THE GRUNEWALD-O'HALLORAN CONJECTURE FOR NILPOTENT LIE ALGEBRAS OF RANK ≥ 1

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Abstract. Grunewald and O'Halloran conjectured in 1993 that every complex nilpotent Lie algebra is the degeneration of another, non isomorphic, Lie algebra. We prove the conjecture for the class of nilpotent Lie algebras admitting a semisimple derivation, and also for 7-dimensional nilpotent Lie algebras. The conjecture remains open for characteristically nilpotent Lie algebras of dimension grater than or equal to 8.

1. Introduction

The study of the algebraic varieties of Lie algebras, solvable, and nilpotent Lie algebras of dimension n turned out to be a very hard subject. The theory of deformations of algebras started with a series of papers by Gerstenhaber, the first being [G]. Since then a lot of efforts has been done (see for instance [NR1, R, NR2, V, C1, K]), however many natural questions remain unsolved. For example, the determination of the irreducible components of the variety of nilpotent Lie algebras seems today out of reach.

Two well known conjectures about the variety of nilpotent Lie algebras remain open. The oldest, known as Vergne's conjecture and stated after her seminal work [V], states that there are no rigid complex nilpotent Lie algebras in the algebraic variety $\mathcal{L}_n$ of complex Lie algebras of dimension n. Meaning that there are no nilpotent Lie algebras with open orbit in $\mathcal{L}_n$, that is such that their isomorphisms classes are open in $\mathcal{L}_n$. The other one, stated by Grunewald and O'Halloran [GO2], claims that every complex nilpotent Lie algebra is the degeneration of another, non isomorphic, Lie algebra and in particular is non rigid. The second conjecture is a priori stronger than the first one. In this paper we address the Grunewald-O'Halloran conjecture.

It is well known that, over fields of characteristic zero, geometric rigidity is equivalent to formal rigidity, the latest meaning that all formal deformations are trivial [GS]. However, this does not imply that the Grunewald-O'Halloran conjecture and Vergne's conjecture are equivalent. If so, it would also imply that every non geometrically rigid Lie algebra is the degeneration of another non isomorphic Lie algebra, which is not true already in dimension $n = 3$. In fact the only complex rigid Lie algebra of dimension 3 is the simple Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ and, for instance, the solvable (non nilpotent) Lie algebra $\mathfrak{r} + \mathbb{C}$, where $\mathfrak{r}$ is the 2-dimensional solvable Lie algebra, is on

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top of the Hasse diagram of degenerations, and in particular it is not the degeneration of any other Lie algebra (see [CD] and [BSt]).

Complex Lie algebras and nilpotent Lie algebras of small dimension are classified and in this cases all the degenerations among them and also which are rigid is known. All degenerations that occur among complex Lie algebras of dimension ≤ 4 are given in [St] and [BSt]. In [GO1] and [Se] all degenerations for complex nilpotent Lie algebras of dimension 5 and 6 are given and more recently, in [B], some degenerations for some 5-step and 6-step complex nilpotent Lie algebras of dimension 7 are given. Results on the different varieties and on rigidity in low dimensions may be found in [CD, C2]. In [AG] and [AGGV] the components of the varieties of nilpotent Lie algebras of dimension 7 and 8 are given.

Carles [C1] investigated the structure of rigid Lie algebras over algebraically closed fields of characteristic zero. In particular he proved that nilpotent Lie algebras of rank ≥ 1 are never rigid and moreover nilpotent Lie algebras with a codimension 1 ideal of rank ≥ 1 are also never rigid. That is, Vergne's conjecture holds for this class, remaining open for characteristically nilpotent Lie algebras for which all its ideals of codimension 1 are also characteristically nilpotent.

In the paper [GO2], the authors constructed nontrivial linear deformations for large classes of nilpotent Lie algebras and left open the question of which of those deformations correspond to degenerations. Their construction of linear deformations of a given Lie algebra $\mathfrak{g}$, relies on the existence of a codimension 1 ideal $\mathfrak{h}$ of $\mathfrak{g}$ with a semisimple derivation $D \in \mathrm{Der}(\mathfrak{h})$, and applies not only to nilpotent Lie algebras. In general, the deformations constructed do not correspond to a degeneration. A fixed ideal $\mathfrak{h}$ may produce many non equivalent deformations, some of which may correspond to a degeneration and some may not.

In this paper we prove the Grunewald-O'Halloran conjecture for two classes of algebras: nilpotent Lie algebras of rank ≥ 1 and 7-dimensional nilpotent Lie algebras. The conjecture then remains open for characteristically nilpotent Lie algebras of dimension greater or equal to 8.

More precisely, given a nilpotent Lie algebra with a semisimple derivation with construct a linear deformation of it and we are able to show that it corresponds to a degeneration. The first result is the following.

