

NICHOLS ALGEBRAS WITH MANY CUBIC RELATIONS

I. HECKENBERGER, A. LOCHMANN, AND L. VENDRAMIN

ABSTRACT. Nichols algebras of group type with many cubic relations are classified under a technical assumption on the structure of Hurwitz orbits of the third power of the underlying indecomposable rack. All such Nichols algebras are finite-dimensional and their Hilbert series have a factorization into quantum integers. Also, all known finite-dimensional elementary Nichols algebras turn out to have many cubic relations. The technical assumption of our theorem can be removed if a conjecture in the theory of cellular automata can be proven.

CONTENTS

1. Introduction	1
2. Racks	4
3. On the cycle structure of racks	6
4. Quotients of Hurwitz orbits	8
5. Cellular automata over homogeneous spaces	13
6. Plagues on Hurwitz orbits	15
7. Proof of Theorem 6.3	17
8. Nichols algebras with many cubic relations	40
9. Computation of the 2-cocycles and the main theorem	44
References	46

1. INTRODUCTION

Let (V, c) be a braided vector space. A fundamental question in the theory of Hopf algebras is whether the Nichols algebra $\mathfrak{B}(V)$ (which heavily depends on c) is finite-dimensional. If V is of diagonal type, then $\mathfrak{B}(V)$ is known to have a restricted PBW basis and the PBW generators are parametrized by Lyndon words. The multidegrees of these Lyndon words can be regarded as positive roots in a generalized root system, which has the symmetry of a Weyl groupoid. Moreover, the existence of the restricted PBW basis implies that the Hilbert series of $\mathfrak{B}(V)$ is rational.

If the braiding c is of group type, then the structure of $\mathfrak{B}(V)$ is much less understood. Moreover, only a few finite-dimensional examples are known

which are not of diagonal type. One common feature of them is the factorization of the Hilbert series of $\mathfrak{B}(V)$ as

$$\mathcal{H}_{\mathfrak{B}(V)}(t) = \prod_{i=1}^{k_1} (a_i)_t \prod_{i=1}^{k_2} (b_i)_{t^2}$$

for some $k_1, k_2 \geq 0$, $a_1, \dots, a_{k_1}, b_1, \dots, b_{k_2} \geq 2$, where $(a)_{t^b} = 1 + t^b + t^{2b} + \dots + t^{(a-1)b}$ for all $a, b \geq 1$. We say that $\mathcal{H}_{\mathfrak{B}(V)}(t)$ is *t-integral of depth two*. The *t-integrality* of all known Hilbert series of Nichols algebras of group type was the starting point of a classification program in [5] and [6]. It turned out that under additional technical restrictions the *t-integrality* of the Hilbert series is equivalent to an inequality on the dimensions of the homogeneous components of low degree of the Nichols algebra, and that a complete list can be given.

Let G be a group, \mathbb{k} be a field and $V \in {}_{\mathbb{k}G}^G \mathcal{YD}$. Then the Nichols algebra $\mathfrak{B}(V)$ is called of group type. We say that the Nichols algebra $\mathfrak{B}(V)$ is *elementary* if V is finite-dimensional, absolutely irreducible and if its support

$$\text{supp } V = \{x \in G \mid V_x \neq 0\}$$

generates the group G . A fundamental problem in the theory of Nichols algebras is the classification of finite-dimensional elementary Nichols algebras. In fact, often it is not important to know G . Let H be a group. We say that two Yetter-Drinfeld modules $V \in {}_{\mathbb{k}G}^G \mathcal{YD}$, $W \in {}_{\mathbb{k}H}^H \mathcal{YD}$ are *bg-equivalent*¹ if there exists a bijection $\varphi : \text{supp } V \rightarrow \text{supp } W$ and a linear isomorphism $\psi : V \rightarrow W$ such that

$$\psi(V_g) = W_{\varphi(g)}, \quad \psi(gv) = \varphi(g)\psi(v)$$

for all $g, x \in \text{supp } V$, $v \in V$. Two Nichols algebras of group type are called *bg-equivalent* if their degree one parts are bg-equivalent. Then it is more convenient to ask for all finite-dimensional elementary Nichols algebras up to bg-equivalence. The answer to this problem is unknown to a large extent. However, there are indications that the following conjecture is possibly true:

Conjecture 1.1.² *All finite-dimensional elementary Nichols algebras are t-integral of depth two. In particular, any such Nichols algebra is bg-equivalent to one of those listed in Table 9.1.*

In this paper we extend the results in [5] and [6]. More precisely, we reduce the classification problem of elementary Nichols algebras with *t-integral* Hilbert series of depth two to a problem for cellular automata on braid group orbits.

Let us discuss some details of our approach. Assume first that the Hilbert series of $\mathfrak{B}(V)$ is *t-integral* of depth two. In [6, Section 2.1] it was shown

¹g refers to the grading by the group and b refers to the braiding

²This conjecture was posed in the Oberwolfach mini-workshop “Nichols algebras and Weyl groupoids” in October 2012.

that then the inequality

$$\dim \ker (1 + c_{12} + c_{12}c_{23}) \geq \frac{1}{3} \dim V((\dim V)^2 - 1)$$

holds. Now, if this inequality holds for V , we say that $\mathfrak{B}(V)$ has many cubic relations. Our intention is to classify all Nichols algebras with many cubic relations and to prove that their Hilbert series are t -integral of depth two. If $\text{supp } V$ is a braided rack, then this claim was proven in [6]. Here we attack the general problem.

Recall that the braid group \mathbb{B}_3 can be presented by generators σ_1 and σ_2 and relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$. The group \mathbb{B}_3 acts on $(\text{supp } V)^3$ by $\sigma_1 \cdot (x, y, z) = (x \triangleright y, x, z)$ and $\sigma_2 \cdot (x, y, z) = (x, y \triangleright z, y)$ for all $x, y, z \in \text{supp } V$, where \triangleright means conjugation. The orbits of the action of \mathbb{B}_3 are called Hurwitz orbits. We assume that the quotients of the Hurwitz orbits of $(\text{supp } V)^3$ by the action of the center of the braid group have only σ_1 -cycles of length ≤ 4 . (The braidedness condition on racks means that the Hurwitz orbits of $(\text{supp } V)^3$ themselves have only σ_1 -cycles of length ≤ 3 .) This way we cover the braided case and the cases $\text{supp } V = \text{Aff}(5, i)$ with $i = 2, 3$, for which finite-dimensional Nichols algebras are known to exist. On the other hand, we have to deal with infinitely many Hurwitz orbits in contrast to [6].

Looking at homogeneous components, $\dim \ker (1 + c_{12} + c_{12}c_{23})$ can be estimated from above by a purely combinatorial way. Choose a subset Y of $(\text{supp } V)^3$ and take an element $\bar{\alpha} \in \bigoplus_{(x,y,z) \in Y} V_x \otimes V_y \otimes V_z$. Let $k(Y, \bar{\alpha})$ be the set of all $\alpha \in \ker (1 + c_{12} + c_{12}c_{23})$ with projection $\bar{\alpha}$ to its homogeneous parts with degree in Y . If $x, y, z \in \text{supp } V$ and two of

$$\{(x, y, z), \sigma_2 \cdot (x, y, z), \sigma_1\sigma_2 \cdot (x, y, z)\}$$

are in Y then for any $\alpha \in k(Y, \bar{\alpha})$ the summand with the third degree is uniquely determined. Thus $k(Y, \bar{\alpha}) \subseteq k(Y', \bar{\alpha}')$, where Y' is the union of Y and the third degree and $\bar{\alpha}'$ is the extension of $\bar{\alpha}$. This procedure of enlarging Y can be regarded as a cellular automaton on $(\text{supp } V)^3$. If $Y = (\text{supp } V)^3$ then $k(Y, \bar{\alpha}) = \bar{\alpha}$ or $k(Y, \bar{\alpha}) = \emptyset$. Hence, if a given subset $Y \subseteq (\text{supp } V)^3$ can be enlarged this way to $(\text{supp } V)^3$, then the projection of $\ker (1 + c_{12} + c_{12}c_{23})$ to the sum of homogeneous parts of degree $(x, y, z) \in Y$ is injective. Thus an important question is the following: for a given Hurwitz orbit \mathcal{O} , provide (the size of) a smallest subset Y which can be enlarged by the above process to \mathcal{O} . The size of such a Y yields surprisingly often a sharp upper bound for $\dim \ker (1 + c_{12} + c_{12}c_{23})$. We call the quotient $|Y|/|\mathcal{O}|$ the *immunity* of \mathcal{O} .

Our classification of finite-dimensional elementary Nichols algebras with many cubic relations has three steps. First we solve the evolution problem on the Hurwitz orbits in X^3 for all indecomposable racks X such that the quotient by the center has only σ_1 -cycles of length ≤ 4 . This is one of our main results. Then we conclude a small upper bound for the size of possible racks. Using additional information on the cycle structure of racks and the classification of indecomposable racks of size ≤ 35 , we finally obtain all racks satisfying the inequality on the immunity of X^3 . For those racks

we then study the possible cocycles coming from Yetter-Drinfeld structures of the enveloping group G_X and find all elementary Nichols algebras with many cubic relations, under the standing assumption on the Hurwitz orbit structure. This finishes our classification.

The paper is organized as follows. In Section 2 we review the basic definitions concerning racks and we prove our first main result about the structure of indecomposable racks. In Section 3 we provide an obstruction on the cycle structure of indecomposable crossed sets. Section 4 is devoted to the theory of coverings of Hurwitz orbits. In Section 5 we define cellular automata on G -spaces for arbitrary groups G and study particular examples. In Section 7 we prove our second main result formulated in Section 6: an explicit upper bound for the immunity of a class of Hurwitz orbits in terms of local data. Finally, in Sections 8 and 9 we classify elementary Nichols algebras with many cubic relations under our standing assumption on Hurwitz orbits.

2. RACKS

We recall basic notions and facts about racks. For additional information we refer to [1]. A *rack* is a pair (X, \triangleright) , where X is a non-empty set and $\triangleright : X \times X \rightarrow X$ is a map (considered as a binary operation on X) such that

- (1) the map $\varphi_i : X \rightarrow X$, where $x \mapsto i \triangleright x$, is bijective for all $i \in X$, and
- (2) $i \triangleright (j \triangleright k) = (i \triangleright j) \triangleright (i \triangleright k)$ for all $i, j, k \in X$ (i.e. \triangleright is *self-distributive*).

A rack (X, \triangleright) , or shortly X , is a *quandle* if $i \triangleright i = i$ for all $i \in X$. A *crossed set* X is a quandle such that $x \triangleright y = y$ if and only if $y \triangleright x = x$ for all $x, y \in X$. A *subrack* of a rack X is a non-empty subset $Y \subseteq X$ such that (Y, \triangleright) is also a rack. The *inner group* of a rack X is the group generated by the permutations φ_i of X , where $i \in X$. We write $\text{Inn}(X)$ for the inner group of X . For any rack X , the *enveloping group of X* is the group G_X given by the generating set X and the relations $xy = (x \triangleright y)x$ for all $x, y \in X$. A rack X is called *injective* if the canonical map $X \rightarrow G_X$ is injective.

We say that a rack X is *indecomposable* if the inner group $\text{Inn}(X)$ acts transitively on X . Also, X is *decomposable* if it is not indecomposable. The *profile* of an indecomposable rack X is the cycle structure of the permutation φ_x for some $x \in X$. For example, a profile $1^a 2^b 3^c$ means that for any $x \in X$, φ_x is a product of b disjoint transpositions, c disjoint 3-cycles and $|X| = a + b + c$.

Example 2.1. A group G is a rack with $x \triangleright y = xyx^{-1}$ for all $x, y \in G$. If a subset $X \subseteq G$ is stable under conjugation by the elements of X , then it is a subrack of G . In particular, the conjugacy class g^G of any $g \in G$ is a rack.

Remark 2.2. For a union X of conjugacy classes of a group G , the subgroup of G generated by X is a quotient of the enveloping group of X . This implies that a rack is injective if and only if it is isomorphic to a union of conjugacy classes of a group.

Example 2.3. Let A be an abelian group, and $X = A$. For any $g \in \text{Aut}(A)$ we have a rack structure on X given by

$$x \triangleright y = (1 - g)x + gy$$

for all $x, y \in X$. This rack is called the affine rack associated to the pair (A, g) and will be denoted by $\text{Aff}(A, g)$. In particular, let p be a prime number, q a power of p and $\alpha \in \mathbb{F}_q \setminus \{0, 1\}$. We write $\text{Aff}(\mathbb{F}_q, \alpha)$, or simply $\text{Aff}(q, \alpha)$, for the affine rack $\text{Aff}(A, g)$, where $A = \mathbb{F}_q$ and g is the automorphism given by $x \mapsto \alpha x$ for all $x \in \mathbb{F}_q$.

Example 2.4. Let p be a prime number. The affine rack $\text{Aff}(p, -1)$ is called dihedral rack and it will be denoted by \mathbb{D}_p .

Let X be a finite indecomposable injective rack and let $x \in X$. In [5, Section 2.3] integers k_n for $n \in \mathbb{N}_{\geq 2}$ were defined by

$$k_n = |\{y \in X \mid \underbrace{x \triangleright (y \triangleright (x \triangleright (y \triangleright \cdots)))}_{n \text{ elements}} = y, \\ \underbrace{x \triangleright (y \triangleright (x \triangleright (y \triangleright \cdots)))}_{j \text{ elements}} \neq y \text{ for all } j \in \{1, 2, \dots, n-1\}\}|.$$

Since X is indecomposable, the numbers k_n do not depend on x . Recall that for all $x \in X$ the permutation φ_x has precisely $1 + k_2$ fixed points. Furthermore, $1 + k_2 + k_3 + \cdots$ is just the cardinality of X . We frequently use the notation

$$k'_m = k_{m+1} + k_{m+2} + k_{m+3} + \cdots$$

where $m \in \mathbb{N}$. The number of points moved by φ_x is then $k'_2 = k_3 + k'_3$.

Lemma 2.5. Let X be a crossed set and let $u \in X$. Then the subset $S = \{x \in X \mid u \triangleright x = x, x \neq u\}$ is a subquandle of X .

Proof. Let $x, y \in X$ with $u \triangleright x = x, u \triangleright y = y$. Then

$$u \triangleright (x \triangleright y) = (u \triangleright x) \triangleright (u \triangleright y) = x \triangleright y.$$

Further, $x \triangleright u = u$ since X is a crossed set and hence $\varphi_x(S) \subseteq S$. Clearly, $\varphi_x|_S$ is injective. Finally, $\varphi_x^{-1}(S \cup \{u\}) \subseteq S \cup \{u\}$ since

$$u \triangleright (\varphi_x^{-1}(y)) = \varphi_{u \triangleright x}^{-1}(u \triangleright y) = \varphi_x^{-1}(y)$$

and $\varphi_x^{-1}(u) = u$. Thus $\varphi_x|_S$ is invertible with inverse $\varphi_x^{-1}|_S$. \square

Proposition 2.6. Let X be an indecomposable crossed set and let $u \in X$. Let $T = \{x \in X \mid u \triangleright x \neq x\}$. Then for all $t \in T$ there exists $t' \in T$ with $t \triangleright t' \neq t'$.

Proof. Let $S = X \setminus (T \cup \{u\})$ and

$$T_0 = \{t \in T \mid \text{supp } \varphi_t \subseteq S \cup \{u\}\}.$$

We have to show that $T_0 = \emptyset$.

Assume that $T_0 \neq \emptyset$. Since X is indecomposable, there exist $x \in T_0, y \in X \setminus T_0, z \in X$ such that $z \triangleright x = y$. As $x \triangleright z \neq z$ and $x \in T_0$, we conclude

that $z \in S \cup \{u\}$. As $S \cup \{u\}$ is a subquandle of X by Lemma 2.5 and $x = \varphi_z^{-1}(y)$, it follows that $y \in T \setminus T_0$. But $\text{supp } \varphi_{z \triangleright x} = \varphi_z(\text{supp } \varphi_x) \subseteq S \cup \{u\}$, a contradiction to $z \triangleright x = y \in T \setminus T_0$. This proves the proposition. \square

Theorem 2.7. *Let X be an indecomposable crossed set. Assume that the profile of X is $1^{a_1} 2^{a_2} \dots k^{a_k}$. Then*

$$k_2 \leq \sum_{j \geq 2} a_j(k'_2 - 2) \quad \text{and} \quad |X| \leq \sum_{j \geq 2} a_j(k'_2 - 2) + k'_2 + 1.$$

Proof. Let $u \in X$, $T = \{x \in X \mid u \triangleright x \neq x\}$ and $S = X \setminus (T \cup \{u\})$. Then S is a subquandle of X by Lemma 2.5. Let S_1, \dots, S_n be the $\text{Inn}(S)$ orbits of S . Then $S \triangleright S_j = S_j$ for all j , and $S \triangleright T = T$ because S is a subquandle and $S \triangleright \{u\} = \{u\}$. Since X is indecomposable, there can only be elements of T to connect the $\text{Inn}(S)$ orbits S_j with each other, with $\{u\}$ and with T .

Let $x \in T$, $s \in S_j$ and $y \in X \setminus S_j$ be such that $x \triangleright y = s$. Applying $\text{Inn}(S)$ one can see that for all $s' \in S_j$ there must be elements $x' \in T$, $y' \in X \setminus S_j$ with $x' \triangleright y' = s'$. Hence, if one element of S_j is connected to $X \setminus S_j$ via a single element of T , then every element of S_j is. We conclude that each of the k_2 elements of S has to be moved by at least one φ_x for an $x \in T$.

Each φ_x , $x \in T$, moves k'_2 elements within X . One of these elements must be u , and by Proposition 2.6 another one must be an element of T . So there are at most $k'_2 - 2$ elements which may be moved by φ_x from S to X for each $x \in T$, and $k'_2(k'_2 - 2)$ altogether. Let

$$Y = \{(x, s) \mid x \in T, s \in S, x \triangleright s \neq s\}.$$

For any $(x, s) \in Y$, $\varphi_u(x) \triangleright s = \varphi_u(x) \triangleright \varphi_u(s) = \varphi_u(x \triangleright s) \neq \varphi_u(s) = s$, and $\varphi_u(x) \neq x$. So each $s \in S$ appears not only once as a second component in some $(x, s) \in Y$, but at least j times, where $j = \min\{m \in \mathbb{N} \mid \varphi_u^m(x) = x\}$. For each j , there are ja_j such $x \in T$, which provide (since $x \triangleright u \neq u$) at most $ja_j(k'_2 - 2)$ pairs $(x, s) \in Y$. In these pairs we may have up to $a_j(k'_2 - 2)$ different elements from S as a second component, and hence $k_2 \leq \sum_{j \geq 2} a_j(k'_2 - 2)$. This proves the first formula of the claim. The second now follows immediately from $|X| = 1 + k_2 + k'_2$. \square

Corollary 2.8. *Under the assumptions of Theorem 2.7, $k_2 \leq \frac{1}{2} k'_2(k'_2 - 2)$ and $|X| \leq \frac{1}{2} k'^2_2 + 1$.*

Proof. Use the inequality $\sum_{j \geq 2} a_j \leq \frac{1}{2} \sum_{j \geq 2} ja_j = k'_2/2$. \square

3. ON THE CYCLE STRUCTURE OF RACKS

Lemma 3.1. *Let $d \in \mathbb{N}$, $a, b \in \mathbb{N}_{\geq 2}$ with $\gcd(a, b) = 1$ and let $X = \{1, \dots, d\}$ be an indecomposable crossed set. Assume that $\varphi_1^a(x) = x$ or $\varphi_1^b(x) = x$ for*

all $x \in X$. Let

$$\begin{aligned} S &= \{x \in X \mid 1 \triangleright x = x, x \neq 1\}, \\ T_a &= \{x \in X \mid \varphi_1^a(x) = x, 1 \triangleright x \neq x\}, \\ T_b &= \{x \in X \mid \varphi_1^b(x) = x, 1 \triangleright x \neq x\}. \end{aligned}$$

Then X is the disjoint union of $\{1\}, S, T_a$ and T_b and we have

$$\begin{aligned} 1 \triangleright 1 &= 1, & 1 \triangleright S &= S, & 1 \triangleright T_a &= T_a, & 1 \triangleright T_b &= T_b, \\ S \triangleright 1 &= 1, & S \triangleright S &= S, & S \triangleright T_a &= T_a, & S \triangleright T_b &= T_b, \\ T_a \triangleright T_b &= T_b, & T_b \triangleright T_a &= T_a. \end{aligned}$$

Proof. The action of 1 is clear by definition of S, T_a and T_b . X is a crossed set, therefore $s \triangleright 1 = 1$ holds for each $s \in S$. Moreover, $S \triangleright S = S$ by Lemma 2.5. Let $s \in S$ and $t \in T_a$ be arbitrary. Then $s \triangleright t = s \triangleright \varphi_1^a(t) = \varphi_1^a(s \triangleright t)$ because 1 and s commute. Hence, $s \triangleright t \in T_a \cup S \cup \{1\}$. But φ_s is bijective, so $s \triangleright t \in T_a$; similar for $t \in T_b$. Now let $x \in T_a$ and $y \in T_b$ be arbitrary. Assume $x \triangleright y \in T_a \cup S \cup \{1\}$. Then apply φ_1^a to get $x \triangleright \varphi_1^a(y) = x \triangleright y$, hence $\varphi_1^a(y) = y$, which is a contradiction; so $T_a \triangleright T_b = T_b$ and similarly $T_b \triangleright T_a = T_a$. \square

Proposition 3.2. *Let $r, d \in \mathbb{N}_{\geq 1}$, $a_1, \dots, a_r, b \in \mathbb{N}_{\geq 2}$ with $\gcd(a_1 \cdots a_r, b) = 1$ and $d \geq a_1 + \cdots + a_r + b + 1$. For all $i \in \{1, \dots, r\}$ let σ_i be an a_i -cycle and let τ be a b -cycle in \mathbb{S}_d with pairwise disjoint supports. Then there is no indecomposable crossed set $X = \{1, \dots, d\}$ with $\varphi_1 = \sigma_1 \cdots \sigma_r \tau$.*

Proof. Assume to the contrary that X is an indecomposable crossed set with $\varphi_1 = \sigma_1 \cdots \sigma_r \tau$. Then $1 \notin \text{supp } \sigma_i$ for all i and $1 \notin \text{supp } \tau$. Let $y_1, \dots, y_b \in X$ with $1 \triangleright y_i = y_{i+1}$ for all $1 \leq i < b$ and $1 \triangleright y_b = y_1$. Then

$$\tau = (y_1 \cdots y_b).$$

Let

$$T = \bigcup_{i=1}^r \text{supp } \sigma_i, \quad S = X \setminus (\{1\} \cup T \cup \text{supp } \tau).$$

Then Lemma 3.1 applies with $a = a_1 \cdots a_r$, $T_a = T$, $T_b = \text{supp } \tau$. We consider four cases.

