An analysis of a regular black hole interior

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We analyze the thermodynamical properties of the regular static and spherically symmetric black hole model presented by Mboyne and Kazanas. Equations for the thermodynamical quantities valid for an arbitrary density profile are deduced, and from them we show that the model is thermodynamically unstable. Evidence is also presented pointing to its dynamical instability. The gravitational entropy of this solution based on the Weyl curvature conjecture is calculated, following the recipe given by Rudjord, Gr∅n and Sigbj∅rn, and it is shown to have the expected behaviour for a good description of the gravitational entropy.

I. INTRODUCTION

The existence of singularities in some solutions of General Relativity is one of the most important unresolved issues in our description of the gravitational field. In fact, singularities are an undesirable feature of any theory of gravitation: they can be naturally considered as a source of lawlessness (see for instance [1]), because the space-time description breaks down "there", and physical laws presuppose space-time. There are several ways in which singularities can be avoided (see [2] for the case of Cosmology), both at the classical and at the quantum level. Since there is no widely-accepted theory of quantum gravity, the issue of the singularity has been addressed from classical considerations in a number of ways, one of which is the limiting curvature hypothesis, based on the fact that, at a singularity, invariants constructed with the Riemann tensor generally diverge. Hence, a suitable way to eliminate singularities would be to impose a limiting principle on the curvature, associated to a fundamental scale (which may be the Planck scale). In practice, this hypothesis (first proppposed in [3]) is implemented by imposing that any solution of the field equations reduce to a definite nonsingular solution (for instance de Sitter space-time) when some of the invariants attain their limiting values, in such a way that SEC (one of the hypothesis of the singularity theorems [4–6]) is violated.

A theory based on this hypothesis has been implemented by modificating the dynamics of General Relativity by the addition of nonlinear terms in [7]. However, there is another way, suggested by Gliner [8], in which de Sitter geometry can be attained for

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high curvatures, which consists in assuming that the microphysics of high-density matter is such that a phase transition must occur leading any system under such extreme conditions to the de Sitter geometry. Models of this type were also considered by Sakharov [9], Zeldovich [10], and Bardeen, who suggested that a fluid satisfying the condition $p = -\rho$ could be the final state of gravitational collapse [11]. In fact it was Bardeen who in 1968 obtained the first regular solution of Einstein field equations having an event horizon, in the presence of an electromagnetic field, parametrized by the mass m and the charge e. This space-time metric behaves as the Schwarzschild metric for large radii and, is de Sitter's towards the core of the object.

Another exact solution of the Einstein field equations containing at the core a de Sitter fluid was found by Dymnikova [12]. This solution represents a nonsingular black hole, and the stress-energy tensor which is the source of the geometry joins the central de Sitter region to the standard vacuum at infinity, with an anisotropic vacuum layer in between the two. This unsatisfactory anisotropic vacuum layer was replaced by Mboyne and Kazanas (MK) [13] with a region filled with matter under both radial and tangential pressures, described by an equation of state (EOS) that relates the radial pressure to the density and smoothly reproduces vacuumlike behavior near $r = 0$, tending to a polytrope at larger r, low ρ values [1].

The main goal of the present work is to study the MK model for a regular black hole in which the singularity is avoided precisely by the second method mentioned above. In particular, a detailed study of the thermodynamical aspects of the matter which is the source of the geometry and also of the gravitational field will be given [2]. It will be shown that the solution is thermodinamically unstable, and we shall exhibit evidence supporting the dynamical instability of the model. The issue of the gravitational entropy of this model will be also addressed, following the ideas presented in [16].

We shall begin in Sect.II with a short review of the regular black hole solution advanced in [13]. Its thermodynamical properties will be discussed in Sect.III, and the stability will be scrutinized in Sect.IV. In Sect.V, the gravitational entropy of the MK solution will be calculated. These results will be analyzed in Sect.VI.

[1] Different types of regular black hole solutions were reviewed recently in [14], where regular charged black holes, with the exterior described by the Reissner-Nordstrom solution were discussed as well.

[2] Other features of the model were studied in [15].

II. A MODEL OF A REGULAR BLACK HOLE

The model introduced in [13] is a regular static black hole, with a matter source that smoothly goes from a de Sitter behaviour near the origin to Schwarszchild's spacetime outside the object. A spacetime-metric well-adapted to examine the properties of this system is [15]:

$$ds^2 = -B(r)dt^2 + \left(1 - \frac{2m(r)}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{1}$$

The EOS proposed by Mbonye and Kazanas [13] is given by:

$$p_{\mathsf{r}}(\rho) = \left[\alpha - (\alpha+1)\left(\frac{\rho}{\rho_{\mathsf{max}}}\right)^2\right]\left(\frac{\rho}{\rho_{\mathsf{max}}}\right)\rho, \tag{2}$$

where p_r is the radial pressure (to be distinguished from the tangential pressure, which is needed in these models to satisfy the Tolman-Oppenheimer-Volkoff equation, [17]), and $\alpha = 2.2135$ [3]. Such an equation describes the behaviour of matter that changes smoothly from a normal behaviour to a core of an "exotic fluid" with an EOS that approaches $p_r = -\rho$ when $r \to 0$. The plot of Eq. (2) is shown in Fig 1.

In order to solve Einstein's equations, MK chose the density profile suggested by Dymnikova [12]:

$$\rho(r) = \rho_{\mathsf{max}} e^{-8\frac{r^3}{R^3}}, \tag{3}$$

where $R^3 = 8R_{\mathrm{S}} r_0^2$, R_S is the Schwarszchild radius,

$$r_0 = \left(\frac{3}{8\pi\rho_{\mathsf{max}}}\right)^{1/2}, \tag{4}$$

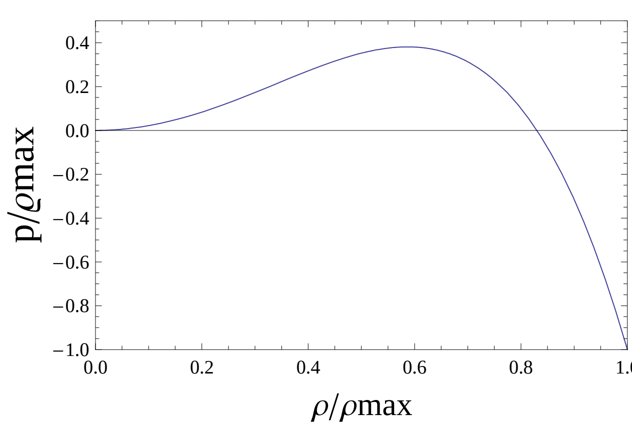

Figure 1. Plot of Eq. (1), which gives the relation between the radial pressure and the density.

[3] This value of α is chosen to yield a sound speed bounded by the speed of light [13].