Theorem 1. *If $\mathfrak{n}$ is a complex nilpotent Lie algebra with a nontrivial semisimple derivation, then $\mathfrak{n}$ is the degeneration of another, non isomorphic, Lie algebra.*

The first characteristically nilpotent Lie algebras, that is without any semisimple derivation, appear in dimension 7. By Theorem 1, the Grunewald-O'Halloran conjecture holds in dimension < 7. Complex nilpotent Lie algebras of dimension 7 are classified; there are infinitely many isomorphism classes and infinitely many of them are characteristically nilpotent. We refer to the classification by Magnin [M] and work out this family on a case by case basis. In all cases we construct a linear deformation that corresponds to a degeneration. The second result is the following.

Theorem 2. *Every complex nilpotent Lie algebra of dimension ≤ 7, is the degeneration of another, non isomorphic, Lie algebra.*

In this paper all Lie algebras will be over the complex numbers.

2. Linear deformations and degenerations

Let $\mathcal{L}_n$ be the algebraic variety of complex Lie algebras of dimension n, that is the algebraic variety of Lie brackets μ on $\mathbb{C}^n$ ($\mathcal{L}_n \subseteq \mathbb{C}^{n^3}$). Given a complex Lie algebra $\mathfrak{g} = (\mathbb{C}^n, \mu)$, we shall refer to it indistinctly by $\mathfrak{g}$, $(\mathfrak{g}, \mu)$ or μ. The group $GL_n = GL_n(\mathbb{C})$ acts on $\mathcal{L}_n$ by 'change of basis':

$$g \cdot \mu(x, y) = g(\mu(g^{-1}x, g^{-1}y)), \qquad g \in GL_n.$$

Thus the orbit $\mathcal{O}(\mu)$ of μ in $\mathcal{L}_n$, is the isomorphism class of μ.

A Lie algebra μ is said to degenerate to a Lie algebra λ, denoted by $\mu \to_{\deg} \lambda$, if $\lambda \in \overline{\mathcal{O}(\mu)}$, the Zariski closure of $\mathcal{O}(\mu)$. If $\lambda \not\simeq \mu$, then λ is in the boundary of the orbit $\mathcal{O}(\mu)$ but outside it. Since the Zariski closure of $\mathcal{O}(\mu)$ coincides with its closure in the relative topology of $\mathbb{C}^{n^3}$, if $g : \mathbb{C}^\times \to GL_n$, $t \mapsto g_t$, is continuous and $\lim_{t \mapsto 0} g_t \cdot \mu = \lambda$, then $\mu \to_{\deg} \lambda$. The degeneration $\mu \to_{\deg} \lambda$ is said to be realized by a 1-PSG, if g_t is a 1-parameter subgroup as a morphism of algebraic groups. Recall that if g_t is a 1-PSG, then g_t is diagonalizable with eigenvalues t^{m_i} for some integers m_i.

A linear deformation of a Lie algebra μ is, for the aim of this paper, a family μ_t, $t \in \mathbb{C}^\times$, of Lie algebras such that

$$\mu_t = \mu + t\phi,$$

where ϕ is a skew-symmetric bilinear form on $\mathbb{C}^n$. It turns out that μ_t is a linear deformation of μ if and only if ϕ is a Lie algebra bracket which in addition is a 2-cocycle of μ.

If a given a linear deformation μ_t of μ is such that $\mu_t \in \mathcal{O}(\mu_1)$ for all $t \in \mathbb{C}^\times$, then $\mu_1 \to_{\deg} \mu$. In fact, for each $t \in \mathbb{C}^\times$ there exist $g_t \in GL_n$ such that $g_t^{-1} \cdot \mu_1 = \mu_t$, then $\lim_{t \mapsto 0} g_t^{-1} \cdot \mu_1 = \lim_{t \mapsto 0} \mu_t = \mu$. Hence, in order to show that $\mu_1 \to_{\deg} \mu$, one only needs to prove that for each $t \in \mathbb{C}^\times$ there exist $g_t \in GL_n$ such that

$$\mu_1(g_t(x), g_t(y))) = g_t(\mu_t(x, y)), \quad \text{for all } x, y \in \mathbb{C}^n. \tag{2.1}$$

2.1. **Construction of linear deformations.** We recall now the construction of linear deformations in [GO2].

Let $(\mathfrak{g}, \mu)$ be a given Lie algebra of dimension n and let $\mathfrak{h}$ be a codimension 1 ideal of $\mathfrak{g}$ with a semisimple derivation D. For any element X of $\mathfrak{g}$ outside $\mathfrak{h}$, $\mathfrak{g} = \langle X \rangle \oplus \mathfrak{h}$. The bilinear form μ_D on $\mathfrak{g}$ defined by $\mu_D(X, z) = D(z)$ and $\mu_D(y, z) = 0$, for $y, z \in \mathfrak{h}$, is a 2-cocycle for μ and a Lie bracket. Hence,

$$\mu_t = \mu + t\mu_D, \tag{2.2}$$

is a linear deformation of μ. If $\mathfrak{g}$ is nilpotent, then μ_t is always solvable but not nilpotent. In particular, μ_t is not isomorphic to μ for all $t \in \mathbb{C}^\times$. The construction described above can be carried out also for any derivation D, not necessarily semisimple. However, one can not assure that μ_t is not isomorphic to μ in this case.

2.2. **Degenerations from deformations.** Under certain hypothesis on the derivation D, the deformation constructed above does correspond to a degeneration.