Case 1. $y_1 \triangleright 1 \in S$. Let $s = y_1 \triangleright 1$. Then

$$\varphi_s = \varphi_{y_1 \triangleright 1} = \varphi_{y_1} \triangleright \varphi_1 = \varphi_{y_1} \sigma_1 \cdots \sigma_r \varphi_{y_1}^{-1} \cdot (y_1 y_1 \triangleright y_2 \cdots y_1 \triangleright y_b).$$

As $s \in S$, we know from Lemma 3.1 that $s \triangleright \text{supp } \tau = \text{supp } \tau$ and hence $y_1 \in \text{supp } \tau$ implies that $y_1 \triangleright y_i \in \text{supp } \tau$ for all $i \in \{1, \dots, b\}$. Thus $y_1 \triangleright \text{supp } \tau = \text{supp } \tau$. Applying φ_1 and using the transitivity of the action of φ_1 on $\text{supp } \tau$ we conclude that $y_i \triangleright \text{supp } \tau = \text{supp } \tau$ for all $i \in \{1, \dots, b\}$. Thus $\text{supp } \tau$ is X -stable by Lemma 3.1 which contradicts the indecomposability of X .

Case 2. $y_1 \triangleright 1 \in T$. Since $y_1 \in \text{supp } \tau$, Lemma 3.1 yields that $y_1 \triangleright T = T$ and hence $\varphi_{y_1}^{-1}(T) = T$, a contradiction to $1 \in \varphi_{y_1}^{-1}(T)$.

Case 3. $y_1 \triangleright 1 \in \text{supp } \tau$, $\varphi_{y_1}^{a_1 \cdots a_r}(1) = 1$. Let $a = a_1 \cdots a_r$. Then $\varphi_{y_1}^a \varphi_1 \varphi_{y_1}^{-a} = \varphi_1$, and hence $\varphi_{y_1}^a \tau \varphi_{y_1}^{-a} = \tau$. Since $\varphi_{y_1}(y_1) = y_1$, we conclude that $\varphi_{y_1}^a(y) = y$ for all $y \in \text{supp } \tau$. Conjugation by φ_1 yields

$$(3.1) \quad \varphi_y^a(y') = y' \quad \text{for all } y, y' \in \text{supp } \tau.$$

By assumption, $y_1 \triangleright 1 \in \text{supp } \tau$. Further,

$$\varphi_{y_1 \triangleright 1} = \varphi_{y_1} \varphi_1 \varphi_{y_1}^{-1} = \varphi_{y_1} \sigma_1 \cdots \sigma_r \varphi_{y_1}^{-1} \cdot (y_1 y_1 \triangleright y_2 \cdots y_1 \triangleright y_b).$$

Since $b \nmid a$, we conclude that $\varphi_{y_1 \triangleright 1}^a(y_1) \neq y_1$ which is a contradiction to Equation (3.1).

Case 4. $y_1 \triangleright 1 \in \text{supp } \tau$, $\varphi_{y_1}^b(1) = 1$. By applying φ_1 and using Lemma 3.1 we obtain that $y \triangleright 1 \in \text{supp } \tau$ and $\varphi_y^b(1) = 1$ for all $y \in \text{supp } \tau$. In particular, for all $y \in \text{supp } \tau$ there exist $\sigma_{y,1}, \dots, \sigma_{y,r}, \tau_y \in \mathbb{S}_d$ with $1 \in \text{supp } \tau_y$ such that $\varphi_y = \sigma_{y,1} \cdots \sigma_{y,r} \tau_y$. Since

$$(3.2) \quad \varphi_{y_1 \triangleright 1} = \prod_{i=1}^r (\varphi_{y_1} \sigma_i \varphi_{y_1}^{-1}) \cdot (y_1 y_1 \triangleright y_2 \cdots y_1 \triangleright y_b)$$

and $y_1 \triangleright T = T$ by Lemma 3.1, we conclude that $\text{supp } \sigma_{y_1 \triangleright 1, i} \subseteq T$ for all $i \in \{1, \dots, r\}$. By conjugation of Equation (3.2) with φ_1 and using $1 \triangleright T = T$ and the transitivity of φ_1 on $\text{supp } \tau$ it follows that

$$(3.3) \quad \text{supp } \sigma_{y,1} \cdots \sigma_{y,r} = T \quad \text{for all } y \in \text{supp } \tau.$$

Let $x \in T$, $y \in \text{supp } \tau$, $z \in X$ with $x \triangleright y = z$. Then $z \in \text{supp } \tau$ by Lemma 3.1. Further,

$$\sigma_{z,1} \cdots \sigma_{z,r} \tau_z = \varphi_z = \varphi_{x \triangleright y} = \varphi_x (\sigma_{y,1} \cdots \sigma_{y,r}) \varphi_x^{-1} \cdot \varphi_x \tau_y \varphi_x^{-1}$$

and hence $\sigma_{z,1} \cdots \sigma_{z,r} = \varphi_x (\sigma_{y,1} \cdots \sigma_{y,r}) \varphi_x^{-1}$. Thus $\varphi_x(T) = T$. We conclude that $T \triangleright T = T$ and hence $X \triangleright T = T$ by Lemma 3.1, a contradiction to the indecomposability of X . \square

4. QUOTIENTS OF HURWITZ ORBITS

For the general theory of braid groups we refer to [8]. Recall that the braid group in three strands can be presented as

$$\mathbb{B}_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \rangle.$$

Let X be a finite rack and $(x, y, z) \in X^3$. Then the braid group \mathbb{B}_3 acts on X^3 via the Hurwitz action:

$$\sigma_1 \cdot (x, y, z) = (x \triangleright y, x, z), \quad \sigma_2 \cdot (x, y, z) = (x, y \triangleright z, y)$$

for all $x, y, z \in X$. The orbits of this action are called *Hurwitz orbits*. We write $\mathcal{O} = \mathcal{O}(x, y, z)$ for the Hurwitz orbit of (x, y, z) . According to Brieskorn [3], this action of \mathbb{B}_3 on X^3 was implicitly considered by Hurwitz in [7].

Lemma 4.1. *Let X be a rack and $\mathcal{O} \subseteq X^3$ a Hurwitz orbit. Then the map $\mathcal{O} \rightarrow G_X$, $(x, y, z) \mapsto xyz$ is constant.*

Proof. According to the definition of the Hurwitz action, it suffices to show that $xy = (x \triangleright y)x$ for all $x, y \in X$. The latter holds by the definition of G_X . \square

Lemma 4.2. *Let X be an injective rack, $\mathcal{O} \subseteq X^3$ a Hurwitz orbit and let $(x, y, z) \in \mathcal{O}$. Then $|(x, y, z)^{\langle \sigma_1 \rangle} \cap (x, y, z)^{\langle \sigma_2 \rangle}| = 1$.*

Proof. The elements of $(x, y, z)^{\langle \sigma_1 \rangle} \cap (x, y, z)^{\langle \sigma_2 \rangle}$ are of the form (x, y', z) with $y' \in X$. For all such triples, $xyz = xy'z$ in G_X by Lemma 4.1. Thus $y = y'$ in G_X and hence $y = y'$ in X since X is injective. \square

Let $(x, y, z), (x', y', z') \in X^3$. We define the following relation on X^3 :

$$(4.1) \quad (x, y, z) \sim (x', y', z') \Leftrightarrow \Delta^m \cdot (x, y, z) = (x', y', z') \text{ for some } m \in \mathbb{Z},$$

where $\Delta = (\sigma_1 \sigma_2)^3$. Clearly, \sim is an equivalence relation. We write $\overline{\mathcal{O}}(x, y, z)$ for the equivalence class containing (x, y, z) . The set $\overline{\mathcal{O}}$ of equivalence classes of \mathcal{O} is called a *Hurwitz orbit quotient*.

Let $x = \sigma_2^{-1} \sigma_1^{-1} Z(\mathbb{B}_3)$ and $y = \sigma_1 \sigma_2 \sigma_1 Z(\mathbb{B}_3)$. Then, by construction, the group

$$\langle x, y \mid x^3 = y^2 = 1 \rangle \simeq \mathbf{PSL}(2, \mathbb{Z}) \simeq \mathbb{B}_3 / \langle \Delta \rangle$$

acts on $\overline{\mathcal{O}}$, see for example [8, Appendix A]. Since \mathbb{B}_3 acts transitively on \mathcal{O} , the action of $\mathbf{PSL}(2, \mathbb{Z})$ on $\overline{\mathcal{O}}$ is transitive, that is, $\overline{\mathcal{O}}$ is a homogeneous space.

An *xy-cycle* in a $\mathbf{PSL}(2, \mathbb{Z})$ -space (in particular, in a Hurwitz orbit quotient) is a minimal non-empty subset C such that $xy \cdot v \in C$ and $(xy)^{-1} \cdot v \in C$ for all $v \in C$. We write $C_{xy}(v)$ for the *xy-cycle* containing a fixed element v . Similarly, one defines *yx-cycles* $C_{yx}(v)$.

We intend to determine all finite homogeneous $\mathbf{PSL}(2, \mathbb{Z})$ -spaces (up to isomorphism) such that any *xy-cycle* has at most 4 elements. Finite homogeneous $\mathbf{PSL}(2, \mathbb{Z})$ -spaces up to isomorphism are known to be in bijection with conjugacy classes of finite index subgroups of the modular group $\mathbf{PSL}(2, \mathbb{Z})$, which are intensively studied, see e.g. [9]. We present such sets in terms of their Schreier graphs with respect to the generators x and y of $\mathbf{PSL}(2, \mathbb{Z})$. Here a Schreier graph of a subgroup $H \subseteq \mathbf{PSL}(2, \mathbb{Z})$ will be an oriented labeled graph with vertices corresponding to the left H -cosets. In the interpretation as a $\mathbf{PSL}(2, \mathbb{Z})$ -space, the vertices correspond to the points of the space. In the Schreier graph, an x -arrow points from any coset gH (equivalently, point of the $\mathbf{PSL}(2, \mathbb{Z})$ -space) to the coset xgH , and a y -arrow points from any coset gH to the coset ygH . Instead of a double arrow labeled by y we display a single dashed line. We then omit the label x on the x -arrows. Later on we will add further labels on the graph which is used to study Schreier graphs of finite index subgroups of coverings of $\mathbf{PSL}(2, \mathbb{Z})$.

Proposition 4.3. *Let M be a finite homogeneous $\mathbf{PSL}(2, \mathbb{Z})$ -space such that any *xy-cycle* in M has at most 4 elements. Then the Schreier graph of M is one of the graphs in Figures 7.1–7.18.*

We denote the homogeneous $\mathbf{PSL}(2, \mathbb{Z})$ -spaces and their Schreier graphs appearing in Proposition 4.3 by Σ_{nX} , where n denotes the number of vertices of the graph, and X is a capital letter which serves as a further distinction.

Proof. The calculations are somewhat lengthy but elementary. It is reasonable to start with the classification of all homogeneous spaces with a point fixed by x , that is, with Schreier graphs containing an oriented loop. The restriction on the cycles yields that only the Schreier graphs in Figures 7.1, 7.2, 7.5, 7.6, 7.11, and 7.12 are possible. We explain this step in detail.

Let v_1 be a point of M with $xv_1 = v_1$. If $yv_1 = v_1$, then the Schreier graph of M is in Figure 7.1. Otherwise, let $v_2 = yv_1$. If $xv_2 = v_2$, then the Schreier graph of M is in Figure 7.2. Assume now that $xv_2 \neq v_2$. Since $x^3v_2 = v_2$, we have at least two more points $v_3 = xv_2$ and $v_4 = xv_3$, and then $xv_4 = v_2$. Recall that each vertex of the Schreier graph has precisely one incoming and one outgoing black arrow and one outgoing dashed line, which implies that v_3, v_4 can neither be equal to an existing point nor to each other. If $yv_3 = v_4$, then the Schreier graph of M is in Figure 7.5. If $yv_4 = v_4$, then let $v = yv_3$. The xy -cycle through v contains the vertices $v, xyv = v_4, xyv_4 = v_2, xyv_2 = v_1$, and $xyv_1 = v_3$. Since it can contain at most 4 vertices, we conclude that $v = v_3$. Similarly, $yv_3 = v_3$ implies that $yv_4 = v_4$. Then the Schreier graph of M is in Figure 7.6. Assume now that $yv_3 \neq v_3$ and $yv_4 \neq v_4$. Then we have at least two more points, $v_5 = yv_3$ and $v_6 = yv_4$. Since $(xy)^m v_6 = v_6$ for some $m \in \{1, 2, 3, 4\}$ and since $xyv_6 = v_2$, $(xy)^2 v_6 = v_1$, $(xy)^3 v_6 = v_3$, we obtain that $v_6 = xyv_3 = v_5$. Since $x^3 v_5 = v_5$, there is a point $v_7 = xv_6$ of M , and $xv_7 = v_5$. If $yv_7 = v_7$, then the Schreier graph of M is in Figure 7.11. Finally, let $v_8 = yv_7$. Then $v_8, v_5 = xyv_8, v_4 = xyv_5$, and $v_7 = xyv_4$ are pairwise distinct, and hence $xyv_7 = v_8$ by assumption on the length of xy -cycles. Thus $xv_8 = v_8$. The Schreier graph of M is then in Figure 7.12.

Next, one proceeds with the homogeneous spaces having a point fixed by xy but no point fixed by x . The corresponding Schreier graphs are Figures 7.3 and 7.7. Next, one can determine those homogeneous spaces which have a point fixed by y , but no points fixed by x or xy . The corresponding Schreier graphs are Figures 7.4, 7.8, 7.9, 7.13, and 7.14. Finally, only Schreier graphs are left which have no oriented loops and no y -edges starting and ending in the same oriented triangle. Such graphs can be classified for example by looking at the number of y -edges between two oriented triangles. Under the restriction on the length of xy -cycles, one gets the Figures 7.10, 7.15, 7.16, 7.17, and 7.18. \square

For our analysis, finite homogeneous \mathbb{B}_3 -spaces are relevant. We obtain such spaces as coverings of homogeneous $\mathbf{PSL}(2, \mathbb{Z})$ -spaces.

Definition 4.4. Let T be a $\mathbf{PSL}(2, \mathbb{Z})$ -space. Consider T as a \mathbb{B}_3 -space on which $Z(\mathbb{B}_3)$ acts trivially. A covering of T is a triple (p, S, T) , where S

is a \mathbb{B}_3 -space, and $p : S \rightarrow T$ is a surjective \mathbb{B}_3 -equivariant map such that $p(s_1) = p(s_2)$ for $s_1, s_2 \in S$ implies that $s_1 = \Delta^m s_2$ for some $m \in \mathbb{Z}$.

Remark 4.5. Let T be a $\mathbf{PSL}(2, \mathbb{Z})$ -space and let (p, S, T) be a covering of T . If the action on T is transitive then the action on S is transitive.

Definition 4.6. A covering (p, S, T) is said to be trivial if $p : S \rightarrow T$ is bijective. We say that a covering $p : S \rightarrow T$ is finite if S (and hence T) is a finite set.

Since Δ is contained in (and in fact generates) the center $Z(\mathbb{B}_3)$, the following lemma is straightforward.

Lemma 4.7. Let (p, S, T) be a covering of a homogeneous $\mathbf{PSL}(2, \mathbb{Z})$ -space T . Then $|p^{-1}(v)| = |p^{-1}(w)|$ for all points v, w of T .

Let (p, S, T) be a covering of a homogeneous $\mathbf{PSL}(2, \mathbb{Z})$ -space T . We will always write N for the number of elements of the fibers.

Remark 4.8. Let T be a homogeneous $\mathbf{PSL}(2, \mathbb{Z})$ -space. Coverings (p, S, T) of T can be displayed as labeled Schreier graphs with respect to the generators $\sigma_2^{-1}\sigma_1^{-1}$ and $\sigma_1\sigma_2\sigma_1$ of \mathbb{B}_3 . Hence, by definition of x and y , the generators $\sigma_2^{-1}\sigma_1^{-1}$ and $\sigma_1\sigma_2\sigma_1$ correspond to labeled x - and y -arrows, respectively, in the labeled Schreier graph. Since the covering is a homogeneous space and the sequence

$$Z(\mathbb{B}_3) \rightarrow \mathbb{B}_3 \rightarrow \mathbf{PSL}(2, \mathbb{Z})$$

is exact, the fiber over any $v \in T$ consists of a $\langle \Delta \rangle$ -orbit. We may fix a point $v[0] \in S$ in the fiber over a point $v \in T$ and enumerate all other points of the fiber by $v[i] = \Delta^i v[0]$ for all $i \in \{0, 1, \dots, N-1\}$, where N is the size of the fiber. We also write $v[*]$ for the complete fiber $p^{-1}(v)$ over v . By choosing a spanning tree of the Schreier graph of T and the images of $v[0]$ along the arrows of the spanning tree, one obtains the images of $v[i]$ for all i since Δ is central. The remaining arrows $v_i \rightarrow v_j$ (those not on the spanning tree) in the graph of T then have to obtain labels indicating the index shift in the fiber: a label s tells that $v_i[k]$ is mapped to $v_j[k + s \pmod{N}]$ for all k . Then, up to the choice of the spanning tree, any covering of T is uniquely determined by the labels of the x - and y -edges.

Observe that, since $\Delta = (\sigma_1\sigma_2)^3 = (\sigma_1\sigma_2\sigma_1)^2$, the sum of the labels in any x -triangle is -1 and the sum of the two labels of a y -edge is 1 . The y -edges we interpret as double arrows and put the label of the arrow close to its destination.

For any xy -cycle (or yx -cycle) C in T , the label of C is the sum of the labels of x - and y -edges of the cycle.

Definition 4.9. Let \mathcal{O} be a covering of a $\mathbf{PSL}(2, \mathbb{Z})$ -space. The graph of the covering is the labeled Schreier graph as explained in Remark 4.8. The arrows correspond to the action of $\sigma_2^{-1}\sigma_1^{-1}$ and $\sigma_1\sigma_2\sigma_1$.

