and $[\mathfrak{a}, \mathfrak{a} \oplus \mathfrak{h}] = 0$. If in addition*

- *$\mathfrak{n}$ admits an inner product such that $\operatorname{ad} A|_{\mathfrak{n}}$ are semisimple operators (over $\mathbb{C}$) for any $A \in \mathfrak{a}$,*
- *no $\operatorname{ad} A|_{\mathfrak{n}}$ has all its eigenvalues purely imaginary,*
- *there exists an element A in $\mathfrak{a}$ such that all the eigenvalues of $\operatorname{ad} A|_{\mathfrak{n}}$ have positive real parts,*
- *$\mathfrak{h}$ admits an inner product with $\operatorname{Ric} < 0$ such that there exists a Cartan decomposition $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}$ with $\mathfrak{k}$ orthogonal to $\mathfrak{p}$,*

then $\mathfrak{g}$ admits an inner product with negative Ricci curvature.

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2. Preliminaries and notation

2.1. **Lie algebras.** We recall some background from [15] we will need along the paper. Let $\mathfrak{g} = (\mathbb{R}^m, [\cdot, \cdot])$ be a Lie algebra of dimension m, that is, the underlying linear space of $\mathfrak{g}$ is (identified with) $\mathbb{R}^m$ and $[\cdot, \cdot]$ belongs to the space of Lie brackets $\mathcal{L}_m \subset \Lambda^2(\mathbb{R}^m)^* \otimes \mathbb{R}^m$, defined as

$$\mathcal{L}_m := \{\mu : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m : \mu \text{ bilinear, skew-symmetric and satisfies Jacobi}\}.$$

$\mathcal{L}_m$ is also called the variety of Lie algebras of dimension m. We consider the following action of $\mathrm{GL}_m(\mathbb{R})$ on $\mathcal{L}_m$:

$$(g\cdot\mu)(X,Y)=g\mu(g^{-1}X,g^{-1}Y),\qquad g\in \mathrm{GL}_m(\mathbb{R}),\ \mu\in\mathcal{L}_m,\ X,Y\in\mathfrak{g}.$$

Note that $(\mathbb{R}^m,\mu)$ is isomorphic to $(\mathbb{R}^m,g\cdot\mu)$ for any $g\in\mathrm{GL}_m(\mathbb{R})$, though, $(\mathbb{R}^m,\mu)$ is not isomorphic to $(\mathbb{R}^m,\mu_o)$ for μ_o in the boundary of the orbit $\mathrm{GL}_m(\mathbb{R})\cdot\mu$. Since $\mathcal{L}_m\subset\Lambda^2\mathbb{R}^{m\star}\otimes\mathbb{R}^m$ is defined by polynomials equations, any μ_0 in the closure is also a Lie bracket. We will say that μ_o is a *degeneration* of μ or that μ degenerates to μ_o if $\mu_o\in\overline{\mathrm{GL}_m(\mathbb{R})\cdot\mu}$. Note that by continuity, many of the properties of μ_o are shared by μ. In particular if $(\mathbb{R}^m,\mu_o)$ admits a metric with negative (or positive) sectional or Ricci curvature, so does $(\mathbb{R}^m,\mu)$ (see [14, Remark 6.2] or [12, Proposition 1]).

Proposition 2.1. *Suppose $\mu,\lambda\in\mathcal{L}_m$ and that λ is in the closure of the orbit $\mathrm{GL}_m(\mathbb{R})\cdot\mu$. If the Lie algebra $(\mathbb{R}^m,\lambda)$ admits an inner product of negative Ricci curvature, then so does the Lie algebra $(\mathbb{R}^m,\mu)$.*

Moreover, if we fix an inner product on $\mathfrak{g}=(\mathbb{R}^m,\mu)$, or equivalently, an orthonormal basis, then the orbit $\mathrm{GL}(\mathfrak{g})\cdot\mu$ parameterizes, from a different point of view, the set of all inner products on $\mathfrak{g}$. Indeed,

$$(\mathfrak{g},g\cdot\mu,\langle\cdot,\cdot\rangle)\text{ is isometric to }(\mathfrak{g},\mu,\langle g\cdot,g\cdot\rangle)\ \text{ for any } g\in\mathrm{GL}(\mathfrak{g}).\tag{1}$$

Let $(\mathfrak{g},[\cdot,\cdot],\langle\cdot,\cdot\rangle)$ be a metric Lie algebra and $H\in\mathfrak{g}$ the only element such that $\langle H,X\rangle=\operatorname{tr}\operatorname{ad}X$ for any $X\in\mathfrak{g}$, usually called the mean curvature vector, and let B denotes the symmetric map defined by the Killing form of $(\mathfrak{g},[\cdot,\cdot])$ (i.e. $\langle BX,X\rangle=\operatorname{tr}(\operatorname{ad}X)^2$). The Ricci operator of $(\mathfrak{g},[\cdot,\cdot],\langle\cdot,\cdot\rangle)$ is given by (see for instance [11, Appendix]):

$$\operatorname{Ric}=M-\tfrac12 B-S(\operatorname{ad}H),\tag{2}$$

where, $S(\operatorname{ad}H)=\frac12(\operatorname{ad}H+(\operatorname{ad}H)^t)$ is the symmetric part of $\operatorname{ad}H$ and M is the symmetric operator defined by

$$\langle MX,X\rangle=-\tfrac12\sum\langle[X,X_i],X_j\rangle^2+\tfrac14\sum\langle[X_i,X_j],X\rangle^2,\qquad\forall X\in\mathfrak{g},\tag{3}$$

where $\{X_i\}$ is any orthonormal basis of $(\mathfrak{g},\langle\cdot,\cdot\rangle)$. Note that if $\mathfrak{g}$ is nilpotent, then $\operatorname{Ric}=M$.

If $\mathfrak{g}$ is a solvable Lie algebra and we consider an orthogonal decomposition

$$\mathfrak{g}=\mathfrak{a}\oplus\mathfrak{n},\tag{4}$$

where $\mathfrak{n}$ is the nilradical of $\mathfrak{g}$ (i.e. maximal nilpotent ideal), the expression of Ric is much simpler when $\mathfrak{a}$ is abelian (see [10]). Indeed, we get

$$\begin{aligned}
&\langle\operatorname{Ric}A,A\rangle=-\operatorname{tr}S(\operatorname{ad}A|_{\mathfrak{n}})^2,\\
&\langle\operatorname{Ric}A,X\rangle=-\tfrac12\operatorname{tr}(\operatorname{ad}A|_{\mathfrak{n}})^t\operatorname{ad}X|_{\mathfrak{n}}\\
&\langle\operatorname{Ric}X,X\rangle=-\tfrac12\sum\langle[X,X_i],X_j\rangle^2+\tfrac14\sum\langle[X_i,X_j],X\rangle^2\\
&\qquad\qquad+\tfrac12\sum\langle[\operatorname{ad}A_i|_{\mathfrak{n}},(\operatorname{ad}A_i|_{\mathfrak{n}})^t]X,X\rangle-\langle[H,X],X\rangle,
\end{aligned}\tag{5}$$

for all $A\in\mathfrak{a}$ and $X\in\mathfrak{n}$, where $\{A_i\}$, $\{X_i\}$, are any orthonormal basis of $\mathfrak{a}$ and $\mathfrak{n}$, respectively. If in addition $\operatorname{ad}A$ are normal operators for all $A\in\mathfrak{a}$, then we get that $\operatorname{tr}(\operatorname{ad}A|_{\mathfrak{n}})^t\operatorname{ad}X|_{\mathfrak{n}}=0$ (see [10, (25) and Prop. 4.3]) and therefore

$$\begin{aligned}
&\langle\operatorname{Ric}A,A\rangle=-\operatorname{tr}S(\operatorname{ad}A|_{\mathfrak{n}})^2,\qquad\langle\operatorname{Ric}A,X\rangle=0\\
&\langle\operatorname{Ric}X,X\rangle=-\tfrac12\sum\langle[X,X_i],X_j\rangle^2+\tfrac14\sum\langle[X_i,X_j],X\rangle^2-\langle[H,X],X\rangle.
\end{aligned}\tag{6}$$

2.2. **Some representations of $\mathfrak{u}(m)$.** For each $n \geq 2$ let (π_n, V_n) be the representation of $\mathrm{SU}(m)$ where $V_n = \mathcal{P}_{m,n}(\mathbb{C})$ is the space of homogeneous polynomials in m variables of degree n seen as a real vector space and the action is given by

$$(\pi_n(g)P)(z_1, \dots, z_n) = P(g^{-1}\begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}).$$

This gives us, by differentiation, a representation of its Lie algebra $\mathfrak{su}(m)$ that will also be denoted by (π_n, V_n). Moreover, since $\mathfrak{u}(m) = \mathfrak{su}(m) \oplus \mathbb{R}Z$, $Z = \imath Id$, all the above representations can be extended to $\mathfrak{u}(m)$ by letting Z act as the identity map on V_n. We will denote these representations of $\mathfrak{u}(m)$ also by (V_n, π_n). Note that these representations are not irreducible in general.

Consider $\mathfrak{g} = \mathfrak{su}(m)$ as a Lie subalgebra of $\mathfrak{gl}(m)$ with basis

$$\beta = \{H_1, \dots H_{m-1},\, X_{i,j},\, Y_{i,j}, \quad 1 \leq i < j \leq m\}.$$

Here,

$$H_l = \imath(E_{i,i} - E_{i+1,i+1}),\ l = 1, \dots m-1,$$

$$X_{i,j} = E_{i,j} - E_{j,i},\ 1 \leq i < j \leq m, \tag{7}$$

$$Y_{i,j} = \imath(E_{i,j} + E_{j,i}),\ 1 \leq i < j \leq m,$$

where, as usual, $E_{i,j}$ is the $m \times m$ matrix with zero entries except for the i,j which is 1. We note that this basis is constructed using the root vectors as in [5, Chap. III, Theorem 6.3 (2)], or in [7] pp. 353. In fact, the complexification of $\mathfrak{g}$, $\mathfrak{sl}(m,\mathbb{C})$ is type A_{m-1}, has roots

$$\alpha_{i,j}(H) = e_i(H) - e_j(H), \qquad 1 \leq i \neq j \leq m, \tag{8}$$

where $e_k(\sum_{l=1}^{m} h_l E_{l,l}) = h_k$ and the corresponding roots vectors are $E_{i,j}$ (see [5] pp. 187). Hence $\mathfrak{su}(m)$ decomposes as $\mathfrak{k} \oplus \imath\mathfrak{p}$ where $\mathfrak{k} = \mathrm{Span}\{\sum X_{i,j}\}$, $\mathfrak{p} = \mathrm{Span}\{\sum \imath H_l \oplus \sum \imath Y_{i,j}\}$ and $\mathfrak{sl}(m,\mathbb{R}) = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition of the non-compact dual of $\mathfrak{su}(m)$, $\mathfrak{sl}(m,\mathbb{R})$ (see [5] V, §2).