Proposition 2.1. *Let $\mathfrak{n}$ be a nilpotent Lie algebra with an ideal $\mathfrak{h}$ of codimension 1 admitting a nontrivial semisimple derivation D. If D is the restriction of a semisimple derivation $\tilde{D}$ of $\mathfrak{n}$ such that it is nontrivial on a direct invariant complement of $\mathfrak{h}$, then $\mathfrak{n}$ is the degeneration of another, non isomorphic, Lie algebra. Moreover, the degeneration can be realized by a 1-PSG.*

Proof. Let $\mathfrak{n} = (\mathfrak{n}, \mu)$. Let X be an eigenvector of $\tilde{D}$ complementary to $\mathfrak{h}$ and let $\lambda_0 \neq 0$ be its eigenvalue. We may assume that $\lambda_0 = 1$ (by considering $\tilde{D}/\lambda_0$ and D/λ_0 instead of $\tilde{D}$ and D).

Let $\lambda_1, \dots, \lambda_k$ be the different eigenvalues of D and let $\mathfrak{h} = \mathfrak{h}_{\lambda_1} \oplus \cdots \oplus \mathfrak{h}_{\lambda_k}$ be the corresponding graded decomposition of $\mathfrak{h}$, that is $\mu(\mathfrak{h}_{\lambda_i}, \mathfrak{h}_{\lambda_j}) \subseteq \mathfrak{h}_{\lambda_i+\lambda_j}$.

Hence,

$$\mathfrak{n} = (\langle X \rangle \oplus \mathfrak{h}, \mu)$$

where both summands of $\mathfrak{n}$ are $\tilde{D}$-invariant and $\mu(X, \mathfrak{h}_{\lambda_j}) \subseteq \mathfrak{h}_{1+\lambda_j}$.

Let $\mu_t = \mu + t\mu_D$ be the linear deformation constructed as in (2.2), which is given by

$$\mu_t(X, y_j) = \mu(X, y_j) + t\lambda_j y_j, \quad \text{if } y_j \in \mathfrak{h}_{\lambda_j}, \text{ for } 1 \leq j \leq k.$$
$$\mu_t(y_i, y_j) = \mu(y_i, y_j), \quad \text{if } y_i \in \mathfrak{h}_{\lambda_i} \text{ and } y_j \in \mathfrak{h}_{\lambda_j}, \text{ for } 1 \leq i, j \leq k.$$

Let $g_t \in GL_n$, where $n = \dim \mathfrak{n}$, be defined by

$$g_t|_{\langle X \rangle} = tI \qquad \text{and} \qquad g_t|_{\mathfrak{h}_{\lambda_i}} = t^{\lambda_i} I, \quad \text{for } i = 1 \dots k.$$

It is not difficult to check that (2.1) is satisfied. In fact, if $y_i \in \mathfrak{h}_{\lambda_i}$ and $y_j \in \mathfrak{h}_{\lambda_j}$ for $1 \leq i, j \leq k$, then

$$\begin{aligned} g_t(\mu_t(X, y_j)) &= g_t(\mu(X, y_j) + \lambda_j t y_j) = t^{1+\lambda_j}\mu(X, y_j) + \lambda_j t^{\lambda_j+1} y_j, \\ \mu_1(g_t(X), g_t(y_j)) &= \mu_1(tX, t^{\lambda_j} y_j) = t^{1+\lambda_j}\mu(X, y_j) + \lambda_j t^{\lambda_j+1} y_j, \end{aligned}$$

and

$$\begin{aligned} g_t(\mu_t(y_i, y_j)) &= g_t(\mu(y_i, y_j)) = t^{\lambda_i+\lambda_j}\mu(y_i, y_j), \\ \mu_1(g_t(y_i), g_t(y_j)) &= \mu_1(t^{\lambda_i} y_i, t^{\lambda_j} y_j) = t^{\lambda_i+\lambda_j}\mu(y_i, y_j). \end{aligned}$$

Therefore, being μ_1 solvable, μ is the degeneration of another, non isomorphic, Lie algebra. □

In the above proposition the ideal $\mathfrak{h}$ is given, but clearly any such ideal will work. Hence, if $\tilde{D}$ is a derivation of $\mathfrak{n}$ that preserves an ideal $\mathfrak{h}$ and such that its restriction to $\mathfrak{h}$ is semisimple, we get for $\mathfrak{n}$ the same conclusion of Proposition 2.1. This is the statement in Theorem 1.

