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Nichols algebras over groups with finite root system of rank two III

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A B S T R A C T

We compute the finite-dimensional Nichols algebras over the sum of two simple Yetter–Drinfeld modules V and W over non-abelian epimorphic images of a certain central extension of the dihedral group of eight elements or $\mathbf{SL}(2,3)$, and such that the Weyl groupoid of the pair (V,W) is finite. These central extensions appear in the classification of non-elementary finite-dimensional Nichols algebras with finite Weyl groupoid of rank two. We deduce new information on the structure of primitive elements of finite-dimensional Nichols algebras over groups.

Contents

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Introduction

In [6] an approach to a particular instance of the classification of finite-dimensional Nichols algebras was initiated. Assume that U is the direct sum of two absolutely simple Yetter–Drinfeld modules V and W and that G is generated by the support of U. If the Nichols algebra of U is finite-dimensional, then the Weyl groupoid of (V, W) is finite, and this groupoid can be calculated, see [2,10,3,7,6]. If the square of the braiding between V and W is the identity, then by a result of Graña the Nichols algebra of U is the tensor product of the Nichols algebras of V and W. So we are interested in the remaining cases. For a particular class of groups and Yetter–Drinfeld modules, it was possible to construct and classify those U with a finite-dimensional Nichols algebra.

A breakthrough for the approach in [6] was achieved in [9, Thm. 4.5], where it was proved that if the square of the braiding between V and W is not the identity and $\mathfrak{B}(V \oplus W)$ is finite-dimensional, then G is a non-abelian quotient of one of four groups which can be described explicitly. Again, the main tool was the theory of Weyl groupoids of tuples of simple Yetter–Drinfeld modules.

The theorem in [9] has very far reaching consequences. One of these consequences is the existence of good bounds for the dimensions of V and W, see [9, Cor. 4.6]. Another consequence is the possibility to obtain new examples of finite-dimensional Nichols algebras. This can be done if the Weyl groupoids (and of course the Nichols algebras) appearing in the context of [9] are studied.

In this work we study finite-dimensional Nichols algebras over two of the groups appearing in [9, Thm. 4.5]. One of these groups is the so-called group T, which is a certain central extension of the group $\mathbf{SL}(2,3)$, see Section 2.1. The other is Γ_4, a central extension of the dihedral group of eight elements, see Section 5.1. To study these Nichols algebras we recognize some pairs (V, W) of Yetter–Drinfeld modules over non-abelian epimorphic images of T and Γ_4 admitting a Cartan matrix of finite type. Then we determine when the Yetter–Drinfeld modules $(\operatorname{ad} V)^m(W)$ and $(\operatorname{ad} W)^m(V)$ are absolutely simple or zero for all $m \in \mathbb{N}$. In fact, around one third of the paper consists of these calculations, which form the most technical but highly important part of this work. With the results of the calculations in the pocket, we can compute the Cartan matrices, the reflections and the Weyl groupoid of the pairs (V, W). This allows us to determine the structure of Nichols algebras over non-abelian epimorphic images of T and Γ_4.

As a consequence, we will obtain two (new families of) finite-dimensional Nichols algebras, see Theorems 2.9 and 5.5. One of these families of Nichols algebras has a root system of type G_2 and dimension

$$\begin{cases} 6^3 72^3 & \text{if } \operatorname{char}\mathbb{K} \neq 2, \\ 3^3 36^3 & \text{if } \operatorname{char}\mathbb{K} = 2. \end{cases}$$

The others have a root system of type B_2 and dimension

$$\begin{cases} 8^2 64^2 & \text{if } \operatorname{char}\mathbb{K} \neq 2, \\ 4^2 64^2 & \text{if } \operatorname{char}\mathbb{K} = 2. \end{cases}$$

As a byproduct of our study of the Nichols algebras associated to epimorphic images of T and Γ_4 we improve the application given in [9, Cor. 4.6]. More precisely, under the assumptions of [9, Thm. 4.5] we conclude that the support of the sum of the two simple Yetter–Drinfeld modules is isomorphic (as a quandle) to one of the five quandles listed in Theorem 7.1.

This work and the results of [6] and [9] are an important part of the classification of Nichols algebras admitting a finite root system of rank two achieved in [8].

The paper is organized as follows. Section 1 is devoted to state some general facts about adjoint actions and braidings. In Section 2 we review basic facts about the group T and state the main results concerning Nichols algebras over non-abelian epimorphic images of T, see Proposition 2.8 and Theorem 2.9. These results are proved in Sections 3 and 4. In Section 5 we review the basic facts concerning the group Γ_4 and state our main result about Nichols algebras over non-abelian epimorphic images of Γ_4, see Theorem 5.5. This theorem is then proved in Section 6. Finally, in Section 7 the application mentioned in the previous paragraph is deduced.

1. Some preliminaries

Fix a field $\mathbb{K}$. We use the notations and the definitions given in [9, Sect. 2.1] mostly without recalling them again. However, we recall the definition of the Cartan matrix of a pair of Yetter–Drinfeld modules. Let G be a group and let $V, W \in {}^G_G\mathcal{YD}$. If $(\operatorname{ad} V)^p(W) = 0$ and $(\operatorname{ad} W)^q(V) = 0$ for some $p, q \in \mathbb{N}_0$ then one defines the *Cartan matrix* $(a_{ij}^M) \in \mathbb{Z}^{2\times 2}$ of M by

$$\begin{aligned} a_{11}^M &= a_{22}^M = 2, \\ a_{12}^M &= -\sup\{m \in \mathbb{N}_0 \colon\ (\operatorname{ad} V)^m(W) = 0\}, \\ a_{21}^M &= -\sup\{m \in \mathbb{N}_0 \colon\ (\operatorname{ad} W)^m(V) = 0\}. \end{aligned}$$

In [9], a sufficient criterion for the non-vanishing of

$$(\operatorname{ad} W)^m(V) \subseteq \mathfrak{B}(V \oplus W)$$

for $m \in \mathbb{N}$ was formulated in terms of elements of G satisfying some properties. We can use this idea to obtain a condition on the braiding of $V \oplus W$ under some assumptions on $(\operatorname{ad} W)^m(V)$ and $(\operatorname{ad} W)^{m+1}(V)$ for some $m \in \mathbb{N}$. The following proposition (and its proof) is analogous to [9, Prop. 5.5]. Before reading it, we strongly recommend to read [9, Prop. 5.5] and its proof.

Proposition 1.1. *Let G be a group and let $V, W \in {}^G_G\mathcal{YD}$. Let $m \in \mathbb{N}$, $i \in \{1, \dots, m\}$, $r_1, \dots, r_m, p_1, \dots, p_m \in \operatorname{supp} W$ and $s, p_{m+1} \in \operatorname{supp} V$. Assume that $(p_1, \dots, p_{m+1}) \in \operatorname{supp} Q_m(r_1, \dots, r_m, s)$, $Q_{m+1}(p_i, r_1, \dots, r_m, s) = 0$, and that*

$$p_{i+1} \triangleright p_i \neq p_i, \qquad p_j \triangleright p_i = p_i \quad \text{for all } j \text{ with } i+1 < j \leq m+1, \tag{1.1}$$

$$p_i \notin \{p_j \mid 1 \leq j \leq m,\ j \neq i\} \cup \bigl\{(p_{j+1} \cdots p_{m+1})^{-1} \triangleright p_j \bigm| 1 \leq j < i\bigr\}. \tag{1.2}$$

Then $\dim W_{p_i} = 1$ *and* $p_i w = -w$ *for all* $w \in W_{p_i}$.

Proof. By the definition of Q_{m+1}, the set $\operatorname{supp} Q_{m+1}(p_i, r_1, \dots, r_m, s)$ consists of tuples of the form

$$\begin{aligned} &\bigl(p_i \triangleright p'_1, \dots, p_i \triangleright p'_{j-1}, p_i, p'_j, \dots, p'_{m+1}\bigr), \\ &\bigl(p_i \triangleright p'_1, \dots, p_i \triangleright p'_{j-1}, p_i p'_j \cdots p'_{m+1} \triangleright p_i, p_i \triangleright p'_j, \dots, p_i \triangleright p'_{m+1}\bigr) \end{aligned} \tag{1.3}$$

with $1 \leq j \leq m+1$, where $(p'_1, \dots, p'_{m+1}) \in \operatorname{supp} Q_m(r_1, \dots, r_m, s)$. This for $j = i$ and the assumption imply that

$$\bigl(p_i \triangleright p_1, \dots, p_i \triangleright p_{i-1}, p_i, p_i, p_{i+1}, \dots, p_{m+1}\bigr)$$

appears among the tuples in (1.3). Comparing this tuple with all other possible tuples similarly to the proof of [9, Prop. 5.5], one obtains that it appears precisely twice among the tuples in (1.3): In the first line for $j = i$ and for $j = i+1$, where $p'_k = p_k$ for all $k \in \{1, 2, \dots, m+1\}$. The two tuples correspond to the summands

$$c_{i-1\,i} \cdots c_{23} c_{12} (\mathrm{id} \otimes \pi_{p_1} \otimes \cdots \otimes \pi_{p_m} \otimes \pi_{p_{m+1}})(u)$$

and

$$c_{i\,i+1} \cdots c_{23} c_{12} (\mathrm{id} \otimes \pi_{p_1} \otimes \cdots \otimes \pi_{p_m} \otimes \pi_{p_{m+1}})(u)$$

of $\varphi_{m+1}(u)$ for any $u \in W_{p_i} \otimes Q_m(r_1, \dots, r_m, s)$. As $Q_{m+1}(p_i, r_1, \dots, r_m, s) = 0$ by assumption, we obtain that

$$(\mathrm{id} + c_{i,i+1}) c_{i-1\,i} \cdots c_{23} c_{12} (\mathrm{id} \otimes \pi_{p_1} \otimes \cdots \otimes \pi_{p_m} \otimes \pi_{p_{m+1}})(u) = 0$$

for all $u \in W_{p_i} \otimes Q_m(r_1, \dots, r_m, s)$. Then there exists $w_0 \in W_{p_i} \setminus \{0\}$ such that $(\mathrm{id} + c)(w \otimes w_0) = 0$ for all $w \in W_{p_i}$. Since $c(w \otimes w_0) = p_i w_0 \otimes w$ for all $w \in W_{p_i}$, we conclude that $\dim W_{p_i} = 1$ and $p_i w = -w$ for all $w \in W_{p_i}$. $\square$

Lemma 1.2. *Let G be a group and let $V, W \in {}^G_G\mathcal{YD}$. Assume that there exist $q_V, q_W \in \mathbb{K}$ such that $xv = q_V v$ and $yw = q_W w$ for all $x, y \in G$, $v \in V_x$, $w \in W_y$. Then*

$$c_{W,V} c_{V,W}(v \otimes w) = q_V^{-1} xyv \otimes q_W^{-1} xyw$$

for all $x, y \in G$, $v \in V_x$, $w \in W_y$.

Proof. A direct computation yields

$$c_{W,V} c_{V,W}(v \otimes w) = c_{W,V}(xw \otimes v) = xyx^{-1}v \otimes xw.$$

Since $x^{-1}v = q_V^{-1}v$ and $xyw = q_W xw$, the lemma follows. □

One of the key step towards our main result depends on the calculation of the Yetter–Drinfeld modules $(\operatorname{ad} V)^m(W)$ and $(\operatorname{ad} W)^m(V)$ for some Yetter–Drinfeld modules V and W and for all $m \in \mathbb{N}$. For that purpose, the following lemma is useful.

Lemma 1.3. *(See [6, Thm. 1.1].) Let V and W be Yetter–Drinfeld modules over a Hopf algebra H with bijective antipode. Let $\varphi_0 = 0$ and $\varphi_m \in \operatorname{End}(V^{\otimes m} \otimes W)$ be given by*

$$\varphi_m = \mathrm{id} - c_{V^{\otimes(m-1)}\otimes W, V} c_{V, V^{\otimes(m-1)}\otimes W} + (\mathrm{id} \otimes \varphi_{m-1}) c_{1,2}$$

for all $m \geq 1$, and let $X_0^{V,W} = W$, and

$$X_m^{V,W} = \varphi_m(V \otimes X_{m-1}) \subseteq V^{\otimes m} \otimes W$$

for all $m \geq 1$. Then $(\operatorname{ad} V)^n(W) \simeq X_n^{V,W}$ for all $n \in \mathbb{N}_0$.

In the paper, we will use the Yetter–Drinfeld modules $X_m^{V,W}$ for calculations, but in the applications we usually turn back to the more suggestive module $(\operatorname{ad} V)^m(W)$.

The basic theory of Weyl groupoids and Nichols algebras of [2] and [7] is reviewed in [6, Sect. 2].

2. Nichols algebras over epimorphic images of T

2.1. Preliminaries

Recall that the group T is

$$T = \langle \zeta \rangle \times \langle \chi_1, \chi_2, \chi_3, \chi_4 \mid \chi_i \chi_j = \chi_{i \triangleright j} \chi_i,\ i, j \in \{1, 2, 3, 4\} \rangle,$$

where $\triangleright$ is defined by

$\triangleright$	1	2	3	4
1	1	4	2	3
2	3	2	4	1
3	4	1	3	2
4	2	3	1	4

The table describes the structure of the quandle associated to the vertices of the tetrahedron, see [1, §1]. By [4, Lems. 2.17 and 2.18],

$$\chi_1^3 = \chi_2^3 = \chi_3^3 = \chi_4^3 \tag{2.1}$$

is a central element of T. Moreover, the center of T is $Z(T) = \langle \chi_1^3, \chi_1\chi_2\chi_3, \zeta\rangle$.

The group T can be presented by generators χ_1, χ_2, ζ with relations

$$\zeta\chi_1 = \chi_1\zeta, \qquad \zeta\chi_2 = \chi_2\zeta, \qquad \chi_1\chi_2\chi_1 = \chi_2\chi_1\chi_2, \qquad \chi_1^3 = \chi_2^3. \tag{2.2}$$

Then $\chi_3 = \chi_2\chi_1\chi_2^{-1}$ and $\chi_4 = \chi_1\chi_2\chi_1^{-1}$ in T, and the elements χ_1, χ_2, χ_3, χ_4 form a conjugacy class of T. The group T is isomorphic to the enveloping group of the quandle $\chi_1^T \cup \zeta^T$.

Remark 2.1. The $\chi_1^T \cup \zeta^T$ is the disjoint union of the trivial quandle with one element and the quandle associated to the vertices of the tetrahedron.

In what follows, let G be a non-abelian quotient of the group T. Equivalently, the elements χ_i, $1 \leq i \leq 4$, represent pairwise different elements of G. Let $z \in Z(G)$ and $x_1 \in G$ such that $G = \langle z, x_1^G\rangle$ and there exists a quandle isomorphism $f : \chi_1^T \cup \{\zeta\} \to x_1^G \cup \{z\}$ with $f(\chi_1) = x_1$. For all $2 \leq i \leq 4$ let $x_i := f(\chi_i)$. The quandle isomorphism f induces a surjective group homomorphism $T \to G$.

Lemma 2.2. *The following hold:*

(1) $G^z = G$.
(2) $G^{x_1} = \langle x_1, x_2x_3, z\rangle$.

Proof. The first claim is trivial since z is central. Let us prove (2). Since $T^{\chi_1} = \langle \chi_1, \chi_2\chi_3, \zeta\rangle$ by [4, Lem. 5.5], we obtain that $\langle x_1, x_2x_3, z\rangle \subseteq G^{x_1}$. Moreover, T^{χ_1} has index four in T. From $|x_1^G| = 4$ we conclude that G^{x_1} has index four in G. Hence $G^{x_1} = \langle x_1, x_2x_3, z\rangle$. □

Lemma 2.3. *Let* $V, W \in {}^G_G\mathcal{YD}$ *such that* $\operatorname{supp} V = z^G$, $\operatorname{supp} W = x_1^G$, *and* $(\operatorname{ad} V)(W) \neq 0$. *Then* $(\operatorname{ad} W)^3(V) \neq 0$. *If* $(\operatorname{ad} W)^3(V)$ *is irreducible, then* $\dim W = 4$ *and* $x_1w = -w$ *for all* $w \in W_{x_1}$.