Definition 4.10. For $i \in \{1, 2\}$, a σ_i -cycle of a homogeneous \mathbb{B}_3 -space is a minimal non-empty subset which is closed under the action of σ_i . We say that a \mathbb{B}_3 -space S (or a covering (p, S, T) of a $\mathbf{PSL}(2, \mathbb{Z})$ -space T) has simply intersecting cycles if any given σ_1 -cycle c_1 and σ_2 -cycle c_2 in S intersect in at most one point.

Example 4.11. By Lemma 4.2, if $\mathcal{O} \subseteq X^3$ is a Hurwitz orbit, where X is an injective rack, then \mathcal{O} has simply intersecting cycles.

Remark 4.12. For any covering (p, S, T) of a $\mathbf{PSL}(2, \mathbb{Z})$ -space T , the image of a σ_1 -cycle in S is an xy -cycle in T , and the image of a σ_2 -cycle in S is a yx -cycle in T .

Lemma 4.13. Let T be a $\mathbf{PSL}(2, \mathbb{Z})$ -space and let (p, S, T) be a covering of T with simply intersecting cycles. Let v be a vertex of the graph of T .

- (1) If there exists an x -loop on v with label a then $3a \equiv -1 \pmod{N}$.
- (2) If there exists a y -loop on v with label a then $2a \equiv 1 \pmod{N}$.

Proof. It follows from Remark 4.8. □

The following four lemmas are easy consequences of the definition of simply intersecting cycles.

Lemma 4.14. Let T be a $\mathbf{PSL}(2, \mathbb{Z})$ -space and let (p, S, T) be a covering of T with simply intersecting cycles. Let $v \in T$ and $w \in C_{yx}(v) \cap C_{xy}(v)$. If $w \neq v$, then (p, S, T) is not trivial.

Lemma 4.15. Let T be a $\mathbf{PSL}(2, \mathbb{Z})$ -space and let (p, S, T) be a covering of T with simply intersecting cycles. Let $v \in T$ and $N = |p^{-1}(v)|$. Let $\lambda \in \mathbb{Z}_N$ and $\mu \in \mathbb{Z}_N$ be the labels of the xy - and yx -cycle containing v , respectively. Then $\langle \lambda \rangle \cap \langle \mu \rangle = 0$ as subgroups of \mathbb{Z}_N .

Lemma 4.16. Let T be a $\mathbf{PSL}(2, \mathbb{Z})$ -space and let (p, S, T) be a covering of T with simply intersecting cycles. Let $v \in T$ and assume that $xv = v$ or $yv = v$ and that $\mathbf{PSL}(2, \mathbb{Z})v \neq \{v\}$. Then the labels of the xy - and yx -cycles containing v are 0.

Lemma 4.17. Let T be a $\mathbf{PSL}(2, \mathbb{Z})$ -space and let (p, S, T) be a covering of T with simply intersecting cycles. Let $v, w \in T$, $N = |p^{-1}(v)|$ and assume that $v \neq w$ and that v, w are on the same xy - and the same yx -cycle. Let λ and μ be the labels of the xy - and yx -path from v to w , respectively. Then $\lambda \not\equiv \mu \pmod{N}$.

Corollary 4.18. Let T be a $\mathbf{PSL}(2, \mathbb{Z})$ -space. Let $v, w \in T$ and assume that $v \neq w$, $yv = v$, $yw = w$ and that v, w are on the same xy -cycle. Then T has no coverings with simply intersecting cycles.

Proof. Assume to the contrary that (p, S, T) is a covering of T with simply intersecting cycles. Let $N = |p^{-1}(v)|$ and let a and b be the labels of the y -loops at v and w , respectively. By Lemma 4.13(2) we obtain that $2a \equiv 1 \pmod{N}$ and $2b \equiv 1 \pmod{N}$ and hence $a \equiv b \pmod{N}$ and N is odd.

Since $yv = v$ and $yw = w$, v and w are on the same yx -cycle and the xy - and yx -paths from v to w have the same labels. This is a contradiction to Lemma 4.17. \square

5. CELLULAR AUTOMATA OVER HOMOGENEOUS SPACES

As explained in the introduction, for the classification of Nichols algebras with many cubic relations we are going to determine upper bounds for immunities of Hurwitz orbits. The principle of the method to calculate these upper bounds is well-known from the theory of cellular automata. Therefore we intend to formulate our techniques in the language of cellular automata.

The following definition is inspired by [4, Definition 1.4.1], where the very similar definition of a cellular automaton over a group G was given. Let G be a group acting transitively on a set Ω and let A be a set. Let A^Ω be the set of all functions from Ω to A .

Definition 5.1. *Let S be a set, let $(g_s)_{s \in S}$ be a family of elements in G , and let $\mu : A^S \rightarrow A$ be a map. Then the map $\tau : A^\Omega \rightarrow A^\Omega$ such that*

$$(5.1) \quad \tau(f)(w) = \mu((f(g_s \cdot w))_{s \in S})$$

for all $f \in A^\Omega$, $w \in \Omega$, is called a cellular automaton over (G, Ω) with alphabet A . The infinite sequence $(\tau^n(f))_{n \geq 0}$ is called the evolution of f .

Remark 5.2. A good interpretation of a cellular automaton over (G, Ω) is the following. For any $w \in \Omega$, consider the family of points $(g_s \cdot w)_{s \in S}$ as the neighborhood of w . Then for any function $f \in A^\Omega$, the value of $\tau(f)$ at w is obtained from the values of f in the neighborhood of w according to the rule determined by μ .

We will only consider cellular automata over (G, Ω) with the alphabet $A = \mathbb{Z}_2$. For any function $f \in \mathbb{Z}_2^\Omega$ let

$$\text{supp } f = \{w \in \Omega \mid f(w) = 1\}.$$

Conversely, the characteristic function of a set $I \subseteq \Omega$ is

$$(5.2) \quad \chi_I \in \mathbb{Z}_2^\Omega, \quad x \mapsto \begin{cases} 1 & \text{if } x \in I, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 5.3. *Let τ be a cellular automaton over (G, Ω) with alphabet \mathbb{Z}_2 . We say that τ is monotonic if*

- (1) $\text{supp } f \subseteq \text{supp } \tau(f)$ for all $f \in \mathbb{Z}_2^\Omega$, and
- (2) $\text{supp } \tau(f) \subseteq \text{supp } \tau(g)$ for all $f, g \in \mathbb{Z}_2^\Omega$ with $\text{supp } f \subseteq \text{supp } g$.

In what follows we will only study monotonic cellular automata over homogeneous spaces.

Definition 5.4. *Let τ be a monotonic cellular automaton over (G, Ω) with alphabet \mathbb{Z}_2 . For any two subsets $I, J \subseteq \Omega$ with $I \subseteq J$ we say that I spreads to J , if $J \subseteq \text{supp } \tau^n(\chi_I)$ for some $n \in \mathbb{N}$. A subset $I \subseteq \Omega$ is a quarantine if*

$\tau(\chi_I) = \chi_I$. A subset $I \subseteq \Omega$ is a plague if the smallest quarantine containing I is Ω . The cardinality of a plague I is also called its size.

Remark 5.5. Let τ be a monotonic cellular automaton over (G, Ω) with alphabet \mathbb{Z}_2 . If a subset I spreads to another subset J of Ω , then any subset $I' \subseteq \Omega$ with $I \subseteq I'$ spreads to J .

Assume that Ω has only finitely many points. Then a subset I of Ω is a plague if and only if it spreads to Ω . In this case, any subset of Ω containing I is a plague.

Plagues of Hurwitz orbits were already introduced in [6, Definition 3]. That definition is a special case of a plague of the monotonic cellular automaton in the following example, which is the main example of our interest.

Example 5.6. Let $G = \mathbb{B}_3$ and let Ω be a finite homogeneous G -space. For example, if X is a finite rack, then G acts on X^3 by $\sigma_1 \cdot (x, y, z) = (x \triangleright y, x, z)$ and $\sigma_2 \cdot (x, y, z) = (x, y \triangleright z, y)$, and Ω can be taken as an orbit of this action. Let

$$(g_s)_{s \in \{1, \dots, 7\}} = (1, \sigma_2, \sigma_1 \sigma_2, \sigma_2^{-1} \sigma_1^{-1}, \sigma_1^{-1}, \sigma_2^{-1}, \sigma_1) \in \mathbb{B}_3^7.$$

As an alphabet take $A = \mathbb{Z}_2$. We consider the configuration given in Figure 5.1. (In Figure 5.1 the black arrow indicates the action of σ_1 , and the dashed arrow the action of σ_2 .) Then $x_s = g_s \cdot x_1$ for all $s \in \{1, 2, \dots, 7\}$. Define $\mu : A^7 \rightarrow A$ by

$$\begin{aligned} \mu(f_1, f_2, \dots, f_7) &= f_1 \vee f_2 f_3 \vee f_4 f_5 \vee f_6 f_7 \\ &= 1 - (1 - f_1)(1 - f_2 f_3)(1 - f_4 f_5)(1 - f_6 f_7), \end{aligned}$$

where $f_1, \dots, f_7 \in A$, and \vee denotes logical or. Then the map τ defined by μ and $(g_s)_{s \in \{1, \dots, 7\}}$ is a monotonic cellular automaton over (\mathbb{B}_3, Ω) . A plague of this automaton is literally the same as a plague in the sense of [6, Definition 3].

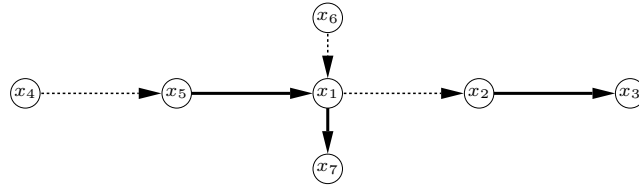


FIGURE 5.1. Neighbors of x_1

Problem 5.7. Find a plague of minimal size for the cellular automata of Example 5.6.

Example 5.8. A cellular automaton in the very traditional sense is just a cellular automaton over $(\mathbb{Z}^n, \mathbb{Z}^n)$ with $n \in \{1, 2\}$, where the action is given by the usual free action of \mathbb{Z}^n on itself and in $(g_s)_{s \in S}$ only $n \in \{1, -1\}$ $((n_1, n_2) \in \mathbb{Z}^2$ with $|n_1|, |n_2| \leq 1$, respectively,) appear.

5.1. Examples over $(\mathbb{Z}, \mathbb{Z}_m)$. In this subsection we present a family of examples of automata useful for studying the following process.

Let $m \in \mathbb{N}_{\geq 2}$, $f \in \mathbb{Z}_2^{\mathbb{Z}_m}$, $r \in \mathbb{N}$, and $a_1, \dots, a_r \in \mathbb{Z}_m \setminus \{0\}$. The group $G = \mathbb{Z}$ acts transitively on $\Omega = \mathbb{Z}_m$. Let $A = \mathbb{Z}_2$, $S = \{0, 1, \dots, r\}$, and $(g_s)_{s \in S} = (0, -a_1, -a_2, \dots, -a_r) \in G^S$. Define $\mu : A^S \rightarrow A$ by

$$\mu(f_0, f_1, \dots, f_r) = \begin{cases} 1 & \text{if } f_0 = 1 \text{ or } f_1 = f_2 = \dots = f_r = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The map $\tau : \mathbb{Z}_2^{\mathbb{Z}_m} \rightarrow \mathbb{Z}_2^{\mathbb{Z}_m}$ defined by μ and $(g_s)_{s \in S}$ is then a monotonic cellular automaton. By definition,

$$(5.3) \quad \text{supp } \tau(f) = \text{supp } f \cup \{w \in \Omega \mid f(x - a_1) = \dots = f(x - a_r) = 1\}$$

for all $f \in \mathbb{Z}_2^{\mathbb{Z}_m}$. We study now plagues for special cases of this cellular automaton.

Example 5.9. *Let $r = 1$ and $a_1 = \lambda$. The cellular automaton is determined by the rule*

$$(5.4) \quad \text{supp } \tau(f) = \text{supp } f \cup \{x \in \mathbb{Z}_m \mid f(x - \lambda) = 1\}.$$

Let $\Gamma = \langle \lambda \rangle$ and let I be a set of representatives for Ω/Γ . Then I is a plague.

Example 5.10. *Let $\lambda \in \mathbb{Z}_m$, $r = 3$, $a_1 = 1$, $a_2 = \lambda + 1$, and $a_3 = -\lambda$. Let $\Gamma = \langle \lambda \rangle$ and let I be the union of a set of representatives for Ω/Γ with Γ . For example $I = \langle \lambda \rangle \cup \{1, 2, \dots, \lambda - 1\}$. If $\text{supp } f$ contains a coset $a + \Gamma$, where $a + 1 \in I$, then $\text{supp } f$ spreads to $a + 1 + \Gamma$. Thus I is a plague.*

Example 5.11. *Let $\lambda \in \Omega \setminus \{0, 1\}$, $r = 2$, $a_1 = \lambda$, $a_2 = \lambda - 1$. Let $I = \{0, 1, \dots, (m-1)/2\}$ if m is odd and $\{0, 1, \dots, m/2 - 1\}$ if m is even. Then I is a plague of size $\leq (m+1)/2$. It is in general not minimal, for example for $m \geq 3$, $\lambda = 2$ the set $\{0, 1\}$ is a plague.*

6. PLAGUES ON HURWITZ ORBITS

Example 5.6 describes cellular automata over braid group orbits. In order to prove Theorem 6.3, we have to find small plagues for these automata.

By [6, §1.4], the *immunity* of a Hurwitz orbit Σ is the ratio $s/|\Sigma|$, where s is the size of a smallest plague for Σ with respect to the cellular automaton in Example 5.6. We write $\text{imm}(\Sigma)$ for the immunity of Σ .

Let us formulate the cellular automaton in Example 5.6 in terms of the generators $\sigma_2^{-1}\sigma_1^{-1}$ and $\sigma_1\sigma_2\sigma_1$ of the group \mathbb{B}_3 . Note that $x = \sigma_2^{-1}\sigma_1^{-1}\langle \Delta \rangle$ and $y = \sigma_1\sigma_2\sigma_1\langle \Delta \rangle$ in $\mathbf{PSL}(2, \mathbb{Z})$. Let f be a \mathbb{Z}_2 -valued function on the braid group orbit Ω and let $P = \text{supp } f$. To obtain the support of $\tau(f)$, proceed as follows.

We use the notation regarding Ω and its Schreier graph introduced in Remark 4.8. Let p be a point in the braid group orbit quotient $\Omega/\langle \Delta \rangle$. Let

I be a subset of \mathbb{Z}_N and let $p[I] = \{p[i] \mid i \in I\}$ be the corresponding subset of the fiber over p . Consider the three neighboring subsets

$$(\sigma_2^{-1}\sigma_1^{-1})^{-1} \cdot p[I], \quad \sigma_1\sigma_2\sigma_1 \cdot p[I], \quad \text{and} \quad \sigma_2^{-1}\sigma_1^{-1} \cdot p[I+1]$$

of $p[I]$. They are displayed in Figure 6.1 and are denoted by $x_1[I-c]$, $x_2[I+a]$ and $x_3[I+b+1]$, respectively, where, for example, $I+a$ means the translation of I by a , i.e., $I+a = \{i+a \mid i \in I\}$. In this setting we call $p[I]$ a *pivot*.

By Example 5.6, if P contains the subsets $x_1[I-c] = \sigma_1\sigma_2 \cdot p[I]$ and $\sigma_2 \cdot x_1[I-c] = x_2[I+a]$, then $\text{supp } \tau(f)$ contains

$$\sigma_1 \cdot x_2[I+a] = \sigma_1\sigma_2\sigma_1\sigma_2 \cdot p[I] = \sigma_2^{-1}\sigma_1^{-1}\Delta \cdot p[I] = x_3[I+b+1].$$

Similarly, if any two of the neighboring subsets $x_1[I-c]$, $x_2[I+a]$, and $x_3[I+b+1]$ of $p[I]$ are contained in P , then the third is a subset of $\text{supp } \tau(f)$. Moreover, $\text{supp } \tau(f)$ is the smallest subset of Ω containing $\text{supp } f$ and all sets constructed this way for some point p and some subset $I \subseteq \mathbb{Z}_N$.

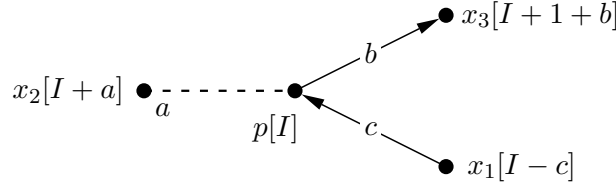


FIGURE 6.1. The cellular automaton on braid group orbits

We intend to obtain an upper bound for the immunity of each Hurwitz orbit using the cycle structure at each vertex. For that purpose, we define

$$(6.1) \quad (\omega'_{ij})_{i,j \geq 1} = \begin{pmatrix} 1 & 1/3 & 11/24 & 1/2 & 1/2 & \cdots \\ 1/3 & 1/3 & 1/3 & 1/3 & 1/3 & \cdots \\ 11/24 & 1/3 & 7/24 & 7/24 & 7/24 & \cdots \\ 1/2 & 1/3 & 7/24 & 1/4 & 1/4 & \cdots \\ 1/2 & 1/3 & 7/24 & 1/4 & 1/4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The notation for $\mathbf{PSL}(2, \mathbb{Z})$ -spaces was fixed below Proposition 4.3. For the coverings of a $\mathbf{PSL}(2, \mathbb{Z})$ -space Σ_* , where $*$ stands for any possible index, we use the notation $\Sigma_*^{N;a,b,\dots}$. Here, N is the size of the fiber over a point of Σ_* , and a, b, \dots are the values of the individual labels determining the covering. These labels are chosen by the method described in Remark 4.8 and can be read off from the corresponding figure. We again use the same notation for the homogeneous space and its Schreier graph.

Let Σ be a homogeneous \mathbb{B}_3 -space and let $v \in \Sigma$. Assume that v belongs to a σ_1 -cycle of length i , and also to a σ_2 -cycle of length j . Then we write

$c(v) = (i, j)$. Let $\omega : \Sigma \rightarrow \mathbb{Q}$ be the map defined by

$$(6.2) \quad \omega(v) = \begin{cases} \omega'_{ij} + \frac{1}{30} = \frac{13}{40} & \text{if } \Sigma = \Sigma_{4A}^{5;3,2} \text{ and } v \in v_1[*], \\ \omega'_{ij} + \frac{1}{12} = \frac{1}{3} & \text{if } \Sigma = \Sigma_{6A}^{4;2,2} \text{ and } v \in v_3[*], \\ \omega'_{ij} + \frac{1}{24} = \frac{1}{3} & \text{if } \Sigma \text{ is the trivial covering of } \Sigma_{12C}, \\ \omega'_{ij} & \text{otherwise,} \end{cases}$$

where $c(v) = (i, j)$, and $v_i[*]$ is defined in Remark 4.8.

Remark 6.1. If we would not add the three exceptional coverings in the definition of ω , then Theorem 6.3 below would not hold. We could compensate this by changing the infinite matrix $(\omega'_{ij})_{i,j \geq 1}$, but then we would have only a weak upper bound for the immunity of a Hurwitz orbit. In that case we would not have enough information to reduce the proof of Theorem 9.3 to the study of few small racks.

Definition 6.2. Let Σ be a homogeneous \mathbb{B}_3 -space. The weight of Σ is defined as

$$(6.3) \quad \omega(\Sigma) = \frac{1}{|\Sigma|} \sum_{v \in \Sigma} \omega(v).$$

The main goal of this section is to formulate a good upper bound for the immunity of Hurwitz orbits.

Theorem 6.3. Let Σ be a covering with simply intersecting cycles of a finite homogeneous $\mathbf{PSL}(2, \mathbb{Z})$ -space $\overline{\Sigma}$. Assume that any xy -cycle of $\overline{\Sigma}$ has at most four elements. Then $\text{imm}(\Sigma) \leq \omega(\Sigma)$.

We prove Theorem 6.3 in Section 7.

7. PROOF OF THEOREM 6.3

This section contains a case-by-case analysis of the coverings, immunities and weights for the Schreier graphs of finite homogeneous $\mathbf{PSL}(2, \mathbb{Z})$ -spaces $\overline{\Sigma}$ such that any xy -cycle of $\overline{\Sigma}$ has at most four elements. The main goal of this section is to prove Theorem 6.3.

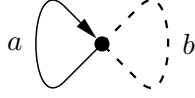
7.1. The graph Σ_{1A} .