Proof of Theorem 1. The semisimple derivation D of $\mathfrak{n}$ preserves the (characteristic) ideal $[\mathfrak{n}, \mathfrak{n}]$. Let V be a D-invariant complement of $[\mathfrak{n}, \mathfrak{n}]$ and let $\{X_1, \dots, X_r\}$ be a basis of V formed by eigenvectors of D. Since V generates $\mathfrak{n}$ as a Lie algebra (see for instance [J], page 29) and D is nontrivial, D is nontrivial on V and we may assume that X_1 is an eigenvector with nonzero eigenvalue. Now let $\mathfrak{h} = \langle X_2, \dots, X_r\rangle \oplus [\mathfrak{n}, \mathfrak{n}]$. Clearly $\mathfrak{h}$ is an ideal of $\mathfrak{n}$ of codimension 1, D preserves $\mathfrak{h}$, $D|\mathfrak{h}$ is semisimple and D is nontrivial on X_1. Therefore, by Proposition 2.1, $\mathfrak{n}$ is the degeneration of a Lie algebra non isomorphic to $\mathfrak{n}$. $\square$

3. The conjecture in dimension 7

All nilpotent Lie algebras of dimension < 7 have semisimple derivations. Therefore the Grunewald-O'Halloran conjecture holds in this case. Moreover, in dimensions 2, 3, 4, 5 and 6 where there are only a finite number of isomorphism classes, the Hasse diagram of degenerations has an algebra on top of it, degenerating to all others [GO1, Se]. Being degeneration transitive, an algebra degenerating to the top one degenerates to all the others as well.

Example 3.1. The 6-dimensional nilpotent Lie algebra 12346_E in [Se], that we rename μ, defined by

$$\begin{aligned} &\mu(e_1, e_2) = e_3, && \mu(e_1, e_3) = e_4, && \mu(e_1, e_4) = e_5, \\ &\mu(e_2, e_3) = e_5, && \mu(e_2, e_5) = e_6, && \mu(e_3, e_4) = -e_6, \end{aligned} \tag{3.1}$$

degenerates to all other nilpotent Lie algebras of dimension 6 [Se].

We construct a solvable linear deformation of μ that degenerates to it, and therefore to all other 6-dimensional nilpotent Lie algebras. To this end consider the ideal $\mathfrak{h} = \langle e_2, e_3, e_4, e_5, e_6\rangle$ and the derivation D of $\mathfrak{h}$ defined by

$$D(e_2) = e_2, \quad D(e_3) = 0, \quad D(e_4) = 2e_4, \quad D(e_5) = e_5, \quad D(e_6) = 2e_6.$$

This produces the 2-cocycle μ_D, defined by

$$\mu_D(e_1, e_2) = e_2, \quad \mu_D(e_1, e_4) = 2e_5, \quad \mu_D(e_1, e_5) = e_5, \quad \mu_D(e_1, e_6) = 2e_6.$$

The corresponding deformation of μ, $\mu_t = \mu + t\mu_D$, is then given by

$$\begin{aligned} &\mu_t(e_1, e_2) = e_3 + te_2, && \mu_t(e_1, e_3) = e_4, && \mu_t(e_1, e_4) = e_5 + 2te_4, \\ &\mu_t(e_1, e_5) = te_5, && \mu_t(e_1, e_6) = 2te_6, && \mu_t(e_2, e_3) = e_5, \\ &\mu_t(e_2, e_5) = e_6, && \mu_t(e_3, e_4) = -e_6, \end{aligned}$$

and in particular μ_1 is given by

$$\begin{aligned} &\mu_1(e_1, e_2) = e_3 + e_2, && \mu_1(e_1, e_3) = e_4, && \mu_1(e_1, e_4) = e_5 + 2e_4, \\ &\mu_1(e_1, e_5) = e_5, && \mu_1(e_1, e_6) = 2e_6, && \mu_1(e_2, e_3) = e_5, \\ &\mu_1(e_2, e_5) = e_6, && \mu_1(e_3, e_4) = -e_6. \end{aligned}$$

Let $g_t \in GL_6$ be the 1-PSG given by

$$g_t = \begin{pmatrix} t &&&&& \\ & t^2 &&&& \\ && t^3 &&& \\ &&& t^4 && \\ &&&& t^5 & \\ &&&&& t^7 \end{pmatrix}.$$

It is easy to verify that, for all $t \neq 0$, $g_t^{-1} \cdot \mu_1 = \mu_t$ and thus $\mu_1 \to_{\deg} \mu$.

Remark 3.2. By considering different linear deformations, we found that each nilpotent Lie algebra of dimension < 7 is the degeneration of many others, non isomorphic, Lie algebras. Many of those degenerations can be realized by a 1-PSG, but others can not.

Now we come to the 7-dimensional case. To this end we consider the classification by Magnin [M] of all (indecomposable) characteristically nilpotent Lie algebras of dimension 7. Here, representatives of all isomorphism clases are given as a continuous 1-parameter family and seven isolated algebras:

$$\mathfrak{g}_{7,0.1} \quad \mathfrak{g}_{7,0.2} \quad \mathfrak{g}_{7,0.3} \quad \mathfrak{g}_{7,0.4(\lambda)} \quad \mathfrak{g}_{7,0.5} \quad \mathfrak{g}_{7,0.6} \quad \mathfrak{g}_{7,0.7} \quad \mathfrak{g}_{7,0.8}$$

Remark 3.3. There are no decomposable characteristically nilpotent Lie algebras of dimension 7.