Proof. First, $(x_4, z) \in \operatorname{supp} Q_1(x_4, z)$ since $c_{V,W}c_{W,V} \neq \mathrm{id}$. Therefore [9, Prop. 5.5] implies that $(x_1, x_4, z) \in \operatorname{supp} Q_2(x_1, x_4, z)$. Since

$$x_2 \notin \{x_1, x_4, (x_4 z)^{-1} \triangleright x_1 = x_3\},$$

[9, Prop. 5.5] with $i = 2$ yields

$$(x_2 \triangleright x_1, x_2, x_4, z) \in \operatorname{supp} Q_3(x_2, x_1, x_4, z).$$

In particular, $(\operatorname{ad} W)^3(V) \neq 0$. It is easy to check that $x_2x_1x_4 = x_1x_2x_3$, and hence the non-central element $x_1x_1x_4$ is not conjugate to the central element $x_2x_1x_4$ in G. Therefore $Q_3(x_1, x_1, x_4, z) = 0$ by the irreducibility of $(\operatorname{ad} W)^3(V)$, and hence Proposition 1.1 with the parameters $m = 2$, $i = 1$, and $(p_1, p_2, p_3) = (r_1, r_2, s) = (x_1, x_4, z)$ yields the claim. □

Let $W = M(x_1, \sigma)$ be a Yetter–Drinfeld module over G for some absolutely irreducible representation σ of G^{x_1}. The centralizer $G^{x_1} = \langle x_1, x_2x_3, z\rangle$ is abelian and hence $\deg \sigma = 1$. Let $\epsilon = \sigma(x_2x_3)$.

Remark 2.4. Let $w_1 \in W_{x_1}$ such that $w_1 \neq 0$. Then $w_1, w_2 := \sigma(x_1)^{-1}x_4w_1$, $w_3 := \sigma(x_1)^{-1}x_2w_1$, $w_4 := \sigma(x_1)^{-1}x_3w_1$ is a basis of W. The degrees of these vectors are x_1, x_2, x_3 and x_4, respectively. Furthermore, $x_iw_j = q_{ij}w_{i\triangleright j}$, where

$$q_{ij} = \begin{pmatrix} \sigma(x_1) & \sigma(x_1) & \sigma(x_1) & \sigma(x_1) \\ \sigma(x_1) & \sigma(x_1) & \sigma(x_1)^3\epsilon^{-1} & \sigma(x_1)^{-1}\epsilon \\ \sigma(x_1) & \sigma(x_1)^{-1}\epsilon & \sigma(x_1) & \sigma(x_1)^3\epsilon^{-1} \\ \sigma(x_1) & \sigma(x_1)^3\epsilon^{-1} & \sigma(x_1)^{-1}\epsilon & \sigma(x_1) \end{pmatrix}.$$

For example, one can easily compute that

$$x_2w_4 = \sigma(x_1)^{-1}x_2x_3w_1 = \sigma(x_1)^{-1}\epsilon w_1.$$

Then, since $x_1^3 = x_2^3$ is central,

$$\begin{aligned} x_2w_3 &= \sigma(x_1)^{-1}x_2x_2w_1 = \sigma(x_1)^{-1}x_2^{-1}x_2^3w_1 \\ &= \sigma(x_1)^{-1}x_2^{-1}x_1^3w_1 = \sigma(x_1)^2x_2^{-1}w_1 = \sigma(x_1)^3\epsilon^{-1}w_4. \end{aligned}$$

Remark 2.5. Assume that $\sigma(x_1) = -1$. Since $x_1^4 = (x_2x_3)^2$, we obtain that $\epsilon^2 = 1$. Then the action of G on W is given by the following table:

W	w_1	w_2	w_3	w_4
x_1	$-w_1$	$-w_4$	$-w_2$	$-w_3$
x_2	$-w_3$	$-w_2$	$-\epsilon w_4$	$-\epsilon w_1$
x_3	$-w_4$	$-\epsilon w_1$	$-w_3$	$-\epsilon w_2$
x_4	$-w_2$	$-\epsilon w_3$	$-\epsilon w_1$	$-w_4$
z	$\sigma(z)w_1$	$\sigma(z)w_2$	$\sigma(z)w_3$	$\sigma(z)w_4$

Let $V = M(z, \rho)$ for some absolutely irreducible representation ρ of the centralizer $G^z = G$. The following lemma tells us that we will only need to study those representations of degree at most two.

Lemma 2.6. *Assume that* $(\operatorname{ad} W)(V) \subseteq \mathfrak{B}(V \oplus W)$ *and* $(\operatorname{ad} W)^2(V)$ *are absolutely simple Yetter–Drinfeld modules over* G. *Then* $\dim V \leq 2$.

Proof. By [6, Lem. 1.7],

$$X_1^{W,V} = \varphi_1(W \otimes V) = \mathbb{K}G\{\varphi_1(w_1 \otimes v) \mid v \in V\}.$$

Moreover, a direct computation yields

$$\varphi_1(w_i \otimes v) = w_i \otimes v - c_{V,W}c_{W,V}(w_i \otimes v) = w_i \otimes \big(v - \sigma(z)x_i v\big) \tag{2.3}$$

for $i \in \{1, 2, 3, 4\}$ and $v \in V$. Since $X_1^{W,V} \simeq (\operatorname{ad} W)(V) \neq 0$, there exists $v \in V$ such that the tensor $w_1 \otimes (v - \sigma(z)x_1 v) \in (W \otimes V)_{x_1 z}$ is non-zero. Let $v_0 := v - \sigma(z)x_1 v$. Since $X_1^{W,V}$ is absolutely simple and the centralizer of $x_1 z$ is abelian, $(X_1^{W,V})_{x_1 z}$ is one-dimensional. Therefore there exist $\alpha_1, \alpha_2 \in \mathbb{K}^\times$ such that

$$\alpha_1^4 = \alpha_2^2, \qquad x_1 v_0 = \alpha_1 v_0, \qquad x_2 x_3 v_0 = \alpha_2 v_0. \tag{2.4}$$

By [6, Lem. 1.7],

$$X_2^{W,V} = \varphi\big(W \otimes X_1^{W,V}\big) = \mathbb{K}G\big\{\varphi_2(w_1 \otimes w_1 \otimes v_0), \varphi_2(w_2 \otimes w_1 \otimes v_0)\big\}.$$

Let $y := \varphi_2(w_2 \otimes w_1 \otimes v_0) \in X_2^{W,V}$. A direct calculation yields

$$\begin{aligned} y &= \varphi_2(w_2 \otimes w_1 \otimes v_0) \\ &= w_2 \otimes w_1 \otimes v_0 - x_3 z w_2 \otimes x_2 w_1 \otimes x_2 v_0 + x_2 w_1 \otimes w_2 \otimes \big(v_0 - \sigma(z)x_2 v_0\big), \end{aligned}$$

and hence $y \in (W \otimes W \otimes V)_{x_2 x_1 z}$ is non-zero. Since $(\operatorname{ad} W)^2(V)$ is absolutely simple and the centralizer of $x_2 x_1 z$ is the abelian group

$$G^{x_2 x_1 z} = x_3 G^{x_2 x_3 z} x_3^{-1} = \langle x_2 x_1 z, x_4, z\rangle,$$

there exists $\xi \in \mathbb{K}$ such that $x_4 y = \xi y$. The second tensor factors w_1, $x_2 w_1$, and w_2 in y are linearly independent and $2 \triangleright 1 = 3$. Hence, by comparing the third tensor factors, we conclude that there exists $\alpha_3 \in \mathbb{K} \setminus \{0\}$ such that

$$\alpha_3\big(v_0 - \sigma(z)x_2 v_0\big) = x_4 v_0. \tag{2.5}$$

By the presentation for T given in (2.2) and by the irreducibility of V, it is enough to show that $S := \operatorname{span}_{\mathbb{K}}\{v_0, x_2 v_0\}$ is stable under the action of x_1 and x_2. First, $x_1 v_0 \in S$

since $x_1v_0 = \alpha_1v_0$. Equations $x_1x_2 = x_4x_1$ and (2.5) imply that $x_1x_2v_0 = x_4x_1v_0 = \alpha_1x_4v_0 \in S$. Finally, applying x_2 to Eq. (2.5) and using $x_2x_4 = x_4x_1$ we conclude that $x_2^2v_0 \in S$. □

Lemma 2.7. *Assume that $\mathbb{K}$ is algebraically closed. Let (ρ, U) be an irreducible representation of $\mathbb{K}G$ of degree 2. Then $\mathrm{char}(\mathbb{K}) \neq 2$, $\rho(z) \in \mathbb{K}^\times$ and there exist $\alpha, \beta \in \mathbb{K}^\times$ with $\beta^2 + \beta + 1 = 0$ and a basis of U such that*

$$\rho(x_1) = \begin{pmatrix} \alpha & -\alpha^2\beta^2 \\ 0 & \alpha\beta \end{pmatrix}, \qquad \rho(x_2) = \begin{pmatrix} 0 & -\alpha^2\beta \\ 1 & -\alpha\beta^2 \end{pmatrix},$$
$$\rho(x_3) = \begin{pmatrix} \alpha\beta & 0 \\ \beta^2 & \alpha \end{pmatrix}, \qquad \rho(x_4) = \begin{pmatrix} -\alpha\beta^2 & -\alpha^2 \\ \beta & 0 \end{pmatrix}, \tag{2.6}$$

with respect to this basis. Further, $\rho(x_1x_2x_3) = -\alpha^3 \mathrm{id}_U$.

Proof. Let $v_0 \in U \setminus \{0\}$ and let $\alpha_1, \alpha_2 \in \mathbb{K}^\times$ such that $x_1v_0 = \alpha_1v_0$ and $x_2x_3v_0 = \alpha_2v_0$. Then $\alpha_1^4 = \alpha_2^2$. Since $\deg \rho = 2$ and G is generated by x_1, x_2, and the central element z, $U = \mathrm{span}_\mathbb{K}\{v_0, x_2v_0\}$ and $x_3v_0 = \beta_1v_0 + \beta_2x_2v_0$ and $x_4v_0 = \beta_3v_0 + \beta_4x_2v_0$ for some $\beta_1, \beta_2, \beta_3, \beta_4 \in \mathbb{K}$. Writing $x_2x_3v_0 = \alpha_2v_0$ as $\alpha_2^{-1}x_3v_0 = x_2^{-1}v_0$ and using (2.1) we conclude that

$$x_2(x_2v_0) = x_2^{-1}x_1^3v_0 = \alpha_1^3\alpha_2^{-1}x_3v_0 = \alpha_1^3\alpha_2^{-1}(\beta_1v_0 + \beta_2x_2v_0).$$

Therefore

$$\rho(x_1) = \begin{pmatrix} \alpha_1 & \alpha_1\beta_3 \\ 0 & \alpha_1\beta_4 \end{pmatrix}, \qquad \rho(x_2) = \begin{pmatrix} 0 & \alpha_1^3\alpha_2^{-1}\beta_1 \\ 1 & \alpha_1^3\alpha_2^{-1}\beta_2 \end{pmatrix},$$
$$\rho(x_3) = \begin{pmatrix} \beta_1 & 0 \\ \beta_2 & \alpha_1 \end{pmatrix}, \qquad \rho(x_4) = \begin{pmatrix} \beta_3 & \alpha_2 \\ \beta_4 & 0 \end{pmatrix}.$$

Since $\det \rho(x_1) = \det \rho(x_3)$ and $x_3x_2 = x_1x_3$, we obtain that $\alpha_1\beta_4 = \beta_1$ and

$$\beta_2\beta_4 = 1, \qquad \beta_2\beta_3 + \beta_1 = 0, \qquad \alpha_1\alpha_2^{-1}\beta_1^2 = \beta_3, \qquad \alpha_1\alpha_2^{-1}\beta_2(\alpha_1 + \beta_1) = \beta_4.$$

Let $\alpha := \alpha_1$ and $\beta := \beta_4$. Then the above equations are equivalent to

$$\alpha_2 = -\alpha^2, \qquad \beta_1 = \alpha\beta, \qquad \beta_2 = \beta^2, \qquad \beta_3 = -\alpha\beta^2, \qquad \beta^2 + \beta + 1 = 0.$$

Hence we conclude (2.6). Since $\rho(x_1x_2x_3)v_0 = \alpha_1\alpha_2v_0 = -\alpha^3v_0$, we obtain that $\rho(x_1x_2x_3) = -\alpha^3\mathrm{id}_U$ from $x_1x_2x_3 \in Z(G)$, the absolute irreducibility of ρ, and Schur's Lemma.

Assume that $\mathrm{char}\,\mathbb{K} = 2$. Then $v = \alpha v_0 + x_2v_0 \in U$ is a ρ-invariant vector. This is a contradiction to the irreducibility of (ρ, U). □

2.2. Main results

Let G, $z, x_1, \dots, x_4$, V and W be as in Section 2.1. Our aim is to prove Proposition 2.8 and Theorem 2.9 below.

Proposition 2.8. *Let $V = M(z,\rho)$ and $W = M(x_1,\sigma)$ be absolutely simple Yetter–Drinfeld modules over G. Assume that (V,W) admits all reflections, the Weyl groupoid $\mathcal{W}(V,W)$ is finite, and the Cartan matrix of (V,W) is non-diagonal and of finite type. Then* $\deg\rho = 1$.

The proof of Proposition 2.8 will be given in Section 3.

Recall that $(k)_t = 1 + t + \cdots + t^{k-1}$ for all $k \in \mathbb{N}$.

Theorem 2.9. *Let $V = M(z,\rho)$ and $W = M(x_1,\sigma)$ be absolutely simple Yetter–Drinfeld modules over G. Assume that $c_{W,V}c_{V,W} \neq \mathrm{id}_{V\otimes W}$. The following are equivalent:*

(1) *The Nichols algebra $\mathfrak{B}(V \oplus W)$ is finite-dimensional.*
(2) *The pair (V,W) admits all reflections and $\mathcal{W}(V,W)$ is finite.*
(3) $\deg\rho = 1$, *and* $(\rho(x_1)\sigma(z))^2 - \rho(x_1)\sigma(z) + 1 = 0$, $\sigma(x_1) = -1$, $\sigma(x_2x_3) = 1$, $\rho(x_1z)\sigma(z) = 1$.

In this case, $\mathcal{W}(V,W)$ is standard with Cartan matrix of type G_2. If $\operatorname{char}\mathbb{K} \neq 2$ *then*

$$\begin{aligned}&\mathcal{H}_{\mathfrak{B}(V\oplus W)}(t_1,t_2)\\ &\quad = (6)_{t_1}(6)_{t_1t_2^3}(6)_{t_1^2t_2^3}(2)_{t_2}^2(3)_{t_2}(6)_{t_2}(2)_{t_1t_2}^2(3)_{t_1t_2}(6)_{t_1t_2}(2)_{t_1t_2^2}^2(3)_{t_1t_2^2}(6)_{t_1t_2^2},\end{aligned}$$

and $\dim\mathfrak{B}(V\oplus W) = 6^3 72^3 = 80\,621\,568$. *If* $\operatorname{char}\mathbb{K} = 2$ *then*

$$\mathcal{H}_{\mathfrak{B}(V\oplus W)}(t_1,t_2) = (3)_{t_1}(3)_{t_1t_2^3}(3)_{t_1^2t_2^3}(2)_{t_2}^2(3)_{t_2}^2(2)_{t_1t_2}^2(3)_{t_1t_2}^2(2)_{t_1t_2^2}^2(3)_{t_1t_2^2}^2,$$

and $\dim\mathfrak{B}(V\oplus W) = 3^3 36^3 = 1\,259\,712$.