Let us fix a basis of V_n,

$$\beta_1 = \{p_{j_1,\dots,j_m},\ \imath p_{j_1,\dots,j_m}, \quad j_i \in \mathbb{N}_0,\ \ j_1 + \dots + j_m = n\}. \tag{9}$$

where $p_{j_1,\dots,j_m} = z_1^{j_1} \dots z_m^{j_m} \in \mathcal{P}_{m,n}(\mathbb{C})$. Note that dimension of V_n is $d = 2\binom{n+m-1}{m-1}$. Concerning the action, to get explicit formulas we use the fact that the algebra is acting by derivations and

$$H_l \cdot z_k = \begin{cases} -\imath z_k, & k = l, \\ \imath z_k & k = l+1, \\ 0, & k \neq l, l+1, \end{cases} \quad X_{i,j} \cdot z_k = \begin{cases} -z_j, & k = i, \\ z_i, & k = j, \\ 0, & k \neq i,j, \end{cases} \quad Y_{i,j} \cdot z_k = \begin{cases} -\imath z_j, & k = i, \\ -\imath z_i, & k = j, \\ 0, & k \neq i,j. \end{cases}$$

In this way, we get for example that for $s = 1, \imath$

$$X_{i,j} \cdot s z_k^n = \begin{cases} -n\, s z_i^{n-1} z_j, & k = i, \\ n\, s z_j^{n-1} z_i, & k = j, \\ 0, & k \neq i,j. \end{cases}$$

Note that the subset

$$\mathcal{S} = \{z_k^n, \imath z_k^n,\ \ k = 1, \dots m\} \subset \beta_1, \tag{10}$$

has the property that for every $p \in \mathcal{S}$, $H_l \cdot p \in \mathrm{Span}(\mathcal{S})$ and when they are non zero $X_{i,j} \cdot p \notin \mathrm{Span}(\mathcal{S})$ and $Y_{i,j} \cdot p \notin \mathrm{Span}(\mathcal{S})$. It is easy to see that for $n = 1$, $\mathcal{S} = \beta_1$.

3. Ricci negative inner product on $\mathfrak{u}(m) \ltimes V_n$

Using the same ideas as in [15] we will show in the following that $\mathfrak{u}(m) \ltimes V_n$ degenerates into a solvable Lie algebra that admits a inner product with negatively defined Ricci operator and hence so does the starting algebra. Note that since the case of $m = 2$ have already been consider in [15], we may assume that $m \geq 3$.

Let $\mathfrak{g} = \mathfrak{u}(m) \ltimes V_n$ and for each $t > 0$ define $\phi_t \in \mathfrak{gl}(\mathfrak{g})$ such that

$$\begin{aligned} &\phi_t(Z) = Z, \qquad \phi_t(H_l) = H_l,\ l = 1, \dots m-1, \\ &\phi_t(X_{i,j}) = tX_{i,j},\ 1 \leq i < j \leq m, \qquad \phi_t(Y_{i,j}) = tY_{i,j},\ 1 \leq i < j \leq m, \\ &\phi_t(sp_{j_1,\dots,j_m}) = \begin{cases} t\, sp_{j_1,\dots,j_m}, & \text{if } j_l = n \ \text{ for some } l, s = 1, \imath, \\ t^2\, sp_{j_1,\dots,j_m}, & \text{if } j_l \neq n \ \forall\, l, s = 1, \imath. \end{cases} \end{aligned} \tag{11}$$

Note that in V_n we get $\phi_t(p) = t\,p$ if $p \in \mathcal{S}$ and $\phi_t(p) = t^2\, p$ if $p \notin \mathcal{S}$.

We have that $[\cdot,\cdot]_t = \phi_t.[\cdot,\cdot]$ is given by

$$\begin{aligned} &[H_l, X_{i,j}]_t = [H_l, X_{i,j}],\ \ [H_l, Y_{i,j}]_t = [H_l, Y_{i,j}],\ \ \forall\, i,j,l, \\ &[X_{i,j}, Y_{k,l}]_t = -\tfrac{1}{t^\epsilon}[X_{i,j}, Y_{k,l}],\ \ \epsilon = 1 \text{ if } i,j \neq k,l, \epsilon = 2 \text{ if } i,j = k,l \\ &[Y_{i,j}, Y_{k,l}]_t = -\tfrac{1}{t}[Y_{i,j}, Y_{i,j}],\ \ [X_{i,j}, X_{k,l}]_t = -\tfrac{1}{t}[X_{i,j}, X_{k,l}],\ \ \forall\, i,j,l, \\ &[Z, p]_t = p, \quad [H_l, p]_t = [H_l, p] \quad \forall\, p \in \beta_1, \forall\, l, \\ &[X_{i,j}, p]_t = [X_{i,j}, p], \quad [Y_{i,j}, p]_t = [Y_{i,j}, p],\ \ \forall\, i,j, \forall\, p \in \mathcal{S}, \\ &[X_{i,j}, p]_t = \tfrac{1}{t^2}[X_{i,j}, p],\ \ [Y_{i,j}, p]_t = \tfrac{1}{t^2}[Y_{i,j}, p], \forall\, i,j, \forall\, p \notin \mathcal{S}, \end{aligned} \tag{12}$$

To see this we can calculate the brackets using the explicit matrix realization (see also [7] pp. 353) or one can use the relations

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k},\ \ [\mathfrak{k}, \imath\mathfrak{p}] \subset \imath\mathfrak{p},\ \ [\imath\mathfrak{p}, \imath\mathfrak{p}] \subset \mathfrak{k}.$$

Therefore, $\mu = \lim_{t\to\infty} [\cdot,\cdot]_t = \lim_{t\to\infty} \phi_t.[\cdot,\cdot]$ is well defined and it is given by

$$\begin{aligned} &\mu(H_l, X_{i,j}) = [H_l, X_{i,j}],\ \ \mu(H_l, Y_{i,j}) = [H_l, Y_{i,j}],\ \ \forall\, i,j,l, \\ &\mu(Z, p) = p, \quad \mu(H_l, p) = [H_l, p] \quad \forall\, p \in \beta_1, \forall\, l, \\ &\mu(X_{i,j}, p) = [X_{i,j}, p], \quad \mu(Y_{i,j}, p) = [Y_{i,j}, p],\ \ \forall\, i,j, \forall\, p \in S, \end{aligned} \tag{13}$$

As in the $n = 2$ case, $\mathfrak{h}_\infty = (\mathbb{R}^r, \mu)$, where $r = m^2 + d$ is a solvable Lie algebra with nilradical

$$\mathfrak{n} = \mathrm{Span}\{X_{i,j}, Y_{i,j}, \beta_1,\ \ 1 \leq i < j \leq m\}$$

whose center $\mathfrak{z}$ is contained in V_n and therefore it satisfies the first condition of [12, Theorem 2] but not the second one.

Note that we have simplify the notation since $\mathfrak{h}, \mathfrak{h}_\infty, [\cdot,\cdot], \mu_t, \mu$, etc depend on n and m.

Lemma 3.1. *If $n \neq 1$, $\mathfrak{h}_\infty$ admits an inner product with negative Ricci curvature.*

Proof. Let $\langle\cdot,\cdot\rangle$ be the inner product that makes

$$\beta = \{Z, H_l, X_{i,j}, Y_{i,j}, \beta_1\, 1 \leq m-1, 1 \leq i < j \leq m\}$$

an orthonormal basis. First note that $\mathfrak{h}_\infty = \mathfrak{a} \oplus \mathfrak{n}$ as in (4) where $\mathfrak{a} = \mathrm{Span}\{Z, H_l, 1 \le l \le m-1\}$ is abelian and for each $1 \le l \le m-1$, $\mathrm{ad}_\mu(H_l)_{|\mathfrak{n}}$ is a skew-symmetric operator. Indeed, if $\mu(H_l, X_{i,j}) \ne 0$ then $l \in \{i, i-1, j, j-1\}$ and in that case we have

$$\mu(H_l, X_{i,j}) = [H_l, X_{i,j}] = \pm Y_{i,j} \ \text{ and } \ \mu(H_l, Y_{i,j}) = [H_l, Y_{i,j}] = \mp X_{i,j}.$$

On the other hand if $\mu(H_l, sp_{j_1,\dots,j_m}) \ne 0$ for $s = 1, \imath$ then $j_l \ne 0 \ne j_{l+1}$ and

$$\mu(H_l, p_{j_1,\dots,j_m}) = [H_l, p_{j_1,\dots,j_m}] = H_l \cdot p_{j_1,\dots,j_m} = -(j_l - j_{l+1})\imath p_{j_1,\dots,j_m}$$

$$\mu(H_l, \imath p_{j_1,\dots,j_m}) = [H_l, \imath p_{j_1,\dots,j_m}] = H_l \cdot \imath p_{j_1,\dots,j_m} = (j_l - j_{l+1}) p_{j_1,\dots,j_m}.$$

It is easy to check that the mean curvature vector is $H = (\dim V_n)Z = d\,Z$ and since $\mathfrak{a}$ is acting by normal operators on $\mathfrak{n}$, $\langle \mathrm{Ric}_\mu \mathfrak{a}, \mathfrak{n}\rangle = 0$ (see (5)). Finally, straightforward calculation shows that β is a basis of eigenvectors of Ric_μ and

$$\begin{aligned}
&\langle \mathrm{Ric}_\mu Z, Z\rangle = -d, \qquad \langle \mathrm{Ric}_\mu H_l, H_l\rangle = 0\\
&\langle \mathrm{Ric}_\mu X_{i,j}, X_{i,j}\rangle = \langle \mathrm{Ric}_\mu Y_{i,j}, Y_{i,j}\rangle = -2n^2,\\
&\langle \mathrm{Ric}_\mu p, p\rangle = -(m-1)n^2 - d, \quad p \in \mathcal{S}\\
&\langle \mathrm{Ric}_\mu p, p\rangle = k_n n^2 - d,\ p = sp_{j_1,\dots,j_m} \text{ and } j_l = n-1 \text{ for some } l, s = 1, \imath,\\
&\langle \mathrm{Ric}_\mu p, p\rangle = -d,\ p = sp_{j_1,\dots,j_m} \text{ and } j_l \ne n, n-1, \forall\, l, s = 1, \imath, (n \ge 3),
\end{aligned}$$

where $k_n = 2$ if $n = 2$ and $k_n = 1$ for $n \ge 3$. Note that if $n = 2$, β is not a nice basis.