Proof of Theorem 2. We start by considering the family $\mathfrak{g}_{7,0.4(\lambda)}$, which is defined by

$$\begin{aligned} &\mu(e_1,e_2)=e_3, \quad &&\mu(e_1,e_3)=e_4, \quad &&\mu(e_1,e_4)=e_6+\lambda e_7,\\ &\mu(e_1,e_5)=e_7, &&\mu(e_1,e_6)=e_7, &&\mu(e_2,e_3)=e_5,\\ &\mu(e_2,e_4)=e_7, &&\mu(e_2,e_5)=e_6, &&\mu(e_3,e_5)=e_7. \end{aligned}$$

Take the ideal $\mathfrak{h}=\langle e_2,e_3,e_4,e_5,e_6,e_7\rangle$ and $D\in \mathrm{Der}(\mathfrak{h})$ defined by

$$D(e_2)=e_2, \quad D(e_3)=0, \quad D(e_4)=0, \quad D(e_5)=e_5, \quad D(e_6)=2e_6, \quad D(e_7)=e_7.$$

The corresponding 2-cocycle μ_D is given by

$$\mu_D(e_1,e_2)=e_2, \quad \mu_D(e_1,e_5)=e_5, \quad \mu_D(e_1,e_6)=2e_6, \quad \mu_D(e_1,e_7)=e_7,$$

and the corresponding deformation $\mu_t=\mu+t\mu_D$ of μ is given by

$$\begin{aligned} &\mu_t(e_1,e_2)=e_3+te_2, \quad &&\mu_t(e_1,e_3)=e_4, \quad &&\mu_t(e_1,e_4)=e_6+\lambda e_7,\\ &\mu_t(e_1,e_5)=e_7+te_5, &&\mu_t(e_1,e_6)=e_7+2te_6, &&\mu_t(e_1,e_7)=te_7,\\ &\mu_t(e_2,e_3)=e_5, &&\mu_t(e_2,e_4)=e_7, &&\mu_t(e_2,e_5)=e_6,\\ &\mu_t(e_3,e_5)=e_7. \end{aligned}$$

Consider now $g_t=g_t(\lambda)\in GL_7$ given by

$$g_t=\begin{pmatrix} t & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & t & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & t^2 & 0 & 0 & 0\\ \frac14\left(\frac{t^2-1}{t}\right) & \left(1-\lambda+\frac{\lambda}{t}-\frac{1}{t^2}\right) & 0 & 0 & t & 0 & 0\\ 0 & 0 & \frac14\left(\frac{1-t^2}{t}\right) & \frac12\left(1-t^2\right) & 0 & t & 0\\ 0 & 0 & \left(t-\lambda t+\lambda-\frac1t\right) & \left(\frac12 t^2-\lambda t^2+\lambda t-\frac12\right) & \left(\lambda t-t-\lambda+\frac1t\right) & 0 & t^2 \end{pmatrix}.$$

The calculations below show that $g_t^{-1}\cdot\mu_1=\mu_t$ and thus $\mu_1\to_{deg}\mu$.

$$\begin{aligned}
\bullet\ g_t\mu_t(e_1,e_2) &= g_t(e_3+te_2)\\
&= te_3+\frac{1}{4}\left(\frac{1-t^2}{t}\right)e_6+\left(t-\lambda t+\lambda-\frac{1}{t}\right)e_7+te_2+t\left(1-\lambda+\frac{\lambda}{t}-\frac{1}{t^2}\right)e_5\\
&= te_2+te_3+\left(t-\lambda t+\lambda-\frac{1}{t}\right)e_5+\frac{1}{4}\left(\frac{1-t^2}{t}\right)e_6+\left(t-\lambda t+\lambda-\frac{1}{t}\right)e_7\\
\mu_1(g_te_1,g_te_2) &= \mu_1\left(te_1+\frac{1}{4}\left(\frac{t^2-1}{t}\right)e_5, e_2+\left(1-\lambda+\frac{\lambda}{t}-\frac{1}{t^2}\right)e_5\right)\\
&= t(e_3+e_2)+t\left(1-\lambda+\frac{\lambda}{t}-\frac{1}{t^2}\right)(e_7+e_5)-\frac{1}{4}\left(\frac{t^2-1}{t}\right)e_6\\
&= te_2+te_3+\left(t-\lambda t+\lambda-\frac{1}{t}\right)e_5+\frac{1}{4}\left(\frac{1-t^2}{t}\right)e_6+\left(t-\lambda t+\lambda-\frac{1}{t}\right)e_7
\end{aligned}$$

$$\begin{aligned}
\bullet\ g_t\mu_t(e_1,e_3) &= g_te_4\\
&= t^2e_4+\frac{1}{2}(1-t^2)e_6+\left(\frac{1}{2}t^2-\lambda t^2+\lambda t-\frac{1}{2}\right)e_7\\
\mu_1(g_te_1,g_te_3) &= \mu_1\left(te_1+\frac{1}{4}\left(\frac{t^2-1}{t}\right)e_5, te_3+\frac{1}{4}\left(\frac{1-t^2}{t}\right)e_6+\left(t-\lambda t+\lambda-\frac{1}{t}\right)e_7\right)\\
&= t^2e_4+\frac{1}{4}t\left(\frac{1-t^2}{t}\right)(e_7+2e_6)+t\left(t-\lambda t+\lambda-\frac{1}{t}\right)e_7-\frac{1}{4}(t^2-1)e_7\\
&= t^2e_4+\frac{1}{4}(1-t^2)e_6+\left(\frac{1}{4}-\frac{1}{4}t^2+t^2-\lambda t^2+\lambda t-1-\frac{1}{4}t^2+\frac{1}{4}\right)e_7\\
&= t^2e_4+\frac{1}{4}(1-t^2)e_6+\left(\frac{1}{2}t^2-\lambda t^2+\lambda t-\frac{1}{2}\right)e_7
\end{aligned}$$