We will prove Theorem 2.9 in Section 4.

3. Proof of Proposition 2.8

Let $V = M(z,\rho)$ and $W = M(x_1,\sigma)$ as in Section 2.1. We write $X_n = X_n^{V,W}$ and $\varphi_n = \varphi_n^{V,W}$ for all $n \in \mathbb{N}_0$ if no confusion can arise. We now prepare the proof of Proposition 2.8. Assume that $\deg\rho = 2$, ρ is given by (2.6) of Lemma 2.7 with respect to a basis $\{v_0, x_2v_0\}$ of V, and that the characteristic of $\mathbb{K}$ is not 2.

Assume that σ is an absolutely irreducible representation of G^{x_1} with $\sigma(x_1) = -1$. Then $\sigma(x_2x_3)^2 = 1$. The action of G on W is described in Remark 2.5. We first compute $(\operatorname{ad}V)(W) \simeq X_1^{V,W}$. By [6, Lem. 1.7],

$$X_1^{V,W} = \varphi_1(V \otimes W) = \mathbb{K}G\{\varphi_1(v_0 \otimes w_1), \varphi_1(x_2 v_0 \otimes w_1)\}.$$

We record explicit formulas for later use in the following lemma.

Lemma 3.1. *Assume that $\sigma(x_1) = -1$. Then the following hold:*

$$\varphi_1(v_0 \otimes w_1) = \big(1 - \sigma(z)\alpha\big) v_0 \otimes w_1, \tag{3.1}$$

$$\varphi_1(x_2 v_0 \otimes w_1) = \big(1 - \sigma(z)\alpha\beta\big) x_2 v_0 \otimes w_1 + \sigma(z)\alpha^2\beta^2 v_0 \otimes w_1. \tag{3.2}$$

Further $w_1' := \varphi_1(x_2 v_0 \otimes w_1) \in (V \otimes W)_{x_1 z}$ is non-zero and hence $X_1^{V,W} \neq 0$.

Proof. Eq. (3.1) follows by a direct computation using Remark 2.5 and (2.6). Let us prove Eq. (3.2). Using Remark 2.5 and (2.6) we obtain

$$\begin{aligned} c_{W,V} c_{V,W}(x_2 v_0 \otimes w_1) &= c_{W,V}(z w_1 \otimes x_2 v_0) \\ &= \sigma(z) x_1 x_2 v_0 \otimes w_1 \\ &= \sigma(z)\big(-\alpha^2\beta^2 v_0 + \alpha\beta x_2 v_0\big) \otimes w_1. \end{aligned}$$

Since $\varphi_1 = \mathrm{id} - c_{W,V} c_{V,W}$, this implies Eq. (3.2). □

Lemma 3.2. *Assume that $\sigma(x_1) = -1$. Then $X_1^{V,W}$ is absolutely simple if and only if $(1-\sigma(z)\alpha)(1-\sigma(z)\alpha\beta) = 0$. In this case, $X_1^{V,W} \simeq M(x_1 z, \sigma_1)$, where σ_1 is an absolutely irreducible representation of $G^{x_1 z} = G^{x_1}$ with*

$$\sigma_1(x_1) = -\alpha^2\beta\sigma(z), \qquad \sigma_1(x_1 x_2 x_3) = \epsilon\alpha^3, \qquad \sigma_1(z) = \sigma(z)\rho(z).$$

Proof. Since $\operatorname{supp} X_1^{V,W} = (x_1 z)^G$ and the centralizer $G^{x_1 z} = G^{x_1}$ is abelian, $X_1^{V,W}$ is absolutely simple if and only if $\dim(X_1^{V,W})_{x_1 z} = 1$. Recall that $(X_1^{V,W})_{x_1 z} = \operatorname{span}_{\mathbb{K}}\{\varphi_1(v_0 \otimes w_1), \varphi_1(x_2 v_0 \otimes w_1)\}$. Thus Lemma 3.1 implies that $X_1^{V,W}$ is absolutely simple if and only if $(1 - \sigma(z)\alpha)(1 - \sigma(z)\alpha\beta) = 0$.

Let $w_1' = \varphi_1(x_2 v_0 \otimes w_1)$. Using Eqs. (3.2) and (2.6) we compute

$$x_1 w_1' = \big(\alpha^2\beta^2 - \alpha^3\sigma(z) - \alpha^3\beta^2\sigma(z)\big) v_0 \otimes w_1 - \alpha\beta\big(1 - \alpha\beta\sigma(z)\big) x_2 v_0 \otimes w_1.$$

If $\sigma(z)\alpha = 1$ then

$$x_1 w_1' = -\alpha^2 v_0 \otimes w_1 - \alpha\beta(1-\beta) x_2 v_0 \otimes w_1 = -\alpha\beta w_1',$$

and if $\sigma(z)\alpha\beta = 1$ then $x_1 w_1' = -\alpha^2\beta v_0 \otimes w_1 = -\alpha w_1'$. In both cases we conclude that $x_1 w_1' = -\alpha^2\beta\sigma(z) w_1'$. Since $z, x_1 x_2 x_3 \in Z(G)$, $\sigma(x_1 x_2 x_3) = -\epsilon$, and $\rho(x_1 x_2 x_3) = -\alpha^3$ by Lemma 2.7, σ_1 has the claimed properties. □

Lemma 3.3. *Assume that* $\sigma(x_1) = -1$ *and* $(1-\sigma(z)\alpha)(1-\sigma(z)\alpha\beta) = 0$. *Then* $X_2^{V,W} = 0$ *if and only if* $\rho(z) = -1$.

Proof. From [6, Lem. 1.7] we conclude that

$$X_2^{V,W} = \mathbb{K}G\{\varphi_2(v_0 \otimes w_1'), \varphi_2(x_2v_0 \otimes w_1')\}.$$

Since $x_1w_1' = -\alpha^2\beta\sigma(z)w_1'$ by Lemma 3.2 and $x_1x_2v_0 = \alpha\beta x_2v_0 - \alpha^2\beta^2v_0$, the vanishing of $X_2^{V,W}$ is equivalent to the vanishing of $\varphi_2(x_2v_0 \otimes w_1')$.

We first compute

$$\begin{aligned} c_{X_1,V}c_{V,X_1}(x_2v_0 \otimes w_1') &= \rho(z)\sigma(z)c_{X_1,V}(w_1' \otimes x_2v_0) \\ &= \rho(z)^2\sigma(z)(-\alpha^2\beta^2v_0 + \alpha\beta x_2v_0) \otimes w_1'. \end{aligned} \tag{3.3}$$

Assume first that $\sigma(z)\alpha\beta = 1$. Then $w_1' = \varphi_1(x_2v_0 \otimes w_1) = \alpha\beta v_0 \otimes w_1$. Since $\varphi_2 = \mathrm{id} - c_{X_1,V}c_{V,X_1} + (\mathrm{id} \otimes \varphi_1)c_{1,2}$, Eq. (3.3) implies that

$$\varphi_2(x_2v_0 \otimes w_1') = \alpha\beta\rho(z)(1+\rho(z))v_0 \otimes w_1' + (1-\rho(z)^2)x_2v_0 \otimes w_1'.$$

Assume now that $\sigma(z)\alpha = 1$. Then $w_1' = (1-\beta)x_2v_0 \otimes w_1 + \alpha\beta^2v_0 \otimes w_1$ by Lemma 3.1. Using Eq. (3.3) one obtains that

$$\varphi_2(x_2v_0 \otimes w_1') = (1+\rho(z))(\alpha\beta^2\rho(z)v_0 \otimes w_1' + (1-\beta\rho(z))x_2v_0 \otimes w_1').$$

In both cases, $\varphi_2(x_2v_0 \otimes w_1') = 0$ if and only if $\rho(z) = -1$. □

Proof of Proposition 2.8. Since (V,W) admits all reflections and $\mathcal{W}(V,W)$ is finite, $(\mathrm{ad}\,W)^m(V)$ is absolutely simple or zero for all $m \in \mathbb{N}_0$ by [7, Thm. 7.2(3)]. The Cartan matrix of (V,W) is non-diagonal and hence $(\mathrm{ad}\,W)(V) \neq 0$. Then Lemma 2.3 implies that $a_{2,1}^{(V,W)} \leq -3$ and $\sigma(x_1) = -1$. Therefore $a_{1,2}^{(V,W)} = -1$ and $a_{2,1}^{(V,W)} = -3$ by assumption. Hence $X_m^{V,W} = 0$ if and only if $m \geq 2$ and $X_m^{W,V} = 0$ if and only if $m \geq 4$ by the definition of the entries of the Cartan matrix $A^{(V,W)}$. Further, $\deg\rho \leq 2$ by Lemma 2.6.

Suppose that $\deg\rho = 2$. Then

$$(\alpha\sigma(z) - 1)(\alpha\beta\sigma(z) - 1) = 0 \tag{3.4}$$

by Lemma 3.2, and $\rho(z) = -1$ by Lemma 3.3. From $a_{1,2}^{(V,W)} = -1$ we obtain that $R_1(V,W) = (V^*, X_1^{V,W})$. Since $\operatorname{supp} X_1^{V,W} = (x_1z)^G \simeq \chi_1^T$ and $\operatorname{supp} V^* \simeq \operatorname{supp} V$ as quandles, Lemma 2.3 implies that $(\mathrm{ad}\,X_1^{V,W})^3(V^*)$ is non-zero. Then $(\mathrm{ad}\,X_1^{V,W})^3(V^*)$ is absolutely simple by [7, Thm. 7.2(3)]. Now Lemma 2.3 implies that $\sigma_1(x_1z) = -1$. By Lemma 3.2,

$$-1 = \sigma_1(x_1z) = -\alpha^2\beta\sigma(z)^2\rho(z) = \alpha^2\beta\sigma(z)^2.$$

On the other hand $\alpha^2\beta\sigma(z)^2 \in \{\beta,\beta^2\}$ by Eq. (3.4). This contradicts to $1+\beta+\beta^2=0$. Therefore $\deg\rho = 1$. $\square$

4. Proof of Theorem 2.9

As in Section 2.1, let $V = M(z,\rho)$ and $W = M(x_1,\sigma)$, and assume that $\deg\rho = 1$. Then $\rho(x_1) = \rho(x_2) = \rho(x_3) = \rho(x_4)$ since x_1, x_2, x_3, x_4 are conjugate elements. We write $X_n = X_n^{V,W}$ and $\varphi_n = \varphi_n^{V,W}$ for all $n \in \mathbb{N}_0$ if no confusion can arise.

Lemma 4.1. *Assume that $\sigma(x_1) = -1$. Then $X_1^{V,W}$ is non-zero if and only if $\rho(x_1)\sigma(z) \neq 1$. In this case, $X_1^{V,W}$ is absolutely simple and $X_1^{V,W} \simeq M(x_1 z, \sigma_1)$, where σ_1 is an absolutely irreducible representation of the centralizer $G^{x_1 z} = G^{x_1}$ and*

$$\sigma_1(x_1) = -\rho(x_1), \qquad \sigma_1(x_1x_2x_3) = -\epsilon\rho(x_1)^3, \qquad \sigma_1(z) = \rho(z)\sigma(z).$$

For $i \in \{1,2,3,4\}$ let $w_i' := v \otimes w_i$. Then w_1', w_2', w_3', w_4' is a basis of $X_1^{V,W}$. The degrees of these basis vectors are x_1z, x_2z, x_3z and x_4z, respectively.

Proof. By [6, Lem. 1.7], $X_1^{V,W} = \varphi_1(V \otimes W) = \mathbb{K}G\varphi_1(v \otimes w_1)$. Then

$$\varphi_1(v \otimes w_1) = (\mathrm{id} - c_{W,V}c_{V,W})(v \otimes w_1) = \big(1 - \rho(x_1)\sigma(z)\big)v \otimes w_1. \tag{4.1}$$

Hence $v \otimes w_1 \in (V \otimes W)_{x_1 z}$ is non-zero if and only if $\rho(x_1)\sigma(z) \neq 1$. Further,

$$x_1 w_1' = x_1 v \otimes x_1 w_1 = -\rho(x_1) v \otimes w_1 = -\rho(x_1) w_1'.$$

The remaining claims on σ_1 follow from the absolute irreducibility of V and W and the facts that $x_1x_2x_3, z \in Z(G)$ and $X_1^{V,W} \subseteq V \otimes W$. $\square$

Remark 4.2. To compute the action of G on $X_1^{V,W}$ one has to note that

$$x_i w_j' = x_i(v \otimes w_j) = x_i v \otimes x_i w_j = \rho(x_1) v \otimes x_i w_j$$

and then use the action of G on W of Remark 2.5.

Lemma 4.3. *Assume that $\sigma(x_1) = -1$ and $\rho(x_1)\sigma(z) \neq 1$. Then $X_2^{V,W} = 0$ if and only if $(1+\rho(z))(1-\rho(x_1z)\sigma(z)) = 0$.*

Proof. Let $w_1' = v \otimes w_1$. Then $X_2^{V,W} = \varphi_2(V \otimes X_1) = \mathbb{K}G\varphi_2(v \otimes w_1')$ by [6, Lem. 1.7]. Since $\varphi_2 = \mathrm{id} - c_{X_1,V}c_{V,X_1} + (\mathrm{id} \otimes \varphi_1)c_{1,2}$, we first compute

$$c_{X_1,V}c_{V,X_1}\big(v \otimes w_1'\big) = c_{X_1,V}\big(zw_1' \otimes v\big) = x_1zv \otimes zw_1' = \rho(z)\rho(x_1z)\sigma(z)v \otimes w_1'.$$

Now using Eq. (4.1) we compute

$$\begin{aligned}(\mathrm{id}\otimes\varphi_1)c_{1,2}\big(v\otimes w_1'\big)&=(\mathrm{id}\otimes\varphi_1)c_{1,2}(v\otimes v\otimes w_1)\\&=(\mathrm{id}\otimes\varphi_1)\big(\rho(z)v\otimes v\otimes w_1\big)\\&=\rho(z)\big(1-\rho(x_1)\sigma(z)\big)v\otimes v\otimes w_1.\end{aligned}$$

Hence $\varphi_2(v\otimes w_1')=(1+\rho(z))(1-\rho(x_1z)\sigma(z))v\otimes w_1'$. This implies the claim. □

Now we compute the adjoint actions $(\operatorname{ad}W)^m(V)$ for $m\in\{2,3,4\}$. We write $X_n=X_n^{W,V}$ and $\varphi_n=\varphi_n^{W,V}$ for all $n\in\mathbb{N}_0$ if no confusion can arise.

Remark 4.4. By [6, Lem. 1.7],

$$X_1^{W,V}=\varphi_1(W\otimes V)=\mathbb{K}G\varphi_1(w_1\otimes v).$$

Moreover, for all $i\in\{1,2,3,4\}$ we obtain that

$$\varphi_1(w_i\otimes v)=\big(1-\rho(x_1)\sigma(z)\big)w_i\otimes v. \tag{4.2}$$

Lemma 4.5. *Assume that $\rho(x_1)\sigma(z)\neq 1$ and $\sigma(x_1)=-1$. Then $X_1^{W,V}$ is absolutely simple. Moreover, $X_1^{W,V}\simeq M(x_1z,\rho_1)$, where ρ_1 is an absolutely irreducible representation of $G^{x_1z}=G^{x_1}$ with*

$$\rho_1(x_1)=-\rho(x_1),\qquad \rho_1(x_1x_2x_3)=-\epsilon\rho(x_1)^3,\qquad \rho_1(z)=\rho(z)\sigma(z).$$

For all $i\in\{1,2,3,4\}$ let $v_i'=w_i\otimes v$. Then $v_i'\in(X_1^{W,V})_{x_iz}$ for all i, and v_1',v_2',v_3',v_4' form a basis of $X_1^{W,V}$.