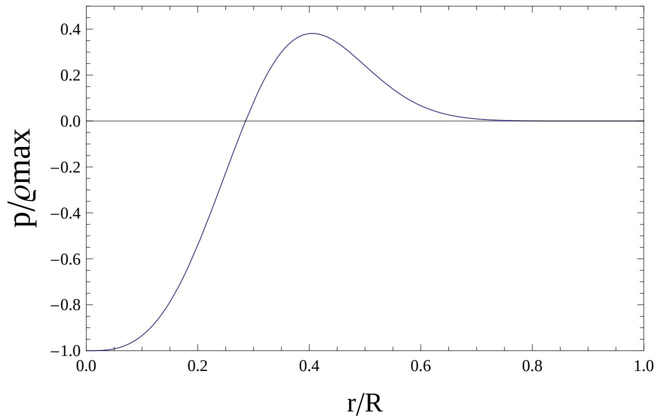

Figure 2. Pressure as a function of radial coordinate inside the regular black hole considered in the text.

and $\rho_{\max}$ is of the order of the Planck density. This density undergoes a smooth transition from the de Sitter state at the center to the vacuum state at infinity, with an intermediate region of non-inflationary material, a situation that was anticipated in [18]. It follows from Eq. (3) that all the mass is contained inside a sphere of radius R within the black hole, which corresponds to the surface of the object.

The plot of the radial pressure as a function of the radial coordinate is shown in Fig. 2. It can be seen that the pressure follows the equation $p = -\rho$ at the core of the object, and it goes to zero at $r/R = 1$, the surface of the matter region. The radial pressure has another zero located at $r/R = 0.28$, two inflexion points at $r/R = 0.26$ and $r/R = 0.5$, and an absolute maximun at $r/R = 0.4$. The exotic matter (for which the pressure decreases with decreasing r) occupies the region $r/R < 0.4$ [4].

The solution of the Einstein field equations $G^\mu_\nu = -8\pi T^\mu_\nu$, with $T^\mu_\nu = \mathrm{diag}(\rho(r), p_r(r), p_\perp(r), p_\perp(r))$, for the metric given in Eq. (1) and the EOS (2) takes the form [15]:

$$B(r) = \exp \int_{r_0}^{r} \frac{2}{r'^2}\left[m(r') + 4\pi r'^3 p_{r'}\right] \times \left[\frac{1}{\left(1 - \frac{2m(r')}{r'}\right)}\right] dr', \tag{5}$$

where:

$$m(r) = 4\pi \int_0^r \rho(r') r'^2 dr', \tag{6}$$

is the ADM mass.

[4] Although this profile resembles the qualitative plot for the radial pressure in a gravastar (see [17]), it must be noted that the MK model has a horizon, which is absent in gravastars.

Equations (5) and (6) describe the geometry of the space-time of the regular black hole introduced in [13]. Since the EOS and the density as a function of r were specified in [13], the tangential pressure in terms of the given quantities follows from the Einstein field equations. Explicitly:

$$p_\perp = p_\mathrm{r} + \frac{r}{2} p'_\mathrm{r} + \frac{1}{2}\left(p_\mathrm{r} + \rho\right)\left[\frac{m(r) + 4\pi r^3 p_\mathrm{r}}{r - 2m(r)}\right]. \tag{7}$$

The behaviour of the tangential pressure with r is given in Fig. 3. It is important to emphasize at this point that the MK model differs from the usual approach (see for instance [19]), in the sense that, while in the former one the EOS and the explicit form of $\rho(r)$ were given, in the latter two relations between the quantities p_r, $p_\perp$ and ρ are advanced. As shown in the Appendix, the choice in the MK model has important consequences in the analysis of the perturbations of the field equations.

III. THERMODYNAMICS OF THE MATTER

Following [13], we shall start by assuming that the body has reached a static equilibrium configuration (i.e inside the black hole, the pull of the gravitational field is balanced by the repulsion exerted by the exotic matter). It will also be assumed that matter is in thermodynamic equilibrium. As shown below, the results obtained under these hypotheses turn out to be incorrect, since they imply that the system is both dynamically and thermodynamically unstable.

The temperature of the matter as a function of the radius can be estimated from the laws of standard thermodynamics:

$$T dS = d(\rho V) + p\, dV. \tag{8}$$

It follows from this equation, along with Eq. (2) and (3)

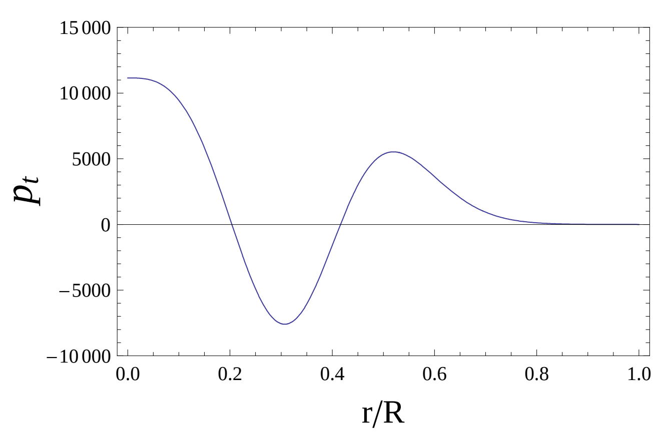

Figure 3. Plot of the tangential pressure as a function of radial coordinate, given by Eq. (7).

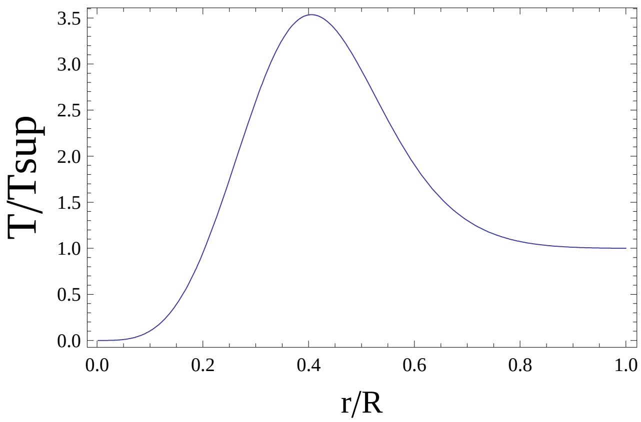

Figure 4. Temperature as a function of radial coordinate inside the black hole, given by Eq. (9). $T_{\rm sup}$ stands for the temperature of the the matter field at $r = R$.

that:

$$\frac{T}{T_{\sf sup}} = \left[1 + \alpha e^{-8r^3/R^3} - (\alpha+1)e^{-24r^3/R^3}\right]^{4/3} e^{\Xi(r)}, \tag{9}$$

where $T_{\rm sup}$ stands for the temperature of the the matter field at $r = R$, and:

$$\Xi(r) = \frac{2}{3}\int_0^{e^{-8r^3/R^3}} \frac{\alpha d(\rho/\rho_{\sf max})}{1+\alpha(\rho/\rho_{\sf max}) - (\alpha+1)(\rho/\rho_{\sf max})^3}. \tag{10}$$

Fig. 4 displays the temperature of the matter as a function of the radius. We note that the temperature tends to absolute zero close to the core. As for the pressure, there is only one maximum at $r/R = 0.4$. The inflexion points of $T(r)$ occur at $r/R = 0.31$ and at $r/R = 0.57$.