Lemma 7.1. Every covering of Σ_{1A} in Figure 7.1 with simply intersecting cycles is trivial.

Proof. Lemma 4.13 implies that $3a \equiv -1 \pmod{N}$ and $2b \equiv 1 \pmod{N}$. By Lemma 4.16, $a + b \equiv 0 \pmod{N}$ and hence $N = 1$. \square

The following lemma is trivial, but we state it for completeness.

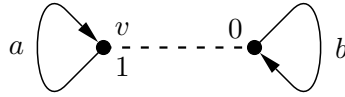
Lemma 7.2. Let Σ be the trivial covering of the homogeneous $\mathbf{PSL}(2, \mathbb{Z})$ -space Σ_{1A} . Then $\text{imm}(\Sigma) = 1 = \omega(\Sigma)$.

FIGURE 7.1. Schreier graph Σ_{1A} and its coverings

7.2. The graph Σ_{2A} .

Lemma 7.3. *The graph Σ_{2A} in Figure 7.2 has no coverings with simply intersecting cycles.*

Proof. Lemma 4.13 implies that $3a \equiv 3b \equiv -1 \pmod{N}$. Further, from Lemma 4.16 on v we obtain $a + b + 1 \equiv 0 \pmod{N}$ and the claim follows. \square

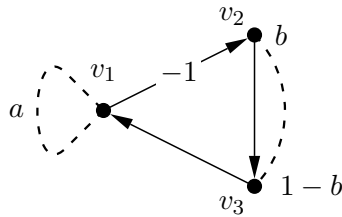
FIGURE 7.2. Schreier graph Σ_{2A} and its coverings

7.3. The graph Σ_{3A} .

Lemma 7.4. *Any covering of Σ_{3A} in Figure 7.3 with simply intersecting cycles satisfies $2a \equiv 1 \pmod{N}$, $a \equiv b \pmod{N}$.*

Proof. Lemma 4.13 on v_1 implies that $2a \equiv 1 \pmod{N}$. Since v_1 belongs to cycles with label $a - b$, $a \equiv b \pmod{N}$ by Lemma 4.16. \square

Example 7.5. *The trivial covering of Σ_{3A} in Figure 7.3 is isomorphic to the Hurwitz orbit with three elements of [6, Figure 8].*

FIGURE 7.3. Schreier graph Σ_{3A} and its coverings

Lemma 7.6. *Let Σ be a covering of Σ_{3A} with simply intersecting cycles. Then $\text{imm}(\Sigma) \leq 1/3 = \omega(\Sigma)$.*

Proof. First we prove that $P = v_2[*]$ is a plague. With $v_3[*]$ as a pivot we see that P spreads to $v_1[*]$. Then, with $v_1[*]$ as a pivot, it also spreads to $v_3[*]$ and hence $\text{imm}(\Sigma) \leq 1/3$. Now we prove $\omega(\Sigma) = 1/3$. By Lemma 7.4, the cycle structure on each vertex of Σ is the following:

$v_1[i] : \text{two 2-cycles,}$

$v_2[i] \text{ and } v_3[i] : \text{one 2-cycle and a cycle of length } |\langle b \rangle|,$

for all $i \in \mathbb{Z}_N$. Then $\omega(\Sigma) = 1/3$ and this proves the claim. \square

7.4. The graph Σ_{3B} .

Lemma 7.7. *The graph Σ_{3B} in Figure 7.4 has no coverings with simply intersecting cycles.*

Proof. This follows from Corollary 4.18. \square

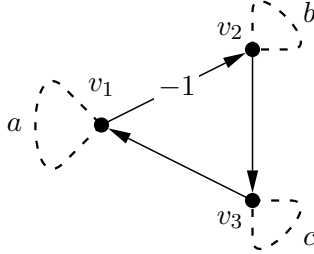


FIGURE 7.4. Schreier graph Σ_{3B} and its coverings

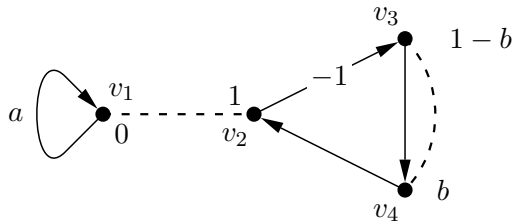


FIGURE 7.5. Schreier graph Σ_{4A} and its coverings

7.5. The graph Σ_{4A} .

Lemma 7.8. *Any covering of Σ_{4A} in Figure 7.5 with simply intersecting cycles satisfies $3a \equiv -1 \pmod{N}$, $a + b \equiv 0 \pmod{N}$ and $N > 1$.*

Proof. Since $v_2 \in C_{xy}(v_1) \cap C_{yx}(v_1)$, Lemma 4.14 implies that $N > 1$. Since v_1 is on an xy -cycle with label $a + b$, by Lemmas 4.13(1) and 4.16 on v_1 the claim follows. \square

Example 7.9. *Let Σ be the covering of Σ_{4A} with $N = 2$ and $a = b = 1$. Then Σ can be presented as*

$$\sigma_1 = (v_1 \ v_3 \ \Delta v_2)(\Delta v_1 \ \Delta v_3 \ v_2), \quad \sigma_2 = (v_1 \ v_2 \ v_4)(\Delta v_1 \ \Delta v_2 \ \Delta v_4),$$

and Σ is isomorphic to the Hurwitz orbit of eight elements of [6, Figure 10]. The isomorphism is given by the bijection

$$\begin{pmatrix} v_1 & \Delta v_1 & v_2 & \Delta v_2 & v_3 & \Delta v_3 & v_4 & \Delta v_4 \\ E & D & B & G & H & A & C & F \end{pmatrix}.$$

Lemma 7.10. *Let Σ be a covering of Σ_{4A} with simply intersecting cycles. Then $\text{imm}(\Sigma) \leq \frac{N+1}{4N}$.*

Proof. Let $I = \{0\} \subseteq \mathbb{Z}_N$. We claim that $P = v_3[*] \cup v_1[I]$ is a plague. With $v_4[*]$ as a pivot, P spreads to $v_2[*]$. Since the neighbors of $v_1[\lambda]$ are $v_1[\lambda - a]$, $v_1[\lambda + a + 1]$ and $v_2[\lambda + 1]$, pivoting from $v_1[I + a]$ implies that P spreads to $v_1[I + 2a + 1]$. We describe this situation with the following table:

$$\begin{array}{c|cc} \text{pivot} & v_4[*] & v_1[I + a] \\ \hline & v_2[*] & v_1[I + 2a + 1] \end{array}$$

From Lemma 7.8 we obtain that $\langle 2a + 1 \rangle = \langle -a \rangle = \mathbb{Z}_N$ and hence Example 5.9 implies that P spreads to $v_1[*]$. Now with $v_2[*]$ as a pivot we see that P spreads to Σ and we are done. Thus the claim follows. \square

Lemma 7.11. *Let Σ be a covering of Σ_{4A} with simply intersecting cycles. Then $\text{imm}(\Sigma) \leq \omega(\Sigma)$.*

Proof. By Lemma 7.8, the cycle structure on each vertex of Σ is the following:

$$(7.1) \quad v_1[i] \text{ and } v_2[i] : \text{ two 3-cycles,}$$

$$(7.2) \quad v_3[i] \text{ and } v_4[i] : \text{ one 3-cycle and a cycle of length } |\langle 1 - b \rangle|,$$

for all $i \in \mathbb{Z}_N$. Assume first that $N = 3k + 1$. Then $a = k$, $b = 2k + 1$ and $k \geq 1$. Since $3(b - 1) \equiv -2 \pmod{N}$ and $b - 1$ is even, we get

$$|\langle 1 - b \rangle| = |\langle 2 \rangle| = \begin{cases} N & \text{if } k \text{ is even,} \\ \frac{N}{2} & \text{if } k \text{ is odd.} \end{cases}$$

By (7.2), $\omega(v_3[i]) = \omega(v_4[i]) = 1/3$ if $k = 1$, and $\omega(v_3[i]) = \omega(v_4[i]) = 7/24$ if $k > 1$, for all $i \in \mathbb{Z}_N$. Therefore

$$\omega(\Sigma) = \begin{cases} 5/16 & \text{if } k = 1, \\ 7/24 & \text{if } k > 1. \end{cases}$$

If $k = 1$ then $N = 4$ and $\text{imm}(\Sigma) \leq \frac{N+1}{4N} = 5/16 = \omega(\Sigma)$. Otherwise, if $k > 1$, then $N > 6$ and hence

$$\text{imm}(\Sigma) \leq \frac{N+1}{4N} < \frac{7}{24} = \omega(\Sigma).$$

Now assume that $N = 3k + 2$. Then $a = 2k + 1$, $b = k + 1$ and $k \geq 0$. Further, $3(b - 1) \equiv -2 \pmod{N}$ and hence

$$|\langle 1 - b \rangle| = \begin{cases} \frac{N}{2} & \text{if } k \text{ is even,} \\ N & \text{if } k \text{ is odd.} \end{cases}$$

By (7.2),

$$\omega(v_3[i]) = \begin{cases} 11/24 & \text{if } k = 0, \\ 7/24 & \text{if } k > 0, \end{cases}$$

for all $i \in \mathbb{Z}_N$, and therefore

$$\omega(\Sigma) = \begin{cases} 3/8 & \text{if } k = 0, \\ 7/24 & \text{if } k > 0. \end{cases}$$

If $k = 0$ then $N = 2$ and $\text{imm}(\Sigma) \leq \frac{N+1}{4N} = 3/8 = \omega(\Sigma)$. If $k = 1$ then $N = 5$,

$$\text{imm}(\Sigma) \leq \frac{N+1}{4N} = 3/10 = (4 \cdot 7/24 + 1/30)/4 = \omega(\Sigma).$$

Finally, if $k > 1$, then $\text{imm}(\Sigma) \leq \frac{N+1}{4N} \leq 7/24 = \omega(\Sigma)$ for $N \geq 6$. \square

Example 7.12. Let $x_1 = (1\ 2\ 3\ 4\ 5)$, $x_2 = (1\ 2\ 4\ 5\ 3)$ and $x_3 = (1\ 4\ 3\ 5\ 2)$ be 5-cycles in \mathbb{A}_5 . Let X be the rack associated to the conjugacy class of x_1 in \mathbb{A}_5 . The covering $\Sigma_{4A}^{5;3,2}$ can be realized as the Hurwitz orbit of $(x_1, x_2, x_3) \in X^3$.

Lemma 7.13. Let X be an injective indecomposable rack. Assume that X^3 contains a Hurwitz orbit isomorphic to the covering $\Sigma_{4A}^{5;3,2}$. If $(x, y, z) \in v_1[*]$ then $(x \triangleright y) \triangleright y = z$.

Proof. By inspection,

$$(7.3) \quad \sigma_1 v_1[i] = v_3[i], \quad \sigma_2 v_1[i] = v_2[i + 4],$$

$$(7.4) \quad \sigma_1 v_2[i] = v_1[i + 3], \quad \sigma_2 v_2[i] = v_4[i + 1],$$

$$(7.5) \quad \sigma_1 v_3[i] = v_2[i + 2], \quad \sigma_2 v_3[i] = v_3[i + 4],$$

$$(7.6) \quad \sigma_1 v_4[i] = v_4[i + 4], \quad \sigma_2 v_4[i] = v_1[i].$$

Let $1, 2, 3 \in X$ with $v_1[0] = (1, 2, 3) \in \Sigma_{4A}^{5;3,2} \subseteq X^3$. Set $1 \triangleright 2 = 4$, $1 \triangleright 3 = 5$ and $1 \triangleright 4 = 6$. (The elements $1, 2, 3, 4, 5, 6 \in X$ are not necessarily pairwise distinct.) We claim that $(1 \triangleright 2) \triangleright 1 = 2$. Indeed, using (7.3) we obtain $v_3[0] = \sigma_1 v_1[0] = (1 \triangleright 2, 1, 3)$. By (7.5), $v_2[2] = \sigma_1 v_3[0] = ((1 \triangleright 2) \triangleright 1, 1 \triangleright 2, 3)$. Since $v_1[0] = \sigma_1 v_2[2]$, we obtain $(1 \triangleright 2) \triangleright 1 = 2$. Similarly, using formulas (7.3)–(7.6), straightforward computations show that:

i	$v_1[i]$	$v_2[i]$	$v_3[i]$	$v_4[i]$
0	(1, 2, 3)	(3, 4, 5)	(4, 1, 3)	(1, 3, 4)
1	(3, 5, 6)	(6, 4, 1)	(4, 3, 6)	(3, 6, 4)
2	(6, 1, 2)	(2, 4, 3)	(4, 6, 2)	(6, 2, 4)
3	(2, 3, 5)	(5, 4, 6)	(4, 2, 5)	(2, 5, 4)
4	(5, 6, 1)	(1, 4, 2)	(4, 5, 1)	(5, 1, 4)

Since $\sigma_2 \cdot (1, 4, 2) = \sigma_2 v_2[4] = v_4[0] = (1, 3, 4)$, we get $(1 \triangleright 2) \triangleright 2 = 4 \triangleright 2 = 3$. Since Δ acts transitively on $v_1[*]$, the lemma follows. \square

7.6. The graph Σ_{4B} .

Lemma 7.14. *The graph Σ_{4B} in Figure 7.6 has no coverings with simply intersecting cycles.*

Proof. This follows from Corollary 4.18. \square

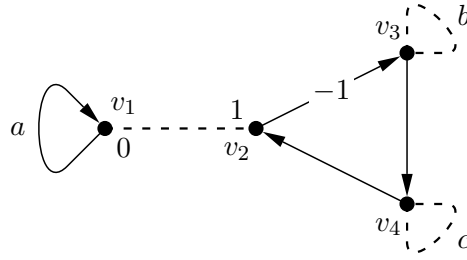


FIGURE 7.6. Schreier graph Σ_{4B} and its coverings

7.7. The graph Σ_{6A} .

Lemma 7.15. *Any covering of Σ_{6A} in Figure 7.7 with simply intersecting cycles satisfies $N \neq 1$, $a + b \equiv 0 \pmod{N}$ and $2b \not\equiv 1 \pmod{N}$.*

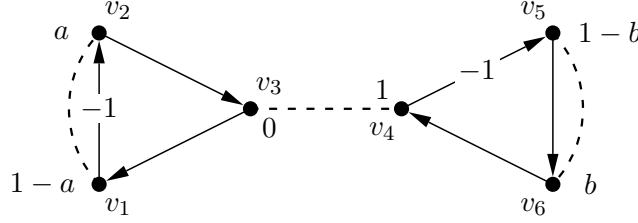
Proof. The xy -cycles and their labels are: $(v_1 v_3 v_5 v_4)$ with $a + b$, (v_2) with $-a$ and (v_6) with $-b + 1$. The yx -cycles are: $(v_2 v_4 v_6 v_3)$ with $a + b$, (v_5) with $-b + 1$ and (v_1) with $-a$. Lemma 4.15 on v_3 implies $a + b \equiv 0 \pmod{N}$. Since v_3 and v_4 are on the same cycle, $N > 1$ by Lemma 4.14 and $a + 1 \not\equiv b \pmod{N}$ by Lemma 4.17. \square

Lemma 7.16. *Let Σ be a covering of Σ_{6A} with simply intersecting cycles, and let*

$$m = \min \left\{ \frac{|\langle 1 + a \rangle| - 1}{6N}, \frac{1}{6|\langle -a \rangle|} \right\}.$$

Then

$$\text{imm}(\Sigma) \leq \begin{cases} 1/3 & \text{if } a \in \{0, -1\}, \\ \frac{1}{6|\langle a+1 \rangle|} + \frac{1}{6} + m & \text{otherwise.} \end{cases}$$

FIGURE 7.7. Schreier graph Σ_{6A} and its coverings

Proof. Let $\Sigma = \Sigma_{6A}^{N;a,b}$ as in Figure 7.7. There exists a unique isomorphism from Σ to $\Sigma_{6A}^{N;b-1,1+a}$ with

$$\begin{aligned} v_1 &\mapsto v_5, & v_3 &\mapsto v_4[1], & v_5 &\mapsto v_1[1], \\ v_2 &\mapsto v_6[1], & v_4 &\mapsto v_3, & v_6 &\mapsto v_2. \end{aligned}$$

The labels $-a$ and $1-b$ are permuted under this isomorphism. Hence without loss of generality we may assume $|\langle -a \rangle| \leq |\langle 1-b \rangle|$. By Lemma 7.15 we have $b \equiv -a \pmod{N}$ and the cycle structure of Σ is the following:

$$(7.7) \quad v_1[i] \text{ and } v_2[i] : \text{ one 4-cycle and a cycle of length } |\langle -a \rangle|,$$

$$(7.8) \quad v_3[i] \text{ and } v_4[i] : \text{ two 4-cycles,}$$

$$(7.9) \quad v_5[i] \text{ and } v_6[i] : \text{ one 4-cycle and a cycle of length } |\langle 1+a \rangle|$$

for all $i \in \mathbb{Z}_N$. First assume that $a \in \{0, -1\}$. A straightforward computation shows that $v_1[*] \cup v_5[*]$ is a plague of size $2N$. In this case, $\text{imm}(\Sigma) \leq 1/3$.

Now assume that $a \notin \{0, -1\}$. Let I (resp. J) be a set of representatives for $\mathbb{Z}_N/\langle -a \rangle$ (resp. $\mathbb{Z}_N/\langle a+1 \rangle$). We claim that $P = v_1[*] \cup v_2[I] \cup v_5[J]$ is a plague. First we compute

$$\begin{array}{c|cc} \text{pivot} & v_2[*] & v_1[I] \\ \hline & v_3[*] & v_2[I+a] \end{array}$$

and hence P spreads to $v_2[*]$ by Example 5.9. Now we compute

$$\begin{array}{c|cc} \text{pivot} & v_3[*] & v_6[J] \\ \hline & v_4[*] & v_5[J+1+a] \end{array}$$

and hence P spreads to $v_1[*] \cup v_2[*] \cup v_3[*] \cup v_4[*] \cup v_5[*]$ by Example 5.9. Then we conclude that P is a plague.

As an alternative, let J be a set of representatives for $\mathbb{Z}_N/\langle 1+a \rangle$, and let $K = \langle 1+a \rangle$ and $I = J \cup K$. We prove that $P = v_1[*] \cup v_5[I]$ is a plague. With $v_2[*]$ as a pivot we see that P spreads to $v_3[*]$, with $v_4[I]$ it spreads to $v_6[I]$, and with $v_5[(I-1) \cap (I+a)]$ it spreads further to $v_4[I \cap (I+a+1)]$. Finally, with $v_6[(I-1) \cap (I+a) \cap (I-a-1)]$ as a pivot, P further spreads

to $v_5[(I-1) \cap (I+a) \cap (I-a-1)]$. By restricting our attention to the fiber over v_5 , we conclude from Example 5.10 that P spreads to $v_5[*]$. Since $v_1[*] \cup v_5[*]$ is a plague of Σ , the claim of the lemma follows. \square

Lemma 7.17. *Let Σ be a covering of Σ_{6A} with simply intersecting cycles. Then $\text{imm}(\Sigma) \leq \omega(\Sigma)$.*

Proof. By Lemma 7.15 we have $b \equiv -a \pmod{N}$ and $N > 1$. As in the proof of Lemma 7.16, we may assume $|\langle -a \rangle| \leq |\langle 1+a \rangle|$. We split the proof into several cases according to the order of $\langle -a \rangle$. Assume first that $|\langle -a \rangle| = 1$. Then $a = 0$ and $|\langle 1+a \rangle| = N \geq 2$. Hence

$$\omega(\Sigma) \geq 2(1/2 + 1/4 + 1/4)/6 = 1/3 = \text{imm}(\Sigma)$$

by Lemma 7.16.

Now assume that $|\langle -a \rangle| = 2$. Then $a = N/2$, N is even, and $N \geq 4$. (If $N = 2$ then $a = 1$, which contradicts to $|\langle -a \rangle| \leq |\langle 1+a \rangle|$.) As before, we need to consider three cases: $N = 4$, $N = 6$ and $N \geq 8$. Then

N	$ \langle 1+a \rangle $	$\omega(v_1), \omega(v_2)$	$\omega(v_3), \omega(v_4)$	$\omega(v_5), \omega(v_6)$
4	4	1/3	1/4	1/4
6	3	1/3	1/4	7/24
≥ 8	≥ 5	1/3	1/4	1/4

and therefore (observe that the case $N = 4$ is exceptional)

$$\omega(\Sigma) = \begin{cases} (3 \cdot 1/3 + 3 \cdot 1/4)/6 = 7/24 & \text{if } N = 4, \\ 7/24 & \text{if } N = 6, \\ 5/18 & \text{if } N \geq 8. \end{cases}$$

Further, $|\langle a+1 \rangle| = N/2$ if a is odd and $|\langle a+1 \rangle| = N$ if a is even. Thus Lemma 7.16 yields that $\text{imm}(\Sigma) \leq 7/24$ for $N = 4$, $\text{imm}(\Sigma) \leq 1/4 + 1/6N$ for $N \geq 6$, both if a is odd and if a is even. This implies the claim.