Proof. Since $c_{V,W}:X_1^{V,W}\to X_1^{W,V}$ is an isomorphism in ${}^G_G\mathcal{YD}$, the claim follows from Lemma 4.1. □

Remark 4.6. Assume that $\rho(x_1)\sigma(z)\neq 1$ and $\sigma(x_1)=-1$. For $j\in\{1,2,3,4\}$ let $v_j'=w_j\otimes v$. Remark 2.5 implies that the action of G on $X_1^{W,V}$ is given by $zv_j'=\rho(z)\sigma(z)v_j'$ for all $j\in\{1,2,3,4\}$ and

$$x_iv_j'=\begin{cases}-\rho(x_1)v_{i\triangleright j}' & \text{if } i=1 \text{ or } j=1 \text{ or } i=j,\\ -\epsilon\rho(x_1)v_{i\triangleright j}' & \text{otherwise.}\end{cases}$$

By [6, Lem. 1.7],

$$X_2^{W,V}=\varphi_2\big(W\otimes X_1^{W,V}\big)=\mathbb{K}G\big\{\varphi_2\big(w_1\otimes v_1'\big),\varphi_2\big(w_2\otimes v_1'\big)\big\}.$$

The two generators are computed in the following lemma.

Lemma 4.7. *Assume that* $\rho(x_1)\sigma(z) \neq 1$ *and* $\sigma(x_1) = -1$*. Then*

$$\varphi_2\big(w_1 \otimes v_1'\big) = 0, \tag{4.3}$$

$$\varphi_2\big(w_2 \otimes v_1'\big) = w_2 \otimes v_1' - \epsilon\rho(x_1)\sigma(z)w_1 \otimes v_3' - \big(1 - \rho(x_1)\sigma(z)\big)w_3 \otimes v_2'. \tag{4.4}$$

Moreover, $v_1'' := x_2\varphi_2(w_2 \otimes v_1') \in (W \otimes W \otimes V)_{x_2x_3z}$ *is non-zero.*

Proof. We first prove that $\varphi_2(w_1 \otimes v_1') = 0$. Lemma 4.5 implies that

$$c_{X_1,W}c_{W,X_1}\big(w_1 \otimes v_1'\big) = c_{X_1,W}\big(x_1v_1' \otimes w_1\big) = -\rho_1(x_1)\sigma(z)w_1 \otimes v_1'.$$

Then we compute

$$\begin{aligned}(\mathrm{id} \otimes \varphi_1)c_{1,2}\big(w_1 \otimes v_1'\big) &= (\mathrm{id} \otimes \varphi_1)c_{1,2}(w_1 \otimes w_1 \otimes v) \\ &= (\mathrm{id} \otimes \varphi_1)(x_1w_1 \otimes w_1 \otimes v) \\ &= -\big(1 - \rho(x_1)\sigma(z)\big)w_1 \otimes v_1'\end{aligned}$$

using Eq. (4.2). Since $\rho_1(x_1) = -\rho(x_1)$ by Lemma 4.5, we conclude that $\varphi_2(w_1 \otimes v_1') = 0$. Now we prove Eq. (4.4). First we use Lemma 4.5 and Remark 4.6 to compute

$$c_{X_1,W}c_{W,X_1}\big(w_2 \otimes v_1'\big) = x_3zw_2 \otimes x_2v_1' = \epsilon\rho(x_1)\sigma(z)w_1 \otimes v_3'.$$

Then using Eq. (4.2) we obtain that

$$(\mathrm{id} \otimes \varphi_1)c_{1,2}\big(w_2 \otimes v_1'\big) = -w_3 \otimes \big(1 - \rho(x_1)\sigma(z)\big)v_2'.$$

These equations imply Eq. (4.4). Now $\varphi_2(w_2 \otimes v_1') \in (W \otimes W \otimes V)_{x_2x_1z}$ is non-zero, and hence $x_2\varphi_2(w_2 \otimes v_1') \in (W \otimes W \otimes V)_{x_2x_3z}$ is non-zero. □

Remark 4.8. Eq. (4.4) and Remarks 2.5 and 4.6 imply that

$$v_1'' = \rho(x_1)w_2 \otimes v_3' - \rho(x_1)^2\sigma(z)w_3 \otimes v_4' - \epsilon\rho(x_1)\big(1 - \rho(x_1)\sigma(z)\big)w_4 \otimes v_2'. \tag{4.5}$$

Lemma 4.9. *Assume that* $\rho(x_1)\sigma(z) \neq 1$ *and* $\sigma(x_1) = -1$*. Then* $X_2^{W,V}$ *is absolutely simple if and only if*

$$\epsilon = 1, \qquad \big(\rho(x_1)\sigma(z)\big)^2 - \rho(x_1)\sigma(z) + 1 = 0. \tag{4.6}$$

In this case, $X_2^{W,V} \simeq M(x_2x_3z, \rho_2)$*, where* ρ_2 *is an absolutely irreducible representation of* $G^{x_2x_3z} = G^{x_1}$ *with*

$$\rho_2(x_1) = -\rho(x_1)^2\sigma(z), \qquad \rho_2(x_1x_2x_3) = \rho(x_1)^3, \qquad \rho_2(z) = \rho(z)\sigma(z)^2.$$

Let $v_2'' = \rho(x_1)\sigma(z)^2 x_4 v_1''$, $v_3'' = \rho(x_1)\sigma(z)^2 x_2 v_1''$, and $v_4'' = \rho(x_1)\sigma(z)^2 x_3 v_1''$. Then the set $\{v_1'', v_2'', v_3'', v_4''\}$ is a basis of $X_2^{W,V}$. The degrees of these elements are x_2x_3z, x_1x_4z, x_1x_2z, and x_1x_3z, respectively.

Proof. By Remark 4.8, Lemma 4.5 and Remark 4.6,

$$x_1 v_1'' = \rho(x_1)^2 w_4 \otimes v_2' - \rho(x_1)^3\sigma(z) w_2 \otimes v_3' - \epsilon\rho(x_1)^2\big(1-\rho(x_1)\sigma(z)\big) w_3 \otimes v_4'.$$

Assume that $X_2^{W,V}$ is absolutely simple. Then $(X_2^{W,V})_{x_2x_3z}$ is 1-dimensional, since the centralizer $G^{x_2x_3z} = G^{x_1}$ is abelian. Hence v_1'' and x_1v_1'' are linearly dependent. Relating the coefficients of $w_3 \otimes v_4'$ and $w_4 \otimes v_2'$ of v_1'' and $x_1 v_1''$, respectively, and using that $\epsilon^2 = 1$, we conclude that $(\rho(x_1)\sigma(z))^2 - \rho(x_1)\sigma(z) + 1 = 0$. Relating the coefficients of $w_3 \otimes v_4'$ and $w_2 \otimes v_3'$ of v_1'' and x_1v_1'', respectively, we conclude that $\epsilon = 1$.

Conversely, (4.6) implies that $x_1 v_1'' = -\rho(x_1)^2\sigma(z)v_1''$. Since $x_1x_2x_3, z \in Z(G)$ and $z'u = \sigma(z')^2\rho(z')u$ for all $z' \in Z(G)$, $u \in W\otimes W\otimes V$, and since $G^{x_2x_3z} = \langle x_1, x_1x_2x_3, z\rangle$, the Yetter–Drinfeld module $X_2^{W,V}$ is absolutely simple if and only if (4.6) holds. The above calculations also prove the formulas for ρ_2. The last claim follows easily, since $x_2x_3z \in x_1^{-1}Z(G)$. □

Remark 4.10. Assume that

$$\sigma(x_1) = -1, \qquad \epsilon = 1, \qquad \big(\rho(x_1)\sigma(z)\big)^2 - \rho(x_1)\sigma(z) + 1 = 0.$$

Let v_j'' for $j \in \{1,2,3,4\}$ be as in Lemma 4.9. The action of G on $X_2^{W,V}$ is given by

$$z v_j'' = \rho(z)\sigma(z)^2 v_j'', \qquad x_i v_j'' = -\rho(x_1)^2\sigma(z) v_{i\triangleright j}'' \tag{4.7}$$

for all $i, j \in \{1,2,3,4\}$. Indeed, for $j = 1$ this follows from the definition of v_j'' and from $\rho(x_1)^3\sigma(z)^3 = -1$. Further, $v_j'' = \rho(x_1)\sigma(z)^2 x_{1\triangleright j} v_1''$ for $j \in \{2,3,4\}$. Hence (4.7) for $i = 1$ and $j > 1$ follows from $x_1 x_{1\triangleright j} = x_{1\triangleright(1\triangleright j)} x_1$. For $i = j > 1$, (4.7) follows from $x_i x_{1\triangleright i} = x_{1\triangleright i} x_1$. For $i > 1$, $j = 1 \triangleright i$, (4.7) follows from $i \triangleright (1 \triangleright i) = x_1$ and $x_i x_{1\triangleright j} = x_2 x_3$ and from $\rho_2(x_2x_3) = -\rho(x_1)\sigma(z)^{-1}$. Finally, (4.7) for $i > 1$, $j = 1 \triangleright (1 \triangleright i)$ follows from $x_i^3 v_j'' = \rho_2(x_1^3) v_j''$, $i \triangleright (i \triangleright (i \triangleright j)) = j$, and from the equations $(-\rho(x_1)^2\sigma(z))^3 = \rho(x_1)^3 = \rho_2(x_1^3)$.

Recall that $v_1'' \in (X_2^{W,V})_{x_2x_3z}$. By [6, Lem. 1.7],

$$X_3^{W,V} = \varphi_3\big(W \otimes X_2^{W,V}\big) = \mathbb{K}G\big\{\varphi_3\big(w_1 \otimes v_1''\big), \varphi_3\big(w_2 \otimes v_1''\big)\big\}.$$

Therefore we need to compute $\varphi_3(w_1 \otimes v_1'')$ and $\varphi_3(w_2 \otimes v_1'')$.

Lemma 4.11. *Assume that $(\rho(x_1)\sigma(z))^2 - \rho(x_1)\sigma(z) + 1 = 0$, $\sigma(x_1) = -1$, and $\epsilon = 1$. Then the following hold:*

$$\varphi_2\big(w_2 \otimes v'_2\big) = 0, \tag{4.8}$$

$$\varphi_2\big(w_2 \otimes v'_3\big) = \rho(x_1)^{-1} v''_1, \tag{4.9}$$

$$\varphi_2\big(w_2 \otimes v'_4\big) = -\sigma(z) v''_3, \tag{4.10}$$

$$\varphi_2\big(w_1 \otimes v'_3\big) = -\sigma(z) v''_4, \tag{4.11}$$

$$\varphi_2\big(w_1 \otimes v'_2\big) = \rho(x_1)\sigma(z)^2 v''_3, \tag{4.12}$$

$$\varphi_2\big(w_1 \otimes v'_4\big) = \rho(x_1)^{-1} v''_2. \tag{4.13}$$

Proof. By Lemma 4.7, $\varphi_2(w_1 \otimes v'_1) = 0$. Applying x_4 to this equation we obtain Eq. (4.8), where we used (4.7). To prove (4.9) we compute

$$v''_1 = x_2\varphi_2\big(w_2 \otimes v'_1\big) = \varphi_2\big(x_2 w_2 \otimes x_2 v'_1\big) = \rho(x_1)\varphi_2\big(w_2 \otimes v'_3\big)$$

using Remarks 2.5 and 4.6, and Eq. (4.9) follows. Now apply x_2 to Eq. (4.9) to obtain Eq. (4.10).

To prove Eq. (4.11) apply x_3 to (4.9) and use Lemma 4.9 and (4.10). Similarly, acting with x_1 on Eq. (4.11) we obtain Eq. (4.12). Finally, acting with x_1 on Eq. (4.12) we obtain Eq. (4.13). □

Lemma 4.12. *Assume that* $(\rho(x_1)\sigma(z))^2 - \rho(x_1)\sigma(z) + 1 = 0$, $\sigma(x_1) = -1$, *and* $\epsilon = 1$. *Then* $\varphi_3(w_2 \otimes v''_1) = 0$.

Proof. Using Remark 2.5 and Lemma 4.9 we obtain that

$$c_{X_2,W} c_{W,X_2}\big(w_2 \otimes v''_1\big) = c_{X_2,W}\big(x_2 v''_1 \otimes w_2\big) = -\rho(x_1)^2\sigma(z)^2 w_4 \otimes v''_3.$$

Using Eq. (4.5) we compute

$$\begin{aligned}(\mathrm{id} \otimes \varphi_2)c_{1,2}\big(w_2 \otimes v''_1\big) = {} & -\rho(x_1) w_2 \otimes \varphi_2\big(w_2 \otimes v'_3\big) \\ & + \rho(x_1)^2\sigma(z) w_4 \otimes \varphi_2\big(w_2 \otimes v'_4\big) \\ & + \rho(x_1)\big(1 - \rho(x_1)\sigma(z)\big) w_1 \otimes \varphi_2\big(w_2 \otimes v'_2\big).\end{aligned}$$

Since $\varphi_3 = \mathrm{id} - c_{X_2,W} c_{W,X_2} + (\mathrm{id} \otimes \varphi_2)c_{1,2}$, Eqs. (4.8)–(4.10) imply that

$$\begin{aligned}\varphi_3\big(w_2 \otimes v''_1\big) = {} & w_2 \otimes v''_1 - \rho(x_1)^2\sigma(z)^2 w_4 \otimes v''_3 \\ & - \rho(x_1) w_2 \otimes \rho(x_1)^{-1} v''_1 + \rho(x_1)^2\sigma(z) w_4 \otimes \big(-\sigma(z) v''_3\big).\end{aligned}$$

Thus the claim follows. □

Lemma 4.13. *Assume that* $(\rho(x_1)\sigma(z))^2 - \rho(x_1)\sigma(z) + 1 = 0$, $\sigma(x_1) = -1$, *and* $\epsilon = 1$. *Let* $v'''_1 = \varphi_3(w_1 \otimes v''_1)$. *Then*

$$v_1''' = \rho(x_1)\sigma(z)\big(w_1 \otimes v_1'' + w_2 \otimes v_2'' + w_3 \otimes v_3'' + w_4 \otimes v_4''\big) \tag{4.14}$$

is a non-zero element of $(W \otimes W \otimes W \otimes V)_{x_1x_2x_3z}$.