An expression for the entropy (up to an additive constant) as a function of the coordinates can be obtained by substituing (9) in (8):

$$S \equiv \frac{(\rho+p)}{T} V. \tag{11}$$

The result is plotted in Fig. 5. The entropy goes to zero as $r \to 0$ in accordance to Nerst's theorem, and it has a maximum close to $r/R = 0.4$, in the region of highest density of normal matter. The entropy density of the matter inside the regular black hole can also be calculated, and is given by:

$$\frac{s}{s_{\sf R/2}} = \frac{(\rho/\rho_{\sf max})[1+\alpha(\rho/\rho_{\sf max}) - (\alpha+1)(\rho/\rho_{\sf max})^3]^{-1/3}}{0.2076 e^{(2/3)\int_0^\rho \alpha d\rho[\rho_{\sf max}+\alpha\rho-(\alpha+1)(\rho^3/\rho^2_{\sf max})]^{-1}}}. \tag{12}$$

The plots of the entropy density as a function of ρ and r are shown in Figs. 6 and 7. It can be seen that the entropy density diverges at the origin as a consequence of the vanishing volume. The entropy density as a function of the density has one inflexion point at $\rho/\rho_{\rm max} = 0.73$ which is located at $r/R = 0.34$.

The radial sound speed as a function of the energy density follows from $v_r^2 = dp/d\rho$ (with $c = 1$), yielding:

$$v_r^2 = 2e^{-8r^3/R^3}\left[\alpha - 2(\alpha+1)e^{-16r^3/R^3}\right]. \tag{13}$$

The result is shown in Fig. 8. From Figs. 2 and 8, we see that the sound speed is zero at the value of r/R for which the pressure is maximum. In addition, the sound speed takes complex values in the region $r/R < 0.4$, in accordance with the negative slope of the equation of state as a function of the density (see Eq. 2 and Fig. 1). The fact that sound waves do not propagate in the region $r/R < 0.4$ is a consequence of the exotic behaviour of the fluid there: a variation in pressure causes an expansion rather than a compression.

To discuss the thermodynamical equilibrium we shall need below the Helmholtz free energy, given by $F = U - TS = -pV$. Explicitly:

$$\frac{F}{F_{0.2R}} = \frac{-1}{0.018}\left[\alpha - (\alpha+1)\left(\frac{\rho}{\rho_{\rm max}}\right)^2\right]\left(\frac{\rho}{\rho_{\rm max}}\right)^2 \frac{4\pi}{3}\left(\frac{r}{R}\right)^3. \tag{14}$$

The plot of the Helmholtz free energy F as a function of the radial coordinate is shown in Fig. 9.

Armed with the functions discussed in this section, we shall discuss in the next the issue of thermodynamical and dynamical equilibrium.

IV. EQUILIBRIUM

IV.1. Thermodynamical equilibrium

A condition for a system to be in stable thermodynamical equilibrium is that, for a given value of entropy and volume, the energy must be minimum. This is equivalent to impose on the the specific heat at constant volume the

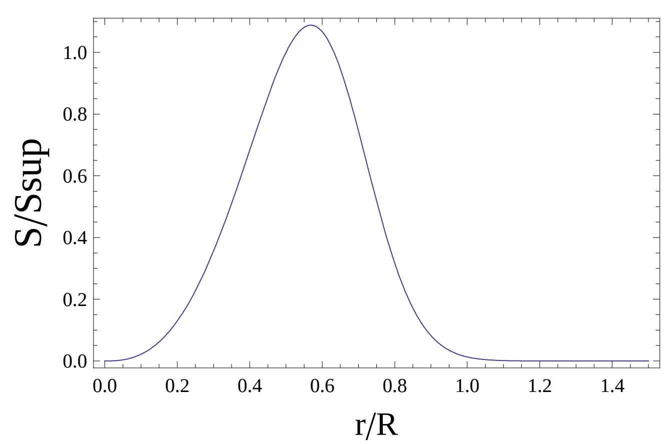

Figure 5. Entropy of the matter field as a function of radial coordinate inside the black hole. $S_{\rm sup}$ stands for the entropy of the the matter field at $r = R$.

condition:

$$C_V > 0. \tag{15}$$

The dependence of C_V with the radial coordinate can be calculated from:

$$C_V = T\left(\frac{dS}{dT}\right)_V. \tag{16}$$

Using the expressions deduced in the previous sections, we obtain:

$$\frac{C_\mathrm{V}}{V} = \frac{1}{2T}\frac{\rho_\mathrm{max}\left(1+\left[\alpha-(\alpha+1)\left(\frac{\rho}{\rho_\mathrm{max}}\right)^2\right]\frac{\rho}{\rho_\mathrm{max}}\right)}{\left[\alpha-2(\alpha+1)\left(\frac{\rho}{\rho_\mathrm{max}}\right)^2\right]} \tag{17}$$

For $r = R$, ρ is zero, and Eq.(17) yields:

$$\frac{C_\mathrm{Vsup}}{V_\mathrm{sup}} = \frac{\rho_\mathrm{max}}{2\alpha T_\mathrm{sup}}. \tag{18}$$

Introducing this expression as a constant of normalization in Eq.(17), it follows that:

$$\frac{C_\mathrm{V}}{C_\mathrm{Vsup}} = \left(\frac{r}{R}\right)^3 \frac{\alpha\left(1+\left[\alpha-(\alpha+1)\left(\frac{\rho}{\rho_\mathrm{max}}\right)^2\right]\frac{\rho}{\rho_\mathrm{max}}\right)}{\left[\alpha-2(\alpha+1)\left(\frac{\rho}{\rho_\mathrm{max}}\right)^2\right]}\frac{T_\mathrm{sup}}{T}. \tag{19}$$

The plot of the specific heat at constant volume is shown in Fig. 10. The discontinuous behaviour of C_V (with the sign change) at $r/R = 0.4$ is typical of a second-order phase transition, suggesting that the system is thermodynamically unstable. Similarly, we can calculate the specific heat at constant pressure, defined by:

$$C_\mathrm{p} = T\left(\frac{dS}{dT}\right)_\mathrm{p}. \tag{20}$$

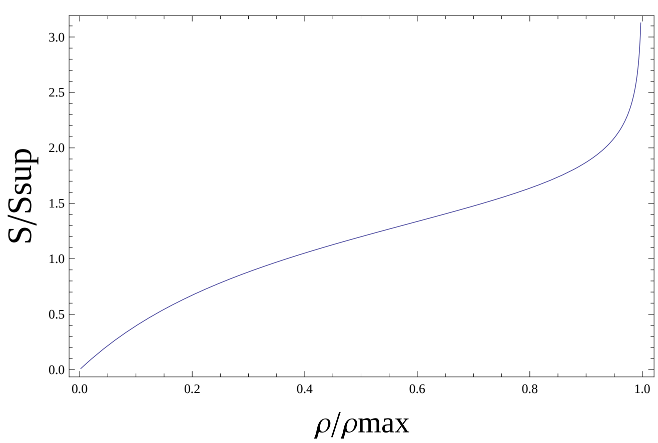

Figure 6. Entropy density of the matter inside the black hole as a function of the density. S_sup stands for the entropy of the matter field at $r = R$.