Now assume that $|\langle -a \rangle| = 3$. Clearly, $3a \equiv 0 \pmod{N}$ and 3 divides N . We claim that $4 \leq |\langle 1+a \rangle|$. Indeed, if $|\langle 1+a \rangle| = 3$ then $3(1+a) \equiv 0 \pmod{N}$ and $N = 3$, $a = 2$, $b = 1$, by Lemma 7.15. Hence $|\langle 1+a \rangle| = 0$, which is a contradiction. Therefore, by (7.7)–(7.9) and Lemma 7.16, we obtain $\text{imm}(\Sigma) \leq 19/72 = \omega(\Sigma)$.

Finally, assume that $|\langle -a \rangle| \geq 4$. Using (7.7)–(7.9) and Lemma 7.16 we obtain $\omega(\Sigma) = 1/4 \geq \text{imm}(\Sigma)$. \square

Example 7.18. *Let $X = \text{Aff}(5, 2)$ or $X = \text{Aff}(5, 3)$. The Hurwitz orbit of $(1, 1, 2) \in X^3$ is isomorphic to the covering $\Sigma_{6A}^{4;2,2}$.*

Lemma 7.19. *Let X be an injective indecomposable rack, and $x \in X$. Assume that X has at least one Hurwitz orbit isomorphic to the covering $\Sigma_{6A}^{4;2,2}$. Then*

- (1) φ_x contains a 4-cycle and $k_4 \geq 4$.
- (2) If $(x, y, z) \in v_3[*]$ and $w = x \triangleright y$, then $w \triangleright ((w \triangleright x) \triangleright x) = z$.

Proof. We may assume that $X = \{1, 2, \dots, d\}$. By inspection,

$$(7.10) \quad \sigma_1 v_1[i] = v_3[i+2], \quad \sigma_2 v_1[i] = v_1[i+2],$$

$$(7.11) \quad \sigma_1 v_2[i] = v_2[i+2], \quad \sigma_2 v_2[i] = v_4[i+1],$$

$$(7.12) \quad \sigma_1 v_3[i] = v_5[i], \quad \sigma_2 v_3[i] = v_2[i+2],$$

$$(7.13) \quad \sigma_1 v_4[i] = v_1[i], \quad \sigma_2 v_4[i] = v_6[i+1],$$

$$(7.14) \quad \sigma_1 v_5[i] = v_4[i+2], \quad \sigma_2 v_5[i] = v_5[i+3],$$

$$(7.15) \quad \sigma_1 v_6[i] = v_6[i+3], \quad \sigma_2 v_6[i] = v_3[i].$$

Without loss of generality we may assume that $v_1[0] = (1, 2, 3) \in X^3$, where $|\{1, 2, 3\}| \leq 3$. Set $4 = 1 \triangleright 2$, $5 = 1 \triangleright 4$, $6 = 4 \triangleright 5$, $7 = 2 \triangleright 4$, $8 = 7 \triangleright 1$ and $9 = \varphi_2^{-1}(8)$. Using formulas (7.10)–(7.15), a straightforward computation shows that:

i	$v_1[i]$	$v_2[i]$	$v_3[i]$	$v_4[i]$	$v_5[i]$	$v_6[i]$
0	(1, 2, 3)	(4, 7, 1)	(7, 1, 2)	(2, 8, 3)	(8, 7, 2)	(7, 2, 8)
1	(9, 4, 7)	(3, 2, 9)	(2, 9, 4)	(4, 8, 7)	(8, 2, 4)	(2, 4, 8)
2	(1, 3, 2)	(7, 4, 1)	(4, 1, 3)	(3, 8, 2)	(8, 4, 3)	(4, 3, 8)
3	(9, 7, 4)	(2, 3, 9)	(3, 9, 7)	(7, 8, 4)	(8, 3, 7)	(3, 7, 8)

Since there are no loops, we obtain $|\{2, 3, 4, 7\}| = 4$. Furthermore, using (7.10)–(7.15) we obtain that $\varphi_8 = (2 \ 7 \ 3 \ 4)\tau$ for some $\tau \in \mathbb{S}_d$, $\tau|_{\{2, 3, 4, 7\}} = \text{id}$. In particular, $|\{2, 3, 4, 7, 8\}| = 5$. Moreover, $[1 \ 3 \ 8 \ 7]$ is a Hurwitz orbit in X^2 , and acting with φ_8 on this we obtain three other Hurwitz orbits $[? \ 4 \ 8 \ 3]$, $[? \ 2 \ 8 \ 4]$, $[? \ 7 \ 8 \ 2]$. Hence $k_4 \geq 4$. Finally, $3 \triangleright 7 = 4$, $7 \triangleright 1 = 8$, $8 \triangleright ((8 \triangleright 7) \triangleright 7) = 8 \triangleright (3 \triangleright 7) = 8 \triangleright 4 = 2$, and hence $w \triangleright ((w \triangleright x) \triangleright x) = z$ for $(x, y, z) = (7, 1, 2)$. \square

7.8. The graph Σ_{6B} .

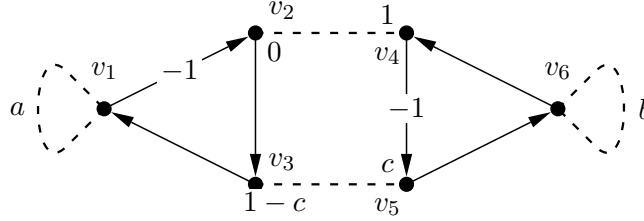
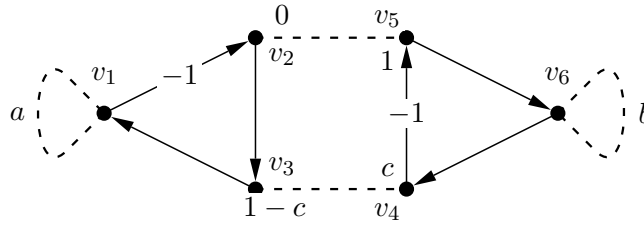
Lemma 7.20. *There is no covering of Σ_{6B} in Figure 7.8 with simply intersecting cycles.*

Proof. The xy -cycles and their labels are $(v_1 \ v_2 \ v_5)$ with $a - c$ and $(v_3 \ v_6 \ v_4)$ with $c + b$. The yx -cycles and their labels are $(v_1 \ v_4 \ v_3)$ with $a - c$ and $(v_2 \ v_5 \ v_6)$ with $c + b$. By Lemma 4.14 on v_3 and v_4 , $N > 1$. Lemma 4.13 on v_1 and v_6 implies that $2a \equiv 2b \equiv 1 \pmod{N}$ and hence $a \equiv b \pmod{N}$ and N is odd. By Lemma 4.16 on v_1 and v_6 , $b + c \equiv 0 \pmod{N}$ and $a - c \equiv 0 \pmod{N}$. Then $a \equiv -b \pmod{N}$ and hence $1 \equiv -1 \pmod{N}$. This is a contradiction to $N > 2$. \square

7.9. The graph Σ_{6C} .

Lemma 7.21. *There is no covering of Σ_{6C} in Figure 7.9 with simply intersecting cycles.*

Proof. This follows from Corollary 4.18. \square

FIGURE 7.8. Schreier graph Σ_{6B} and its coveringsFIGURE 7.9. Schreier graph Σ_{6C} and its coverings

7.10. The graph Σ_{6D} .

Lemma 7.22. *Let Σ be a covering of Σ_{6D} in Figure 7.10 with simply intersecting cycles. Then*

- (1) $\langle a + b \rangle \cap \langle -a + 1 \rangle = 0$,
- (2) $\langle a + b \rangle \cap \langle -b \rangle = 0$,
- (3) $\langle -a + 1 \rangle \cap \langle -b \rangle = 0$.

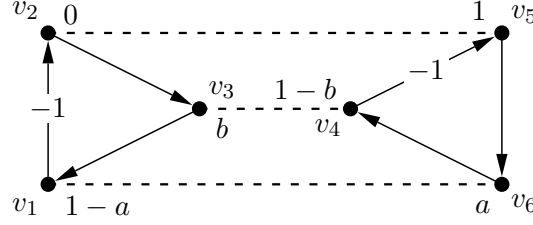
Proof. The xy -cycles and their labels are: $(v_1 v_4)$ with label $a + b$, $(v_2 v_6)$ with $-a + 1$ and $(v_3 v_5)$ with $-b$. The yx -cycles are: $(v_1 v_5)$ with label $-a + 1$, $(v_2 v_4)$ with $-b$ and $(v_3 v_6)$ with $a + b$. Now apply Lemma 4.15. \square

Example 7.23. *The trivial covering of Σ_{6D} is isomorphic to the Hurwitz orbit of six elements of [6, Figure 9].*

Lemma 7.24. *Let Σ be a covering of Σ_{6D} with simply intersecting cycles. Let $c \in \{1 - a, -b, a + b\}$ and let $n = |\langle c \rangle|$. Then*

$$\text{imm}(\Sigma) \leq \frac{N + N/n + n - 1}{6N}.$$

Proof. There exists an isomorphism $\Sigma_{6D}^{N;a,b} \rightarrow \Sigma_{6D}^{N;1-a-b,a-1}$, $v_1 \mapsto v_2$. This isomorphism induces the permutation $(1\ 3\ 2)$ on the triple $(1 - a, -b, a + b)$. Therefore we may assume that $c \equiv -b \pmod{N}$.

FIGURE 7.10. Schreier graph Σ_{6D} and its coverings

The cycle structure at each vertex of Σ is the following:

$$(7.16) \quad v_1[i] \text{ and } v_6[i] : \text{cycles of length } 2|\langle a+b \rangle| \text{ and } 2|\langle -a+1 \rangle|,$$

$$(7.17) \quad v_2[i] \text{ and } v_5[i] : \text{cycles of length } 2|\langle -a+1 \rangle| \text{ and } 2|\langle -b \rangle|,$$

$$(7.18) \quad v_3[i] \text{ and } v_4[i] : \text{cycles of length } 2|\langle a+b \rangle| \text{ and } 2|\langle -b \rangle|,$$

for all $i \in \mathbb{Z}_N$. Let I be the union of $\langle b \rangle$ with a set of representatives for $\mathbb{Z}_N/\langle b \rangle$. Then $|I| = n + N/n - 1$. We claim that $P = v_1[*] \cup v_5[I]$ is a plague. Using the sequence of pivots $v_6[I]$, $v_3[I+b]$, $v_2[I-1]$, $v_5[(I+b) \cap I]$, and $v_4[(I+1) \cap (I+b+1) \cap (I-b)]$, we see that P spreads to

$$v_4[I+1] \cup v_2[I+b] \cup v_3[I] \cup v_6[(I+1) \cap (I+b+1)] \cup v_5[(I+1) \cap (I+b+1) \cap (I-b)].$$

By looking at the evolution of χ_P at $v_5[*]$, Example 5.10 implies that P spreads to $v_5[*]$. Since $P \cup v_5[*]$ is a plague, also P is a plague. This implies the lemma. \square

Lemma 7.25. *Let $n \in \mathbb{N}$ with $2 \leq n \leq N/2$. Then*

$$\frac{6N}{n} + 6n - 6 \leq 3N + 6.$$

Proof. The inequality is equivalent to $(n-2)(2n-N) \leq 0$ and hence the claim follows. \square

Lemma 7.26. *Let Σ be a covering of Σ_{6D} with simply intersecting cycles. Then $\text{imm}(\Sigma) \leq \omega(\Sigma)$.*

Proof. There is an isomorphism $\Sigma_{6D}^{N;a,b} \rightarrow \Sigma_{6D}^{N;b+1,a-1}$, $v_1 \mapsto v_4$ and this induces the permutation $(1\ 2)$ on $(1-a, -b, a+b)$. Using this and the isomorphism in the proof of Lemma 7.24, we may assume that

$$0 \leq [1-a] \leq [-b] \leq [a+b] < N,$$

where for any $x \in \mathbb{Z}$ we let $[x] \in \{0, 1, \dots, N-1\}$ such that $[x] \equiv x \pmod{N}$.

Suppose first that $a = 1$ and $b = 0$. Then, using Lemma 7.24 with $c = 1-a$, we obtain $\text{imm}(\Sigma) \leq 1/3 = \omega(\Sigma)$.

Assume now that $a \equiv 1 \pmod{N}$, $b \not\equiv 0 \pmod{N}$ and $a + b \not\equiv 0 \pmod{N}$. Then $N \geq 3$ by Lemma 7.22. Further, by (7.16)–(7.18),

$$\omega(\Sigma) = \frac{(1/3)4N + (1/4)2N}{6N} = \frac{11}{36}.$$

Let $n = |\langle -b \rangle|$. Then $2 \leq c \leq N/2$ by Lemma 7.22. Hence Lemma 7.24 for $c = -b$ and Lemma 7.25 imply that $\text{imm}(\Sigma) \leq (9N + 6)/36N$ and hence $\text{imm}(\Sigma) \leq \omega(\Sigma)$ since $N \geq 3$.

Finally we assume that $a \not\equiv 1 \pmod{N}$, $b \not\equiv 0 \pmod{N}$, and $a + b \not\equiv 0$. Let $c \in \{1 - a, -b, a + b\}$ and $n = |\langle c \rangle|$. Because of Lemma 7.22 we have $n \notin \{N, N/2\}$ and hence $n \leq N/3$, $N - 3n \geq 0$. Moreover, by choosing $c \in \{1 - a, -b, a + b\}$ appropriately, we may assume that $n \geq 4$. Since all vertices of Σ belong to cycles of length ≥ 4 , we obtain that $\omega(\Sigma) = 1/4$. Then Lemma 7.24 implies that

$$\text{imm}(\Sigma) \leq \frac{N + N/n + n - 1}{6N} = \frac{3Nn - (Nn - 2N - 2n^2 + 2n)}{12Nn} \leq \frac{1}{4}$$

because $Nn - 2N - 2n^2 + 2n = (N - 2n)(n - 4) + 2(N - 3n) \geq 0$. \square

7.11. The graph Σ_{7A} .

Lemma 7.27. *Let Σ be a covering of Σ_{7A} in Figure 7.11 with simply intersecting cycles. Then $N = 7$ and $(a, b, c) = (2, 4, 3)$.*

Proof. The xy cycles and their labels are: $(v_1 v_3 v_6 v_2)$ with label $a - c + 1$ and $(v_5 v_4 v_7 v_3)$ with label $b + c$. The yx -cycles are: $(v_1 v_2 v_5 v_4)$ with label $a - c + 1$ and $(v_6 v_7 v_3 v_4)$ with label $b + c$. Since v_1 and v_2 are on the same xy - and yx -cycles, $N > 1$ by Lemma 4.14. Lemma 4.13 on v_1 and v_7 implies that $3a \equiv -1 \pmod{N}$ and $2b \equiv 1 \pmod{N}$. By Lemma 4.16 on v_1 and v_7 we obtain $b \equiv -c \pmod{N}$ and $a \equiv c - 1 \pmod{N}$. From this the claim follows. \square

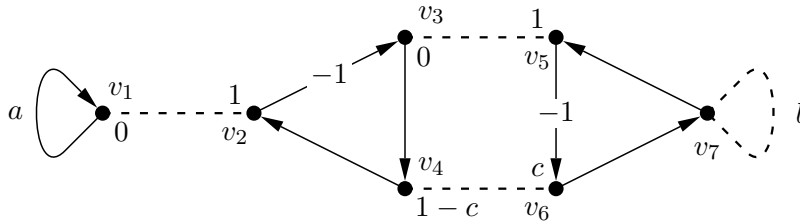


FIGURE 7.11. Schreier graph Σ_{7A} and its coverings

Lemma 7.28. *Let Σ be a covering of Σ_{7A} with simply intersecting cycles. Then $\text{imm}(\Sigma) \leq \omega(\Sigma)$.*

Proof. First we prove that $\text{imm}(\Sigma) \leq 13/49$. By Lemma 7.27, $N = 7$ and $(a, b, c) = (2, 4, 3)$. Let $I = \{0, \dots, 5\}$. We claim that $P = v_1[I] \cup v_3[*]$ is a plague of size 13. Use the pivots $v_2[I]$, $v_1[(I-3) \cap (I+2)]$, $v_4[(I-3) \cap (I+2)]$, $v_5[I \cap (I-2)]$, $v_3[(I-3) \cap (I-1) \cap (I+2)]$ and $v_7[(I-2) \cap I \cap (I+1) \cap (I+3)]$ to let P spread to

$$\begin{aligned} &v_4[I] \cup v_2[(I-2) \cap (I+3)] \cup v_6[(I-2) \cap I] \\ &\cup v_7[(I-2) \cap I] \cup v_5[\{0, 1, 3, 5\}] \cup v_5[\{2, 4, 6\}]. \end{aligned}$$

Then P spreads to $v_5[*]$. With $v_3[I-1]$ and with $v_1[3]$ as pivots, P further spreads to $v_2[I] \cup v_1[6]$. Thus P spreads to $v_1[*] \cup v_3[*]$. The above calculations with $I = \mathbb{Z}_N$ prove that P is a plague. Hence $\text{imm}(\Sigma) \leq 13/49$. By Lemma 7.27, the cycle structure of Σ is

$$\begin{aligned} &v_1[i] \text{ and } v_2[i] : \text{two 4-cycles,} \\ &v_3[i], v_4[i], v_5[i], v_6[i] : \text{one 3-cycle and one 4-cycle,} \\ &v_7[i] : \text{two 3-cycles,} \end{aligned}$$

for all $i \in \mathbb{Z}_N$. Hence $\omega(\Sigma) = 47/168 > 13/49 \geq \text{imm}(\Sigma)$. \square

7.12. The graph Σ_{8A} .

Lemma 7.29. *Let Σ be the covering of Σ_{8A} with simply intersecting cycles. Then $\text{imm}(\Sigma) \leq \omega(\Sigma)$.*

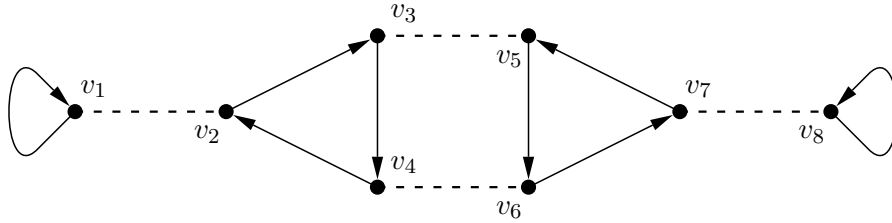
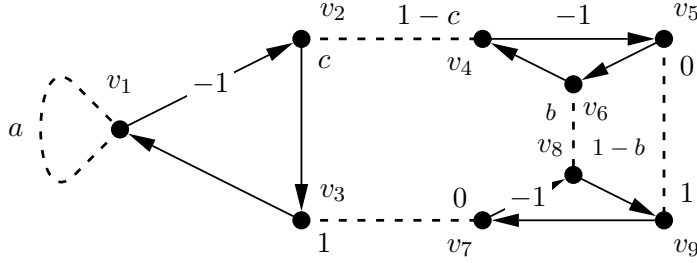


FIGURE 7.12. Schreier graph Σ_{8A} and its coverings

Proof. A straightforward computation shows that $v_1[*] \cup v_3[*]$ is a plague of size $2N$ and hence $\text{imm}(\Sigma) \leq 1/4$. Since all cycles have length ≥ 4 , the claim follows. \square

7.13. The graph Σ_{9A} .

Lemma 7.30. *Let Σ be a covering of Σ_{9A} in Figure 7.13 with simply intersecting cycles. Then $N > 1$, $2a \equiv 1 \pmod{N}$, N is odd, $c \equiv a + 1 \pmod{N}$ and $\langle b - 1 \rangle \cap \langle a + b \rangle = 0$.*

FIGURE 7.13. Schreier graph Σ_{9A} and its coverings

Proof. The xy -cycles and their labels are: $(v_1 v_2 v_5 v_7)$ with label $a - c + 1$, $(v_6 v_9)$ with $-b + 1$ and $(v_3 v_8 v_4)$ with $b + c - 1$. The yx -cycles are: $(v_1 v_4 v_9 v_3)$ with $a - c + 1$, $(v_2 v_7 v_6)$ with $b + c - 1$ and $(v_5 v_8)$ with $-b + 1$. Lemmas 4.13 and 4.16 on v_1 imply $2a \equiv 1 \pmod{N}$ and $c \equiv a + 1 \pmod{N}$. Since v_2 and v_7 are on the same xy -cycle and the same yx -cycle, we conclude that $N > 1$ by Lemma 4.14. By Lemma 4.15 on v_6 the claim follows. \square

Lemma 7.31. *Let Σ be a covering of Σ_{9A} with simply intersecting cycles. Then*

$$\text{imm}(\Sigma) \leq \begin{cases} 8/27 & \text{if } a + b \equiv 0 \pmod{N} \text{ and } b \equiv 1 \pmod{N}, \\ 11/45 & \text{if } a + b \not\equiv 0 \pmod{N} \text{ and } b \not\equiv 1 \pmod{N}, \\ 7/27 & \text{otherwise.} \end{cases}$$

Proof. There are four cases to consider. Assume first that $b \equiv 1 \pmod{N}$ and $a + b \equiv 0 \pmod{N}$. Lemma 7.30 implies then that $N = 3$, $a \equiv 2 \pmod{N}$, $b \equiv 1 \pmod{N}$, and $c \equiv 0 \pmod{N}$. Let $I = \{0, 1\}$. Then $P = v_1[I] \cup v_2[*] \cup v_5[*]$ is a plague of size 8. Indeed,

pivot	$v_4[*]$	$v_1[I + 1]$	$v_3[I - 1]$	$v_8[I + 1]$	$v_2[I \cap (I - 1)]$	$v_5[I + 1]$
	$v_6[*]$	$v_3[I + 1]$	$v_7[I - 1]$	$v_9[I + 2]$	$v_4[I \cap (I + 1)]$	$v_4[I + 2]$

and hence, since $(I \cap (I + 1)) \cup (I + 2) = \mathbb{Z}_N$, P spreads to $v_4[*]$. From this the claim follows.