$$\begin{aligned}
\bullet\ g_t\mu_t(e_1,e_4) &= g_t(e_6+\lambda e_7)\\
&= te_6+\lambda t^2e_7\\
\mu_1(g_te_1,g_te_4) &= \mu_1\left(te_1+\frac{1}{4}\left(\frac{t^2-1}{t}\right)e_5, t^2e_4+\frac{1}{2}(1-t^2)e_6+\left(\frac{1}{2}t^2-\lambda t^2+\lambda t-\frac{1}{2}\right)e_7\right)\\
&= t^3(e_6+\lambda e_7)+\frac{1}{2}t(1-t^2)(e_7+2e_6)+t\left(\frac{1}{2}t^2-\lambda t^2+\lambda t-\frac{1}{2}\right)e_7\\
&= (t^3+t-t^3)e_6+\left(\lambda t^3+\frac{1}{2}t-\frac{1}{2}t^3+\frac{1}{2}t^3-\lambda t^3+\lambda t^2-\frac{1}{2}t\right)e_7\\
&= te_6+\lambda t^2e_7
\end{aligned}$$

$$\begin{aligned}
\bullet\ g_t\mu_t(e_1,e_5) &= g_t(e_7+te_5)\\
&= t^2e_7+t^2e_5+t\left(\lambda t-t-\lambda+\frac{1}{t}\right)e_7\\
&= t^2e_5+(\lambda t^2-\lambda t+1)e_7\\
\mu_1(g_te_1,g_te_5) &= \mu_1\left(te_1+\frac{1}{4}\left(\frac{t^2-1}{t}\right)e_5, te_5+\left(\lambda t-t-\lambda+\frac{1}{t}\right)e_7\right)\\
&= t^2(e_7+e_5)+t\left(\lambda t-t-\lambda+\frac{1}{t}\right)e_7\\
&= t^2e_5+(t^2+\lambda t^2-t^2-\lambda t+1)e_7\\
&= t^2e_5+(\lambda t^2-\lambda t+1)e_7
\end{aligned}$$

$$\bullet\ g_t\mu_t(e_1,e_6) = g_t(e_7+2te_6)$$
$$= t^2e_7+2t^2e_6$$
$$\mu_1(g_te_1,g_te_6) = \mu_1\left(te_1+\frac{1}{4}\left(\frac{t^2-1}{t}\right)e_5, te_6\right)$$
$$= t^2(e_7+2e_6)$$
$$= t^2e_7+2t^2e_6$$

$$\bullet\ g_t\mu_t(e_1,e_7) = g_t(te_7)$$
$$= t^3e_7$$
$$\mu_1(g_te_1,g_te_7) = \mu_1\left(te_1+\frac{1}{4}\left(\frac{t^2-1}{t}\right)e_5, t^2e_7\right)$$
$$= t^3e_7$$

$$\bullet\ g_t\mu_t(e_2,e_3) = g_te_5$$
$$= te_5+\left(\lambda t-t-\lambda+\frac{1}{t}\right)e_7$$
$$\mu_1(g_te_2,g_te_3) = \mu_1\left(e_2+\left(1-\lambda+\frac{\lambda}{t}-\frac{1}{t^2}\right)e_5, te_3+\frac{1}{4}\left(\frac{1-t^2}{t}\right)e_6+\left(t-\lambda t+\lambda-\frac{1}{t}\right)e_7\right)$$
$$= te_5-t\left(1-\lambda+\frac{\lambda}{t}-\frac{1}{t^2}\right)e_7$$
$$= te_5+\left(\lambda t-t-\lambda+\frac{1}{t}\right)e_7$$

$$\bullet\ g_t\mu_t(e_2,e_4) = g_te_7$$
$$= t^2e_7$$
$$\mu_1(g_te_2,g_te_4) = \mu_1\left(e_2+\left(1-\lambda+\frac{\lambda}{t}-\frac{1}{t^2}\right)e_5, t^2e_4+\frac{1}{4}(1-t^2)e_6+\left(\frac{1}{2}t^2-t+\lambda+\frac{1}{t}\right)e_7\right)$$
$$= t^2e_7$$

$$\bullet\ g_t\mu_t(e_2,e_5) = g_te_6$$
$$= te_6$$
$$\mu_1(g_te_2,g_te_5) = \mu_1\left(e_2+\left(1-\lambda+\frac{\lambda}{t}-\frac{1}{t^2}\right)e_5, te_5+\left(\lambda t-t-\lambda+\frac{1}{t}\right)e_7\right)$$
$$= te_6$$

$$\bullet\ g_t\mu_t(e_3,e_5) = g_te_7$$
$$= t^2e_7$$
$$\mu_1(g_te_3,g_te_5) = \mu_1\left(te_3+\frac{1}{4}\left(\frac{1-t^2}{t}\right)e_6+\left(t-\lambda t+\lambda-\frac{1}{t}\right)e_7, te_5+\left(\lambda t-t-\lambda+\frac{1}{t}\right)e_7\right)$$
$$= t^2e_7$$