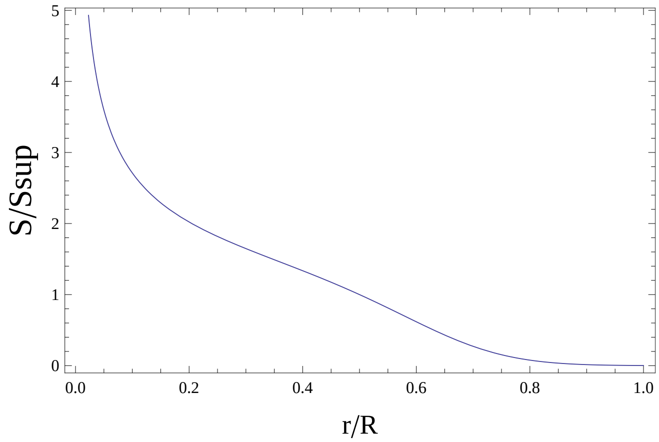

Figure 7. Entropy density of the matter inside the black hole as a function of radial coordinate. S_sup stands for the entropy of the matter field at $r = R$.

The result is:

$$C_\mathrm{p}/C_\mathrm{psup} = \frac{\alpha}{7}\frac{f_1 f_2}{\left(\frac{T}{T_\mathrm{sup}}\right)\left[\alpha-2(\alpha+1)\left(\frac{\rho}{\rho_\mathrm{max}}\right)^2\right]_\mathrm{p}}, \tag{21}$$

where:

$$f_1 = 8\left(\frac{r}{R}\right)^3 - 1 - \frac{\rho}{\rho_\mathrm{max}}\left[\alpha-(\alpha+1)\left(\frac{\rho}{\rho_\mathrm{max}}\right)^2\right], \tag{22}$$

$$f_2 = 1 + \frac{\rho}{\rho_\mathrm{max}}\left[\alpha-(\alpha+1)\left(\frac{\rho}{\rho_\mathrm{max}}\right)^2\right]. \tag{23}$$

The plot of of the specific heat at constant pressure, shown in Fig. 11 indicates, as in the case of C_V, that the system is thermodinamically unstable.

We found that both the specific heat at constant volume and constant pressure are not defined for $r/R = 0.4$, where normal matter changes into exotic matter. The

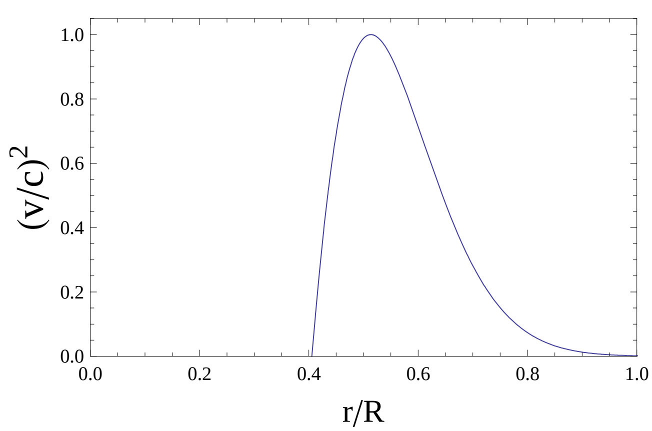

Figure 8. Radial sound speed as a function of radial coordinate.

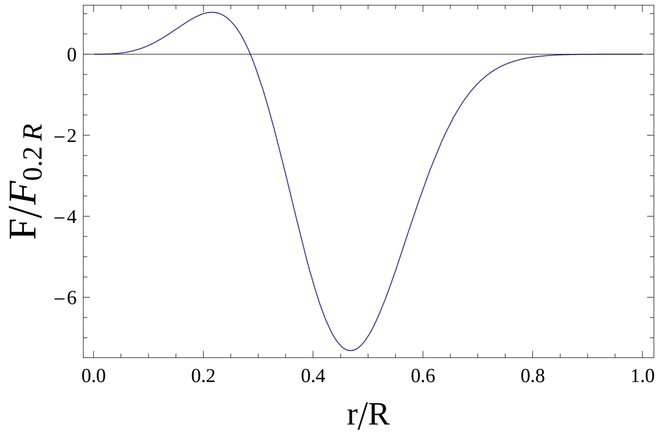

Figure 9. Helmholtz free energy as a function of radial coordinate. $F_{0.2R}$ stands for the Helmholtz free energy at $r/R = 0.2$.

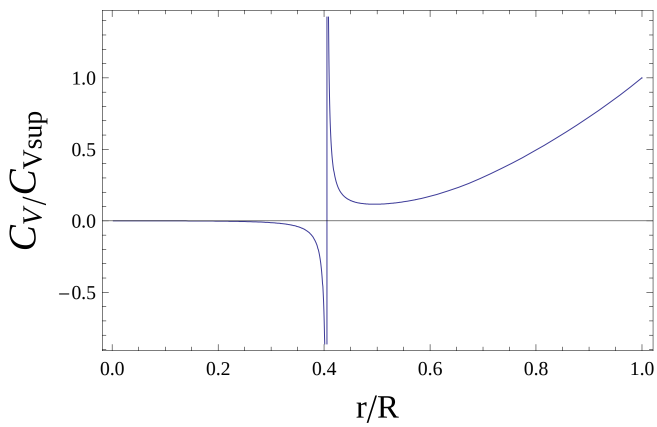

Figure 10. Specific heat at constant volume as a function of radial coordinate in the black hole interior. $C_{V\mathrm{sup}}$ stands for the specific heat at constant volume at $r = R$.

change of sign of C_V at $r/R = 0.4$ can be understood in terms of Eq.(16), the regions where the temperature increases and decreases, and the density profile given in Eq. (3). In Fig. 11 we can see that $C_p = 0$ at $r/R = 0.57$; this point coincides with one of the inflexion points of the temperature. C_p is positive for $r/R < 0.4$ and for $r/R > 0.57$. In the region $0.4 < r/R < 0.57$, C_p is negative. These results show that the specific heats are not defined for the same value of r/R where the sound speed equals to zero, which reinforces the existence of a region of instability for the normal matter field.

We arrive at the same conclusion by the examination of the plot of the the isothermal compressibility κ_T, defined by:

$$\kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T, \tag{24}$$

as a function of r/R. The equilibrium condition is in this case $\kappa_T > 0$.