To get a negative Ricci operator, we change the basis in V_n by rescaling the elements in $\mathcal{S}$. Let

$$\beta_2 = \{a\, z_j^n, b\,\imath z_j^n,\ 1 \le j \le m\} \cup (\beta_1 \smallsetminus \mathcal{S}) \tag{14}$$

and denote by f the corresponding diagonal element in $\mathfrak{gl}(\mathfrak{h}_\infty)$. Note that for this rescaling $\mathrm{tr}(\mathrm{ad}_\mu(H)^t\, \mathrm{ad}_\mu(X)) = 0$ for any $X \in \mathfrak{n}$, $H \in \mathfrak{a}$ still holds and hence $\langle \mathrm{Ric}_\mu \mathfrak{a}, \mathfrak{n}\rangle = 0$. Also, $[\mathrm{ad}_{f\cdot\mu} H_l|_{\mathfrak{n}}, (\mathrm{ad}_{f\cdot\mu} H_l|_{\mathfrak{n}})^t]$ is diagonal for any $1 \le l \le m-1$ and moreover it does not vanish only on $\mathrm{Span}\{z_l^n, \imath z_l^n, z_{l+1}^n, \imath z_{l+1}^n\}$. Direct calculation shows that

$$\begin{aligned}
&\langle \mathrm{Ric}_{f\cdot\mu} Z, Z\rangle = -d,\ \ \langle \mathrm{Ric}_{f\cdot\mu} H_l, H_l\rangle = -n^2(\tfrac{b}{a} - \tfrac{a}{b})^2,\\
&\langle \mathrm{Ric}_{f\cdot\mu} X_{i,j}, X_{i,j}\rangle = \langle \mathrm{Ric}_{f\cdot\mu} Y_{i,j}, Y_{i,j}\rangle = -(a^2+b^2)n^2,\\
&\langle \mathrm{Ric}_{f\cdot\mu} z_j^n, z_j^n\rangle = -a^2n^2(m-1) + c_j\left(\left(\tfrac{b}{a}\right)^2 - \left(\tfrac{a}{b}\right)^2\right) - d, \quad 1 \le j \le m\\
&\langle \mathrm{Ric}_{f\cdot\mu} \imath z_j^n, \imath z_j^n\rangle = -b^2n^2(m-1) + c_j\left(\left(\tfrac{a}{b}\right)^2 - \left(\tfrac{b}{a}\right)^2\right) - d, \quad 1 \le j \le m\\
&\langle \mathrm{Ric}_{f\cdot\mu} p, p\rangle = k_n(a^2+b^2)n^2 - d,\ p = sp_{j_1,\dots,j_m}, j_l = n-1 \text{ for some } l, s = 1, \imath\\
&\langle \mathrm{Ric}_{f\cdot\mu} p, p\rangle = -d, \quad p = sp_{j_1,\dots,j_m}, j_l \ne n, n-1, \forall\, l, s = 1, \imath
\end{aligned} \tag{15}$$

where $c_1 = c_m = \frac{n^2}{2}$ and $c_j = n^2$ for $j \ne 1, m$.

Therefore, to get negative Ricci curvature it is enough to choose $a > b > 0$ such that

$$a^2 + b^2 < \frac{d}{2\,n^2}, \quad -b^2n^2(m-1) + n^2\left(\left(\tfrac{a}{b}\right)^2 - \left(\tfrac{b}{a}\right)^2\right) < d,$$

and this can be done as in [15, (16)]. $\square$

Remark 3.2. Note that

$$k_n n^2 - d = k_n n^2 - 2\tfrac{(n+m-1)(n+m-2)\dots(n+1)}{(m-1)!}$$

is always negative for $m \geq 3, n \geq 2$ and therefore Ric_μ is negative semidefinite. In particular, $\mathrm{Ric}_\mu|_{\mathfrak{n}}$ is negative definite. We would like to point out that in [12, Theorem 2] the authors arrive to a similar situation and they perturb the inner product on $\mathfrak{n}$ so that no element of $\mathfrak{a}$ is acting skew-symmetric and $\mathrm{Ric}_\mu|_{\mathfrak{n}}$ is still negative definite. The fact that allows them to do that is that $\mathfrak{n}$ is abelian so one can still get $\mathrm{Ric}_\mu(\mathfrak{a},\mathfrak{n}) = 0$, which is no true in our case.

Theorem 3.3. *Let (V_n, π_n) be the usual real representation of $\mathfrak{su}(m)$ on the space of complex homogeneous polynomials of degree n in m variables $\mathcal{P}_{m,n}(\mathbb{C})$ extended to $\mathfrak{u}(m)$ by letting the center act as multiples of the identity. Hence the Lie algebra $\mathfrak{u}(m) \ltimes V_n$ admits a inner product with negative Ricci curvatures for all $n, m \geq 2$.*

Remark 3.4. As for $m = 2$, the case when $\mathfrak{su}(m)$ acts on $\mathbb{C}^n$ i.e. the case when $n = 1$, must be study separately since the representation is different and the general defined degeneration given in (11) leads to a solvable Lie algebra with an abelian nilradical. For $m = 2$ it is shown in [15] Lemma 3.4 that this problem can be solved.

4. Examples starting with $\mathfrak{so}(n)$

The same procedure can be applied to get inner products with negative Ricci curvature on $\mathfrak{g} = (\mathbb{R}Z \oplus \mathfrak{u}) \ltimes V$ where $\mathfrak{u} = \mathfrak{so}(m)$ and V is the real representation of $\mathfrak{u}$ in some polynomial space. Summarizing the procedure, we will start by using [5, (2) Theorem 6.3 Ch. III] to get a decomposition of $\mathfrak{u}$ as in (7), then we will choose a basis of the representation V with a special subset $\mathcal{S}$. Using this basis we define a degeneration $\phi_t \in \mathfrak{gl}(\mathfrak{g})$ where $\mathfrak{g}$ is the Lie algebra $\mathfrak{g} = (\mathbb{R}Z \oplus \mathfrak{u}) \ltimes V$ such that $[Z, \mathfrak{u}] = 0$ and $\operatorname{ad} Z|_V = Id$, so that the limit $(\mathfrak{g}_\infty, \mu)$ is a solvable Lie algebra and the action of $\mathfrak{u}$ on V is the same on $\mathcal{S}$ and vanish elsewhere. Finally, by rescaling the basis of $\mathfrak{g}_\infty$ in $\mathcal{S}$ we get a negative Ricci operator on the limit and therefore by Lemma 2.1, $\mathfrak{g}$ admits an inner product with negative Ricci operator.

Let (V, π) be the standard representation of $\mathfrak{so}(m)$ on the space of complex-valued homogeneous polynomials of degree n on $\mathbb{R}^m$ derived from the standard action of the group $\mathrm{SO}(m)$. That is,

$$X \cdot p\,(a_1, \dots, a_m) = \tfrac{d}{dt}|_{t=0}\, p\left(exp(tX)^{-1}\begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix}\right) = -p\left(X\begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix}\right)$$

for any $X \in \mathfrak{so}(m)$, $p \in V$. In [8, Chap. IV,§5, Examples 1,2] it is shown that if $(x_1, \dots, x_m) \in \mathbb{R}^m$, it is convenient to see this polynomials as powers of

$$\begin{aligned} &z_1 = x_1 + \imath x_2,\ z_2 = x_1 - \imath x_2, \dots, z_{m-1} = x_{m-1} + \imath x_m,\ z_m = x_{m-1} - \imath x_m, \quad m \text{ is even} \\ &z_1 = x_1 + \imath x_2,\ z_2 = x_1 - \imath x_2, \dots, z_{m-1} = x_{m-2} - \imath x_{m-1},\ z_m = x_m, \quad m \text{ is odd} \end{aligned} \tag{16}$$

since the weight vectors are

$$(x_1 + \imath x_2)^{k_1}(x_1 - \imath x_2)^{r_1} \dots (x_{2l-1} - \imath x_{2l})^{r_l},\ \textstyle\sum k_i + \sum r_i = n, \text{ for m=2l}$$

$$(x_1 + \imath x_2)^{k_1}(x_1 - \imath x_2)^{r_1} \dots (x_{2l-1} - \imath x_{2l})^{r_l} x_{2l+1}^{k_0},\ \textstyle\sum k_i + \sum r_i = n, \text{ for m=2l+1}$$

Recall that to get a real representation we have to consider powers of $z_1, \imath z_1, \dots, z_m, \imath z_m$ and hence $d = \dim V = 2\binom{n+m-1}{n}$.

Note that if $m = 2l+1$, $\mathfrak{so}(m)$ is a real form of the B_l-type complex Lie algebra $\mathfrak{so}(2l+1, \mathbb{C})$ and for $m = 2l$ the corresponding type is D_l. Therefore, we will study these cases separately. Also recall that we have the following isomorphism $\mathfrak{so}(3) \simeq \mathfrak{su}(2)$, $\mathfrak{so}(4) \simeq \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and $\mathfrak{so}(6) \simeq \mathfrak{su}(4)$ and therefore we may assume that $l \geq 2$.