Proof. Using Remark 2.5 and Lemma 4.9 we first compute

$$c_{X_2,W}c_{W,X_2}\big(w_1 \otimes v_1''\big) = c_{X_2,W}\big(x_1v_1'' \otimes w_1\big) = -\rho(x_1)^2\sigma(z)^2 w_1 \otimes v_1''.$$

Using Eq. (4.5) and Remark 2.5, a straightforward calculation yields

$$\begin{aligned}(\mathrm{id} \otimes \varphi_2)c_{1,2}\big(w_1 \otimes v_1''\big) = {}& -\rho(x_1)w_4 \otimes \varphi_2\big(w_1 \otimes v_3'\big) \\ & + \rho(x_1)^2\sigma(z)w_2 \otimes \varphi_2\big(w_1 \otimes v_4'\big) \\ & + \rho(x_1)\big(1 - \rho(x_1)\sigma(z)\big)w_3 \otimes \varphi_2\big(w_1 \otimes v_2'\big).\end{aligned}$$

Since $\varphi_3 = \mathrm{id} - c_{X_2,W}c_{W,X_2} + (\mathrm{id} \otimes \varphi_2)c_{1,2}$, Eqs. (4.11)–(4.13) yield Eq. (4.14). The rest is clear. □

Lemma 4.14. *Assume that* $(\rho(x_1)\sigma(z))^2 - \rho(x_1)\sigma(z) + 1 = 0$, $\sigma(x_1) = -1$, *and* $\epsilon = 1$. *Then* $X_3^{W,V} \simeq M(x_1x_2x_3z, \rho_3)$, *where* ρ_3 *is an absolutely irreducible representation of* $G^{x_1x_2x_3z} = G$ *with*

$$\rho_3(z) = \rho(z)\sigma(z)^3, \qquad \rho_3(x_1) = \rho_3(x_2) = \rho_3(x_3) = \rho_3(x_4) = \rho(x_1)^2\sigma(z).$$

Proof. The formula for $\rho_3(z)$ follows from $zu = \rho(z)\sigma(z)^3u$ for all $u \in W^{\otimes 3} \otimes V$. By the remark above Lemma 4.11, by Lemma 4.12, and since $x_1x_2x_1^{-1} = x_4$ and $x_2x_1x_2^{-1} = x_3$, it is enough to show that $x_1v_1''' = x_2v_1''' = \rho(x_1)^2\sigma(z)v_1'''$. By Lemma 4.9, $x_1v_1'' = -\rho(x_1)^2\sigma(z)v_1''$. Then

$$x_1v_1''' = x_1\varphi_3\big(w_1 \otimes v_1''\big) = \varphi_3\big(x_1w_1 \otimes x_1v_1''\big) = \rho(x_1)^2\sigma(z)v_1'''.$$

The claim on x_2v_1''' follows from Eqs. (4.14) and (4.7). □

By [6, Lem. 1.7] and since $x_iv_1''' = \rho_3(x_1)v_1'''$ for all $i \in \{1,2,3,4\}$,

$$X_4^{W,V} = \varphi_4\big(W \otimes X_3^{W,V}\big) = \mathbb{K}G\varphi_4\big(w_1 \otimes v_1'''\big).$$

The following lemma will be useful for computing $\varphi_4(w_1 \otimes v_1''')$.

Lemma 4.15. *Assume that* $(\rho(x_1)\sigma(z))^2 - \rho(x_1)\sigma(z) + 1 = 0$, $\sigma(x_1) = -1$, *and* $\epsilon = 1$. *Then*

$$\varphi_3\big(w_1 \otimes v_2''\big) = \varphi_3\big(w_1 \otimes v_3''\big) = \varphi_3\big(w_1 \otimes v_4''\big) = 0. \tag{4.15}$$

Proof. By Lemma 4.12, $\varphi_3(w_2 \otimes v_1'') = 0$. By acting on this equation with x_3 and using Eq. (4.7) one obtains that $\varphi_3(w_1 \otimes v_4'') = 0$. The other two equations follow similarly by acting twice with x_1 on the latter equation. □

Lemma 4.16. *Assume that* $(\rho(x_1)\sigma(z))^2 - \rho(x_1)\sigma(z) + 1 = 0$, $\sigma(x_1) = -1$, *and* $\epsilon = 1$. *Then* $X_4^{W,V} = 0$.

Proof. It is enough to prove that $\varphi_4(w_1 \otimes v_1''') = 0$. Lemma 4.14 implies that

$$c_{X_3,W}c_{W,X_3}\big(w_1 \otimes v_1'''\big) = c_{X_3,W}\big(x_1 v_1''' \otimes w_1\big) = -\rho(x_1)^2\sigma(z)^2 w_1 \otimes v_1'''.$$

Eq. (4.14) and Lemma 4.15 yield that

$$(\mathrm{id} \otimes \varphi_3)c_{1,2}\big(w_1 \otimes v_1'''\big) = -\rho(x_1)\sigma(z) w_1 \otimes v_1'''.$$

Therefore $\varphi_4(w_1 \otimes v_1''') = (\mathrm{id} - c_{X_3,W}c_{W,X_3} + (\mathrm{id} \otimes \varphi_3)c_{1,2})(w_1 \otimes v_1''') = 0$. This proves the lemma. □

We summarize the results of this section in the following proposition.

Proposition 4.17. *Let* $V = M(z,\rho)$ *and* $W = M(x_1,\sigma)$. *Assume that* $\deg\rho = \deg\sigma = 1$ *and that* $\rho(x_1)\sigma(z) \neq 1$, $\sigma(x_1) = -1$ *and* $\rho(x_1 z)\sigma(z) = 1$. *Then the following hold:*

(1) $(\mathrm{ad}\, V)(W)$ *is absolutely simple and* $(\mathrm{ad}\, V)^2(W) = 0$.
(2) *The Yetter–Drinfeld modules* $(\mathrm{ad}\, W)^m(V)$ *are absolutely simple or zero for all* $m \in \mathbb{N}_0$ *if and only if* $(\rho(x_1)\sigma(z))^2 - \rho(x_1)\sigma(z) + 1 = 0$ *and* $\epsilon = 1$. *In this case,* $(\mathrm{ad}\, W)^3(V) \neq 0$ *and* $(\mathrm{ad}\, W)^4(V) = 0$.

Proof. The first two claims follow from Lemmas 4.1 and 4.3. The others are Lemmas 4.5, 4.9, 4.14 and 4.16. □

Before proving Theorem 2.9, we need three more technical lemmas. Recall that if X is a finite-dimensional Yetter–Drinfeld module over G, then the dual space X^* is also a Yetter–Drinfeld module with

$$(gf)(x) = f\big(g^{-1}x\big), \qquad f_{(-1)} \otimes f_{(0)}(y) = h^{-1} \otimes f(y)$$

for all $g, h \in G$, $x \in X$, $y \in X_h$ and $f \in X^*$, where $\delta(f) = f_{(-1)} \otimes f_{(0)}$. Further, if X is simple then so is X^*. In particular, $M(x,\gamma)^* \simeq M(x^{-1},\gamma^*)$ for all $x \in G$ and all finite-dimensional representations ρ of $G^x = G^{x^{-1}}$, where γ^* is the dual representation of γ.

Remark 4.18. Let $x_1' := x_1 z$, $x_2' := x_2 z$, $x_3' := x_3 z$ and $x_4' := x_4 z$. Then $G = \langle x_1', x_2', x_3', x_4', z^{-1}\rangle$ and the map $T \to G$,

$$\chi_1 \mapsto x_1', \qquad \chi_2 \mapsto x_2', \qquad \chi_3 \mapsto x_3', \qquad \chi_4 \mapsto x_4', \qquad \zeta \mapsto z^{-1},$$

is a group homomorphism.

Lemma 4.19. *Assume that* $(\rho(x_1)\sigma(z))^2 - \rho(x_1)\sigma(z) + 1 = 0$, $\sigma(x_1) = -1$, $\sigma(x_2x_3) = 1$, *and* $\rho(x_1z)\sigma(z) = 1$. *Then*

$$R_1(V,W) = \left(V^*, X_1^{V,W}\right)$$

with $V^* \simeq M(z^{-1}, \rho^*)$, *where* ρ^* *is the irreducible representation of* G^z *dual to* ρ, $X_1^{V,W} \simeq M(x_1z, \sigma_1)$, *where* σ_1 *is the irreducible representation of* G^{x_1z} *given in Lemma 4.1, and*

$$\sigma_1(x_1z) = -1, \qquad \sigma_1\left(x_2x_3z^2\right) = 1, \tag{4.16}$$

$$\rho^*(x_1)\sigma_1\left(z^{-1}\right) = 1, \qquad \left(\rho^*(x_1z)\sigma_1\left(z^{-1}\right)\right)^2 - \rho^*(x_1z)\sigma_1\left(z^{-1}\right) + 1 = 0. \tag{4.17}$$

Proof. Since $\sigma(x_1) = -1$, $\rho(x_1)\sigma(z) \neq 1$, and $\rho(x_1z)\sigma(z) = 1$, the description of $R_1(V,W)$ follows from Proposition 4.17 and Lemma 4.1. Further, $\sigma_1(x_1) = -\rho(x_1)$, $\sigma_1(x_2x_3) = \sigma(x_2x_3)\rho(x_1)^2$ and $\sigma_1(z) = \rho(z)\sigma(z)$. Then $\sigma_1(x_1z) = -1$ and

$$\sigma_1\left(x_2x_3z^2\right) = \sigma(x_2x_3)\rho(x_1)^2\rho(z)^2\sigma(z)^2 = 1.$$

Further,

$$\rho^*(x_1)\sigma_1\left(z^{-1}\right) = \rho(x_1)^{-1}\rho\left(z^{-1}\right)\sigma\left(z^{-1}\right) = 1,$$

which proves the first equation in (4.17). Since

$$\rho^*(x_1z)\sigma_1\left(z^{-1}\right) = \rho(x_1z)^{-1}\rho(z)^{-1}\sigma(z)^{-1} = \rho(x_1)\sigma(z),$$

the second equation in (4.17) also holds. □

Remark 4.20. Let $x_1'' = x_1^{-1}$, $x_2'' = x_2^{-1}$, $x_3'' = x_4^{-1}$, $x_4'' = x_3^{-1}$, and $z'' = x_1x_2x_3z$. Then $G = \langle x_1'', x_2'', x_3'', x_4'', z'' \rangle$ and the map $T \to G$,

$$\chi_1 \mapsto x_1'', \qquad \chi_2 \mapsto x_2'', \qquad \chi_3 \mapsto x_3'', \qquad \chi_4 \mapsto x_4'', \qquad \zeta \mapsto z'',$$

is a group homomorphism.

Lemma 4.21. *Assume that* $(\rho(x_1)\sigma(z))^2 - \rho(x_1)\sigma(z) + 1 = 0$, $\sigma(x_1) = -1$, $\sigma(x_2x_3) = 1$, *and* $\rho(x_1z)\sigma(z) = 1$. *Let* x_1'', x_2'', x_3'', x_4'' *and* z'' *be as in Remark 4.20. Then*

$$R_2(V,W) = \left(X_3^{W,V}, W^*\right)$$

with $X_3^{W,V} \simeq M(z'', \rho_3)$*, where* ρ_3 *is the irreducible representation of* G *given in Lemma 4.14,* $W^* \simeq M(x_1'', \sigma^*)$*, where* σ^* *is the irreducible representation of* G^{x_1} *dual to* σ*, and*

$$\sigma^*\left(x_1''\right) = -1, \qquad \sigma^*\left(x_2''x_3''\right) = 1, \tag{4.18}$$

$$\rho_3\left(x_1''z''\right)\sigma^*\left(z''\right) = 1, \qquad \left(\rho_3\left(x_1''\right)\sigma^*\left(z''\right)\right)^2 - \rho_3\left(x_1''\right)\sigma^*\left(z''\right) + 1 = 0. \tag{4.19}$$

Proof. The description of $R_2(V,W)$ follows from Proposition 4.17 and Lemma 4.14. Eq. (4.18) follows from the formulas

$$\sigma^*\left(x_1^{-1}\right) = \sigma(x_1) = -1,$$
$$\sigma^*\left(x_2^{-1}x_4^{-1}\right) = \sigma^*\left((x_4x_2)^{-1}\right) = \sigma(x_2x_3) = 1.$$

Similarly, (4.19) follows from the calculations

$$\rho_3(x_2x_3z)\sigma^*(x_1x_2x_3z) = \rho(x_1)^4\sigma(z)^5\rho(z)\left(-\sigma(z)\right)^{-1} = -\rho(x_1)^3\sigma(z)^3 = 1,$$
$$\rho_3\left(x_1^{-1}\right)\sigma^*(x_1x_2x_3z) = \rho(x_1)^{-2}\sigma(z)^{-1}\left(-\sigma(z)\right)^{-1} = \rho(x_1)\sigma(z),$$

where the last equation is valid because of $(\rho(x_1)\sigma(z))^3 = -1$. □

Lemma 4.22. *Assume that* $(\rho(x_1)\sigma(z))^2 - \rho(x_1)\sigma(z) + 1 = 0$*,* $\sigma(x_1) = -1$*,* $\epsilon = 1$*, and* $\rho(x_1z)\sigma(z) = 1$*. Then*

$$V \simeq M\left(z^{-1}, \rho^*\right) \simeq M(x_1x_2x_3z, \rho_3), \tag{4.20}$$

$$W \simeq M(x_1z, \sigma_1) \simeq M\left(x_1^{-1}, \sigma^*\right) \tag{4.21}$$

as braided vector spaces.

Proof. Let $f : V \to M(z^{-1}, \rho^*)$ be a non-zero linear map. Then

$$(f \otimes f)c_{V,V}(v \otimes v) = (f \otimes f)(zv \otimes v) = \rho(z)f(v) \otimes f(v).$$

On the other hand,

$$c_{V^*,V^*}(f \otimes f)(v \otimes v) = z^{-1}f(v) \otimes f(v) = \rho^*\left(z^{-1}\right)f(v) \otimes f(v).$$

Since $\rho^*(z^{-1}) = \rho(z)$, we conclude that V and $M(z^{-1}, \rho^*)$ are isomorphic as braided vector spaces. Similarly, V and $M(x_1x_2x_3z, \rho_3)$ are isomorphic as braided vector spaces, since $\rho_3(x_1x_2x_3z) = \rho(z)$. Indeed,

$$\rho_3(x_1x_2x_3z) = \left(\rho(x_1)^2\sigma(z)\right)^3\rho(z)\sigma(z)^3 = \left(\rho(x_1)\sigma(z)\right)^6\rho(z) = \rho(z)$$

by Lemma 4.14 and since $(\rho(x_1)\sigma(z))^6 = 1$.

We now prove that $W \simeq X_1^{V,W}$ as braided vector spaces. Then W and $M(x_1 z, \sigma_1)$ are isomorphic as braided vector spaces by Lemma 4.1. Let

$$f \in \operatorname{Hom}\bigl(W, X_1^{V,W}\bigr), \qquad w_i \mapsto w_i' \quad \text{for } i \in \{1,2,3,4\},$$

where $w_i' = v \otimes w_i$ for all i. Then

$$(f \otimes f)c_{W,W}(w_i \otimes w_j) = (f \otimes f)(x_i w_j \otimes w_i) = -f(w_{i \triangleright j}) \otimes f(w_i),$$

and on the other hand

$$c_{X_1,X_1}(f \otimes f)(w_i \otimes w_j) = c_{X_1,X_1}\bigl(w_i' \otimes w_j'\bigr) = x_i z w_j' \otimes w_i'.$$

Since $x_i w_j = -w_{i \triangleright j}$ and $x_i z w_j' = -\rho(x_1 z)\sigma(z) w_{i \triangleright j}' = -w_{i \triangleright j}'$ by Remark 2.5, we conclude that $W \simeq X_1^{V,W}$ as braided vector spaces.

Similarly, W and $M(x_1^{-1}, \sigma^*)$ are isomorphic as braided vector spaces. Indeed, let x_1'', x_2'', x_3'', and x_4'' be as in Remark 4.20. Then $\sigma^*(x_1'') = -1$ and $\sigma^*(x_2'' x_3'') = 1$ by Lemma 4.21, and hence by Remark 2.5 there is a basis $w_1'', w_2'', w_3'', w_4''$ of $M(x_1^{-1}, \sigma^*)$ such that $x_i'' w_j'' = -w_{i \triangleright j}''$ for all $i, j \in \{1,2,3,4\}$. This implies that $W \simeq M(x_1^{-1}, \sigma^*)$ as braided vector spaces. □

Proof of Theorem 2.9. (1) ⇒ (2). Since $\mathfrak{B}(V \oplus W) < \infty$, the pair (V, W) admits all reflections by [2, Cor. 3.18] and the Weyl groupoid is finite by [2, Prop. 3.23].