We can summarize the results of this section by asserting that the discontinuities in the second derivatives of

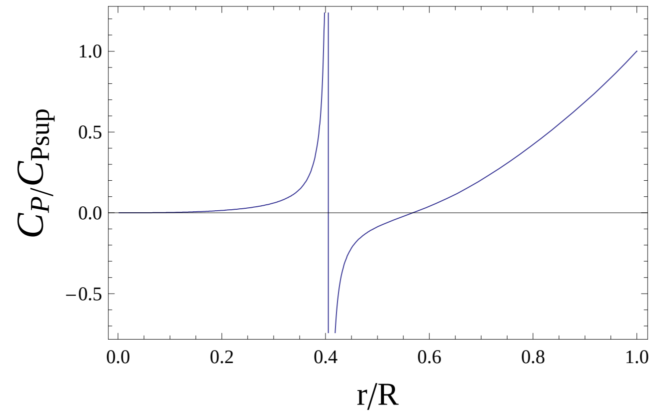

Figure 11. Specific heat at constant pressure as a function of radial coordinate in the black hole interior. $C_{P\mathrm{sup}}$ stands for the specific heat at constant pressure at $r = R$.

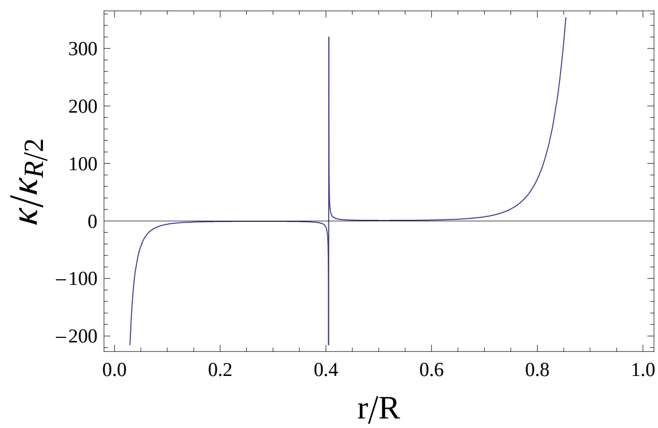

Figure 12. Isothermal compressibility as a function of radial coordinate. $\kappa_{TR/2}$ stands for theothermal compressibility at $r/R = 1/2$.

the state funcions (along with the continuity of the state functions and of their first derivative, as shown in the corresponding plots) indicate that the matter inside the black hole cannot be in thermodynamical equilibrium. This conclusion is reinforced by the plot of the transversal velocity as a function of the r, defined by:

$$v_\perp^2 = \frac{dp_\perp}{d\rho}.$$

The function $v_\perp(r)$ can be calculated from Eqs. (2) and (7). This function is ploted in Fig. 13. The plot evidences not only the instability discussed above, but also a new one, inside the normal matter part of the object.

IV.2. Dynamical equilibrium

To study the dynamical stability of the system, a detailed analysis of the Sturm-Liouville problem associated to the perturbation of the equations of motion for the

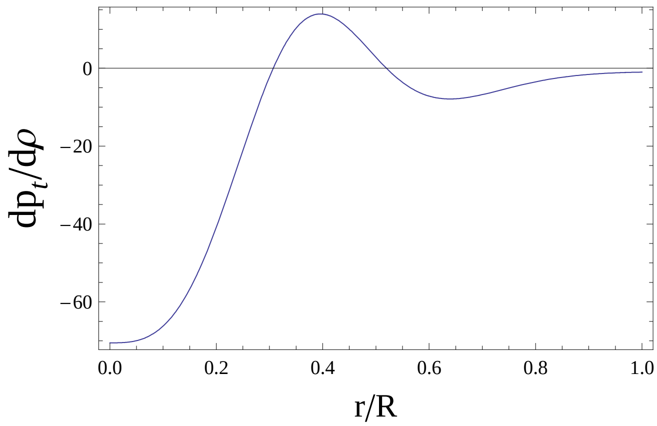

Figure 13. Tangential sound speed as a function the radial coordinate.

fluid and the metric is mandatory. However, for the reasons discussed in the Appendix, we shall content here with some arguments suggesting that the system is dynamically unstable. First, let us recall that in the region of exotic matter there is no propagation of sound waves. The equation for the radial sound speed, in the region of exotic matter, can be put as follows:

$$v_r^2 = -\frac{\Delta p}{\Delta \rho}. \tag{25}$$

From Eq. (25) the pressure as a function of the density takes the form:

$$\Delta p = -v_r^2 \Delta \rho, \tag{26}$$

where v_r^2 represents the square of the radial sound speed and it is always positive. We can see that if the pressure increases, the density decreases; but if the density decreases, the pressure keeps growing. In this process, both pressure and density are related in such a way by Eq. (26) that if the system is perturbed, the fluid never stops expanding. We suggest that the huge accumulation of energy in this process of continuos expansion might lead to divergences that indicate inestability of the system.

V. ENTROPY OF THE GRAVITATIONAL FIELD

Penrose [20] suggested that entropy might be assigned to the gravitational field itself and he proposed that the Weyl curvature tensor can be used to specify it. The behaviour of the Weyl tensor follows what is expected for the gravitational entropy throughout the history of the universe: it is zero in the (homogeneous) Friedmann-Robertson-Walker model and it is large in Schwarzschild's space-time.

Rudjord, Gr∅n and Sigbj∅rn[16] made a recent attempt to develop a classical description of the gravitational entropy based on the construction of a scalar derived from the contraction of the Weyl tensor and the Riemann tensor:

$$P^2 = \frac{C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta}}{R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}} = \frac{W}{K}. \tag{27}$$

This approach is based on matching their description of the entropy of a black hole in the event horizont with the Hawking-Bekenstein entropy[21]. In particular, they calculated the entropy of Schwarzschild black holes and the Schwarzschild-de-Sitter space-time [5].

Rudjord et al. describe the gravitational entropy of a black hole by the surface integral:

$$S = k_{\rm s} \int_\sigma \vec{\Psi} \cdot d\vec{\sigma}, \tag{28}$$

where σ is the horizon of the black hole and the vector field $\vec{\Psi}$ is:

$$\vec{\Psi} = P\vec{e}_r, \tag{29}$$

where $\vec{e}_r$ is a unitary vector in the radial direction. They required Eq. (28) to coincide at the horizon with the Hawking-Bekenstein entropy, thus allowing for the calculation of the constant $k_{\rm s}$[6].

Finally, the entropy density can be determined by means of Gauss's divergence theorem, rewriting Eq. (28) as a volume integral:

$$\mathfrak{s} = k_{\rm s} \mid \nabla \cdot \vec{\Psi} \mid, \tag{30}$$

where the absolute value brackets were added to avoid negative values of entropy.