Let $m = 2l+1$ and for $X \in \mathfrak{so}(2l+1)$ we will use the following notation:

$$X = \left[\begin{array}{ccc|c} A_{1,1} & \dots & A_{1,l} & A_{1,l+1} \\ & \ddots & & \vdots \\ A_{l,1} & \dots & A_{l,l} & A_{l,l+1} \\ \hline A_{l+1,1} & \dots & A_{l+1,l} & 0 \end{array}\right] \tag{17}$$

where $A_{i,j}$ is a 2×2 matrix if $i, j \leq l$ and $A_{i,l+1}$ is a column matrix with 2 rows. Clearly, $A_{j,i} = -A^t_{i,j}$. Using [5, (2) Theorem 6.3 Ch. III] and [8, Example 2, pp. 63] we obtain that a basis of $\mathfrak{so}(m)$ is given by

$$\beta_o = \{H_i, X^{\pm}_{kj}, X_r, Y^{\pm}_{kj}, Y_r,\ i, r \leq l, k < j \leq l\}.$$

where all this matrices have only two non zero blocks as follows:

$$H_i = \begin{bmatrix} \ddots & & & \\ & A_{i,i} & & \\ & & -A_{i+1,i+1} & \\ & & & \ddots \end{bmatrix},\ i \leq l-1, \quad H_l = \begin{bmatrix} \ddots & & \\ & A_{l,l} & \\ & & 0 \end{bmatrix}, \text{ where } A_{r,r} = \left[\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right]$$

$$X^{\pm}_{kj} = \begin{bmatrix} \ddots & & & \\ & & A^{\pm}_{k,j} & \\ & A^{\pm}_{j,k} & & \\ & & & \ddots \end{bmatrix}, -(A^-_{j,k})^t = A^-_{k,j} = \left[\begin{smallmatrix} 2 & \\ & 2 \end{smallmatrix}\right], -(A^+_{j,k})^t = A^+_{k,j} = \left[\begin{smallmatrix} 2 & \\ & -2 \end{smallmatrix}\right], k < j \neq l+1$$

$$Y^{\pm}_{kj} = \begin{bmatrix} \ddots & & & \\ & & A^{\pm}_{k,j} & \\ & A^{\pm}_{j,k} & & \\ & & & \ddots \end{bmatrix}, -(A^-_{j,k})^t = A^-_{k,j} = \left[\begin{smallmatrix} & -2 \\ 2 & \end{smallmatrix}\right], -(A^+_{j,k})^t = A^+_{k,j} = \left[\begin{smallmatrix} & 2 \\ 2 & \end{smallmatrix}\right], k < j \neq l+1$$

$$X_r = \begin{bmatrix} \ddots & & \vdots \\ & & A_{r,l+1} \\ & & \vdots \\ \dots & A_{l+1,r} & \end{bmatrix}, A_{r,l+1} = \left[\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}\right], Y_r = \begin{bmatrix} \ddots & & \vdots \\ & & A_{r,l+1} \\ & & \vdots \\ \dots & A_{l+1,r} & \end{bmatrix}, A_{r,l+1} = \left[\begin{smallmatrix} 0 \\ 2 \end{smallmatrix}\right], 1 \leq r \leq l.$$

We note that if

$$e_i\left(\begin{bmatrix} \begin{smallmatrix} 0 & \imath h_1 \\ -\imath h_1 & 0 \end{smallmatrix} & & & & \\ & \begin{smallmatrix} 0 & \imath h_2 \\ -\imath h_2 & 0 \end{smallmatrix} & & & \\ & & \ddots & & \\ & & & \begin{smallmatrix} 0 & \imath h_l \\ -\imath h_l & 0 \end{smallmatrix} & \\ & & & & 0 \end{bmatrix}\right) = h_i$$

then

$$H_i = \imath H_{e_i - e_{i+1}}, i < l, \quad H_l = \imath H_{e_l}$$

$$X^{\pm}_{kj} = E_{e_k \pm e_j} + E_{-(e_k \pm e_j)},\ k < j \leq l, \quad X_r = E_{e_r} + E_{-e_r}, 1 \leq r \leq l$$

$$Y^{\pm}_{kj} = \imath(E_{e_k \pm e_j} - E_{-(e_k \pm e_j)})\ k < j \leq l, \quad Y_r = \imath(E_{e_r} - E_{-e_r}), 1 \leq r \leq l$$

where E_α are the root vectors given in [8, Example 2, pp. 63] (see (7)). Also note that if $\imath\mathfrak{p} = \mathrm{Span}\{H_i, Y_{k,j}, Y_r,\ i, r \leq l, k < j \leq l\}$ and $\mathfrak{k} = \mathrm{Span}\{X_{k,j}, X_r, k < j \leq l, r \leq l\}$ then $\mathfrak{g}_o = \mathfrak{k} + \mathfrak{p}$ is a Cartan decomposition of the non-compact dual of $\mathfrak{so}(2l+1)$.

Straightforward calculation shows that the non trivial action of $\mathfrak{so}(2l+1)$ on V can be obtained form

$$
H_i \cdot z_r = \begin{cases} -\imath z_r, & r = 2i-1, 2i+2, \\ \imath z_r & r = 2i, 2i+1, \end{cases} \quad i < l, \quad H_l \cdot z_r = \begin{cases} -\imath z_r, & r = 2l-1, \\ \imath z_r & r = 2l, \end{cases}
$$

$$
X^+_{kj} \cdot z_r = \begin{cases} -2z_{2j}, & r = 2k-1, \\ -2z_{2j-1}, & r = 2k, \\ 2z_{2k}, & r = 2j-1, \\ 2z_{2k-1}, & r = 2j, \end{cases} \quad X^-_{kj} \cdot z_r = \begin{cases} -2z_{2j-1}, & r = 2k-1, \\ -2z_{2j}, & r = 2k, \\ 2z_{2k-1}, & r = 2j-1, \\ 2z_{2k}, & r = 2j, \end{cases}
$$

$$
Y^+_{kj} \cdot z_r = \begin{cases} -2\imath z_{2j}, & r = 2k-1, \\ 2\imath z_{2j-1}, & r = 2k, \\ 2\imath z_{2k}, & r = 2j-1, \\ -2\imath z_{2k-1}, & r = 2j, \end{cases} \quad Y^-_{kj} \cdot z_r = \begin{cases} -2\imath z_{2j-1}, & r = 2k-1, \\ 2\imath z_{2j}, & r = 2k, \\ -2\imath z_{2k-1}, & r = 2j-1, \\ 2\imath z_{2k}, & r = 2j, \end{cases}
$$

$$
X_i \cdot z_r = \begin{cases} -2z_m, & r = 2i-1, 2i, \\ z_{2i-1} + z_{2i}, & r = 2l+1, \end{cases} \quad Y_i \cdot z_r = \begin{cases} -2\imath z_m, & r = 2i-1, \\ 2\imath z_m, & r = 2i, \\ -\imath z_{2i-1} + \imath z_{2i}, & r = 2l+1. \end{cases}
$$

Let us fix a basis of V,

$$
\beta_1 = \{ sp_{j_1,\dots,j_m} = s z_1^{j_1} \dots z_m^{j_m}, \quad s = 1, \imath,\ j_i \in \mathbb{N}_0,\ j_i + \dots + j_m = n \}
$$

where z_i are defined in (16) (see (9)) and denote by $\mathcal{S}$ the following subset of β_1

$$
\mathcal{S} = \{ s z_j^n,\ s = 1, \imath,\ j \leq 2l \}.
$$

Consider the Lie algebra $(\mathfrak{g}, [\cdot,\cdot]) = (\mathbb{R}Z \oplus \mathfrak{so}(m)) \ltimes V$ where $\operatorname{ad} Z|_{\mathfrak{so}(m)} = 0$ and $\operatorname{ad} Z|_V = Id$. For each $t > 0$ define $\phi_t \in \mathfrak{gl}(\mathfrak{g})$ such that

$$
\begin{aligned}
&\phi_t(Z) = Z, \quad \phi_t(H_i) = H_i,\ i \leq l, \quad \phi_t(X^\pm_{kj}) = tX^\pm_{kj}, \quad \phi_t(Y^\pm_{kj}) = tY^\pm_{kj},\ k, j \leq l \\
&\phi_t(X_r) = tX_r, \quad \phi_t(Y_r) = tY_r,\ r \leq l, \quad \phi_t(p) = \begin{cases} t\,p, & \text{if } p \in \mathcal{S} \\ t^2\,p & \text{if } p \notin \mathcal{S}. \end{cases}
\end{aligned} \tag{18}
$$

It is not hard to check that the limit $\mathfrak{g}_\infty = (\mathfrak{g}_0, \mu)$, $\mu = \lim\limits_{t\to\infty} [\cdot,\cdot]_t = \lim\limits_{t\to\infty} \phi_t.[\cdot,\cdot]$ is well defined and moreover it is solvable. Straightforward calculation shows that it is given by

$$
\begin{aligned}
&\mu(H_r, X) = [H_r, X],\ \ \mu(H_r, Y) = [H_r, Y],\ \ r \leq l, X = X^\pm_{k,j}, X_i, Y = Y^\pm_{k,j}, Y_i, \\
&\mu(Z, p) = p, \quad \mu(H_r, p) = [H_r, p] \quad \forall\, p \in \beta_1,\ r \leq l, \\
&\mu(X, p) = [X, p], \quad \mu(Y, p) = [Y, p],\ \ \forall\, p \in S,\ X = X^\pm_{k,j}, X_i, Y = Y^\pm_{k,j}, Y_i.
\end{aligned} \tag{19}
$$

Let us fix the inner product $\langle \cdot,\cdot \rangle$ in $\mathfrak{g}_\infty$ such that β_2 is an orthonormal basis, where

$$
\beta_2 = \{ H_i, X^\pm_{kj}, X_r, Y^\pm_{kj}, Y_r, \beta_1,\ i, r \leq l, k < j \leq l \} = \beta_o \cup \beta_1.
$$