(2) ⇒ (3). By [9, Prop. 4.3], after changing the object of $\mathcal{W}(V, W)$ and possibly interchanging V and W, we may assume that $V = M(z, \rho)$ and $W = M(x_1, \sigma)$ satisfy $(\operatorname{ad} V)(W) \neq 0$, $(\operatorname{ad} V)^2(W) = 0$ and $(\operatorname{ad} W)^4(V) = 0$. Further, $\deg \rho = 1$ by Proposition 2.8. By [7, Thm. 7.2(3)], $(\operatorname{ad} W)^m(V)$ is absolutely simple or zero for all $m \in \mathbb{N}_0$. Lemma 2.3 implies that $(\operatorname{ad} W)^2(V) \neq 0$ and $(\operatorname{ad} W)^3(V) \neq 0$. Hence $\sigma(x_1) = -1$ by Lemma 2.3. Since $(\operatorname{ad} V)(W)$ is non-zero, we obtain from Lemma 4.1 that $\rho(x_1)\sigma(z) \neq 1$. Further, $(\operatorname{ad} W)^2(V)$ is absolutely simple, and hence $\sigma(x_2 x_3) = 1$ and $(\rho(x_1)\sigma(z))^2 - \rho(x_1)\sigma(z) + 1 = 0$ by Lemma 4.9. Since $(\operatorname{ad} V)^2(W) = 0$, we obtain that $R_1(V, W) = (V^*, X_1^{V,W})$. Now $\operatorname{supp} X_1^{V,W} \simeq (x_1 z)^G \simeq \chi_1^T$ and $\operatorname{supp} V^* \simeq \operatorname{supp} V$ imply that $(\operatorname{ad} X_1^{V,W})^3(V^*)$ is absolutely simple or zero by [7, Thm. 7.2(3)]. Hence $\sigma_1(x_1 z) = -1$ by Lemma 2.3 and therefore $\rho(x_1 z)\sigma(z) = 1$ by Lemma 4.1.

(3) ⇒ (1). By Proposition 4.17 and Lemmas 4.19 and 4.21 the Weyl groupoid of (V, W) is standard with Cartan matrix of type G_2. Suppose that the Cartan matrix of (V, W) is $A^{(V,W)} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$. Then $s_2 s_1 s_2 s_1 s_2 s_1$ is a reduced decomposition of the longest word in the Weyl group of $A^{(V,W)}$. With respect to this reduced decomposition one obtains

$$\beta_1 = \alpha_2, \qquad \beta_2 = 3\alpha_2 + \alpha_1,$$
$$\beta_3 = 2\alpha_2 + \alpha_1, \qquad \beta_4 = 3\alpha_2 + 2\alpha_1,$$

$$\beta_5 = \alpha_2 + \alpha_1, \qquad \beta_6 = \alpha_1,$$

where $\{\alpha_1, \alpha_2\}$ is the standard basis of $\mathbb{Z}^2$. Since $A^{(V,W)}$ is of finite Cartan type, the set of real roots associated to the pair (V, W) is finite by [6, Cor. 2.4]. By [6, Thm. 2.6],

$$\mathfrak{B}(V \oplus W) \simeq \mathfrak{B}(M_{\beta_6}) \otimes \mathfrak{B}(M_{\beta_5}) \otimes \cdots \otimes \mathfrak{B}(M_{\beta_2}) \otimes \mathfrak{B}(M_{\beta_1}),$$

as $\mathbb{N}_0^2$-graded vector spaces, where $\deg M_{\beta_j} = \beta_j$ for all $j \in \{1, \ldots, 6\}$, $M_{\beta_1} = W$, $M_{\beta_6} = V$, and $M_{\beta_2}, M_{\beta_3}, M_{\beta_4}, M_{\beta_5} \subseteq \mathfrak{B}(V \oplus W)$ are certain finite-dimensional simple subobjects in ${}^G_G\mathcal{YD}$. Moreover, Lemmas 4.19, 4.21 and 4.22 imply that

$$M_{\beta_1} \simeq M_{\beta_3} \simeq M_{\beta_5} \simeq W \quad \text{and} \quad M_{\beta_2} \simeq M_{\beta_4} \simeq M_{\beta_6} \simeq V$$

as braided vector spaces. Indeed, by Lemma 4.19 we can apply our theory to the pair $R_1(V, W) = (V^*, X_1^{V,W})$ if we replace z, ρ, x_1, and σ by $z^{-1}, \rho^*, x_1 z$, and σ_1, respectively, and also to the pair $R_2(V, W)$ by Lemma 4.21. Since $s_1 s_2(\alpha_1) = 3\alpha_2 + 2\alpha_1$, we conclude from [6, Thm. 2.6](1), that M_{β_4} is isomorphic to the first entry of $R_1 R_2(V, W)$ in ${}^G_G\mathcal{YD}$, and hence to V as a braided vector space by iterated application of Lemma 4.22. The other isomorphisms follow similarly. Therefore the Nichols algebras of the braided vector spaces M_{β_k}, $1 \le k \le 6$, are finite-dimensional with Hilbert series

$$\mathcal{H}_{\mathfrak{B}(M_{\beta_k})}(t) = \begin{cases} (6)_t & \text{if } \operatorname{char} \mathbb{K} \neq 2, \\ (3)_t & \text{if } \operatorname{char} \mathbb{K} = 2, \end{cases}$$

for all $k \in \{2, 4, 6\}$, see [5, Sect. 3], and

$$\mathcal{H}_{\mathfrak{B}(M_{\beta_l})}(t) = \begin{cases} (2)_t^2 (3)_t (6)_t & \text{if } \operatorname{char} \mathbb{K} \neq 2, \\ (2)_t^2 (3)_t^2 & \text{if } \operatorname{char} \mathbb{K} = 2, \end{cases}$$

for all $l \in \{1, 3, 5\}$, see [1, Thm. 6.15] and [4, Prop. 5.6]. From this the claim follows. □

5. Nichols algebras over epimorphic images of Γ_4

5.1. Preliminaries

Recall from [6, Sect. 3] that the group Γ_n for $n \ge 2$ is isomorphic to the group given by generators a, b, ν and relations

$$ba = \nu ab, \qquad a\nu = \nu^{-1} a, \qquad b\nu = \nu b, \qquad \nu^n = 1.$$

The case $n = 2$ was studied in [6], and the case $n = 3$ appears to be more complicated. Here we concentrate on the case where $n = 4$. By [6, Sect. 3], the center of Γ_4 is $Z(\Gamma_4) = \langle \nu^{-1} b^2, b^4, a^2 \rangle$.

In what follows, let G be a group and let $g, h, \epsilon \in G$. Assume that $G = \langle g, h, \epsilon \rangle$, $\epsilon^2 \neq 1$, and that there is a group homomorphism $\Gamma_4 \to G$ with $a \mapsto g$, $b \mapsto h$ and $\nu \mapsto \epsilon$. Then G is a non-abelian quotient of Γ_4 such that $|g^G| = 4$ and $|h^G| = 2$. Further, $\epsilon^{-1}h^2, h^4, g^2 \in Z(G)$.

Let $V = M(h, \rho)$, where ρ is an absolutely irreducible representation of the centralizer $G^h = \langle \epsilon, h, g^2 \rangle = \langle h \rangle Z(G)$. Then $\deg \rho = 1$ since G^h is abelian. Let $v \in V_h$ with $v \neq 0$. The elements v, gv form a basis of V. The degrees of these basis vectors are h and $ghg^{-1} = \epsilon^{-1}h$, respectively. The support of V is isomorphic to the trivial quandle with two elements.

Remark 5.1. Assume that $\rho(h) = -1$. Then the action of G on V is given by the following table:

V	v	gv
ϵ	$\rho(\epsilon)v$	$\rho(\epsilon)^{-1}gv$
h	$-v$	$-\rho(\epsilon)^{-1}gv$
g	gv	$\rho(g^2)v$

Let $W = M(g, \sigma)$, where σ is an absolutely irreducible representation of the centralizer $G^g = \langle \epsilon^2, \epsilon^{-1}h^2, g \rangle = \langle g \rangle Z(G)$. Then $\deg \sigma = 1$ since G^g is abelian. Let $w \in W_g$ with $w \neq 0$. The elements w, hw, ϵw, ϵhw form a basis of W. The degrees of these basis vectors are g, ϵg, $\epsilon^2 g$ and $\epsilon^3 g$, respectively. The support of W is isomorphic to the dihedral quandle with four elements.

Remark 5.2. Assume that $\sigma(g) = -1$. Then the action of G on W is given by the following table:

W	w	hw	ϵw	ϵhw
ϵ	ϵw	ϵhw	$\sigma(\epsilon^2)w$	$\sigma(\epsilon^2)hw$
h	hw	$\sigma(\epsilon^{-1}h^2)\epsilon w$	ϵhw	$\sigma(\epsilon^2)\sigma(\epsilon^{-1}h^2)w$
g	$-w$	$-\sigma(\epsilon^2)\epsilon hw$	$-\sigma(\epsilon^2)\epsilon w$	$-\sigma(\epsilon^2)hw$

Remark 5.3. Let us describe the quandle structure of $\operatorname{supp}(V \oplus W)$. Of course, the quandle $\operatorname{supp}(V \oplus W)$ is isomorphic to the conjugation quandle $h^{\Gamma_4} \cup g^{\Gamma_4}$. An alternative description for this quandle goes as follows:

As we said before, $\operatorname{supp} V$ is a trivial quandle with two elements and $\operatorname{supp} W$ is a dihedral quandle with four elements. Thus we may assume that the quandle $\operatorname{supp} V$ is isomorphic to the quandle $Y = \{y_1, y_2\}$ given by $y_i \triangleright y_j = y_j$ for all $i, j \in \{1, 2\}$ and that $\operatorname{supp} W$ is the quandle over $Z = \{z_1, z_2, z_3, z_4\}$ given by $z_i \triangleright z_j = z_{2i-j \bmod 4}$ for all $i, j \in \{1, 2, 3, 4\}$. The quandle $\operatorname{supp}(V \oplus W)$ is then isomorphic to the amalgamated sum of Y and Z with respect to the morphisms $\sigma(y) = (z_1\ z_2)$ for all $y \in Y$ and

$$\tau(z) = \begin{cases} (y_1\ y_2\ y_3\ y_4) & \text{if } z = z_1, \\ (y_1\ y_4\ y_3\ y_2) & \text{if } z = z_2. \end{cases}$$

For the notion of amalgamated sum of quandles we refer to [1, Lem. 1.18].

Lemma 5.4. *Let* $V, W \in {}^G_G\mathcal{YD}$ *such that* $\operatorname{supp} V = h^G$ *and* $\operatorname{supp} W = g^G$. *Then the following hold:*

(1) $(\operatorname{ad} W)(V)$ *and* $(\operatorname{ad} W)^2(V)$ *are non-zero.*
(2) *If* $(\operatorname{ad} V)^2(W) = 0$ *then* $\dim V_h = 1$ *and* $hv = -v$ *for all* $v \in V_h$.
(3) *If* $\operatorname{supp}(\operatorname{ad} W)^2(V)$ *is a conjugacy class of* G *then* $\dim W_g = 1$ *and* $gw = -w$ *for all* $w \in W_g$.

Proof. Since g and h do not commute, $(\operatorname{ad} W)(V)$ is non-zero. Since $(g, h) \in \operatorname{supp} Q_1(g, h)$, [9, Prop. 5.5] yields that $(\epsilon^2 g, \epsilon g, h) \in \operatorname{supp} Q_2(\epsilon g, g, h)$. To prove (3) use Proposition 1.1 with $m = i = 1$, $p_1 = g$ and $p_2 = h$. Similarly, (2) follows from Proposition 1.1 with $m = i = 1$, $p_1 = h$ and $p_2 = g$. □

5.2. Main results

Let G, V and W be as in Section 5.1. Our goal is Theorem 5.5 below.

Theorem 5.5. *Let* $V = M(h, \rho)$ *and* $W = M(g, \sigma)$ *be absolutely simple Yetter–Drinfeld modules over* G. *Assume that* $(\mathrm{id} - c_{W,V} c_{V,W})(V \otimes W) \neq 0$. *The following are equivalent:*

(1) *The Nichols algebra* $\mathfrak{B}(V \oplus W)$ *is finite-dimensional.*
(2) *The pair* (V, W) *admits all reflections and* $\mathcal{W}(V, W)$ *is finite.*
(3) $\rho(h) = -1$, $\sigma(g) = -1$, $\rho(\epsilon) = \rho(g^2)\sigma(\epsilon^{-1}h^2)$ *and* $\rho(\epsilon)^2 = -1$.

In this case, $\mathcal{W}(V, W)$ *is standard with Cartan matrix of type* B_2. *Moreover,*

$$\mathcal{H}_{\mathfrak{B}(V\oplus W)}(t_1, t_2) = (1 + t_2)^4 \big(1 + t_2^2\big)^2 (1 + t_1 t_2)^4 \big(1 + t_1^2 t_2^2\big)^2 q\big(t_1 t_2^2\big) q(t_1),$$

where

$$q(t) = \begin{cases} (1 + t)^2 (1 + t^2) & \text{if } \operatorname{char} \mathbb{K} \neq 2, \\ (1 + t)^2 & \text{if } \operatorname{char} \mathbb{K} = 2. \end{cases}$$

In particular,

$$\dim \mathfrak{B}(V \oplus W) = \begin{cases} 8^2 64^2 = 262\,144 & \text{if } \operatorname{char} \mathbb{K} \neq 2, \\ 4^2 64^2 = 65\,536 & \text{if } \operatorname{char} \mathbb{K} = 2. \end{cases}$$

We will prove Theorem 5.5 in Section 6.

6. Proof of Theorem 5.5

As in Section 5.1 let $V = M(h, \rho)$ and $W = M(g, \sigma)$. We assume that $\deg \rho = \deg \sigma = 1$ and that $\sigma(g) = \rho(h) = -1$. As usual, $X_n = X_n^{V,W}$ and $\varphi_n = \varphi_n^{V,W}$ if no confusion can arise.

Lemma 6.1. *Assume that $\sigma(g) = \rho(h) = -1$. Then $X_1^{V,W}$ is absolutely simple if and only if $\rho(\epsilon) = \rho(g^2)\sigma(\epsilon^{-1}h^2)$. In this case, $X_1^{V,W} \simeq M(hg, \sigma_1)$, where σ_1 is the irreducible character of the centralizer $G^{hg} = \langle \epsilon^2, \epsilon^{-1}h^2, hg\rangle$ given by*

$$\sigma_1(hg) = -1, \qquad \sigma_1(\epsilon^2) = \rho(\epsilon^2)\sigma(\epsilon^2), \qquad \sigma_1(\epsilon^{-1}h^2) = \sigma(\epsilon^{-1}h^2)\rho(\epsilon^{-1}h^2).$$

Let $w' := \varphi_1(v \otimes w)$. Then $w' \in (V \otimes W)_{hg}$ is non-zero. Moreover, the set $\{w', hw', \epsilon w', \epsilon hw'\}$ is a basis of $X_1^{V,W}$. The degrees of these basis vectors are hg, ϵhg, $\epsilon^2 hg$ and $\epsilon^3 hg$, respectively.