Let us recall that the Weyl tensor is the traceless part of the Riemann tensor, and is given by:

$$C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} + \frac{2}{n-2}(g_{\alpha[\gamma}R_{\delta]\beta} - g_{\beta[\gamma}R_{\delta]\alpha}) + \tag{31}$$
$$+ \frac{2}{(n-1)(n-2)}R\ g_{\alpha[\gamma}g_{\delta]\beta},$$

where $R_{\alpha\beta\gamma\delta}$ is the Riemann tensor, $R_{\alpha\beta}$ is the Ricci tensor, R is the Ricci scalar, [ ] refers to the antisymmetric part, and n is the number of dimensions of space-time.

The absence of structure in space-time corresponds to a null Weyl conformal curvature ($W = C^{abcd}C_{abcd} = 0$). Hence, the Weyl tensor contains the information of the gravitational field in absence of matter and other non-gravitational fields.

The Weyl scalar for the space-time metric given in Eq. (1) is:

$$W = \frac{4}{3}\pi^2\rho_{\tt max}^2(p_8\, x^8 + p_6\, x^6 + p_5\, x^5 + p_4\, x^4 + p_3\, x^3 + p_2\, x^2 +$$
$$+ p_1\, x + p_0)^2 / \left[\left(3r - \pi R^3\rho_{\rm max} + \pi R^3\rho_{\rm max}\, x\right) r^6 R^6\right], \tag{32}$$

[5] See reference [22] for other space-times

[6] To simplify the notation, we will set the constant $k_{\rm s}$ equal to 1 since it plays in what follows the role of a scale factor.

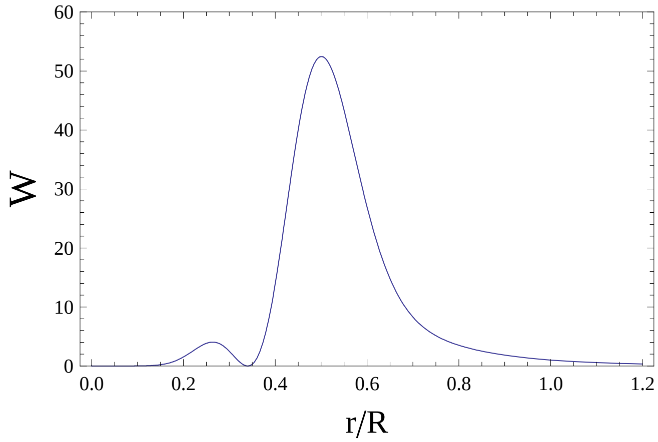

Figure 14. Weyl scalar as a function of radial coordinate.

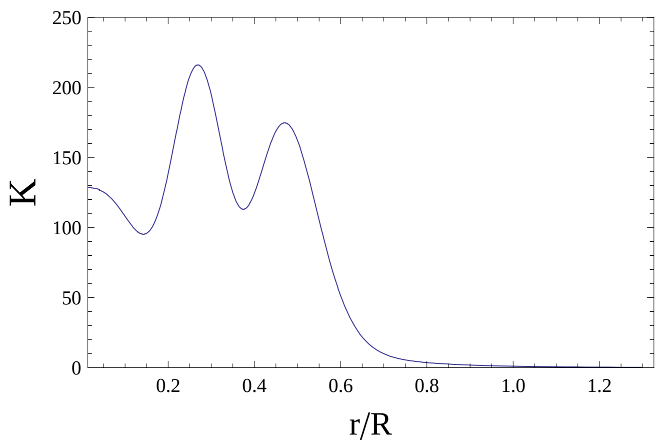

Figure 15. Kretschmann scalar as a function of radial coordinate.

where $x = e^{-8\frac{r^3}{R^3}}$ and the coefficients p_i are:

$$
\begin{aligned}
p_0 &= 3R^6 r - R^9 \pi \rho_{\max}, \\
p_1 &= 2R^9 \pi \rho_{\max} - 3R^6 r + 6r^3 R^6 \pi \rho_{\max} - 24 r^4 R^3, \\
p_2 &= -6r^3 R^6 \pi \rho_{\max} - R^9 \pi \rho_{\max} + 576 r^7 \alpha - \\
&\quad - 192 r^6 \alpha \pi R^3 \rho_{\max} - 2R^6 r^3 \alpha \pi \rho_{\max}, \\
p_3 &= 2R^6 r^3 \alpha \pi \rho_{\max} + 144 r^6 \alpha \pi R^3 \rho_{\max}, \\
p_4 &= (-1152 - 1152\alpha) r^7 - 48\pi R^3 \rho_{\max} (-8\alpha - 8 + \alpha^2) r^6 + \\
&\quad + 2\pi \rho_{\max} R^6 (\alpha + 1) r^3, \\
p_5 &= -336\pi \rho_{\max} R^3 (\alpha + 1) r^6 - 2\pi \rho_{\max} R^6 (\alpha + 1) r^3, \\
p_6 &= 96\pi \rho_{\max} \alpha R^3 (\alpha + 1) r^6, \\
p_8 &= -48\pi R^3 \rho_{\max} (\alpha + 1)^2 r^6.
\end{aligned} \tag{33}
$$

If we let $r \to 0$, the Weyl scalar goes to zero. Since the regular black hole space-time has a de Sitter geometry in the origin, this is the expected limit.

The plot of the Weyl scalar is shown in Fig. 14. We see that for large values of r the Weyl scalar tends asymptotically to zero. In the matter region ($r/R < 1$) it has one absolute maximum at $r/R = 0.5$ and one relative maximum at $r/R = 0.26$. Between these points, $W = 0$ at $r/R = 0.34$. If we analyse the equation of state as a function of the density (Equation 2), we find one inflexion point at $r/R = 0.5$. The absolute maximum of the Weyl scalar seems to be related to the transition point between the two regions of matter.

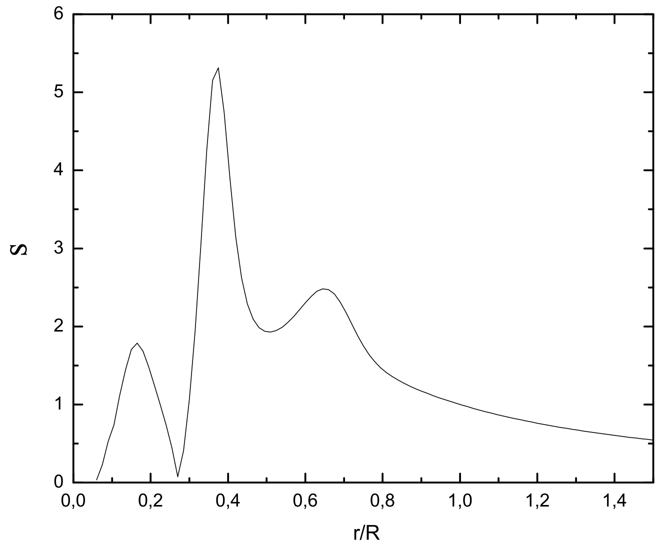

Figure 16. Gravitational entropy density as a function of radial coordinate.