Assume now that $b \not\equiv 1 \pmod{N}$ and that $a + b \equiv 0 \pmod{N}$. Then $|\langle b-1 \rangle| \geq 3$ because N is odd. Let I be a set of representatives for $\mathbb{Z}_N / \langle b-1 \rangle$. We claim that $v_1[*] \cup v_2[*] \cup v_5[I]$ is a plague. We compute

pivot	$v_1[*]$	$v_3[*]$	$v_2[*]$	$v_4[I]$	$v_5[I - 1]$	$v_7[I]$	$v_6[I + b - 1]$
	$v_3[*]$	$v_7[*]$	$v_4[*]$	$v_6[I]$	$v_9[I]$	$v_8[I]$	$v_5[I + b - 1]$

and the claim follows from Example 5.9. Therefore

$$(7.19) \quad \text{imm}(\Sigma) \leq \frac{2N + |I|}{9N} \leq 7/27$$

since $|I| \leq N/3$.

Assume now that $b \equiv 1 \pmod{N}$ and $a + b \not\equiv 0 \pmod{N}$. Let I be a set of representatives for $\mathbb{Z}_N / \langle a + b \rangle$. Then $P = v_4[*] \cup v_8[*] \cup v_2[I]$ is a plague. Indeed, P spreads according to the following table.

pivot	$v_6[*]$	$v_9[*]$	$v_4[I - a - 1]$	$v_8[I - a - b - 1]$	$v_7[I - a - b]$
	$v_5[*]$	$v_7[*]$	$v_6[I - a - 1]$	$v_9[I - a - b]$	$v_3[I - a - b + 1]$

Pivoting from $v_2[I - a - b]$ and $v_3[I - a - b]$, P spreads further to $v_1[I - a - b + 1] \cup v_2[I - a - b]$. As before, since $|I| \leq N/3$, and using Example 5.9 with $\lambda = a + b$, we conclude that P spreads to $v_2[*]$. Hence P is a plague and $\text{imm}(\Sigma) \leq 7/27$.

Finally, assume that $b \not\equiv 1 \pmod{N}$ and $a + b \not\equiv 0 \pmod{N}$. From Lemma 7.30 we obtain that N is odd, $\langle a + b \rangle \cap \langle b - 1 \rangle = 0$, and $c \equiv a + 1 \pmod{N}$. If $|\langle a + b \rangle| = 3$ then $|\langle b - 1 \rangle| \geq 5$ and $v_1[*] \cup v_2[*] \cup v_5[I]$ is a plague, where I is a set of representatives for $\mathbb{Z}_N / \langle b - 1 \rangle$. Indeed, the calculations

pivot	$v_1[*]$	$v_2[*]$	$v_3[*]$	$v_4[I]$	$v_5[I - 1]$	$v_8[I - 1]$	$v_4[I + b - 1]$
	$v_3[*]$	$v_4[*]$	$v_7[*]$	$v_6[I]$	$v_9[I]$	$v_6[I + b - 1]$	$v_5[I + b - 1]$

together with Example 5.9 with $\lambda = 1 - b$ show that P is a plague if $P \cup v_5[*]$ is a plague. However, the latter is easy to check. On the other hand, if $|\langle a + b \rangle| \geq 5$, then let $P = v_4[*] \cup v_8[*] \cup v_2[I]$, where I is a set of representatives for $\mathbb{Z}_N / \langle a + b \rangle$. Then P is a plague. Indeed, the calculations

pivot	$v_6[*]$	$v_9[*]$	$v_4[I - a - 1]$	$v_8[I - a - b - 1]$	$v_7[I - a - b]$
	$v_5[*]$	$v_7[*]$	$v_6[I - a - 1]$	$v_9[I - a - b]$	$v_3[I - a - b + 1]$
pivot				$v_2[I - a - b]$	$v_3[I - a - b]$
				$v_1[I - a - b + 1]$	$v_2[I - a - b]$

together with Example 5.9 with $\lambda = a + b$ show that P is a plague if $P \cup v_2[*]$ is a plague. Again, the latter is easy to check. Since $|I| \leq N/5$, in the last two cases we obtain that

$$\text{imm}(\Sigma) \leq \frac{2N + |I|}{9N} \leq \frac{2N + N/5}{9N} = 11/45.$$

This completes the proof. \square

Lemma 7.32. *Let Σ be a covering of Σ_{9A} with simply intersecting cycles. Then $\text{imm}(\Sigma) \leq \omega(\Sigma)$.*

Proof. The cycle structure at each vertex of Σ is the following:

$$(7.20) \quad v_1[i] \text{ two 4-cycles,}$$

$$(7.21) \quad v_2[i], v_3[i], v_4[i], v_7[i] \text{ one 4-cycle and one cycle of length } 3|\langle a + b \rangle|,$$

$$(7.22) \quad v_5[i], v_9[i] \text{ one 4-cycle and one cycle of length } 2|\langle b - 1 \rangle|,$$

$$(7.23) \quad v_6[i], v_8[i] \text{ cycles of length } 3|\langle a + b \rangle| \text{ and } 2|\langle b - 1 \rangle|,$$

for all $i \in \mathbb{Z}_N$. From (7.20)–(7.23) it is straightforward to compute the weight at every vertex of Σ and $\omega(\Sigma)$:

$$\omega(\Sigma) = \begin{cases} 11/36 & \text{if } a \equiv -b \pmod{N}, b \equiv 1 \pmod{N}, \\ 5/18 & \text{if } a \equiv -b \pmod{N}, b \not\equiv 1 \pmod{N}, \\ 31/108 & \text{if } a \not\equiv -b \pmod{N}, b \equiv 1 \pmod{N}, \\ 1/4 & \text{if } a \not\equiv -b \pmod{N}, b \not\equiv 1 \pmod{N}. \end{cases}$$

From this and from Lemma 7.31 the claim follows. \square

7.14. The graph Σ_{12A} .

Lemma 7.33. *Let Σ be a covering of Σ_{12A} with simply intersecting cycles. Then $\text{imm}(\Sigma) \leq 1/4 = \omega(\Sigma)$.*

Proof. In Σ , all cycles have length ≥ 4 , hence $\omega(\Sigma) = 1/4$. A straightforward computation shows that $v_1[*] \cup v_2[*] \cup v_5[*]$ is a plague of Σ of size $3N$ and hence $\text{imm}(\Sigma) \leq 1/4 = \omega(\Sigma)$. \square

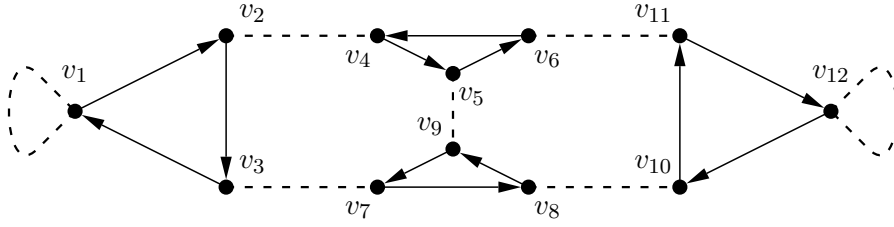


FIGURE 7.14. Schreier graph Σ_{12A}

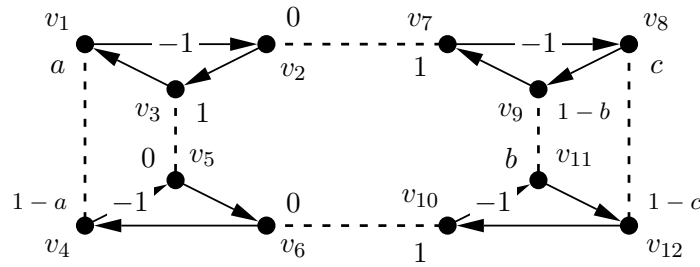


FIGURE 7.15. Schreier graph Σ_{12B} and its coverings

7.15. The graph Σ_{12B} .

Lemma 7.34. *Let Σ be a covering of Σ_{12B} in Figure 7.15 with simply intersecting cycles. Then $N > 1$ and the following conditions are satisfied:*

- (1) $\langle 1 - a \rangle \cap \langle a - c \rangle = 0$,
- (2) $\langle 1 - b \rangle \cap \langle a - c \rangle = 0$,

- (3) $\langle 1 - b \rangle \cap \langle 1 - a \rangle = 0$,
- (4) $\langle 1 - b \rangle \cap \langle b + c \rangle = 0$,
- (5) $\langle a - c \rangle \cap \langle b + c \rangle = 0$,
- (6) $(-c + \langle a - c \rangle) \cap \langle b - 1 \rangle = \emptyset$.

Proof. The yx -cycles and their labels are: $(v_3 v_4)$ with $1 - a$, $(v_8 v_{11})$ with $b + c$, $(v_1 v_7 v_{12} v_6)$ with $a - c$ and $(v_2 v_5 v_{10} v_9)$ with $1 - b$. The xy -cycles are: $(v_1 v_5)$ with $1 - a$, $(v_9 v_{12})$ with $b + c$, $(v_2 v_8 v_{10} v_4)$ with $a - c$ and $(v_3 v_6 v_{11} v_7)$ with $1 - b$. Then Lemma 4.15 implies (1)–(5). Moreover, Lemma 4.14 on v_6 and v_7 implies that $N > 1$. Finally, the claim (6) follows from Lemma 4.17 on v_2 and v_{10} . Indeed, the xy -path starting in $v_2[i]$ for some $i \in \mathbb{Z}_N$, which has length $4k + 2$ with $k \in \mathbb{Z}$, ends in $v_{10}[1 - c + k(a - c)]$, and any yx -path of length $4l + 2$ with the same starting point ends in $v_{10}[1 + l(1 - b)]$. \square

Lemma 7.35. *Let Σ be a covering of Σ_{12B} as in Figure 7.15 with simply intersecting cycles. Then*

$$\text{imm}(\Sigma) \leq \begin{cases} \frac{3N+1}{12N} & \text{if } a \equiv 1 \pmod{N} \text{ or } b + c \equiv 0 \pmod{N}, \\ 1/4 & \text{otherwise.} \end{cases}$$

Proof. The cycle structure at each vertex of Σ is:

- (7.24) $v_1[i], v_4[i] : \text{cycles of length } 4|\langle a - c \rangle| \text{ and } 2|\langle 1 - a \rangle|,$
- (7.25) $v_2[i], v_6[i], v_7[i], v_{10}[i] : \text{cycles of length } 4|\langle 1 - b \rangle| \text{ and } 4|\langle a - c \rangle|,$
- (7.26) $v_3[i], v_5[i] : \text{cycles of length } 2|\langle 1 - a \rangle| \text{ and } 4|\langle 1 - b \rangle|,$
- (7.27) $v_8[i], v_{12}[i] : \text{cycles of length } 2|\langle b + c \rangle| \text{ and } 4|\langle a - c \rangle|,$
- (7.28) $v_9[i], v_{11}[i] : \text{cycles of length } 2|\langle b + c \rangle| \text{ and } 4|\langle 1 - b \rangle|,$

for all $i \in \mathbb{Z}_N$.

Assume first that $a \equiv 1 \pmod{N}$ or $b + c \equiv 0 \pmod{N}$. We claim that there exists a subset $I \subseteq \mathbb{Z}_N$ of size one such that the set $P = v_2[*] \cup v_5[*] \cup v_9[*] \cup v_{10}[I]$ is a plague of size $3N + 1$. We compute

pivot	$v_3[*]$	$v_4[*]$	$v_7[*]$	$v_{11}[I - 1]$	$v_{10}[I]$	$v_{12}[I]$
	$v_1[*]$	$v_6[*]$	$v_8[*]$	$v_{12}[I]$	$v_{11}[I]$	$v_{10}[I + 1]$

and hence P spreads to $v_{10}[*]$ by Example 5.9. Thus P is a plague since $P \cup v_{10}[*]$ is a plague. In this case $\text{imm}(\Sigma) \leq (3N + 1)/12N$.

Assume now that $a \not\equiv 1 \pmod{N}$ and $b + c \not\equiv 0 \pmod{N}$. Let I be a set of representatives for $\mathbb{Z}_N / \langle a - 1 \rangle$ and let J be a set of representatives for $\mathbb{Z}_N / \langle b + c \rangle$. We claim that $v_3[I] \cup v_8[J] \cup v_1[*] \cup v_2[*]$ is a plague. We compute

pivot	$v_3[*]$	$v_4[*]$	$v_5[I - 1]$	$v_1[I + a - 1]$
	$v_5[*]$	$v_6[*]$	$v_4[I]$	$v_3[I + a - 1]$

and therefore P spreads to $v_3[*] \cup v_4[*]$ by Example 5.9. Now the calculations

pivot	$v_2[*]$	$v_6[*]$	$v_9[J]$	$v_{12}[J+b]$
	$v_7[*]$	$v_{10}[*]$	$v_{11}[J+b]$	$v_8[I+b+c]$

show that P spreads to $v_8[*]$ by Example 5.9. Then the claim follows and in this case $\text{imm}(\Sigma) \leq 1/4$ since $|I| + |J| \leq N$. \square

Lemma 7.36. *Let Σ be a covering of Σ_{12B} with simply intersecting cycles. Then $\text{imm}(\Sigma) \leq \omega(\Sigma)$.*

Proof. Using (7.24)–(7.28) we obtain

$$\omega(\Sigma) = \begin{cases} 11/36 & \text{if } a \equiv 1 \pmod{N} \text{ and } b+c \equiv 0 \pmod{N}, \\ 5/18 & \text{if } a \equiv 1 \pmod{N} \text{ and } b+c \not\equiv 0 \pmod{N}, \\ 5/18 & \text{if } a \not\equiv 1 \pmod{N} \text{ and } b+c \equiv 0 \pmod{N}, \\ 1/4 & \text{otherwise.} \end{cases}$$

First notice that if $N = 2$ then $a \equiv 1 \pmod{N}$, $b \equiv 1 \pmod{N}$, and $c \equiv 1 \pmod{N}$ by Lemma 7.34(6). Then $\text{imm}(\Sigma) \leq \frac{3N+1}{12N} = 7/24 < 11/36 = \omega(\Sigma)$ by Lemma 7.35, and the claim holds. So we may assume that $N > 2$.

If $a \equiv 1 \pmod{N}$ or $b+c \equiv 0 \pmod{N}$ then Lemma 7.35 and $N \geq 3$ imply that

$$\text{imm}(\Sigma) \leq \frac{3N+1}{12N} \leq 10/36 \leq \omega(\Sigma).$$

Otherwise $\text{imm}(\Sigma) \leq 1/4 = \omega(\Sigma)$ by Lemma 7.35. \square

7.16. The graph Σ_{12C} .

Lemma 7.37. *Let Σ be a covering of Σ_{12C} in Figure 7.16 with simply intersecting cycles. Then the following hold:*

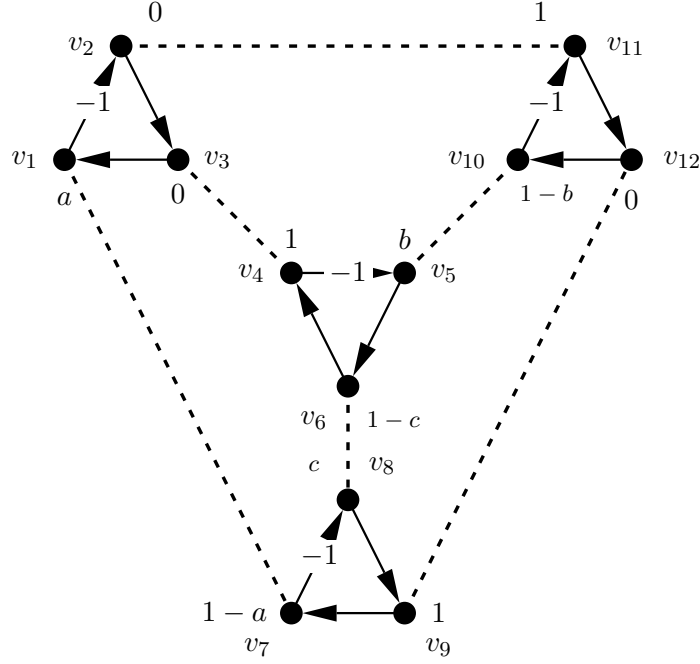
- (1) $\langle -a-c+1 \rangle \cap \langle a+1 \rangle = 0$,
- (2) $\langle a+1 \rangle \cap \langle -b \rangle = 0$,
- (3) $\langle -b \rangle \cap \langle -a-c+1 \rangle = 0$,
- (4) $\langle -b \rangle \cap \langle b+c \rangle = 0$,
- (5) $\langle b+c \rangle \cap \langle -a-c+1 \rangle = 0$,
- (6) $\langle b+c \rangle \cap \langle a+1 \rangle = 0$.