Proof. First notice that $X_1^{V,W} = \varphi_1(V \otimes W) = \mathbb{K}G\varphi_1(v \otimes w)$. A direct calculation using Remarks 5.1 and 5.2 yields

$$w' = \varphi_1(v \otimes w) = (\mathrm{id} - c_{W,V}c_{V,W})(v \otimes w) = v \otimes w - \rho(\epsilon)^{-1}gv \otimes hw.$$

Hence $w' \in (V \otimes W)_{hg}$ is non-zero. Since $G^{hg} = hgZ(G)$ is abelian, we conclude that $X_1^{V,W}$ is absolutely simple if and only if $hgw' \in \mathbb{K}w'$. This is equivalent to $\rho(\epsilon) = \rho(g^2)\sigma(\epsilon^{-1}h^2)$, and then $hgw' = -w'$. The remaining claims on σ_1 follow from $\epsilon^2, \epsilon^{-1}h^2 \in Z(G)$. $\square$

Remark 6.2. Assume that $\sigma(g) = \rho(h) = -1$ and $\rho(\epsilon) = \rho(g^2)\sigma(\epsilon^{-1}h^2)$. Then action of G on X_1 is given by the following table:

$X_1^{V,W}$	w'	hw'	$\epsilon w'$	$\epsilon hw'$
ϵ	$\epsilon w'$	$\epsilon hw'$	$\sigma_1(\epsilon^2)w'$	$\sigma_1(\epsilon^2)hw'$
h	hw'	$\sigma_1(\epsilon^{-1}h^2)\epsilon w'$	$\epsilon hw'$	$\sigma_1(\epsilon h^2)w'$
g	$-\sigma_1(\epsilon^{-1}h^{-2})\epsilon hw'$	$-\sigma_1(\epsilon^2)\epsilon w'$	$-\sigma_1(\epsilon^{-1}h^{-2})hw'$	$-\sigma_1(\epsilon^2)w'$

Lemma 6.3. *Assume that $\sigma(g) = \rho(h) = -1$ and $\rho(\epsilon) = \rho(g^2)\sigma(\epsilon^{-1}h^2)$. Then $X_2^{V,W} = 0$.*

Proof. Since $G \triangleright (h, hg) = h^G \times (hg)^G$, where $\triangleright$ denotes the diagonal action, we conclude that $X_2^{V,W} = \mathbb{K}G\varphi_2(v \otimes w')$. Thus it is enough to prove that $\varphi_2(v \otimes w') = 0$. We compute:

$$\varphi_2(v \otimes w') = v \otimes w' + \rho(\epsilon)^2 gv \otimes hw' + hv \otimes \varphi_1(v \otimes w) - \rho(\epsilon)^{-1}hgv \otimes \varphi_1(v \otimes hw).$$

Since φ_1 is a G-module map, by acting with h on $w' = \varphi_1(v \otimes w)$ we obtain that $\varphi_1(v \otimes hw) = -hw'$. Since $hgv = -\rho(\epsilon)^{-1}gv$, the claim follows. $\square$

Remark 6.4. The braiding $c_{V,W}$ induces an isomorphism of Yetter–Drinfeld modules over G between $X_1^{V,W}$ and $X_1^{W,V}$. The action of G on $X_1^{W,V}$ can be obtained from the action of G on $X_1^{V,W}$ in Remark 6.2. The element $v' := c_{V,W}(w') \in (W \otimes V)_{hg}$ is non-zero. Moreover,

$$v' = hw \otimes v - \rho(\epsilon)^{-1}\sigma\big(\epsilon^{-1}h^2\big)w \otimes gv = \varphi_1(hw \otimes v). \tag{6.1}$$

To compute $X_2^{W,V}$ we need the following lemma.

Lemma 6.5. *Assume that $\sigma(g) = \rho(h) = -1$ and $\rho(\epsilon) = \rho(g^2)\sigma(\epsilon^{-1}h^2)$. Then the following hold:*

$$\varphi_1(\epsilon w \otimes v) = -\sigma\big(\epsilon h^{-2}\big)hv', \tag{6.2}$$

$$\varphi_1(\epsilon w \otimes gv) = -\sigma\big(\epsilon h^{-2}\big)\rho\big(\epsilon^2\big)\epsilon v', \tag{6.3}$$

$$\varphi_1(w \otimes v) = -\sigma\big(\epsilon^{-1}h^{-2}\big)\rho(\epsilon)^{-1}\epsilon hv', \tag{6.4}$$

$$\varphi_1(w \otimes gv) = -\rho(\epsilon)\sigma\big(\epsilon h^{-2}\big)v'. \tag{6.5}$$

Proof. Since $v' = \varphi_1(hw \otimes v)$, acting on this element with h we obtain Eq. (6.2). Acting on (6.2) with g we obtain Eq. (6.3). To prove Eq. (6.4) act with ϵ on Eq. (6.2). Finally, to prove Eq. (6.5) act with g on Eq. (6.4). □

Lemma 6.6. *Assume that $\sigma(g) = \rho(h) = -1$ and $\rho(\epsilon) = \rho(g^2)\sigma(\epsilon^{-1}h^2)$. Then $X_2^{W,V}$ is absolutely simple if and only if $\rho(\epsilon^2) = -1$. In this case, $X_2^{W,V} \simeq M(\epsilon hg^2, \rho_2)$, where ρ_2 is the irreducible character of $G^{\epsilon hg^2} = G^h$ given by*

$$\rho_2(h) = \rho\big(\epsilon g^{-2}\big), \qquad \rho_2\big(\epsilon^{-1}h^2\big) = \rho\big(\epsilon g^{-4}\big), \qquad \rho_2\big(g^2\big) = \rho\big(g^2\big).$$

Moreover, $\rho_2(\epsilon hg^2) = -1$ and the set $\{v'' := \varphi_2(\epsilon w \otimes v'), gv''\}$ is a basis of $X_2^{W,V}$. The degrees of these basis vectors are ϵhg^2 and $\epsilon^2 hg^2$, respectively.

Proof. Since $G \triangleright (g, hg) \cup G \triangleright (\epsilon^2 g, hg) = g^G \times (hg)^G$, we conclude that

$$X_2^{W,V} = \mathbb{K}G\big\{\varphi_2\big(w \otimes v'\big), \varphi_2\big(\epsilon w \otimes v'\big)\big\}.$$

We first prove that $\varphi_2(w \otimes v') = 0$. First, one obtains from Remarks 6.4 and 6.2, that $gv' = -\sigma_1(\epsilon^{-1}h^{-2})\epsilon hv'$. Moreover,

$$\begin{aligned} c_{X_1,W}c_{W,X_1}\big(w \otimes v'\big) &= c_{X_1,W}\big(gv' \otimes w\big) = -\sigma\big(\epsilon^2\big)\epsilon hw \otimes gv' \\ &= \sigma\big(\epsilon^2\big)\sigma_1\big(\epsilon^{-1}h^{-2}\big)\epsilon hw \otimes \epsilon hv' \\ &= \rho\big(\epsilon^2 g^2\big)\epsilon hw \otimes \epsilon hv' \end{aligned}$$

by Lemma 6.1. Therefore

$$\varphi_2(w\otimes v') = w\otimes v' - \rho(\epsilon^2g^2)\epsilon hw\otimes \epsilon hv' \\ + ghw\otimes \varphi_1(w\otimes v) - \rho(\epsilon)^{-1}\sigma(\epsilon^{-1}h^2)gw\otimes \varphi_1(w\otimes gv).$$

Using Eqs. (6.4) and (6.5) we conclude that

$$\varphi_2(w\otimes v') = w\otimes v' - \rho(\epsilon^2g^2)\epsilon hw\otimes \epsilon hv' + \rho(\epsilon)^{-1}\sigma(\epsilon h^{-2})\epsilon hw\otimes \epsilon hv' - w\otimes v',$$

and hence $\varphi_2(w\otimes v') = 0$.

Now we use Eq. (6.1) to compute:

$$\varphi_2(\epsilon w\otimes v') = \epsilon w\otimes v' - c_{X_2,W}c_{W,X_2}(\epsilon w\otimes v') \\ - \epsilon hw\otimes \varphi_1(\epsilon w\otimes v) + \rho(\epsilon)^{-1}\sigma(\epsilon h^2)w\otimes \varphi_1(\epsilon w\otimes gv).$$

Using Lemma 1.2 and Eqs. (6.2) and (6.3) we conclude that

$$v'' = \epsilon w\otimes v' - \sigma(\epsilon^2)hw\otimes \rho(g^2)\epsilon hv' + \sigma(\epsilon h^{-2})\epsilon hw\otimes hv' - \sigma(\epsilon^2)\rho(\epsilon)w\otimes \epsilon v' \tag{6.6}$$

belongs to $(W\otimes W\otimes V)_{\epsilon hg^2}$ and it is non-zero. Since $G^{\epsilon hg^2} = G^h = hZ(G)$, the module $X_2^{W,V}$ is absolutely simple if and only if $hv'' \in \mathbb{K}v''$. This is equivalent to $\rho(\epsilon^2) = -1$. Then $hv'' = \sigma(\epsilon^{-1}h^2)v'' = \rho(\epsilon g^{-2})v''$. □

Remark 6.7. Assume that $\sigma(g) = \rho(h) = -1$, $\rho(\epsilon) = \rho(g^2)\sigma(\epsilon^{-1}h^2)$, and $\rho(\epsilon^2) = -1$. Then the action of G on $X_2^{W,V}$ is given by:

$X_2^{W,V}$	v''	gv''
ϵ	$\rho(\epsilon)v''$	$\rho(\epsilon^{-1})gv''$
h	$\rho(\epsilon g^{-2})v''$	$\rho(g^{-2})gv''$
g	gv''	$\rho(g^2)v''$

Lemma 6.8. *Assume that $\sigma(g) = \rho(h) = -1$, $\rho(\epsilon) = \rho(g^2)\sigma(\epsilon^{-1}h^2)$, and $\rho(\epsilon^2) = -1$. Then the following hold:*

$$\varphi_2(w\otimes v') = 0, \tag{6.7}$$
$$\varphi_2(w\otimes \epsilon hv') = 0, \tag{6.8}$$
$$\varphi_2(w\otimes \epsilon v') = \rho(\epsilon)\sigma(\epsilon^2)v'', \tag{6.9}$$
$$\varphi_2(w\otimes hv') = \rho(\epsilon^{-1}g^{-2})gv''. \tag{6.10}$$

Proof. In the proof of Lemma 6.6 we have shown that $\varphi_2(w\otimes v') = 0$. Act with g on this equation to obtain Eq. (6.8). To prove Eqs. (6.9) and (6.10), act with h^2 and $g\epsilon$ on $v'' = \varphi_2(\epsilon w\otimes v')$, respectively. □

Lemma 6.9. *Assume that $\sigma(g) = \rho(h) = -1$, $\rho(\epsilon) = \rho(g^2)\sigma(\epsilon^{-1}h^2)$ and $\rho(\epsilon^2) = -1$. Then $X_3^{W,V} = 0$.*

Proof. Since $G \triangleright (g, \epsilon h g^2) = g^G \times (\epsilon h g^2)^G$, we conclude that

$$X_3^{W,V} = \mathbb{K}G\varphi_3\big(w \otimes v''\big).$$

Thus it is enough to prove that $\varphi_3(w \otimes v'') = 0$. From Lemma 6.8 we know that $\varphi_2(w \otimes v') = \varphi_2(w \otimes \epsilon h v') = 0$. Hence Lemma 1.2 implies that

$$\begin{aligned}\varphi_3\big(w \otimes v''\big) = w \otimes v'' - \sigma\big(\epsilon^2\big)hw \otimes gv'' + \sigma\big(\epsilon h^{-2}\big)g\epsilon hw \otimes \varphi_2\big(w \otimes hv'\big)\\ - \sigma\big(\epsilon^2\big)\rho(\epsilon)gw \otimes \varphi_2\big(w \otimes \epsilon v'\big).\end{aligned}$$

Now Eqs. (6.9) and (6.10) imply that $\varphi_3(w \otimes v'') = 0$. □

We summarize the results concerning the adjoints actions in the following proposition.

Proposition 6.10. *Assume that*

$$\rho(h) = \sigma(g) = -1, \qquad \rho(\epsilon) = \rho\big(g^2\big)\sigma\big(\epsilon^{-1}h^2\big).$$

Then the following hold:

(1) $(\operatorname{ad} V)(W)$ *is absolutely simple and* $(\operatorname{ad} V)^2(W) = 0$.
(2) *The Yetter–Drinfeld modules* $(\operatorname{ad} W)^m(V)$ *are absolutely simple or zero for all* $m \in \mathbb{N}_0$ *if and only if* $\rho(\epsilon^2) = -1$. *In this case,* $(\operatorname{ad} W)^2(V) \neq 0$ *and* $(\operatorname{ad} W)^3(V) = 0$.

Proof. The claim follows from Lemmas 6.1, 6.3, 6.6 and 6.9. □

Remark 6.11. Let $\epsilon_1 := \epsilon^{-1}$, $h_1 := h^{-1}$ and $g_1 := hg$. Then $G = \langle \epsilon_1, h_1, g_1 \rangle$, $\epsilon_1^2 \neq 1$, and there is a unique group homomorphism $\varGamma_4 \to G$ such that

$$a \mapsto g_1, \qquad b \mapsto h_1, \qquad \nu \mapsto \epsilon_1.$$

Lemma 6.12. *Assume that* $\sigma(g) = \rho(h) = -1$, $\rho(\epsilon) = \rho(g^2)\sigma(\epsilon^{-1}h^2)$, *and* $\rho(\epsilon^2) = -1$. *Then* $R_1(V, W) = (V^*, X_1^{V,W})$, *where* $V^* \simeq M(h_1, \rho^*)$ *and* ρ^* *is the irreducible representation of* G^h *dual to* ρ, $X_1^{V,W} \simeq M(g_1, \sigma_1)$ *and* σ_1 *is the irreducible representation of* G^{g_1} *given in Lemma 6.1, and*

$$\sigma_1(g_1) = \rho^*(h_1) = -1, \qquad \rho^*(\epsilon_1) = \rho^*\big(g_1^2\big)\sigma_1\big(\epsilon_1^{-1}h_1^2\big), \qquad \rho^*\big(\epsilon_1^2\big) = -1. \tag{6.11}$$

Proof. The description of $R_1(V, W)$ follows from Proposition 6.10(1) and Lemma 6.1. Further,

$$\begin{aligned}\rho^*\big((hg)^2\big)\sigma_1\big(\epsilon h^{-2}\big) &= \rho^*\big(\epsilon^{-1}h^2g^2\big)\sigma\big(\epsilon h^{-2}\big)\rho\big(\epsilon h^{-2}\big)\\ &= \rho(\epsilon)\rho\big(g^{-2}\big)\sigma\big(\epsilon h^{-2}\big)\rho(\epsilon) = \rho(\epsilon).\end{aligned}$$

The remaining equations in (6.11) are easily shown. □

Remark 6.13. Let $\epsilon_2 := \epsilon^{-1}$, $h_2 := \epsilon h g^2$ and $g_2 := g^{-1}$. Then $G = \langle \epsilon_2, h_2, g_2 \rangle$, $\epsilon_2^2 \neq 1$, and there is a unique group homomorphism $\Gamma_4 \to G$ such that

$$a \mapsto g_2, \qquad b \mapsto h_2, \qquad \nu \mapsto \epsilon_2.$$

Lemma 6.14. *Assume that* $\sigma(g) = \rho(h) = -1$, $\rho(\epsilon) = \rho(g^2)\sigma(\epsilon^{-1}h^2)$, *and* $\rho(\epsilon^2) = -1$. *Then* $R_2(V,W) = (X_2^{W,V}, W^*)$, *where* $W^* \simeq M(g_2, \sigma^*)$ *and* σ^* *is the irreducible representation of* G^g *dual to* σ, $X_2^{W,V} \simeq M(h_2, \rho_2)$ *and* ρ_2 *is the irreducible representation of* G^{h_2} *given in Lemma 6.6, and*

$$\sigma^*(g_2) = \rho_2(h_2) = -1, \qquad \rho_2(\epsilon_2) = \rho_2\big(g_2^2\big)\sigma^*\big(\epsilon_2^{-1}h_2^2\big), \qquad \rho_2\big(\epsilon_2^2\big) = -1. \tag{6.12}$$

Proof. The description of $R_2(V,W)$ follows from Proposition 6.10 and Lemma 6.6. Further, $\rho_2(h_2) = -1$ by Lemma 6.6, and

$$\rho_2(\epsilon_2) = \big(\rho_2\big(\epsilon^{-1}h^2\big)\rho_2(h)^{-2}\big) = \rho\big(\epsilon g^{-4}\epsilon^{-2}g^4\big) = \rho\big(\epsilon^{-1}\big),$$
$$\rho_2\big(g^{-2}\big)\sigma^*\big(\epsilon^{-1}h^2g^4\big) = \rho\big(g^{-2}\big)\sigma\big(\epsilon h^{-2}\big) = \rho(\epsilon)^{-1}.$$

Now one easily concludes the claimed formulas on ρ_2. □

Before proving Theorem 5.5 we list some well-known finite-dimensional Nichols algebras related to non-abelian epimorphic images of Γ_4.