The value of r/R for which the Weyl scalar has one relative maximum is close to the point where the pressure is zero. This suggests a relation between the region where matter has a negative pressure and the behaviour of the Weyl scalar. Notice that the inflexion point of the entropy density of the matter coincides with the value of r/R for which the Weyl scalar is zero, that is at $r/R = 0.34$. At this point the gravitational field changes from attractive to repulsive.

We also calculate the Kretschmann scalar. The plot is shown in Fig. 15. Again, we see that outside the matter region the Kretschmann scalar tends asymptotically to zero. If we let $r \to 0$, it goes to:

$$
K = \frac{512\pi^2 {\rho_{\max}}^2}{3}, \tag{34}
$$

as expected for the de Sitter space-time. The Kretschmann scalar is always positive and it has one absolute maximum at $r/R = 0.27$ and one relative maximum at $r/R = 0.47$.

As a cross-check, we see from the plots that the calculated Weyl and Kretschmann scalars satisfy the relation:

$$
R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} \geq C^{\alpha\beta\gamma\delta} C_{\alpha\beta\gamma\delta}, \tag{35}
$$

that holds in every static spherically symmetric space-time [16]. The gravitational entropy density of the MK model can be computed following following Rudjord et. al.'s proposal. The plot of the entropy density is shown in Fig. 16. It is seen that $\mathfrak{s} = 0$ at $r/R = 0.27$, which is close to the transition point from positive to negative pressures. The absolute maximum is at $r/R = 0.37$ and the two relative maxima are at $r/R = 0.16$ and at $r/R = 0.65$. For large values of r/R the gravitational entropy density tends asymptotically to zero and in the core of the object it is also zero. This last result is in accordance with a correct classical description of the entropy of the gravitational field.

VI. CONCLUSIONS

We have shown that the thermodynamical quantities describing the matter that is the source of the regular black hole model proposed by Mbonye and Kazanas indicate that the region of exotic matter is unstable. Some evidence has been presented that points to the dynamical instability of the model. The evidence for this second instability is supported by the plots of v_r and $v_\perp$. In particular, the latter indicates the existence of a second type of dynamical instability, in the normal matter part of the object. These findings should be confirmed by a perturbation analysis (which will be presented elsewhere), based on the equations given in the Appendix.

We have also calculated the Weyl and Krestchmann scalars for the regular MK black hole, and a possible classical estimator of the gravitational entropy proposed by Rudjord, Gr∅n and Sigbj∅rn based on the Weyl curvature conjecture. It was shown that close to the core of the object, the entropy density tends to zero as well as for large values of the radius. Hence, this classical estimator gives a good description of the entropy of the gravitational field in de Sitter and Schwarzschild limits.

APPENDIX: DYNAMICAL STABILITY OF THE INTERIOR REGION

In order to study the stablility of a system with tangential pressures against radial perturbations, both Einstein field equations and the equation of state must be perturbed following the standard procedure (see for instance [23]). The goal is to obtain a differential equation for the radial dependence of the quantity $\xi(r,t)$, which represents a small radial displacement of the fluid respect to its position of equilibrium at time t, that is, $r(r,t) = r_0 + \xi(r,t)$.

In what follows, the unperturbed metric coefficients $A(r)$, $B(r)$, and the unpertubed thermodynamics variables will be denoted by a zero subindex. The corresponding perturbations are denoted by $\delta A(r,t)$, $\delta B(r,t)$, $\delta p_{\rm r}(r,t)$, $\delta p_\perp(r,t)$ and $\delta\rho(r,t)$. A long but straightforward calculation (which parallels that presented in [23]) shows that the equations for the perturbations are given by:

$$8\pi r^2\delta\rho = \frac{\partial}{\partial r}\left(\frac{r\delta A}{A_0^2}\right),$$

$$8\pi\dot{\xi}(\rho_0 + p_{\rm r}^0) = -\frac{1}{r}\frac{\partial\delta A}{\partial t}\frac{1}{A_0^2}, \tag{36}$$

$$8\pi r^2\delta p_{\rm r} = \frac{r}{A_0}\left[\frac{\partial}{\partial r}\left(\frac{\delta B}{B_0}\right) - \frac{dB_0}{dr}\frac{\delta A}{A_0}\right] - \frac{\delta A}{A_0^2}, \tag{37}$$

$$\delta p_\perp = \frac{r}{2}\frac{\partial\delta p_{\rm r}}{\partial r} + \delta p_{\rm r} + \frac{r}{4B_0}\frac{dB_0}{dr}(\delta p_{\rm r} + \delta\rho) + \left[\frac{r}{4}\frac{\partial}{\partial r}\left(\frac{\delta B}{B_0}\right) + \frac{r}{2}\frac{A_0}{B_0}\frac{\partial^2\xi}{\partial t^2}\right](p_{\rm r}^0 + \rho_0).$$

Notice that there are six unknowns in this system, a consequence of the choices made by MK (one EOS and the explicit form of $\rho(r)$ to determine the system).

Equation (36) can be integrated respect to time [7], yielding:

$$\frac{\delta A}{A_0} = -8\pi r(\rho_0 + p_{\rm r}^0)A_0\xi.$$

By isolating δB from Eq. (37) we get:

$$\frac{\partial}{\partial r}\left(\frac{\delta B}{B_0}\right) = 8\pi r A_0\delta p_{\rm r} + \frac{\delta A}{A_0}\left(\frac{1}{B_0}\frac{dB_0}{dr} + \frac{1}{r}\right).$$

The final set of equations takes the form:

$$8\pi r^2\delta\rho = \frac{\partial}{\partial r}\left(\frac{r\delta A}{A_0^2}\right), \tag{38}$$

$$\delta p_{\rm r} + (p_{\rm r}^0)'\xi = \left(\frac{dp_{\rm r}}{d\rho}\right)_0(\delta\rho + \rho_0'\xi), \tag{39}$$

$$\delta p_\perp = \frac{r}{2}\frac{\partial\delta p_{\rm r}}{\partial r} + \delta p_{\rm r} + \frac{r}{4B_0}\frac{dB_0}{dr}(\delta p_{\rm r} + \delta\rho) + \left(\frac{r}{4}\frac{\partial}{\partial r}\left(\frac{\delta B}{B_0}\right) + \frac{r}{2}\frac{A_0}{B_0}\frac{\partial^2\xi}{\partial t^2}\right)(p_{\rm r}^0 + \rho_0), \tag{40}$$

$$\frac{\delta A}{A_0} = -8\pi r(\rho_0 + p_0)A_0\xi, \tag{41}$$

$$\frac{\partial}{\partial r}\left(\frac{\delta B}{B_0}\right) = 8\pi r A_0\delta p_{\rm r} + \frac{\delta A}{A_0}\left(\frac{1}{B_0}\frac{dB_0}{dr} + \frac{1}{r}\right). \tag{42}$$

Standard manipulations using the expression $\xi(r,t) = \sigma(r)\exp i\omega t$ in these equations lead to the following second order ordinary inhomogeneous differential equation for the function $\sigma(r)$:

$$A_\star(r)\sigma''(r) + C_\star(r)\sigma'(r) + [B_\star(r) + \omega D_\star(r)]\,\sigma(r) = \Pi(r), \tag{43}$$

where we have also used $\delta p_\perp(r,t) = \Pi(r)\exp i\omega t$.