Since H_r acts by a skew-symmetric matrix for any $r \leq l$ we can never get negative Ricci operator (see (5)) so, as before, we will change the basis by rescaling it in $\mathcal{S}$ (see (14)). Let

$$
\beta_3 = \{ a\, z_j^n, b\,\imath z_j^n,\ j \leq 2l \} \cup (\beta_1 \smallsetminus \mathcal{S}) \tag{20}
$$

and denote by f the diagonal element in $\mathfrak{gl}(\mathfrak{g}_\infty)$ corresponding to the change of basis from $\beta_2 = \beta_o \cup \beta_1$ to $\beta_o \cup \beta_3$. Direct calculation shows that for $n \geq 3$,

$$
\begin{aligned}
&\langle \operatorname{Ric}_{f\cdot\mu} Z, Z\rangle = -d,\\
&\langle \operatorname{Ric}_{f\cdot\mu} H_r, H_r\rangle = -2n^2(\tfrac{b}{a}-\tfrac{a}{b})^2,\ \ \langle \operatorname{Ric}_{f\cdot\mu} H_l, H_l\rangle = -n^2(\tfrac{b}{a}-\tfrac{a}{b})^2,\\
&\langle \operatorname{Ric}_{f\cdot\mu} X^{\pm}_{kj}, X^{\pm}_{kj}\rangle = \langle \operatorname{Ric}_{f\cdot\mu} Y^{\pm}_{kj}, Y^{\pm}_{kj}\rangle = -8n^2(a^2+b^2), \quad k<j\leq l,\\
&\langle \operatorname{Ric}_{f\cdot\mu} X_r, X_r\rangle = \langle \operatorname{Ric}_{f\cdot\mu} Y_r, Y_r\rangle = -4n^2(a^2+b^2), \quad r\leq l,\\
&\langle \operatorname{Ric}_{f\cdot\mu} z^n_j, z^n_j\rangle = -2n^2a^2(4(l-1)+2) + n^2c_j\left(\left(\tfrac{b}{a}\right)^2-\left(\tfrac{a}{b}\right)^2\right) - d, \quad j\leq 2l,\\
&\langle \operatorname{Ric}_{f\cdot\mu} 1z^n_j, 1z^n_j\rangle = -2n^2b^2(4(l-1)+2) + n^2c_j\left(\left(\tfrac{a}{b}\right)^2-\left(\tfrac{b}{a}\right)^2\right) - d, \quad j\leq 2l,\\
&\langle \operatorname{Ric}_{f\cdot\mu} p, p\rangle = 2n^2(a^2+b^2) - d,\ p\in\mathcal{S}_1,\\
&\langle \operatorname{Ric}_{f\cdot\mu} p, p\rangle = -d, \quad p\notin\mathcal{S} \text{ or } \mathcal{S}_1,
\end{aligned}
\tag{21}
$$

where $c_j = \frac{1}{2}$ for $j = 1, 2$ and $c_j = 1$ for $j \neq 1, 2$ and $\mathcal{S}_1 \subset \beta_1$ is defined so that $p \in \mathcal{S}_1$ iff $p = X \cdot s z^n_r$ for some $X = X^{\pm}_{k,j}, X_r, Y^{\pm}_{k,j}, Y_r \in \beta_o$. Note that for $a > b > 0$ such that

$$
(a^2+b^2) < \tfrac{d}{n^2}, \quad -2b^2(4l-2) + \left(\left(\tfrac{a}{b}\right)^2-\left(\tfrac{b}{a}\right)^2\right) < \tfrac{d}{n^2} \tag{22}
$$

all the constants in (21) are negative numbers. To find such a, b we will proceed as in [15, (16)]. Let $t > 1$ such that $t^2 - \frac{1}{t^2} < \frac{d}{n^2}$ and choose $b > 0$ so that $b^2 < \frac{d}{(1+t^2)n^2}$. It is easy to check that if $a = tb$ then a, b satisfy (22).

Remark 4.1. We note that the elements of $\mathfrak{so}(m)$ given in the basis β_o are not the ones given by [5, (2) Theorem 6.3 Ch. III] since there are some constant that we have changed in order to simplify some calculation and expressions. It is easy to see that this is an equivalent realization

Remark 4.2. Also note that for $n = 2$, β is not a basis of eigenvectors of Ric.

One can use this proof to study the case of $\mathfrak{so}(2l)$. First note that the root structure of $\mathfrak{so}(2l)$ can be read off from the one we have constructed for $\mathfrak{so}(2l+1)$ (see ([8, Example 4 pp. 63]). In fact, one can choose the Cartan subalgebra so that the roots and root vectors of $\mathfrak{so}(2l)$ correspond to the ones that can be restricted from $\mathfrak{so}(2l+1)$. Explicitly, the set of roots is $\Delta = \{\pm e_k \pm e_j,\ j \leq l\}$ and the corresponding root vectors are obtained from the ones in $\mathfrak{so}(2l+1)$ by erasing the last column and row. Hence in the same way as before we get a basis

$$
\tilde{\beta}_o = \{\tilde{H}_i, \tilde{X}^{\pm}_{kj}, \tilde{Y}^{\pm}_{kj},\ i \leq l, k < j \leq l\}
$$

where $\tilde{X}$ is the matrix obtained by erasing the last column and row of X for $X \neq H_l$ and $\tilde{H}_l$ is the matrix we obtain by erasing the last column and row of $\begin{bmatrix} \ddots & & & \\ & A_{l-1,l-1} & & \\ & & A_{l,l} & \\ & & & \ddots \end{bmatrix}$ (see (17)).

To show that $(\mathfrak{g}_o, [\cdot,\cdot]) = (\mathbb{R}Z \oplus \mathfrak{so}(2l)) \ltimes V$ admits an inner product with negative Ricci operator one can follow the proof of $\mathfrak{so}(2l+1)$ using the new $\tilde{H}_l$ and ignoring z_{2l+1}, X_r or Y_r where they appear. Note that we can use the same $\mathcal{S}$. We then get that

$$
\begin{aligned}
&\langle \operatorname{Ric}_{f\cdot\mu} Z, Z\rangle = -d,\\
&\langle \operatorname{Ric}_{f\cdot\mu} H_r, H_r\rangle = -2n^2(\tfrac{b}{a}-\tfrac{a}{b})^2, \quad r\le l,\\
&\langle \operatorname{Ric}_{f\cdot\mu} X^{\pm}_{kj}, X^{\pm}_{kj}\rangle = \langle \operatorname{Ric}_{f\cdot\mu} Y^{\pm}_{kj}, Y^{\pm}_{kj}\rangle = -8n^2(a^2+b^2), \quad k<j\le l,\\
&\langle \operatorname{Ric}_{f\cdot\mu} z^n_j, z^n_j\rangle = -2n^2a^2 4(l-1) + n^2 c_j\left(\left(\tfrac{b}{a}\right)^2 - \left(\tfrac{a}{b}\right)^2\right) - d, \quad j\le 2l,\\
&\langle \operatorname{Ric}_{f\cdot\mu} \imath z^n_j, \imath z^n_j\rangle = -2n^2b^2 4(l-1) + n^2 c_j\left(\left(\tfrac{a}{b}\right)^2 - \left(\tfrac{b}{a}\right)^2\right) - d, \quad j\le 2l,\\
&\langle \operatorname{Ric}_{f\cdot\mu} p, p\rangle = 2n^2(a^2+b^2) - d, \ p\in\mathcal{S}_1,\\
&\langle \operatorname{Ric}_{f\cdot\mu} p, p\rangle = -d, \quad p\notin\mathcal{S} \text{ or } \mathcal{S}_1,
\end{aligned}
\tag{23}
$$

where $c_j = \frac{1}{2}$ for $j=1,2$, $c_j = 1$ for $j\neq 1,2,2l-1$ $c_{2l-1} = \frac{3}{2}$.

Theorem 4.3. *Let (V,π) be the standard real representation of $\mathfrak{so}(m)$ on the space of complex-valued homogeneous polynomials of degree n on $\mathbb{R}^m$. Let $(\mathfrak{g},[\cdot,\cdot]) = (\mathbb{R}Z\oplus\mathfrak{so}(m))\ltimes V$ be the Lie algebra where $[Z,\mathfrak{so}(m)]=0$ and Z acts as the identity on V. Then $\mathfrak{g}$ admits a inner product with negative Ricci curvature for all $n,m\geq 3$.*

5. More examples using $\mathfrak{gl}$

We can follow the same construction using $\mathfrak{sl}(n,\mathbb{R})$ instead of $\mathfrak{su}(n)$ or $\mathfrak{so}(m)$. The calculation are more involved since the operators $\operatorname{ad} H$ are no longer skew-symmetric and the basis in the nilradical is no longer nice but everything works anyway. Let us start by considering $\mathfrak{sl}(m,\mathbb{R})$ as the non-compact dual of $\mathfrak{su}(m)$ considered in 2.2. Recall that when $\mathfrak{g}_0$ is a semisimple Lie algebra of complex matrices stable under θ where $\theta(X) = -\bar{X}^t$ and $\mathfrak{g}_0 = \mathfrak{k}\oplus\mathfrak{p}$ is the corresponding Cartan decomposition such that $\mathfrak{k}\cap\imath\mathfrak{p} = 0$, we get that its complexification $\mathfrak{g} = (\mathfrak{k}\oplus\mathfrak{p})^{\mathbb{C}}$ is also semisimple and $\mathfrak{u} = \mathfrak{k}\oplus\imath\mathfrak{p}$ is a compact real form of $\mathfrak{g}$. In this case $\mathfrak{g}$ is usually called the non-compact dual of $\mathfrak{u}$. In particular, any finite-dimensional complex representation of $\mathfrak{g}_0$ gives rise to a representation of $\mathfrak{u}$ and viceversa by using this decomposition (see [7] pp. 443).

Consider then the real representation of $\mathfrak{sl}(m,\mathbb{R})$ on $\mathcal{P}_{m,n}(\mathbb{C})$ obtained by seeing it as a real vector space. Extend it to $\mathfrak{gl}(m,\mathbb{R}) = \mathbb{R}Z\oplus\mathfrak{sl}(m,\mathbb{R})$, where $Z = Id$ acts as the identity operator. We will denote this representation by (V_n,π_n) as before.

Fix a basis of $\mathfrak{sl}(m,\mathbb{R})$

$$\beta_3 = \{H_1,\dots H_{m-1},\, X_{i,j},\, Y_{i,j},\quad 1\le i<j\le m\}.$$

where,

$$
\begin{aligned}
&H_l = (E_{i,i} - E_{i+1,i+1}),\ l = 1,\dots m-1,\\
&X_{i,j} = E_{i,j} - E_{j,i},\ 1\le i<j\le m,\\
&Y_{i,j} = (E_{i,j} + E_{j,i}),\ 1\le i<j\le m,
\end{aligned}
\tag{24}
$$

note the missing $\imath$. Hence a basis of the semidirect product $\mathfrak{s} = \mathfrak{gl}(m,\mathbb{R})\ltimes V_n$ is given by $\beta = \{Z,\beta_3,\beta_1\}$ (see 9). For $s = 1,\imath$, the action is now given by

$$
H_l\cdot sz_k = \begin{cases} -sz_k, & k = l,\\ sz_k, & k = l+1,\\ 0, & k\neq l,l+1,\end{cases}\quad
X_{i,j}\cdot sz_k = \begin{cases} -sz_j, & k = i,\\ sz_i, & k = j,\\ 0, & k\neq i,j,\end{cases}\quad
Y_{i,j}\cdot sz_k = \begin{cases} -sz_j, & k = i,\\ -sz_i, & k = j,\\ 0, & k\neq i,j.\end{cases}
$$

Using that the algebra is acting by derivations we get that

$$\pi_n(H_l)sp_{j_1,\dots,j_m} = (j_{l-1} - j_l)sp_{j_1,\dots,j_m},$$

the action of $X_{i,j}$ is the same as in the $\mathfrak{su}(m)$ case and

$$Y_{i,k}(sp) = -j_i\, s(z_1^{j_1}\dots z_i^{j_i-1}\dots z_k^{j_k+1}\dots z_m^{j_m}) - j_k\, s(z_1^{j_1}\dots z_i^{j_i+1}\dots z_k^{j_k-1}\dots z_m^{j_m})$$

where $p = z_1^{j_1}\dots z_m^{j_m}$.