Proof. The yx -cycles and their labels are: $(v_1 \ v_{11} \ v_9)$ with label $a+1$, $(v_2 \ v_4 \ v_{10})$ with $-b$, $(v_3 \ v_7 \ v_6)$ with $-a-c+1$ and $(v_5 \ v_8 \ v_{12})$ with $b+c$. The xy -cycles are: $(v_1 \ v_8 \ v_4)$ with $-a-c+1$, $(v_2 \ v_{12} \ v_7)$ with $a+1$, $(v_{11} \ v_3 \ v_5)$ with $-b$ and $(v_9 \ v_{10} \ v_6)$ with $b+c$. Then the claim follows from Lemma 4.15. \square

Example 7.38. *The trivial covering of Σ_{12C} is isomorphic to the Hurwitz orbit of size 12 of [6, Figure 12]. The covering $\Sigma_{12C}^{2;1,0,0}$ is isomorphic to the Hurwitz orbit of size 24 of [6, Figure 14].*

Lemma 7.39. *Assume that $N/n \geq 6$ and $n \geq 5$. Then*

$$n \frac{N/n+1}{2} + \frac{N/n-1}{2} + \frac{N}{n} \leq N.$$

FIGURE 7.16. Schreier graph Σ_{12C} and its coverings

Proof. After multiplication with $2n$ the inequality takes the form $Nn - 3N + n - 2n^2 \geq 0$, or $(N - 6n)(n - 3) + 4n(n - 5) + 3n \geq 0$. \square

Lemma 7.40. *Let Σ be a covering of Σ_{12C} in Figure 7.16 with simply intersecting cycles. Then*

$$\text{imm}(\Sigma) \leq \begin{cases} 1/3 & \text{if } \Sigma \text{ is the trivial covering,} \\ 7/24 & \text{if } \Sigma = \Sigma_{12C}^{2;1,0,0}, \\ 7/24 & \text{if } 1 + a \equiv 1 - a - c \equiv b \equiv 0 \pmod{N}, \\ & b + c \not\equiv 0 \pmod{N}, \\ 5/18 & \text{if } 1 + a \equiv 1 - a - c \equiv 0 \pmod{N}, \\ & b, b + c \not\equiv 0 \pmod{N}, \\ 1/4 & \text{otherwise.} \end{cases}$$

Proof. The cycle structure at each vertex is the following:

$$(7.29) \quad v_1[i], v_7[i] : \text{cycles of length } 3|\langle 1 - a - c \rangle| \text{ and } 3|\langle 1 + a \rangle|,$$

$$(7.30) \quad v_2[i], v_{11}[i] : \text{cycles of length } 3|\langle 1 + a \rangle| \text{ and } 3|\langle -b \rangle|,$$

$$(7.31) \quad v_3[i], v_4[i] : \text{cycles of length } 3|\langle -b \rangle| \text{ and } 3|\langle 1 - a - c \rangle|,$$

$$(7.32) \quad v_5[i], v_{10}[i] : \text{cycles of length } 3|\langle b + c \rangle| \text{ and } 3|\langle -b \rangle|,$$

$$(7.33) \quad v_6[i], v_8[i] : \text{cycles of length } 3|\langle b + c \rangle| \text{ and } 3|\langle 1 - a - c \rangle|,$$

$$(7.34) \quad v_9[i], v_{12}[i] : \text{cycles of length } 3|\langle b + c \rangle| \text{ and } 3|\langle 1 + a \rangle|,$$

for all $i \in \mathbb{Z}_N$. Without loss of generality we may assume that

$$(7.35) \quad |\langle 1 + a \rangle| \leq |\langle 1 - a - c \rangle| \leq |\langle -b \rangle| \leq |\langle b + c \rangle|.$$

We split the proof into five cases according to the number of trivial groups in (7.35). However, two of these cases will be considered simultaneously.

First assume that $1 + a \equiv 1 - a - c \equiv b \equiv b + c \equiv 0 \pmod{N}$. Then $N = 1$ (the trivial covering) or $N = 2$ and $(a, b, c) = (1, 0, 0)$. Since these orbits appear as Hurwitz orbits of braided racks, see Example 7.38, the claim follows from [6, Proposition 10].

Assume now that $1 + a \equiv a + c - 1 \equiv b \equiv 0 \pmod{N}$ and $b + c \not\equiv 0 \pmod{N}$. Then $a \equiv -1 \pmod{N}$, $b \equiv 0 \pmod{N}$, $c \equiv 2 \pmod{N}$, and $N \geq 3$. Let I be a set of representatives for $\mathbb{Z}_N / \langle b + c \rangle$. Then $|I| \leq N/2$. We claim that $P = v_1[*] \cup v_2[*] \cup v_3[*] \cup v_5[I]$ is a plague. With the sequence of pivots $v_1[*]$, $v_2[*]$, $v_3[*]$, $v_4[I]$, $v_5[I - 1]$, $v_6[I]$, $v_7[I + c]$, $v_9[I + c]$, $v_{10}[I + c]$ we see that P spreads to

$$v_7[*] \cup v_{11}[*] \cup v_4[*] \cup v_6[I] \cup v_{10}[I - b] \cup v_8[I + c] \cup v_9[I + c] \cup v_{12}[I + c] \cup v_5[I + b + c].$$

Hence the claim follows from Example 5.9 for the fiber over v_5 .

Now assume that exactly two groups of (7.35) are trivial, i.e., $1 + a \equiv a + c - 1 \equiv 0 \pmod{N}$, and $b, b + c \not\equiv 0 \pmod{N}$. Then $a \equiv -1 \pmod{N}$, $c \equiv 2 \pmod{N}$, and $1 < |\langle b \rangle| \leq |\langle b + c \rangle|$. From Lemma 7.34 we obtain $\langle b \rangle \cap \langle b + c \rangle = 0$ and hence $|\langle b + c \rangle| \geq 3$. Therefore the previous computation with $|I| \leq N/3$ yields the claim.

Finally, assume that $1 - a - c, b, b + c \not\equiv 0 \pmod{N}$. We prove that $\text{imm}(\Sigma) \leq 1/4$. Let $n = |\langle b + c \rangle|$. Instead of $|\langle 1 - a - c \rangle| \leq |\langle b \rangle| \leq |\langle b + c \rangle|$ we assume that $|\langle 1 - a - c \rangle|, |\langle b \rangle| \leq n$ and $n|\langle 1 - a - c \rangle| \neq N$. This is possible since any subgroup of \mathbb{Z}_N is uniquely determined by its order and any two of these three subgroups have trivial intersection by Lemma 7.37. Observe that $n \geq 5$. We claim that there exist two subsets $I, J \subseteq \mathbb{Z}_N$ with $|I| + |J| \leq N$, such that

$$(7.36) \quad P = v_1[*] \cup v_2[*] \cup v_3[I] \cup v_5[J]$$

is a plague. We split the proof into several steps.

Step 1. The following computation shows that if $x + a + c \in I$ and $x + b + c \in I \cap J$, then P spreads to $\{v_5[x]\}$:

pivot	
$v_3[*]$	$v_4[*]$
$v_1[I]$	$v_7[I - a + 1]$
$v_4[I \cap J]$	$v_6[I \cap J]$
$v_5[(I - 1) \cap (J - 1)]$	$v_{10}[(I - b) \cap (J - b)]$
$v_{11}[(I - b - 1) \cap (J - b - 1)]$	$v_{12}[(I - b) \cap (J - b)]$
$v_9[(I - a) \cap (I - b) \cap (J - b)]$	$v_8[(I - a) \cap (I - b) \cap (J - b)]$
$v_6[(I - a - c) \cap (I - b - c) \cap (J - b - c)]$	$v_5[(I - a - c) \cap (I - b - c) \cap (J - b - c)]$

Step 2. We claim that if x satisfies $x + a + c - 1 \in I$, $x + a + c \in I$ and $x + b + c \in I \cap J$, then P spreads to $\{v_3[x], v_5[x]\}$. This follows from Step 1 and the following fact: if $x + a + c - 1 \in I$ and $x \in J$, then P spreads to $\{v_3[I]\}$. The proof is obtained from the following table:

pivot	
$v_3[*]$	$v_4[*]$
$v_6[J]$	$v_8[J + c]$
$v_7[J + c]$	$v_9[J + c]$
$v_1[I]$	$v_7[I + 1 - a]$
$v_8[(I - a) \cap (J + c - 1)]$	$v_6[(I - a - c + 1) \cap J]$
$v_4[(I + 1 - a - c) \cap J]$	$v_3[(I - a - c + 1) \cap J]$

Step 3. Since $\langle a + c - 1 \rangle \cap \langle b + c \rangle = 0$ and $\langle b + c \rangle = \langle N/n \rangle$, it follows that $a + c - 1 \not\equiv 0 \pmod{N/n}$. The order of $a + c - 1$ modulo N/n is the same as modulo N , as $\langle a + c - 1 \rangle \cap \langle b + c \rangle = 0$. Since $n|\langle a + c - 1 \rangle| \neq N$, we obtain that $a + c - 1 \not\equiv -1 \pmod{N/n}$. Using the prime factorization of N and three nonzero parameters, from the conditions on the parameters we also get that $N/n \geq 6$. (The generators of the groups generated by the nonzero parameters have at least two distinct prime factors.)

Step 4. Let J be a set of representatives for $\mathbb{Z}_N / \langle b + c \rangle$. Let

$$I = J \cup \{i + \langle b + c \rangle \mid 0 \leq i \leq (N/n - 1)/2\}.$$

Then $|J| = N/n$ and $|I| \leq n \frac{N/n+1}{2} + \frac{N/n-1}{2}$. By Lemma 7.39, $|I| + |J| \leq N$. So it is enough to show that (7.36) is a plague. For that purpose, we use Example 5.11 with $m = N/n$ and $\lambda = 1 - a - c$. \square

Lemma 7.41. *Let Σ be a covering of Σ_{12C} with simply intersecting cycles. Then $\text{imm}(\Sigma) \leq \omega(\Sigma)$.*

Proof. As in the proof of Lemma 7.40, without loss of generality we may assume that $|\langle 1 + a \rangle| \leq |\langle 1 - a - c \rangle| \leq |\langle -b \rangle| \leq |\langle b + c \rangle|$. A straightforward

$$(7.37)$$

$$\omega(\Sigma) = \begin{cases} 1/3 & \text{if } \Sigma \text{ is the trivial covering of } \Sigma_{12C}, \\ 7/24 & \text{if } 1+a, 1-a-c, b, b+c \equiv 0 \pmod{N}, N=2, \\ 7/24 & \text{if } 1+a, 1-a-c, b \equiv 0 \pmod{N}, b+c \not\equiv 0 \pmod{N}, \\ 41/144 & \text{if } 1+a, 1-a-c \equiv 0 \pmod{N}, b, b+c \not\equiv 0 \pmod{N}, \\ 13/48 & \text{if } 1+a \equiv 0 \pmod{N}, 1-a-c, b, b+c \not\equiv 0 \pmod{N}, \\ 1/4 & \text{if } 1+a, 1-a-c, b, b+c \not\equiv 0 \pmod{N}. \end{cases}$$

□

has at least one Hurwitz orbit \mathcal{O} isomorphic to the covering $\Sigma_{12C}^{1,0,0}$. Then $x \triangleright (x \triangleright y) = z$ for all $(x, y, z) \in \mathcal{O}$.

9

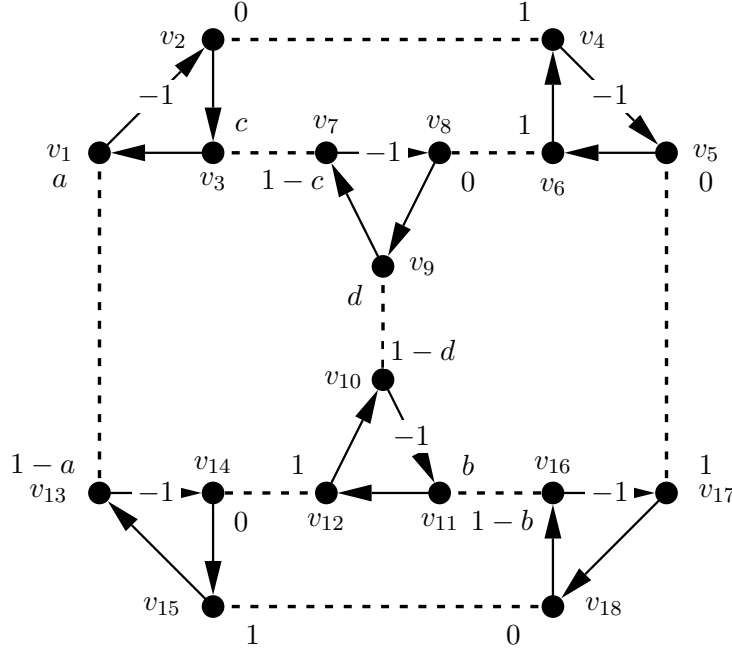


FIGURE 7.17. Schreier graph Σ_{18A}

7.17. The graph Σ_{18A} .

Lemma 7.43. *Let Σ be a covering of Σ_{18A} in Figure 7.17 with simply intersecting cycles. Then*

- $$\begin{aligned} (1) \quad & \langle -a + c + d + 1 \rangle \cap \langle a + 1 \rangle = 0, \\ (2) \quad & \langle -a + c + d + 1 \rangle \cap \langle -c + 1 \rangle = 0, \end{aligned}$$

- (3) $\langle -a + c + d + 1 \rangle \cap \langle -b - d \rangle = 0$,
- (4) $\langle -a + c + d + 1 \rangle \cap \langle b \rangle = 0$,
- (5) $\langle a + 1 \rangle \cap \langle -c + 1 \rangle = 0$,
- (6) $\langle a + 1 \rangle \cap \langle -b - d \rangle = 0$,
- (7) $\langle a + 1 \rangle \cap \langle b \rangle = 0$,
- (8) $\langle -c + 1 \rangle \cap \langle -b - d \rangle = 0$,
- (9) $\langle -b - d \rangle \cap \langle b \rangle = 0$.

Proof. The xy -cycles and their labels are: $(v_1 v_{14} v_{10} v_7)$ with $-a + c + d + 1$, $(v_2 v_5 v_{18} v_{13})$ with $1 + a$, $(v_3 v_8 v_4)$ with $1 - c$, $(v_6 v_9 v_{11} v_{17})$ with $-b - d$ and $(v_{12} v_{15} v_{16})$ with b . The yx -cycles are: $(v_1 v_4 v_{17} v_{15})$ with $1 + a$, $(v_2 v_7 v_6)$ with $1 - c$, $(v_3 v_{13} v_{12} v_9)$ with $-a + c + d + 1$, $(v_5 v_8 v_{10} v_{16})$ with $-b - d$ and $(v_{11} v_{14} v_{18})$ with b . Lemma 4.15 implies the claim. \square

Lemma 7.44. *Let Σ be a covering of Σ_{18A} with simply intersecting cycles. Then $\text{imm}(\Sigma) \leq \omega(\Sigma)$.*

Proof. Let I be a set of representatives for $\mathbb{Z}_N / \langle b \rangle$. We claim that

$$P = v_1[*] \cup v_2[*] \cup v_3[*] \cup v_5[*] \cup v_{11}[I]$$

is a plague. With the sequence of pivots $v_1[*]$, $v_3[*]$, $v_4[*]$, $v_8[*]$, $v_7[*]$, $v_6[*]$, $v_9[*]$ and $v_5[*]$ we obtain that P spreads to

$$v_{13}[*] \cup v_7[*] \cup v_6[*] \cup v_9[*] \cup v_8[*] \cup v_4[*] \cup v_{10}[*] \cup v_{17}[*].$$

Then with the sequence of pivots $v_{16}[I - b]$, $v_{15}[I - b]$, and $v_{12}[I - b]$ we see that P spreads to $v_{18}[I - b] \cup v_{14}[I - b] \cup v_{11}[I - b]$. Hence the claim follows from Example 5.9.

The cycle structure of Σ at the vertices is the following:

$$(7.38) \quad v_1[i], v_{13}[i] : \text{cycles of length } 4|\langle -a + c + d + 1 \rangle| \text{ and } 4|\langle 1 + a \rangle|,$$

$$(7.39) \quad v_2[i], v_4[i] : \text{cycles of length } 4|\langle 1 + a \rangle| \text{ and } 3|\langle 1 - c \rangle|,$$

$$(7.40) \quad v_3[i], v_7[i] : \text{cycles of length } 3|\langle 1 - c \rangle| \text{ and } 4|\langle -a + c + d + 1 \rangle|,$$

$$(7.41) \quad v_5[i], v_{17}[i] : \text{cycles of length } 4|\langle 1 + a \rangle| \text{ and } 4|\langle -b - d \rangle|,$$

$$(7.42) \quad v_6[i], v_8[i] : \text{cycles of length } 4|\langle -b - d \rangle| \text{ and } 3|\langle 1 - c \rangle|,$$

$$(7.43) \quad v_9[i], v_{10}[i] : \text{cycles of length } 4|\langle -b - d \rangle| \text{ and } 4|\langle -a + c + d + 1 \rangle|,$$

$$(7.44) \quad v_{11}[i], v_{16}[i] : \text{cycles of length } 4|\langle -b - d \rangle| \text{ and } 3|\langle b \rangle|,$$

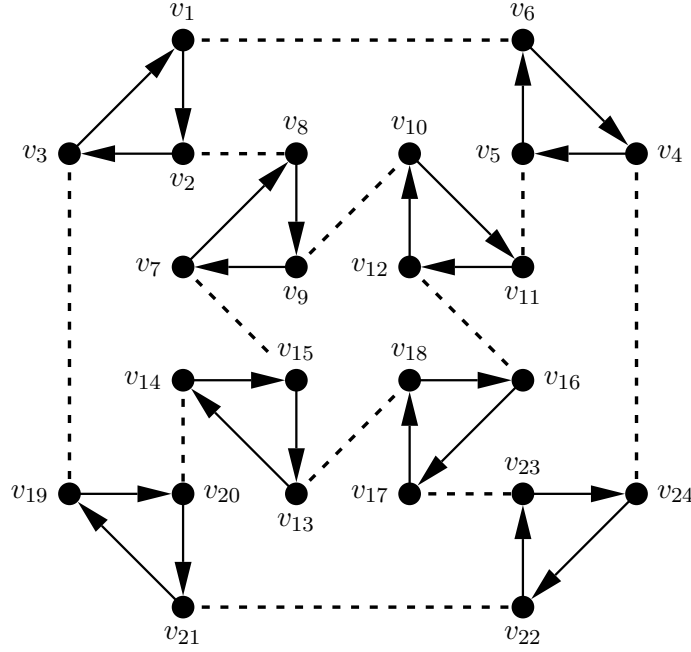
$$(7.45) \quad v_{12}[i], v_{14}[i] : \text{cycles of length } 4|\langle -a + c + d + 1 \rangle| \text{ and } 3|\langle b \rangle|,$$

$$(7.46) \quad v_{15}[i], v_{18}[i] : \text{cycles of length } 4|\langle 1 + a \rangle| \text{ and } 3|\langle b \rangle|,$$

for all $i \in \mathbb{Z}_N$. If $b \equiv 0 \pmod{N}$ and $c \equiv 1 \pmod{N}$ then $\text{imm}(\Sigma) \leq 5/18 = \omega(\Sigma)$. Otherwise, using a graph isomorphism, we may assume that $b \not\equiv 0 \pmod{N}$. Then $\text{imm}(\Sigma) \leq (4N + N/2)/18N = 1/4 \leq \omega(\Sigma)$. \square

7.18. The graph Σ_{24A} .

Lemma 7.45. *Let Σ be a covering of Σ_{24A} with simply intersecting cycles. Then $\text{imm}(\Sigma) \leq 1/4 = \omega(\Sigma)$.*

FIGURE 7.18. Schreier graph Σ_{24A}

Proof. First we prove that $\text{imm}(\Sigma) \leq 1/4$. A straightforward computation shows that $v_1[*] \cup v_2[*] \cup v_3[*] \cup v_4[*] \cup v_7[*] \cup v_{13}[*]$ is a plague. To prove that $\omega(\Sigma) = 1/4$ observe that in every covering all cycles have length ≥ 4 . \square

7.19. The proof of Theorem 6.3. By Proposition 4.3, the Schreier graphs to consider are those in Figures 7.1–7.18. In this section we proved that only the Schreier graphs Σ_{1A} , Σ_{3A} , Σ_{4A} , Σ_{6A} , Σ_{6D} , Σ_{7A} , Σ_{8A} , Σ_{9A} , Σ_{12A} , Σ_{12B} , Σ_{12C} , Σ_{18A} , and Σ_{24A} have coverings with simply intersecting cycles. For each of these coverings we determined an upper bound for the immunity and then we proved that this upper bound can be bounded from above by ω . The corresponding claims are Lemmas 7.2, 7.6, 7.11, 7.17, 7.26, 7.28, 7.29, 7.32, 7.33, 7.36, 7.41, 7.44, and 7.45.

8. NICHOLS ALGEBRAS WITH MANY CUBIC RELATIONS

We refer to [2] for an introduction to Nichols algebras and Yetter-Drinfeld modules. Some elementary facts can also be found in Section 1. Let \mathbb{k} be a field, G be a group, and V be an absolutely irreducible finite-dimensional Yetter-Drinfeld module over the group algebra $\mathbb{k}G$. Recall that V decomposes as $V = \bigoplus_{x \in G} V_x$, where $V_x = \{v \in V \mid \delta(v) = x \otimes v\}$ and $\delta : V \rightarrow \mathbb{k}G \otimes V$ is the left coaction of $\mathbb{k}G$ on V . The *support* of V is the set

$$\text{supp}(V) = \{x \in G \mid V_x \neq 0\}.$$

It is well-known that (V, c) is a braided vector space, where $c \in \text{Aut}(V \otimes V)$ is defined by

$$c(u \otimes v) = gv \otimes u \quad \text{for all } u \in V_g, g \in \text{supp } V, v \in V,$$

and c satisfies the braid relation on $V \otimes V \otimes V$.