This equation defines an *inhomogeneous* Sturm-Liouville (SL) problem, differently from the common

[7] We choose the constant of integration so that $\delta A(r,t) = 0$ if $\delta r = 0$

case which yields a homogeneous one, the inhomogeneity being a direct consequence of the way the MK model solution was obtained, as mentioned before.

Equation (43) is to be solve for $r \in [0,1]$, using the boundary conditions $\sigma(r) = 0$, $\sigma(1) = 0$, and $[\delta\rho(1,t) + \rho_0'(1,t)\xi(1,t)] = 0$ [23]. The coefficients $A_\star(r)$, $B_\star(r)$, $C_\star(r)$, and $D_\star(r)$ are given by:

$$A_\star(r) = \frac{r}{2}\left(p_{\rm r}^0 + \rho_0\right)\left(\frac{dp}{d\rho}\right)_0, \tag{44}$$

$$B_\star(r) = B_1(r) + B_2(r) + B_3(r) + B_4(r), \tag{45}$$

where:

$$B_1(r) = \frac{1}{8\pi r^2} q'(r)\alpha, \tag{46}$$
$$q'(r) = \frac{\partial}{\partial r}\left[-8\pi r^2\left(\rho_0 + p_{\rm r}^0\right)\right],$$
$$\alpha = \frac{r}{2}\frac{\partial}{\partial r}\left(\frac{dp}{d\rho}\right)_0 + \left(\frac{dp}{d\rho}\right)_0 \Sigma(r) + \frac{r}{4B_0}\frac{dB_0}{dr},$$
$$\Sigma(r) = 1 + \frac{r}{4B_0}\frac{dB_0}{dr} + 4\pi r^2 A_0\left(p_{\rm r}^0 + \rho_0\right),$$

$$B_2(r) = \frac{r}{2}\tau(r), \tag{47}$$
$$\tau(r) = \frac{\partial}{\partial r}\left(\frac{dp}{d\rho}\right)_0 \rho_0' - \frac{\partial}{\partial r}\left(p_{\rm r}^0\right)' + \left(\frac{dp}{d\rho}\right)_0 \frac{\partial \rho_0'}{\partial r},$$

$$B_3(r) = \frac{r}{2}\frac{\partial}{\partial r}\left(\frac{1}{8\pi r^2}q'(r)\right)\left(\frac{dp}{d\rho}\right)_0, \tag{48}$$

$$B_4(r) = -8\pi r\left(\rho_0 + p_{\rm r}^0\right)A_0\left[r\frac{d}{dr}\left(\ln B_0\right) + 1\right], \tag{49}$$

$$C_\star(r) = C_1(r) + C_2(r) + C_3(r), \tag{50}$$

$$C_1(r) = \frac{1}{8\pi r^2} q\alpha, \tag{51}$$
$$q = -8\pi r^2\left(\rho_0 + p_{\rm r}^0\right),$$

$$C_2(r) = \frac{r}{2}\left[\left(\frac{dp}{d\rho}\right)_0 \rho_0' - p_0'\right], \tag{52}$$

$$C_3(r) = \frac{1}{8\pi r}\left(-\frac{1}{r}q + q'\right), \tag{53}$$

$$D_\star(r) = -\frac{r}{2}\frac{A_0}{B_0}\left(p_{\rm r}^0 + \rho_0\right). \tag{54}$$

For a given $\Pi(r)$, Eq. (43) might be solved using the Green's function method (see for instance [24]). However,

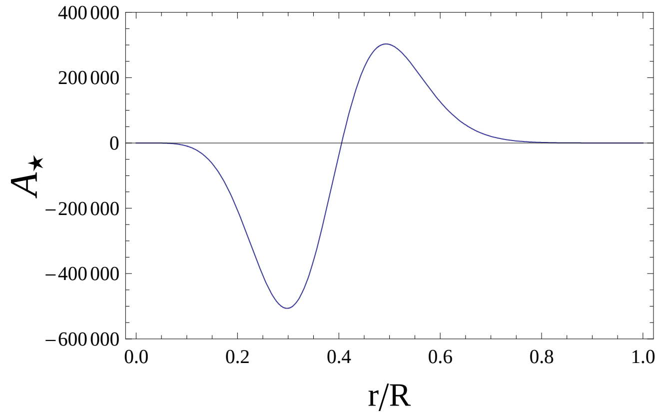

Figure 17. Coefficient $A_\star$ as a function of radial coordinate.

as we shall show next, the SL problem defined by Eq. (43) has a singular point. Consider the equation [25]:

$$-\left(py'\right)' + qy = \lambda w y, \tag{55}$$

which corresponds to the homogeneus SL problem defined on on $J = (a,b)$, with $-\infty \le a \le b \le \infty$. The eigenvalue λ is such that $\lambda \in \mathbb{C}$, and p, q, w and y are functions of x. It is also assumed that:

$$\frac{1}{p}, q, w \in L_{\rm loc}\left(J, \mathbb{C}\right), \tag{56}$$

where $L_{\rm loc}(J,\mathbb{C})$ denotes the linear manifold of functions y satisfying $y \in L([\alpha,\beta],\mathbb{C})$ for all compact intervals $[\alpha,\beta] \subseteq J$. $L(J,\mathbb{C})$ denotes the linear manifold of complex valued Lebesgue measurable functions y defined on J for which: $\int_a^b |y(t)|\,dt \equiv \int_J |y(t)|\,dt \equiv \int_J |y(t)| < \infty$. Here J is any interval of the real line, open, closed, half open, bounded or unbounded. Following [25], we have the next definitions:

- The (finite or infinite) endpoint a is *regular* if

$$\frac{1}{p}, q, w \in L((a,d),\mathbb{C}), \tag{57}$$

  holds for some (hence any) $d \in J$.

- An endpoint is called *singular* if it is not regular,

with similar definitions at $r = b$. To analyse whether $a = 0$ is a regular or singular endpoint for our problem, defined by Eq. (43), we show in Fig. 17 the plot of the coefficient $A_\star$ as a function of radial coordinate. We can see that $A_\star(0) = A_\star(0.4) = 0$, $A_\star(r) < 0$ for $r \in (0, 0.4)$, and $A_\star(r) > 0$ for $r \in (0.4, 1)$. Hence, $\frac{1}{A_\star} \notin L((0,d),\mathbb{C})$ for any $d \in (0,1)$. Therefore, the endpoint $a = 0$ is singular. If an endpoint is singular it could be a *limit point* or a *limit circle*. According to [25], "there is hardly any literature on the LP/LC (limit point-limit circle) dichotomy when all three coefficients are present in the SL equation". In particular, "there seems to be no literature on LP/LC criteria when p changes sign". Due to these complications, we shall attack this problem in a future publication.

ACKNOWLEDGMENTS

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