We apply the same degeneration given in (11) and get the limit $\mathfrak{s}_\infty = (\mathbb{R}^r, \nu)$, where $r = m^2 + d$ and

$$\nu = \lim_{t\to\infty} \phi_t.[\cdot,\cdot],$$

$[\cdot,\cdot]$ the Lie bracket of $\mathfrak{sl}(m)$. Note that we are abusing the notation since everything, $V_n, \pi_n, \mathfrak{s}, \mathfrak{s}_\infty, [\cdot,\cdot]$, etc. depends on n and m.

Direct computation as in (12) shows that

$$\begin{aligned}
&\nu(H_l, X_{i,j}) = [H_l, X_{i,j}], \quad \nu(H_l, Y_{i,j}) = [H_l, Y_{i,j}], \quad \forall\, i, j, l,\\
&\nu(Z, p) = p, \quad \nu(H_l, p) = [H_l, p] \qquad \forall\, p \in \beta_2, \forall\, l,\\
&\nu(X_{i,j}, p) = [X_{i,j}, p], \quad \nu(Y_{i,j}, p) = [Y_{i,j}, p], \quad \forall\, i, j, \forall\, p \in \mathcal{S},
\end{aligned} \tag{25}$$

Note that the bracket on the starting point in quite different from the one in $\mathfrak{su}(m)$ and, in particular, $\mathrm{ad}_\nu(H_l)|_{V_n} = \mathrm{ad}(H_l)|_{V_n}$ are now symmetric operators.

Lemma 5.1. *For $n \neq 1$ the solvable Lie algebra $\mathfrak{s}_\infty = \mathfrak{s}_\infty(m, n)$ admits an inner product with negative Ricci curvature.*

By proposition 2.1 we get that $\mathfrak{s}$ admits a inner product with negative Ricci curvatures. Hence we have the following proposition.

Proposition 5.2. *Let (V_n, π_n) be the usual real representation of $\mathfrak{sl}(m, \mathbb{R})$ on the space of complex homogeneous polynomials of degree n in m variables, extended to $\mathfrak{gl}(m, \mathbb{R})$ by letting the center act as multiples of the identity. Hence the Lie algebra $\mathfrak{gl}(m, \mathbb{R}) \ltimes V_n$ admits an inner product with negative Ricci curvature for any m and $n \geq 2$.*

In the next section we will prove a more general result that implies Proposition 5.2 for $m \geq 3$ so we are going to omit the proof which follows the same lines as the one for $\mathfrak{su}(m)$. It worth to point out that no change of basis is needed for $m \geq 3$ since β is a basis of eigenvectors of Ric_ν with negative eigenvalues for $m \geq 3, n \geq 2$, though the basis is not nice.

Example 5.3. As an example we will go over the example of $\mathfrak{gl}(2, \mathbb{R})$ acting on $\mathcal{P}_{2,2}(\mathbb{C})$, the space of homogeneous complex polynomials of degree 2 in 2 variables seen as a real vector space. As in (24), let

$$H = H_1 = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}, \quad X = X_{1,2} = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}, \quad Y = Y_{1,2} = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}, \tag{26}$$

and $Z = Id$. We fix the orthonormal basis of $\mathfrak{s} = \mathfrak{gl}(2, \mathbb{R}) \ltimes V_2 = (\mathbb{R}^{10}, \nu, \langle\cdot,\cdot\rangle)$

$$\beta = \{Z, H, X, Y, v_1, v_2, v_3, v_4, v_5, v_6\}$$

where

$$v_1 = z_1^2,\ v_2 = \imath z_1^2,\ v_3 = z_1 z_2,\ v_4 = \imath z_1 z_2,\ v_5 = z_2^2,\ v_6 = \imath z_2^2.$$

We have

$$\pi_2(Z) = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}, \qquad \pi_2(H) = \begin{bmatrix} -2 &&&&& \\ & -2 &&&& \\ && 0 &&& \\ &&& 0 && \\ &&&& 2 & \\ &&&&& 2 \end{bmatrix},$$

$$\pi_2(X) = \left[\begin{array}{cc|cc|cc} & & 1 & & & \\ & & & 1 & & \\ \hline -2 & & & & 2 & \\ & -2 & & & & 2 \\ \hline & & -1 & & & \\ & & & -1 & & \end{array}\right], \quad \pi_2(Y) = \left[\begin{array}{cc|cc|cc} & & -1 & & & \\ & & & -1 & & \\ \hline -2 & & & & -2 & \\ & -2 & & & & -2 \\ \hline & & -1 & & & \\ & & & -1 & & \end{array}\right]$$

In this case $\mathcal{S} = \{v_1, v_2, v_5, v_6\}$ (see (10)) so we get that the degeneration is given by $\phi_t \in \mathrm{GL}(\mathfrak{s})$

$$\phi_t|_{\mathfrak{g}} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & t & \\ & & & t \end{bmatrix}, \quad \phi_t|_{W_2} = \begin{bmatrix} t & & & & & \\ & t & & & & \\ & & t^2 & & & \\ & & & t^2 & & \\ & & & & t & \\ & & & & & t \end{bmatrix},$$

and hence, the limit $\mathfrak{s}_\infty = (\mathbb{R}^{10}, \nu, \langle\cdot,\cdot\rangle)$ is a solvable Lie algebra. Its nilradical is $\mathfrak{n} = \mathrm{Span}\{X, Y, V_2\}$ and the center of $\mathfrak{n}$ is $\mathfrak{z} = \mathrm{Span}\{v_3, v_4\}$.

Direct calculation shows that if we change the basis to

$$\{Z, H, X, Y, \tfrac{1}{2}v_1, \tfrac{1}{2}v_2, v_3, v_4, \tfrac{1}{2}v_5, \tfrac{1}{2}v_6\}$$

the corresponding Ricci operator is given by

$$\mathrm{Ric}_{f\cdot\lambda} = \mathrm{Diag}(-6, -24, -2, -2, -7, -7, -4, -4, -7, -7). \tag{27}$$

It can be checked that for $t \geq 4$, $\mathrm{Ric}_{\phi_t \cdot f \cdot [\cdot,\cdot]}$ is negative defined.

Remark 5.4. As in the $\mathfrak{su}(2)$ case we can show that $\mathfrak{s} = \mathfrak{gl}(2,\mathbb{R}) \ltimes \mathbb{C}^2$, that is when $\mathfrak{sl}(2,\mathbb{R})$ acts on $V_1 = \mathbb{C}^2$ seen as a real vector space, also admits an inner product with negative Ricci curvatures. This is the analogous of [15, Lemma 3.4] so as in that case we only need to consider a slightly different degeneration and the right change of basis and therefore we will just give very few details.

In the notation of the above Lemma, consider the metric Lie algebra $\mathfrak{s}_\infty = (\mathbb{R}^8, \nu, \langle\cdot,\cdot\rangle)$, where $\langle\cdot,\cdot\rangle$ is the inner product that makes β an orthonormal basis and the family ϕ_t as in [15, Lemma 3.4]. Direct calculation shows that

$$\mathrm{Ric}_\nu = \begin{bmatrix} -4 & & & & & & & \\ & -12 & & & & & & \\ & & -1 & 1 & & & & \\ & & 1 & -1 & & & & \\ & & & & -5 & & & \\ & & & & & -5 & & \\ & & & & & & -3 & \\ & & & & & & & -3 \end{bmatrix}.$$

By changing the basis to

$$\beta = \{Z, H, X+Y, X-Y, z_1, \imath z_1, z_2, \imath z_2\},$$

we get

$$\mathrm{Ric}_{f\cdot\nu} = \mathrm{Diag}(-4, -12, -8, -12, -2, -2, -6, -6),$$

as desire. Then $\mathfrak{s}_\infty$ and therefore $\mathfrak{s}$ both admits a inner product with negative Ricci curvature.

6. A more general Construction.

In this section, we obtain a generalization of the construction in the previous section in the sense that we consider a more general semidirect products to find examples of non-solvable Lie groups with negative Ricci curvature. We construct Lie algebras $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{a} \oplus \mathfrak{n}$, where $\mathfrak{h}$ is a semisimple Lie algebra without compact factors, $\mathfrak{n}$ is a nilpotent ideal and $\mathfrak{a}$ is abelian. In [9], the Ricci operator for homogeneous spaces has been studied. We are going to use some of their ideas and notation since many of the formulas used there are general.

Definition 6.1. In the following, we will denote by $\mathfrak{g} = \mathfrak{g}(\mathfrak{h}, \mathfrak{a}, \mathfrak{n}) = (\mathfrak{h} \oplus \mathfrak{a}) \ltimes \mathfrak{n}$ a Lie algebra such that

- $\mathfrak{h}$ is semisimple with no compact factors,
- $\mathfrak{a}$ is abelian,
- $\mathfrak{n}$ is nilpotent,
- $[\mathfrak{a}, \mathfrak{h}] = 0$.