Proposition 6.15. *Let* G *be a non-abelian quotient of* Γ_4. *Let* $V = M(h,\rho)$ *and* $W = M(g,\sigma)$, *where* ρ *and* σ *are characters of the centralizers* G^h *and* G^g, *respectively. Assume that* $\sigma(g) = \rho(h) = -1$, $\rho(\epsilon) = \rho(g^2)\sigma(\epsilon^{-1}h^2)$, *and* $\rho(\epsilon)^2 = -1$. *Then* V *and* $(\operatorname{ad} W)^2(V)$ *are of diagonal type. The braiding matrices with respect to the bases* $\{v, gv\}$ *and* $\{v'', gv''\}$ *are*

$$\begin{pmatrix} -1 & \rho(\epsilon) \\ \rho(\epsilon) & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & \rho(\epsilon^{-1}) \\ \rho(\epsilon^{-1}) & -1 \end{pmatrix},$$

respectively. In particular, the Nichols algebras $\mathfrak{B}(V)$ *and* $\mathfrak{B}((\operatorname{ad} W)^2(V))$ *are of Cartan type* $A_1 \times A_1$ *if* $\operatorname{char} \mathbb{K} = 2$ *and* A_2 *if* $\operatorname{char} \mathbb{K} \neq 2$. *Their Hilbert series is*

$$\mathcal{H}_{\mathfrak{B}(V)}(t) = \mathcal{H}_{\mathfrak{B}((\operatorname{ad} W)^2(V))}(t) = \begin{cases} (1+t)^2 & \text{if } \operatorname{char} \mathbb{K} = 2, \\ (1+t)^2(1+t^2) & \text{if } \operatorname{char} \mathbb{K} \neq 2. \end{cases}$$

Proof. The braiding matrices are obtained from a direct calculation using Remarks 5.1 and 6.7. The claim concerning the Hilbert series follows from the definition of the root system [5, Sect. 3] and [5, Thm. 1]. □

Proposition 6.16. *Let G be a non-abelian quotient of Γ_4. Let $V = M(h,\rho)$ and $W = M(g,\sigma)$, where ρ and σ are characters of the centralizers G^h and G^g, respectively. Assume that $\sigma(g) = \rho(h) = -1$, $\rho(\epsilon) = \rho(g^2)\sigma(\epsilon^{-1}h^2)$, and $\rho(\epsilon^2) = -1$. Then the Nichols algebras of W and $(\operatorname{ad} W)(V)$ are finite-dimensional with Hilbert series*

$$\begin{aligned}\mathcal{H}_{\mathfrak{B}(W)}(t) = \mathcal{H}_{\mathfrak{B}((\operatorname{ad} W)(V))}(t) &= (1+t)^4\big(1+t^2\big)^2 \\ &= 1+4t+8t^2+12t^3+14t^4+12t^5+8t^6+4t^7+t^8.\end{aligned}$$

Proof. Let H be the subgroup of G generated by $\operatorname{supp} W$. Then $H = \langle g,\epsilon\rangle$, and there exists a unique surjective group homomorphism $\Gamma_2 \to H$ with $a \mapsto g$, $b \mapsto \epsilon g$, and $\nu \mapsto \epsilon^2$. Consider W as Yetter–Drinfeld module over H by restriction of the G-module structure to H. Since $g^H = \{g,\epsilon^2 g\}$ and $(\epsilon g)^H = \{\epsilon g,\epsilon^3 g\}$, we conclude that $W = V' \oplus W'$, where $V' = \mathbb{K}w + \mathbb{K}\epsilon w$ and $W' = \mathbb{K}hw + \mathbb{K}\epsilon hw$ are simple Yetter–Drinfeld modules over H. Further, $V' \simeq M(g,\rho')$, for some character ρ' of $H^g = \langle g,\epsilon^2\rangle$, and $W' \simeq M(\epsilon g,\sigma')$ for some character σ' of $H^{\epsilon g} = \langle \epsilon g,\epsilon^2\rangle$. Using Remark 5.2 one obtains the following formulas.

$$\rho'(g) = -1, \qquad \rho'\big(\epsilon^2\big) = \sigma\big(\epsilon^2\big), \qquad \sigma'(\epsilon g) = -1, \qquad \sigma'\big(\epsilon^2\big) = \sigma\big(\epsilon^2\big). \tag{6.13}$$

Therefore $\rho'(\epsilon^2(\epsilon g)^2)\sigma'(\epsilon^2 g^2) = \sigma(\epsilon^4) = 1$, and hence $W = V' \oplus W'$ satisfies the assumptions of [6, Thm. 4.6]. Thus $\mathfrak{B}(W)$ is finite-dimensional and has the claimed Hilbert series.

The claim concerning the Nichols algebra $\mathfrak{B}((\operatorname{ad} W)(V))$ is similar. We may replace $(\operatorname{ad} W)(V)$ by $X_1^{V,W}$. Let L be the subgroup of G generated by $\operatorname{supp} X_1^{V,W}$ and consider the unique group homomorphism $\Gamma_2 \to L$ with $a \mapsto hg$, $b \mapsto \epsilon hg$ and $\nu \mapsto \epsilon^2$. As we did in the previous paragraph, [6, Thm. 4.6] yields the Hilbert series of $\mathfrak{B}(X_1^{V,W})$. □

Proof of Theorem 5.5. (1) ⇒ (2). Since $\mathfrak{B}(V \oplus W)$ is finite-dimensional, the pair (V,W) admits all reflections by [2, Cor. 3.18] and the Weyl groupoid is finite by [2, Prop. 3.23].

(2) ⇒ (3). By [9, Prop. 4.3], after changing the object of $\mathcal{W}(V,W)$ and possibly interchanging V and W, we may assume that $V = M(h,\rho)$ and $W = M(g,\sigma)$, such that $\deg\rho = \deg\sigma = 1$, $(\operatorname{ad} V)^2(W) = 0$ and $(\operatorname{ad} W)^4(V) = 0$. By [7, Thm. 7.2(3)], $(\operatorname{ad} W)^m(V)$ is absolutely simple or zero for all $m \in \mathbb{N}_0$. Lemma 5.4 implies that $(\operatorname{ad} W)(V)$ and $(\operatorname{ad} W)^2(V)$ are non-zero and $\rho(h) = \sigma(g) = -1$. Since $(\operatorname{ad} V)(W)$ and $(\operatorname{ad} W)^2(V)$ are absolutely simple, Lemmas 6.1 and 6.6 imply that $\rho(\epsilon) = \rho(g^2)\sigma(\epsilon^{-1}h^2)$ and $\rho(\epsilon^2) = -1$.

(3) ⇒ (1). Proposition 6.10 and Lemmas 6.12 and 6.14 imply that $\mathcal{W}(V,W)$ is standard with Cartan matrix of type B_2. By [6, Cor. 2.7(2)],

$$\mathfrak{B}(V \oplus W) \simeq \mathfrak{B}(V) \otimes \mathfrak{B}\big((\operatorname{ad} W)(V)\big) \otimes \mathfrak{B}\big((\operatorname{ad} W)^2(V)\big) \otimes \mathfrak{B}(W)$$

as $\mathbb{N}_0^2$-graded vector spaces, where

$$\deg V = \alpha_1, \qquad \deg W = \alpha_2,$$
$$\deg\bigl((\operatorname{ad} W)(V)\bigr) = \alpha_1 + \alpha_2, \qquad \deg\bigl((\operatorname{ad} W)^2(V)\bigr) = \alpha_1 + 2\alpha_2.$$

Now the claim on the Hilbert series of $\mathfrak{B}(V \oplus W)$ follows from Propositions 6.15 and 6.16. □

7. An application

In [9] we presented five quandles which are essential to our classification. These quandles are:

$$\begin{aligned} Z_T^{4,1}&: \quad (243)\,(134)\,(142)\,(123)\,\mathrm{id}, \\ Z_2^{2,2}&: \quad (24)\,(13)\,(24)\,(13), \\ Z_3^{3,1}&: \quad (23)\,(13)\,(12)\,\mathrm{id}, \\ Z_3^{3,2}&: \quad (23)\,(45)\,(13)\,(45)\,(12)\,(45)\,(123)\,(132), \\ Z_4^{4,2}&: \quad (24)\,(56)\,(13)\,(56)\,(24)\,(56)\,(13)\,(56)\,(1234)\,(1432). \end{aligned} \tag{7.1}$$

Let us describe these quandles in a different way. Remark 2.1 states that $Z_T^{4,1}$ is isomorphic to the disjoint union of the trivial quandle with one element and the quandle associated with the vertices of the tetrahedron. The quandle $Z_2^{2,2}$ is isomorphic to the dihedral quandle $\mathbb{D}_4$ with four elements. The quandle $Z_3^{3,1}$ is isomorphic to the disjoint union of the trivial quandle with one element and the dihedral quandle $\mathbb{D}_3$ with three elements. Remark 5.3 describes the quandle $Z_4^{4,2}$ as an amalgamated sum of $\mathbb{D}_4$ and the trivial quandle with two elements. Similarly, the quandle $Z_3^{3,2}$ can be presented as an amalgamated sum of $\mathbb{D}_3$ with the trivial quandle of two elements. See [1, §1] for disjoint union and amalgamated sum of quandles.

Theorem 7.1. *Let $\mathbb{K}$ be a field, G be a non-abelian group, and V and W be finite-dimensional absolutely simple Yetter–Drinfeld modules over G such that G is generated by the support of $V \oplus W$. Assume that the pair (V, W) admits all reflections and the Weyl groupoid $\mathcal{W}(V, W)$ of (V, W) is finite. If $c_{W,V}c_{V,W} \neq \mathrm{id}_{V\otimes W}$, then $\operatorname{supp}(V \oplus W)$ is isomorphic to one of the following quandles:*

$$Z_T^{4,1},\ Z_2^{2,2},\ Z_3^{3,1},\ Z_3^{3,2},\ Z_4^{4,2}.$$

Moreover, the group G is isomorphic to a quotient of the enveloping group of the quandle $\operatorname{supp}(V \oplus W)$*:*

Quandle	$Z_T^{4,1}$	$Z_2^{2,2}$	$Z_3^{3,1}$	$Z_3^{3,2}$	$Z_4^{4,2}$
Enveloping group	T	Γ_2	Γ_3	Γ_3	Γ_4

Proof. By [9, Prop. 4.3], after changing the object of $\mathcal{W}(V,W)$ and possibly interchanging V and W we may assume that $(\operatorname{ad} V)(W) \neq 0$, $(\operatorname{ad} V)^2(W) = 0$, and $(\operatorname{ad} W)^4(V) = 0$. Then [9, Thm. 4.4] implies that the group G is a quotient of Γ_n for $n \in \{2,3,4\}$ or a quotient of T.

Suppose first that G is a quotient of Γ_2. Since Γ_2 has conjugacy classes of size one or two [6, Sect. 3] and G is non-abelian, it follows that the quandles appearing after applying reflections to (V,W) are isomorphic to $\operatorname{supp}(V \oplus W) \simeq Z_2^{2,2}$.

Suppose now that G is a quotient of Γ_3. Any conjugacy class of G has size 1, 2, or 3. Assume that an object of $\mathcal{W}(V,W)$ is represented by a pair (V',W') of absolutely irreducible Yetter–Drinfeld modules over G with $|\operatorname{supp} V'| = |\operatorname{supp} W'| = 3$. Then $\operatorname{supp}(V' \oplus W')$ is isomorphic as a quandle to $(gz_1)^G \cup (hgz_2)^G$ for some $z_1, z_2 \in Z(G)$ by [6, Sect. 3.1]. Let $s = gz_1$ and $t = hgz_2$. Since $stst \neq tsts$, [7, Prop. 8.5] implies that $(\operatorname{ad} V')(W')$ is not irreducible, which contradicts [6, Thm. 2.5].

Suppose that G is a quotient of Γ_4. Then all conjugacy classes of G have size 1, 2, or 4 [6, Sect. 3]. After changing the object of $\mathcal{W}(V,W)$ we may assume that $V = M(h,\rho)$, where ρ is a character of G^h and $W = M(g,\sigma)$, where σ is a character of G^g. In particular, $\operatorname{supp}(V \oplus W) \simeq Z_4^{4,2}$ as quandles. By Theorem 5.5, $\rho(h) = \sigma(g) = -1$, $\rho(\epsilon) = \rho(g^2)\sigma(\epsilon^{-1}h^2)$ and $\rho(\epsilon)^2 = -1$. Lemmas 6.12 and 6.14 imply that after applying reflections to (V,W) one obtains new pairs (V',W') such that $\operatorname{supp}(V' \oplus W') \simeq Z_4^{4,2}$ as quandles.

Finally, suppose that G is a quotient of T. By changing the object of $\mathcal{W}(V,W)$ if necessary, we may assume that $V = M(z,\rho)$, where ρ is a representation of G and $W = M(x_1,\sigma)$, where σ is a character of G^{x_1}. In particular $\operatorname{supp}(V \oplus W) \simeq Z_T^{4,1}$ as quandles. From Theorem 2.9 we obtain that $\deg \rho = 1$, $(\rho(x_1)\sigma(z))^2 - \rho(x_1)\sigma(z) + 1 = 0$, $\sigma(x_1) = -1$, $\sigma(x_2x_3) = 1$, and $\rho(x_1z)\sigma(z) = 1$. Lemmas 4.19 and 4.21 imply that all reflections of (V,W) are pairs (V',W') with $\operatorname{supp}(V' \oplus W') \simeq Z_T^{4,1}$ as quandles. This proves the theorem. □

As a combination of our results with the main results in [2] we obtain the following corollary concerning finite-dimensional Nichols algebras.

Corollary 7.2. *Let $\mathbb{K}$ be a field, G be a non-abelian group, and V and W be absolutely simple Yetter–Drinfeld modules over G such that G is generated by* $\operatorname{supp}(V \oplus W)$. *Assume that the Nichols algebra $\mathfrak{B}(V \oplus W)$ is finite-dimensional. If $c_{W,V}c_{V,W} \neq \mathrm{id}_{V\otimes W}$, then* $\operatorname{supp}(V \oplus W)$ *is isomorphic to one of the quandles of Theorem 7.1 and the group G is isomorphic to a quotient of the enveloping group of the quandle* $\operatorname{supp}(V \oplus W)$.

Proof. Since $\mathfrak{B}(V \oplus W) < \infty$, the pair (V,W) admits all reflections by [2, Cor. 3.18] and the Weyl groupoid is finite by [2, Prop. 3.23]. So Theorem 7.1 applies. □

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