The proof for the remaining seven algebras is worked out similarly. The table below contains all relevant data. In each case we indicate the codimension 1 ideal $\mathfrak{h}=\mathfrak{h}_i=\langle e_1,\dots,\hat{e_i},\dots,e_7\rangle$, the semisimple derivation $D\in Der(\mathfrak{h})$ that we choose to construct the linear deformation and the family $g_t\in GL_7$ realizing the degeneration. It is not difficult to check this by hand; we omit the computations. The proof is now complete. □

$\mathfrak{g}$	$\mathfrak{h}$	$D \in Der(\mathfrak{h})$	g_t
$\mathfrak{g}_{7,0.1}$	$\mathfrak{h}_2$	$\begin{pmatrix} 1 &&&&&& \\ & 3 &&&&& \\ && 4 &&&& \\ &&& 5 &&& \\ &&&& 6 && \\ &&&&& 7 & \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2}\left(\frac{t-1}{t}\right) & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{6}\left(\frac{3t^2-5t+2}{t}\right) & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3}\left(\frac{1-t}{t}\right) & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{3}\left(\frac{1-t}{t}\right) & \frac{1}{2}\left(\frac{1-t}{t}\right) & 0 & 1 \end{pmatrix}$
$\mathfrak{g}_{7,0.2}$	$\mathfrak{h}_2$	$\begin{pmatrix} 1 &&&&&& \\ & 0 &&&&& \\ && 1 &&&& \\ &&& 2 &&& \\ &&&& 3 && \\ &&&&& 4 & \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{8}\left(\frac{4t-3t^2-1}{t}\right) & 0 & t & 0 & 0 & 0 \\ \frac{1}{8}\left(\frac{t^2-1}{t^2}\right) & 0 & \frac{1}{2}(1-t) & 0 & t & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(1-t) & 0 & t & 0 \\ 0 & 0 & \frac{1}{8}\left(\frac{1-t^2}{t}\right) & 0 & \frac{1}{2}(1-t) & 0 & t \end{pmatrix}$
$\mathfrak{g}_{7,0.3}$	$\mathfrak{h}_2$	$\begin{pmatrix} 1 &&&&&& \\ & 0 &&&&& \\ && 1 &&&& \\ &&& 2 &&& \\ &&&& 3 && \\ &&&&& 4 & \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t & 0 & 0 & 0 & 0 \\ \frac{1}{4}\left(\frac{t-1}{t}\right) & 0 & 0 & t & 0 & 0 & 0 \\ 0 & \frac{1}{3}(1-t) & 0 & 0 & t & 0 & 0 \\ 0 & 0 & \frac{1}{3}(1-t) & 0 & 0 & t & 0 \\ 0 & 0 & \frac{1}{4}(1-t) & \frac{1}{3}(1-t) & 0 & 0 & t \end{pmatrix}$
$\mathfrak{g}_{7,0.5}$	$\mathfrak{h}_2$	$\begin{pmatrix} 1 &&&&& \\ & 0 &&&& \\ && 1 &&& \\ &&& 3 && \\ &&&& 2 & \\ &&&& -1 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3}\left(\frac{t^2-1}{t}\right) & 0 & t & 0 & 0 & 0 \\ \frac{1}{6}\left(\frac{t^2-1}{t^2}\right) & 0 & \frac{1}{3}\left(\frac{1-t^2}{t}\right) & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{6}\left(\frac{t^2-1}{t}\right) & 0 & 0 & t & 0 \\ 0 & 0 & \frac{1}{3}\left(\frac{t^2-1}{t}\right) & 0 & \frac{5}{6}(t^2-1) & 0 & t \end{pmatrix}$
$\mathfrak{g}_{7,0.6}$	$\mathfrak{h}_1$	$\begin{pmatrix} 1 &&&&&& \\ & 0 &&&&& \\ && 2 &&&& \\ &&& 1 &&& \\ &&&& 3 && \\ &&&&& 2 & \end{pmatrix}$	$\begin{pmatrix} t & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}\left(\frac{1-t^2}{t^2}\right) & \frac{1}{2}\left(\frac{1-t^2}{t}\right) & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}\left(\frac{1-t^2}{t}\right) & 1 & 0 \\ 0 & 0 & \frac{1}{2}\left(\frac{1-t^2}{t}\right) & \frac{3}{2}(1-t^2) & \frac{1}{2}\left(\frac{t^2-1}{t}\right) & 0 & t \end{pmatrix}$