Fix $\langle\cdot,\cdot\rangle$ any inner product on $\mathfrak{g}$ that makes $\mathfrak{h} \oplus \mathfrak{a} \oplus \mathfrak{n}$ an orthogonal decomposition. We note that the mean curvature vector H is orthogonal to $\mathfrak{n}$ and to $\mathfrak{h}$ so $H \in \mathfrak{a}$. Since $\mathfrak{a}$ is abelian, $\mathfrak{h}$ is a subalgebra and $\mathfrak{n}$ is a nilpotent ideal, using formulas from [9, Lemma 4.4], we can show that the Ricci operator of $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ is given by

$$
\begin{aligned}
&\langle \operatorname{Ric} Y, Y\rangle = \langle \operatorname{Ric}_{\mathfrak{h}} Y, Y\rangle - \operatorname{tr} S(\operatorname{ad} Y|_{\mathfrak{n}})^2,\\
&\langle \operatorname{Ric} A, A\rangle = -\operatorname{tr} S(\operatorname{ad} A|_{\mathfrak{n}})^2,\\
&\operatorname{Ric}|_{\mathfrak{n}} = \operatorname{Ric}_{\mathfrak{n}} - S(\operatorname{ad} H|_{\mathfrak{n}}) + \tfrac{1}{2}\sum [\operatorname{ad} Y_i|_{\mathfrak{n}}, (\operatorname{ad} Y_i|_{\mathfrak{n}})^t] + \tfrac{1}{2}\sum [\operatorname{ad} A_i|_{\mathfrak{n}}, (\operatorname{ad} A_i|_{\mathfrak{n}})^t],\\
&\langle \operatorname{Ric} Y, A\rangle = -\operatorname{tr} S(\operatorname{ad} Y|_{\mathfrak{n}}) S(\operatorname{ad} A|_{\mathfrak{n}}),\\
&\langle \operatorname{Ric} Y, X\rangle = -\operatorname{tr} (\operatorname{ad} Y|_{\mathfrak{n}})^t (\operatorname{ad}_{\mathfrak{n}} X),\\
&\langle \operatorname{Ric} A, X\rangle = -\operatorname{tr} (\operatorname{ad} A|_{\mathfrak{n}})^t (\operatorname{ad}_{\mathfrak{n}} X),
\end{aligned}
\tag{28}
$$

where $Y \in \mathfrak{h}$, $A \in \mathfrak{a}$, $X \in \mathfrak{n}$, $\{Y_i\}$ $\{A_i\}$ are orthonormal basis of $\mathfrak{h}$ and $\mathfrak{a}$ respectively and by $\operatorname{Ric}_{\mathfrak{l}}$ we denote the Ricci operator of the Lie subalgebra $\mathfrak{l}$ with the restricted inner product.

Theorem 6.2. *Let $\mathfrak{g} = \mathfrak{g}(\mathfrak{h}, \mathfrak{a}, \mathfrak{n})$ be as above. If in addition,*

1. $\mathfrak{n}$ *admits an inner product such that* $\operatorname{ad} A|_{\mathfrak{n}}$ *are semisimple operators for any* $A \in \mathfrak{a}$*,*
2. *no* $\operatorname{ad} A|_{\mathfrak{n}}$ *has all its eigenvalues purely imaginary,*
3. *there exists an element* A_1 *in* $\mathfrak{a}$ *such that all the eigenvalues of* $\operatorname{ad} A_1|_{\mathfrak{n}}$ *have positive real parts,*
4. $\mathfrak{h}$ *admits an inner product with* $\operatorname{Ric} < 0$ *such that there exists a Cartan decomposition* $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}$ *with* $\mathfrak{k}$ *orthogonal to* $\mathfrak{p}$*,*

then $\mathfrak{g}$ *admits an inner product with negative Ricci curvature.*

By results of [6], it is know that a semisimple Lie group that admits a negative Ricci curved metric can not have any compact factors (see (4)). Note that this theorem gives no new topologies other than the ones obtained in [3].

Also note that if $\mathfrak{h}$ is a non-compact semisimple Lie algebra with Cartan decomposition $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}$ and $\langle \cdot, \cdot \rangle$ is an inner product on $\mathfrak{h}$ such that $\mathfrak{k}$ is orthogonal to $\mathfrak{p}$ (as in (4)) then $\operatorname{Ric}(\mathfrak{k}, \mathfrak{p}) = 0$. In fact, since there exists an inner product on $\mathfrak{h}$ such that $\operatorname{ad} X$ is a symmetric operator for $X \in \mathfrak{p}$ and $\operatorname{ad} Y$ is skew-symmetric for $Y \in \mathfrak{k}$, then the Killing form satisfy $\langle BX, Y\rangle = 0$ for any $X \in \mathfrak{p}$ and $Y \in \mathfrak{k}$. Also, from (3) we get that for any orthonormal basis of $\mathfrak{h}$ $\{X_i\}$,

$$
\langle MX, Y\rangle = -\tfrac{1}{2}\sum \langle [X, X_i], X_j\rangle\langle [Y, X_i], X_j\rangle + \tfrac{1}{4}\sum \langle [X_i, X_j], X\rangle\langle [X_i, X_j], Y\rangle. \tag{29}
$$

Let us chose an $\{X_i\}$, so that the first elements are in $\mathfrak{k}$ and the last ones are in $\mathfrak{p}$. Hence, if $X \in \mathfrak{p}$ and $Y \in \mathfrak{k}$, using that

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k},\ [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p},\ [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k},$$

it is easy to check that all the terms in (29) vanishes and therefore $\operatorname{Ric}(\mathfrak{k}, \mathfrak{p}) = 0$.

Proof. We will consider first the case when $\mathfrak{n}$ is abelian. Let $\langle \cdot, \cdot \rangle$ be an inner product on $\mathfrak{g}$ such that the decomposition $\mathfrak{h} \oplus \mathfrak{a} \oplus \mathfrak{n}$ is orthogonal, $\langle \cdot, \cdot \rangle|_{\mathfrak{h}}$ is an inner product with negative Ricci curvature, $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}$ is the orthogonal Cartan decomposition and $\langle \cdot, \cdot \rangle|_{\mathfrak{n}}$ satisfies that $\mathfrak{a}$ act by normal operators on $\mathfrak{n}$.

Since $[\mathfrak{h}, \mathfrak{a}] = 0$ we can also assume with no loss of generality, that the elements in $\mathfrak{k}$ act on $\mathfrak{n}$ by skew-symmetric operators and $\mathfrak{p}$ act by symmetric ones. Indeed, let H be the complex simply connected Lie group with Lie algebra $\mathfrak{h}^{\mathbb{C}}$ and let H_1 be the connected Lie subgroup of H with Lie algebra $\mathfrak{h}_1 = (\mathfrak{k} + \imath\mathfrak{p})$. Since $\mathfrak{h}_1$ is compact, for any $\langle \cdot, \cdot \rangle_0$ inner product on $\mathfrak{n}$ we get that

$$
\langle X, X'\rangle_1 = \int_{H_1} \langle \pi(Y)(X), \pi(Y)(X')\rangle_0 \, dY, \tag{30}
$$

defines an H_1-invariant hermitian form on $\mathfrak{n}^{\mathbb{C}}$, where π is the representation of H on $\mathfrak{n}^{\mathbb{C}}$ such that $d\pi = \operatorname{ad}|_{\mathfrak{n}}$. Using that H_1 is connected and the fact that $[\mathfrak{h}_1, \mathfrak{a}] = 0$, if $d\pi(A)$ are normal with respect to $\langle \cdot, \cdot \rangle_0$, then they are also normal operators for $(\mathfrak{n}^{\mathbb{C}}, \langle \cdot, \cdot \rangle_1)$ and therefore semisimple. Finally, consider the real part of $\langle \cdot, \cdot \rangle_1$, $\langle \cdot, \cdot \rangle$ which is an inner product with the desired properties.

We therefore get form (28)

$$\tag{31} \begin{aligned} &\langle \operatorname{Ric} Y, Y\rangle = \langle \operatorname{Ric}_{\mathfrak{h}} Y, Y\rangle - \operatorname{tr} S(\operatorname{ad} Y|_{\mathfrak{n}})^2 \\ &\langle \operatorname{Ric} A, A\rangle = -\operatorname{tr} S(\operatorname{ad} A|_{\mathfrak{n}})^2 \\ &\operatorname{Ric}|_{\mathfrak{n}} = -S(\operatorname{ad} H|_{\mathfrak{n}}) + \tfrac{1}{2}\sum[\operatorname{ad} Y_i|_{\mathfrak{n}}, (\operatorname{ad} Y_i|_{\mathfrak{n}})^t] \\ &\langle \operatorname{Ric} Y, A\rangle = -\operatorname{tr} S(\operatorname{ad} Y|_{\mathfrak{n}}) S(\operatorname{ad} A|_{\mathfrak{n}}), \\ &\langle \operatorname{Ric} Y, X\rangle = 0, \qquad \langle \operatorname{Ric} A, X\rangle = 0, \end{aligned}$$

for $Y \in \mathfrak{h}$, $A \in \mathfrak{a}$, $X \in \mathfrak{n}$. Chose an orthonormal basis of $\mathfrak{h}$, $\{Y_1, \dots, Y_r\}$ so that the $Y_j \in \mathfrak{k}$ for $j \leq m$ and $Y_j \in \mathfrak{p}$ for $j > m$. Therefore, $[\operatorname{ad} Y_i|_{\mathfrak{n}}, (\operatorname{ad} Y_i|_{\mathfrak{n}})^t] = 0$.

Let $\{A_1, \dots, A_k\}$ be a basis of $\mathfrak{a}$ so that A_1 is the element as in the statement and $\operatorname{tr} \operatorname{ad} A_i = 0$ for all $i \geq 2$ and take the inner product that makes this an orthonormal basis of $\mathfrak{a}$. Note that up to now the inner product on $\mathfrak{a}$ has no conditions. From this, it is easy to see that $H = \operatorname{tr}(\operatorname{ad} A_1|_{\mathfrak{n}})A_1$.