Definition 8.1. *Let G, H be groups. We say that two Yetter-Drinfeld modules $V \in {}_{\mathbb{k}G}^G\mathcal{YD}$, $W \in {}_{\mathbb{k}H}^H\mathcal{YD}$ are bg-equivalent if there exists a bijection $\varphi : \text{supp } V \rightarrow \text{supp } W$ and a linear isomorphism $\psi : V \rightarrow W$ such that*

$$\psi(V_g) = W_{\varphi(g)}, \quad \psi(gv) = \varphi(g)\psi(v)$$

for all $g, x \in \text{supp } V$, $v \in V$. The pair (ψ, φ) is then called a bg-equivalence between V and W . We also say that the Nichols algebras $\mathfrak{B}(V)$ and $\mathfrak{B}(W)$ are bg-equivalent.

The reason for b and g in the definition is the following.

Lemma 8.2. *Let G, H be groups and let $V \in {}_{\mathbb{k}G}^G\mathcal{YD}$, $W \in {}_{\mathbb{k}H}^H\mathcal{YD}$. If (ψ, φ) is a bg-equivalence between V and W , then $\psi : V \rightarrow W$ is an isomorphism of braided vector spaces, which maps the G -homogeneous components of V to the H -homogeneous components of W .*

Proof. Let $v \in V$, $g \in G$, and $u \in V_g$. Then $\psi(u) \in W_{\varphi(g)}$ and hence

$$\begin{aligned} (\psi \otimes \psi)(c(u \otimes v)) &= \psi(gv) \otimes \psi(u) \\ &= \varphi(g)\psi(v) \otimes \psi(u) \\ &= c(\psi(u) \otimes \psi(v)). \end{aligned}$$

Thus ψ is an isomorphism of braided vector spaces. The rest is clear from the assumptions on ψ . \square

In particular, Nichols algebras of bg-equivalent Yetter-Drinfeld modules are isomorphic as algebras and coalgebras.

Recall from [6, Section 2.1] that a Nichols algebra $\mathfrak{B}(V)$ has *many cubic relations* if

$$(8.1) \quad \dim \ker(1 + c_{12} + c_{12}c_{23}) \geq \frac{1}{3} \dim V ((\dim V)^2 - 1).$$

Let $X = \text{supp}(V)$, $x \in X$ and $e = \dim V_x$. Assume that X is an indecomposable rack of size $d > 1$. Since $V = \bigoplus_{x \in X} V_x$, we conclude that $V^{\otimes 3} = \bigoplus_{\mathcal{O}} V_{\mathcal{O}}^{\otimes 3}$, where the direct sum is taken over all Hurwitz orbits and

$$V_{\mathcal{O}}^{\otimes 3} = \bigoplus_{(x,y,z) \in \mathcal{O}} V_x \otimes V_y \otimes V_z.$$

Each $V_{\mathcal{O}}^{\otimes 3}$ is invariant under the map $1 + c_{12} + c_{12}c_{23}$.

Remark 8.3. By Theorem 6.3, the inequality $\text{imm}(\mathcal{O}) \leq \omega(\mathcal{O})$ holds for $\mathcal{O} \subseteq X^3$ if the quotient $\overline{\mathcal{O}}$ does not have xy -cycles of length ≥ 5 .

Lemma 8.4. *Assume that $\mathfrak{B}(V)$ has many cubic relations. Then the following hold.*

- (1) $\sum_{\mathcal{O}} \text{imm}(\mathcal{O}) \dim V_{\mathcal{O}}^{\otimes 3} \geq \frac{de}{3}((de)^2 - 1)$, where the sum is taken over all Hurwitz orbits of X^3 .
 (2) Assume that $\text{imm}(\mathcal{O}) \leq \omega(\mathcal{O})$ for all $\mathcal{O} \subseteq X^3$. Then

$$\sum_{\mathcal{O}} \omega(\mathcal{O}) \dim V_{\mathcal{O}}^{\otimes 3} \geq \frac{de}{3}((de)^2 - 1).$$

Proof. (1) follows from [6, Proposition 6] and (8.1). (2) follows from (1). \square

Lemma 8.5. Assume that $\mathfrak{B}(V)$ has many cubic relations. Also assume that $\text{imm}(\mathcal{O}) \leq \omega(\mathcal{O})$ for all $\mathcal{O} \subseteq X^3$. Then

$$\sum_{p,q \geq 1} k_p k_q (3\omega'_{p,q} - 1) + \left(\frac{1}{10} + \frac{1}{8}\right) k_3 + \frac{1}{4} k_4 \geq -1.$$

Further, if the covering $\Sigma_{6A}^{4;2,2}$ appears in X^3 then $k_4 \geq 4$, otherwise

$$\sum_{p,q \geq 1} k_p k_q (3\omega'_{p,q} - 1) + \left(\frac{1}{10} + \frac{1}{8}\right) k_3 \geq -1.$$

Proof. The claim on k_4 holds by Lemma 7.19(1). We prove the first inequality of the lemma, the second is analogous. Since $\dim V_{\mathcal{O}}^{\otimes 3} = |\mathcal{O}|e^3$ and $(de)^2 - 1 \geq (de)^2 - e^2$, we may assume that $e = 1$. By Lemma 8.4(2),

$$(8.2) \quad \sum_{v \in X^3} \omega(v) = \sum_{\mathcal{O}} \omega(\mathcal{O}) |\mathcal{O}| \geq d(d^2 - 1)/3.$$

By Lemmas 7.13, 7.19 and 7.42,

$$\begin{aligned} \left| \left\{ v \in X^3 : v \in \Sigma_{4A}^{5;3,2}, v \in v_1[*] \right\} \right| &\leq dk_3, \\ \left| \left\{ v \in X^3 : v \in \Sigma_{6A}^{4;2,2}, v \in v_3[*] \right\} \right| &\leq dk_4, \\ \left| \left\{ v \in X^3 : v \in \Sigma_{12C}^{1;0,0,0} \right\} \right| &\leq dk_3. \end{aligned}$$

By the definition of ω in (6.2), (8.2) and the last three inequalities imply that

$$\sum_{p,q \geq 1} k_p k_q \omega'_{p,q} d + \frac{dk_3}{30} + \frac{dk_4}{12} + \frac{dk_3}{24} \geq \frac{d(d^2 - 1)}{3}.$$

Since $\sum k_p k_q = d^2$, the lemma follows. \square

Corollary 8.6. Assume that $\mathfrak{B}(V)$ has many cubic relations. Also assume that $\text{imm}(\mathcal{O}) \leq \omega(\mathcal{O})$ for all $\mathcal{O} \subseteq X^3$. Then

$$(8.3) \quad (k_3 + k'_3 - 5)^2 + \frac{11}{5} k_3 + k_3'^2 \leq 49.$$

Further, if the covering $\Sigma_{6A}^{4;2,2}$ appears in X^3 then $k_4 \geq 4$, otherwise

$$(8.4) \quad (k_3 + k'_3 - 4)^2 + \frac{1}{5} k_3 + k_3'^2 \leq 40.$$

Proof. The claim follows from Lemma 8.5 and the definition of $\omega'_{p,q}$. Note that $k_4 \leq k'_3$. \square

Remark 8.7. Recall that $k_n \neq 1$ for all $n \geq 3$. In [6], the case where X is braided was completely classified. Thus we may assume that X is not braided. Hence $k'_3 \geq 2$. Corollary 8.6 implies that $k'_3 \leq 6$. Further, if $k'_3 = 2$ then $k_4 \leq 2$ and $0 \leq k_3 \leq 7$. If $k'_3 = 3$ then $0 \leq k_3 \leq 6$. If $k'_3 = 4$ then $0 \leq k_3 \leq 5$. If $k'_3 = 5$ then $0 \leq k_3 \leq 3$. Finally, if $k'_3 = 6$ then $k_3 = 0$. In particular, $k_3 + k'_3 \leq 9$. If the covering $\Sigma_{6A}^{4;2,2}$ does not appear, then we get the same solutions except $(k_3, k'_3) = (5, 4)$ and $(3, 5)$.

Lemma 8.8. *Assume that $\mathfrak{B}(V)$ has many cubic relations. Also assume that $\text{imm}(\mathcal{O}) \leq \omega(\mathcal{O})$ for all $\mathcal{O} \subseteq X^3$. Then $|X| \leq 33$.*

Proof. By Remark 8.7, we may assume that φ_x with $x \in X$ moves at most 9 elements of X . There are 29 possible profiles for φ_x if we ignore the number of fixed points. Using Proposition 3.2 we can exclude 11 profiles and are left with the list $1^a 2, 1^a 3, 1^a 2^2, 1^a 4, 1^a 5, 1^a 2^3, 1^a 24, 1^a 3^2, 1^a 6, 1^a 7, 1^a 2^4, 1^a 2^2 4, 1^a 26, 1^a 4^2, 1^a 8, 1^a 3^3, 1^a 36, 1^a 9$, where $a \geq 1$ is arbitrary. Theorem 2.7 tells that

$$|X| \leq \left(\sum_{j \geq 2} a_j \right) (k'_2 - 2) + k'_2 + 1.$$

Since $k'_2 \leq 9$ is the number of points moved by φ_x , $x \in X$, we conclude that $|X| \leq 31$ if $\sum_{j \geq 2} a_j \leq 3$. The only profile not satisfying the latter inequality is $1^a 2^4$. For this we get $|X| \leq 33$. This yields the claim. \square

Lemma 8.9. *Assume that $\mathfrak{B}(V)$ has many cubic relations, and the rack X satisfies $k'_3(X) \neq 0$. Also assume that $\text{imm}(\mathcal{O}) \leq \omega(\mathcal{O})$ for all $\mathcal{O} \subseteq X^3$. Then X is isomorphic to $\mathbb{D}_5, \mathbb{D}_7, \text{Aff}(5, 2), \text{Aff}(5, 3), \text{Aff}(7, 2)$ or $\text{Aff}(7, 4)$.*

Remark 8.10. In fact, in Lemma 8.9 the rack X has to be isomorphic to $\text{Aff}(5, 2)$ or $\text{Aff}(5, 3)$. The proof of this fact is more complicated and it is not needed here.

Proof. By Lemma 8.8, $|X| \leq 33$. Since indecomposable quandles of size ≤ 35 were classified in [10], a straightforward computer calculation gives a complete list of indecomposable non-braided injective quandles satisfying (8.3). We obtain the list given in the lemma. The following table contains some properties of these racks.

Quandle	Sizes of orbits	Remark
\mathbb{D}_5	1, 24	$k_7 = 6$
$\text{Aff}(5, 2)$ and $\text{Aff}(5, 3)$	1, 24	
\mathbb{D}_7	1, 48	
$\text{Aff}(7, 2)$ and $\text{Aff}(7, 4)$	1, 42, 49	

This completes the proof. \square

Rack	Rank	Dimension	Hilbert series	Remark
$\{1\}$	1	$n \geq 2$	$(n)_t$	
$(12)^{\mathbb{S}_3}$	3	12	$(2)_t^2(3)_t$	
$(12)^{\mathbb{S}_3}$	3	432	$(3)_t(4)_t(6)_t(6)_{t^2}$	$\text{char } \mathbb{k} = 2$
$(123)^{\mathbb{A}_3}$	4	36	$(2)_t^2(3)_t^2$	$\text{char } \mathbb{k} = 2$
$(123)^{\mathbb{A}_4}$	4	72	$(2)_t^2(3)_t(6)_t$	$\text{char } \mathbb{k} \neq 2$
$(123)^{\mathbb{A}_4}$	4	5184	$(6)_t^4(2)_{t^2}^2$	
$(12)^{\mathbb{S}_4}$	6	576	$(2)_t^2(3)_t^2(4)_t^2$	
$(12)^{\mathbb{S}_4}$	6	576	$(2)_t^2(3)_t^2(4)_t^2$	
$(1234)^{\mathbb{S}_4}$	6	576	$(2)_t^2(3)_t^2(4)_t^2$	
$\text{Aff}(5, 2)$	5	1280	$(4)_t^4(5)_t$	
$\text{Aff}(5, 3)$	5	1280	$(4)_t^4(5)_t$	
$\text{Aff}(7, 3)$	7	326592	$(6)_t^6(7)_t$	
$\text{Aff}(7, 5)$	7	326592	$(6)_t^6(7)_t$	
$(12)^{\mathbb{S}_5}$	10	8294400	$(4)_t^4(5)_t^2(6)_t^4$	
$(12)^{\mathbb{S}_5}$	10	8294400	$(4)_t^4(5)_t^2(6)_t^4$	

TABLE 9.1. Finite-dimensional elementary Nichols algebras

9. COMPUTATION OF THE 2-COCYCLES AND THE MAIN THEOREM

Let X be one of the racks $\text{Aff}(5, 2)$ or $\text{Aff}(5, 3)$. Recall that G_X is the enveloping group of X . For all $i \in X$ let x_i be the image of i under the canonical map $X \rightarrow G_X$.

Lemma 9.1. [5, Lemma 5.4] *The centralizer of x_1 in G_X is the cyclic group generated by x_1 .*

Let ρ be an absolutely irreducible representation of $C_{G_X}(x_1)$. By straightforward but lengthy calculations we obtain the following.

Lemma 9.2. *Let $\mathcal{O} \subseteq X^3$ be a Hurwitz orbit of size 24. Then*

$$\dim \ker(1 + c_{12} + c_{12}c_{23})|_{V^{\otimes 3}} \leq \begin{cases} 8 & \text{if } \rho(x_1) = -1, \\ 5 & \text{otherwise.} \end{cases}$$

Theorem 9.3. *Let G be a group, V be a finite-dimensional absolutely irreducible Yetter-Drinfeld module over G , and $X = \text{supp } V$. Assume that X is an indecomposable rack such that no quotient of a Hurwitz orbit in X^3 contains an xy -cycle of length ≥ 5 . Also assume that $\mathfrak{B}(V)$ has many cubic relations.*

(1) *The rack X is isomorphic to one of the following:*

$$\begin{aligned} & (12)^{\mathbb{S}_n} \text{ for } n \in \{3, 4, 5\}, \\ & (1234)^{\mathbb{S}_4}, (123)^{\mathbb{A}_4}, \\ & \text{Aff}(p, \alpha) \text{ for } (p, \alpha) \in \{(5, 2), (5, 3), (7, 3), (7, 5)\}. \end{aligned}$$

(2) The Nichols algebra $\mathfrak{B}(V)$ is *bg-equivalent* to one of the Nichols algebras listed in Table 9.1.

(3) The Hilbert series of $\mathfrak{B}(V)$ is *t-integral of depth two*.

Remark 9.4. Table 9.1 does not contain the information on the 2-cocycle of X corresponding to V . For the racks $\text{Aff}(5, 2)$ and $\text{Aff}(5, 3)$ we use the constant 2-cocycle -1 , see the proof of Theorem 9.3. For the others, see [6, Appendix A].

Proof. In [6, Theorem 11] the theorem was proved under the additional assumption that X is braided. One obtains the same racks except $\text{Aff}(5, 2)$ and $\text{Aff}(5, 3)$. Assume now that X is not braided, i.e., $k'_3(X) \neq 0$. By Lemma 8.9, X is one of the racks $\text{Aff}(5, 2)$, $\text{Aff}(5, 3)$, \mathbb{D}_5 , \mathbb{D}_7 , $\text{Aff}(7, 2)$ and $\text{Aff}(7, 4)$. The racks \mathbb{D}_5 , \mathbb{D}_7 , $\text{Aff}(7, 2)$ and $\text{Aff}(7, 4)$ can be excluded since they have xy -cycles of length ≥ 5 in some Hurwitz orbit quotients.

Let now $X = \text{Aff}(5, 2)$ or $X = \text{Aff}(5, 3)$. Then X^3 decomposes into five Hurwitz orbits of size one and five of size 24. Since $\mathfrak{B}(V)$ has many cubic relations, (8.1) and Lemma 9.2 imply that $\rho(x_1) = -1$. Moreover, in this case (8.1) holds and the Nichols algebras of these V are known to be finite-dimensional, see e.g. [5, Proposition 5.15]. The claim on the Hilbert series is known for all Nichols algebras in Table 9.1. \square

Remark 9.5. In Theorem 9.3 the assumption about the length of an xy -cycle in the quotient of a Hurwitz orbit in X^3 can be replaced by the following weaker assumption, see Theorem 6.3:

$$\text{imm}(\mathcal{O}) \leq \omega(\mathcal{O})$$

for all Hurwitz orbits \mathcal{O} of X^3 . To prove this, we exclude the racks \mathbb{D}_5 , \mathbb{D}_7 , $\text{Aff}(7, 2)$ and $\text{Aff}(7, 4)$. Straightforward calculations yield the information listed in Table 9.2. Hence the inequality of Lemma 8.4(1) cannot be satisfied for the racks \mathbb{D}_7 , $\text{Aff}(7, 2)$ and $\text{Aff}(7, 4)$. The rack \mathbb{D}_5 cannot be excluded with this argument. However, it can be excluded by computing

$$\dim \ker(1 + c_{12} + c_{12}c_{23})|_{V_{\mathcal{O}}^{\otimes 3}},$$

where \mathcal{O} is an orbit of size 24. This is essentially what we did for the racks $\text{Aff}(5, 2)$ and $\text{Aff}(5, 3)$.

We conclude the paper with the following conjecture.

Conjecture 9.6. *The inequality $\text{imm}(\Sigma) \leq \omega(\Sigma)$ holds for all \mathbb{B}_3 -spaces Σ with simply intersecting cycles.*

According to Remark 9.5, Conjecture 9.6 implies Conjecture 1.1 formulated in the introduction.

Rack	Size of orbits	Number	Bound for immunity
\mathbb{D}_5	1	5	1
	24	5	7/24
\mathbb{D}_7	1	7	1
	48	7	14/48
$\text{Aff}(7, \alpha)$ $\alpha = 2, 4$	1	1	1
	42	1	9/42
	49	6	15/49

TABLE 9.2. Some small racks

Acknowledgement. We are grateful to the referee for pointing out some typos and for asking to clarify some proofs. István Heckenberger was supported by German Research Foundation via a Heisenberg professorship. Leandro Vendramin was supported by Conicet and the Alexander von Humboldt Foundation. We used GAP (<http://www.gap-system.org/>) and Rig (<http://code.google.com/p/rig/>) for some computations.

REFERENCES

- [1] N. Andruskiewitsch and M. Graña. From racks to pointed Hopf algebras. *Adv. Math.*, 178(2):177–243, 2003.
- [2] N. Andruskiewitsch and H.-J. Schneider. Pointed Hopf algebras. In *New directions in Hopf algebras*, volume 43 of *Math. Sci. Res. Inst. Publ.*, pages 1–68. Cambridge Univ. Press, Cambridge, 2002.
- [3] E. Brieskorn. Automorphic sets and braids and singularities. In *Braids (Santa Cruz, CA, 1986)*, volume 78 of *Contemp. Math.*, pages 45–115. Amer. Math. Soc., Providence, RI, 1988.
- [4] T. Ceccherini-Silberstein and M. Coornaert. *Cellular automata and groups*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2010.
- [5] M. Graña, I. Heckenberger, and L. Vendramin. Nichols algebras of group type with many quadratic relations. *Adv. Math.*, 227(5):1956–1989, 2011.
- [6] I. Heckenberger, A. Lochmann, and L. Vendramin. Braided racks, Hurwitz actions and Nichols algebras with many cubic relations. *Transform. Groups*, 17(1):157–194, 2012.
- [7] A. Hurwitz. Ueber Riemann’sche Flächen mit gegebenen Verzweigungspunkten. *Math. Ann.*, 39(1):1–60, 1891.
- [8] C. Kassel and V. Turaev. *Braid groups*, volume 247 of *Graduate Texts in Mathematics*. Springer, New York, 2008. With the graphical assistance of Olivier Dodane.
- [9] R. A. Rankin. *Modular forms and functions*. Cambridge University Press, Cambridge, 1977.
- [10] L. Vendramin. On the classification of quandles of low order. *J. Knot Theory Ramifications*, 21(9):1250088, 10, 2012.

PHILIPPS-UNIVERSITÄT MARBURG, FB MATHEMATIK UND INFORMATIK, HANS-MEERWEIN-STRASSE, 35032 MARBURG, GERMANY

E-mail address: heckenberger@mathematik.uni-marburg.de

E-mail address: lochmann@mathematik.uni-marburg.de

E-mail address: lvendramin@dm.uba.ar