$\mathfrak{g}_{7,0.7}$	$\mathfrak{h}_1$	$\begin{pmatrix} 1 & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & 1 & & \\ & & & & 2 & \\ & & & & & 1 \end{pmatrix}$	$\begin{pmatrix} t & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t & 0 & 0 & 0 & 0 \\ (t-1) & 0 & 0 & t^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t & 0 \\ 0 & 0 & (1-t) & (1-t)t & 0 & 0 & t^2 \end{pmatrix}$
$\mathfrak{g}_{7,0.8}$	$\mathfrak{h}_1$	$\begin{pmatrix} 1 & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & 2 & & \\ & & & & 1 & \\ & & & & & 2 \end{pmatrix}$	$\begin{pmatrix} t & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & (1-t^2) & t^3 & t(t^2-1) & 0 & 0 & 0 \\ 0 & 0 & 0 & t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t^3 & 0 \\ 0 & 0 & \frac{1}{2}t^2(1-t) & t(1-t^2) & t^2(1-t^2) & 0 & t^3 \end{pmatrix}$

Remark 3.4. The variety of nilpotent Lie algebras of dimension 7 has two irreducible components, each of which is the closure of the orbits of two families μ_α^1 and μ_α^2, with $\alpha \in \mathbb{C}$ [AG, Main Theorem]. The first family is made of nilpotent Lie algebras of rank ≥ 1, while the second family is made entirely of characteristically nilpotent algebras.

By Theorem 1 and being degeneration transitive, to prove Theorem 2 it suffices to find for each algebra in the second family another non isomorphic Lie algebra degenerating to it. Doing this requires a similar amount of work as in the first part of our proof. One might then save the work we have done for the 7 isolated algebras. However we preferred to construct explicit degenerations to all characteristically nilpotent Lie algebras of dimension 7. A complete picture might be more helpful to others.

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References

[AG] Ancochea-Bermudez, J. and Goze, M., *On the varieties of nilpotent Lie algebras of dimension 7 and 8*, J. Pure Appl. Algebra 77 (1992), 131–140.

[AGGV] Ancochea-Bermudez, J., Gómez-Martin, J., Goze, M., and Valeiras G., *Sur les composantes irréductibles de la varieté des lois d'algèbres de Lie nilpotentes*, J. Pure Appl. Algebra 106 (1996), 11–22.

[B] Burde, D., *Degenerations of 7-dimensional nilpotent Lie algebras*, Comm. Algebra 33 (2005), no. 4, 1259–1277.

[BSt] Burde, D. and Steinhoff C., *Classification of orbit closures of 4-dimensional complex Lie algebras*, J. Algebra 214 (1999), no. 2, 729–739.

[C1] Carles, R., *Sur la structure des algèbres de Lie rigides*, Ann. Inst. Fourier 34 (1984), no. 3, 65–82.

[C2] Carles, R., *Weight systems for complex nilpotent Lie algebras and applications to the varieties of Lie algebras* Publ. Univ. Poitiers, (1996).

[CD] Carles, R., Diakité *Sur les variétés d'algèbres de Lie de dimension* ≤ 7, J. Algebra 91 (1984), no. 1, 53–63.

[G] Gerstenhaber, M., *On the deformations of rings and algebras*, Ann. Math. 74 (1964), no. 1, 59–103.

[GS] Gerstenhaber, M. and Schack, S., *Relative Hochschild cohomology, rigid algebras, and the Bockstein*, J. Pure Appl. Algebra 43 (1986), no. 1, 53–74.

[GO1] Grunewald, F. and O'Halloran, J., *Varieties of nilpotent Lie algebras of dimension less than six*, J. Algebra 112 (1988), no. 2, 315–325.

[GO2] Grunewald, F. and O'Halloran, J., *Deformations of Lie algebras*, J. Algebra 162 (1993), no. 1, 210–224.

[J] Jacobson, N., *Lie algebras*, Dover Publications, Inc., New York (1979).

[K] Khakimdjanov, Y., *Characteristically nilpotent Lie algebras* Math. USSR Sbornik 70 (1990),no. 1, 65–78.

[M] Magnin, L., *Determination of 7-dimensional indecomposable nilpotent complex Lie algebras by adjoining a derivation to 6-dimensional Lie algebras*, Algebr. Represent. Theory 13 (2010), no. 6, 723–753.

[NR1] Nijenhuis, A. and Richardson R.W., *Cohomology and deformations in graded Lie algebras*, Bull. Amer. Math. Soc. 72 (1966), 1–29.

[NR2] Nijenhuis, A. and Richardson R.W., *Deformations of Lie algebra structures*, J. Math. Mech. 17 (1967), 89–105.

[R] Richardson, R.W., *On the rigidity of semi-direct products of Lie algebras*, Pacific. J. Math. 22 (1967), no. 2, 339–344.

[Se] Seeley, C., *Degenerations of 6-dimensional nilpotent Lie algebras over* $\mathbb{C}$, Comm. Algebra 18 (1990), no. 10, 3493–3505.

[St] Steinhoff C., *Klassifikation und Degeneration von Lie Algebren*, Diplomarbeit, Düsseldorf (1997).

[V] Vergne, M., *Cohomologie des algèbres de Lie nilpotentes*, Bull. Soc. Math. France, vol. 98, 81–116.

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