We note that by the choice of the basis, $\langle \operatorname{Ric} Y_j, A_k\rangle = 0$ for any $j \leq m$. Also, since $\{S(\operatorname{ad} A_i|_{\mathfrak{n}})\}$ is a set of symmetric operators that commutes with each other there exists $\beta = \{X_1, \dots X_n\}$ a basis of $\mathfrak{n}$ of common eigenvectors of $\{S(\operatorname{ad} A_i|_{\mathfrak{n}})\}$. Hence, consider a decomposition $\mathfrak{n} = \mathfrak{n}_1 \oplus \cdots \oplus \mathfrak{n}_r$ such that $\operatorname{ad} A_i|_{\mathfrak{n}_j} = a_{ij}\, Id_{\mathfrak{n}_j}$. Now, since $[Y, A] = 0$, $\operatorname{ad} Y|_{\mathfrak{n}}$ preserves the subspaces $\mathfrak{n}_j$ and therefore we obtain that, for any $j > m$,

$$\begin{aligned} \langle \operatorname{Ric} Y_j, A_k\rangle &= \operatorname{tr} S(\operatorname{ad} Y_j|_{\mathfrak{n}}) S(\operatorname{ad} A_k|_{\mathfrak{n}}) = \sum \operatorname{tr}(\operatorname{ad} Y_j|_{\mathfrak{n}_l}) S(\operatorname{ad} A_k|_{\mathfrak{n}_l}) \\ &= \sum \operatorname{tr}(\operatorname{ad} Y_j|_{\mathfrak{n}_l})(a_{kl}\, Id_{\mathfrak{n}_l}) = 0, \end{aligned}$$

since $\mathfrak{h}$ is semisimple. Henceforth, form (31)

$$\begin{aligned} &\langle \operatorname{Ric} Y, Y\rangle = \langle \operatorname{Ric}_{\mathfrak{h}} Y, Y\rangle - \operatorname{tr} S(\operatorname{ad} Y|_{\mathfrak{n}})^2 \\ &\langle \operatorname{Ric} A, A\rangle = -\operatorname{tr} S(\operatorname{ad} A|_{\mathfrak{n}})^2 \\ &\operatorname{Ric}|_{\mathfrak{n}} = -\operatorname{tr}(\operatorname{ad} A_1|_{\mathfrak{n}}) S(\operatorname{ad} A_1|_{\mathfrak{n}}) \\ &\langle \operatorname{Ric} Y, A\rangle = 0, \quad \langle \operatorname{Ric} Y, X\rangle = 0, \quad \langle \operatorname{Ric} A, X\rangle = 0 \end{aligned}$$

is negative definite, as we wanted to show.

If $\mathfrak{n}$ is not abelian, let $\langle\cdot,\cdot\rangle$ be any inner product on $\mathfrak{g}$ such that $(\mathfrak{h} \oplus \mathfrak{a}) \oplus \mathfrak{n}$ is an orthogonal decomposition. For each $t > 0$ consider $\psi_t \in \mathfrak{gl}(\mathfrak{g})$ such that

$$\psi_t|_{\mathfrak{h}\oplus\mathfrak{a}} = Id, \qquad \psi_t|_{\mathfrak{n}} = t\, Id.$$

It is easy to check that $[\cdot,\cdot]_t = \psi_t \cdot [\cdot,\cdot]$ is given by

$$\tag{32} \begin{aligned} &[X_1, X_2]_t = [X_1, X_2] \quad \text{ for } X_i \in \mathfrak{h} \oplus \mathfrak{a},\ i = 1, 2, \\ &[X_1, X_2]_t = \tfrac{1}{t}[X_1, X_2] \quad \text{ for } X_i \in \mathfrak{n},\ i = 1, 2, \\ &[X_1, X_2]_t = [X_1, X_2] \quad \text{ for } X_1 \in \mathfrak{h} \oplus \mathfrak{a},\ X_2 \in \mathfrak{n}. \end{aligned}$$

In the last two equations we have used that $\mathfrak{n}$ is an ideal. Hence, $\lim\limits_{t\to\infty} [\cdot,\cdot]_t = \mu_0$ is well defined and it is given by

$$\mu_0(X_1, X_2) = [X_1, X_2],\ X_1 \in \mathfrak{h} \oplus \mathfrak{a}, X_2 \in \mathfrak{g}, \quad \mu_0(X_1, X_2) = 0,\ X_i \in \mathfrak{n}.$$

Therefore, the limit Lie algebra satisfy the same conditions as in the statement and $\mathfrak{n}$ is now abelian. Using the previous results, the limit Lie algebra admits an inner product with negative Ricci curvature and therefore, by Proposition (2.1) so does $\mathfrak{g}$. $\square$

Remark 6.3. In particular, when $\mathfrak{a} = \mathbb{R}Z$ is acting as a multiple of the identity and we consider one of the inner products on $\mathfrak{sl}(n, \mathbb{R})$ for $n \geq 3$ given in [2] we get the results of the previous section for any real representation. Recall that in [3] it is shown that most of the simple non-compact Lie algebras admits an inner product satisfying the properties in Lemma 6.2 and from there we get a lot of examples.

Remark 6.4. Let $\mathfrak{g}$ be a Lie algebra with Levi decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{s}$. The radical $\mathfrak{s}$ can be decompose as $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$ where $\mathfrak{n}$ is the nilradical of $\mathfrak{g}$. It is not hard to see that there always exists a complement $\mathfrak{a}$ which satisfies $[\mathfrak{a}, \mathfrak{h}] = 0$, therefore that hypothesis is not so restrictive. In fact, since $\mathfrak{s}$ is an ideal and $\mathfrak{h}$ is non-compact semisimple Lie algebra, there exists an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{s}$ and a basis of $\mathfrak{h}$ β, such that $\operatorname{ad} Y : \mathfrak{s} \to \mathfrak{s}$ are symmetric or skew-symmetric operators for any $Y \in \beta$ (see (30)). Also, for any $Y \in \beta$, $\operatorname{ad} Y$ is a derivation of $\mathfrak{s}$ and hence $\operatorname{ad} Y(\mathfrak{s}) \subset \mathfrak{n}$ (see [4, Lemma 2.6]). Let $\mathfrak{a}$ be the orthogonal complement of $\mathfrak{n}$ and hence, since $\mathfrak{n}$ is an ideal, $\operatorname{ad} Y(\mathfrak{n}) \subset \mathfrak{n}$ and therefore for any $A \in \mathfrak{a}, X \in \mathfrak{n}$

$$\langle [YA], X \rangle = \pm \langle A, [YX] \rangle = 0,$$

for any $Y \in \beta$ and therefore for any $Y \in \mathfrak{h}$.

Finally, coming back to the compact case, we can use the same idea as in the previous theorem to get examples with a non-abelian $\mathfrak{n}$. Since we only have a complete description in the case when $\mathfrak{g} = \mathfrak{su}(2)$ we will only state the result for that case.

Theorem 6.5. *Let $\mathfrak{g} = (\mathfrak{su}(2) \oplus Z\mathbb{R}) \ltimes \mathfrak{n}$ be a Lie algebra where $\mathfrak{n}$ is any nilpotent Lie algebra and $[Z, \mathfrak{su}(2)] = 0$. Let $\pi = ad|_{\mathfrak{su}(2)}$ acting on $\mathfrak{n}$ and let $\mathfrak{n} = \mathfrak{n}_1 \oplus \cdots \oplus \mathfrak{n}_k$ be the decomposition of $\mathfrak{n}$ in irreducible components for π. If π is not trivial and Z acts in each $\mathfrak{n}_i$ as a positive multiple of the identity, then $\mathfrak{g}$ admits an inner product with negative Ricci curvature.*

Proof. Let $(\mathfrak{g}, [\cdot, \cdot])$ be the Lie algebra as defined above and endow it with the inner product such that $\|Z\| = 1$ and $Z\mathbb{R} \oplus \mathfrak{h} \oplus \mathfrak{n}$ is an orthogonal decomposition and let $\psi_t \in \mathfrak{gl}(\mathfrak{g})$ as in (32) where $\mathfrak{a} = \mathbb{R}Z$. Hence, as it was shown in the previous theorem, $\mu_o = \lim_{t \to \infty} \psi_t \cdot [\cdot, \cdot]$ is well defined and it is given by

$$\mu_0(X_1, X_2) = [X_1, X_2],\ X_1 \in \mathfrak{su}(2) \oplus \mathbb{R}Z, X_2 \in \mathfrak{g}, \quad \mu_0(X_1, X_2) = 0,\ X_i \in \mathfrak{n}.$$

Note that $(\mathfrak{g}, \mu_o)$ Let $\pi = \operatorname{ad}|_{\mathfrak{su}(2)}$ acting on $\mathfrak{n}$ and decompose the linear space $\mathfrak{n}$ in irreducible components for the action of π. Note that Z act as a positive multiple of the identity in each $\mathfrak{n}_i$. Now we can follow the same proof given in [15] with a few little differences since the mean curvature vector is different (see [15, Remark 3.13]). □

References

[1] I. Dotti, Ricci curvature of left invariant metrics on solvable unimodular Lie groups, *Math. Z.* **180** (1982), 257-263.

[2] I. Dotti, M. L. Leite, Metrics of negative Ricci curvature on SL$(n, \mathbb{R})$, $n \geq 3$. *J. Differential Geom.* **17**, no. 4, (1982), 635-641.

[3] I. Dotti, M. L. Leite, R. Miatello, Negative Ricci curvature on complex semisimple Lie grous, *Geom. Dedicata* **17** (1984), 207-218.

[4] C. Gordon, E. Wilson, Isometry groups of Riemannian solvmanifolds, *Trans. Amer. Math. Soc.* **307** (1988), 245-269.

[5] S. Helgason, Differential geometry, Lie groups and symmetric spaces, *GSM* **34** (1978), AMS.

[6] M. Jablonski, P. Petersen, A step towards the Alekseevskii Conjecture, *Math Ann.* in press.

[7] A. Knapp, Lie groups beyond an introduction, *Prog. Math* **140** (2005), Birkhäuser.

[8] A. Knapp, Representation Theory of Semisimple Groups. *Princ. Land. Math*, Pinceton University Press, Princeton (2001).

[9] R. Lafuente, J. Lauret, Structure of Homogeneous Ricci Solitons and the Alekseevskii Conjecture, *J. Diff. Geom* **98** (2014), 315-147.

[10] J. Lauret, Ricci soliton solvmanifolds, *J. reine angew. Math.* **650** (2011), 1-21.

[11] J. Lauret, C. Will, On the Diagonalization of the Ricci Flow on Lie Groups, *Proc. Amer. Math. Soc.*, **141**, no. 10, (2013), 36513663.

[12] Y. Nikolayevsky, Yu.G. Nikonorov, On solvable Lie groups of negative Ricci curvature, *Math. Z.* **280** (2015), 1-16.

[13] Y. Nikolayevsky, Solvable extensions of negative Ricci curvature of filliform Lie Groups, *Math. Nach.* **289** (2016), 321-331.

[14] C. Will, The space of solsolitons in low dimensions, *Ann. Global Anal. Geom.* **40** (2011), 291-309.

[15] C. Will, Negative Ricci Curvature on some non-solvable Lie groups, *Geom. Dedicata* **186** (2017), no. 1, 181-195.

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