ALGEBRAIC FUNCTIONS IN ŁUKASIEWICZ IMPLICATION ALGEBRAS

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Abstract. In this article we study algebraic functions in $\{\to, 1\}$-subreducts of MV-algebras, also known as Łukasiewicz implication algebras. A function is algebraic on an algebra $\mathbf{A}$ if it is definable by a conjunction of equations on $\mathbf{A}$. We fully characterize algebraic functions on every Łukasiewicz implication algebra belonging to a finitely generated variety. The main tool to accomplish this is a factorization result describing algebraic functions in a subproduct in terms of the algebraic functions of the factors.

We prove a global representation theorem for finite Łukasiewicz implication algebras which extends a similar one already known for Tarski algebras. This result together with the knowledge of algebraic functions allowed us to give a partial description of the lattice of classes axiomatized by sentences of the form $\forall\exists! \bigwedge p \approx q$ within the variety generated by the 3-element chain.

1. Introduction

In this article we study definability of functions by conjunction of equations in the variety of $\{\to, 1\}$-subreducts of MV-algebras, also known as Łukasiewicz implication algebras. Functions defined by conjunctions of equations on a structure $\mathbf{A}$ are called *algebraic functions* on $\mathbf{A}$. They are closed under composition and include the term-functions of $\mathbf{A}$. Algebraic functions were introduced and studied in [7], where the reader can also find characterizations of algebraic functions for algebras in several well-known varieties. In our opinion algebraic functions constitute a natural and intriguing subject of research, as they are *one step* above of term-functions in the hierarchy of functions definable in the first order language of a structure. They also share several properties with term-functions, and thus are natural candidates for operations to expand the original structure with.

We study here the case of Łukasiewicz implication algebras, which constitute the equivalent algebraic semantics of the implicative fragment of the infinite-valued Łukasiewicz logic. As we will see, the structure of the algebraic functions on these algebras is very rich and new interesting operations arise even in the case of the 3-element chain.

As shown in [7], there is a close connection between algebraic functions in algebras in a variety $\mathbb{V}$ and the subclasses of $\mathbb{V}$ axiomatizable by sentences of the form $\forall\exists! \bigwedge p \approx q$. We call these sentences *Equational Function Definition* sentences (EFD-sentences for short), and the classes axiomatized by them *Algebraically Expandable classes* (AE classes for short) [6]. One of the main tools in the study of AE classes are special subdirect representations that preserve EFD-sentences, such as global subdirect products [13] and Boolean products [3]. For instance in [4] it is proved that every finite implication algebra is isomorphic to a global subdirect product with factors in a very simple class, and through this result the AE classes in the variety are completely described. In the current note we extend the representation theorem for implication algebras to all finite Łukasiewicz implication algebras. However the class of factors turns out to be quite more complicated, and we were only able to derive a partial description of the AE classes. This made our usual route (see e.g. [7]) to a characterization of algebraic functions unavailable. Interestingly we found a different approach, by proving a *factorization* result of algebraic functions in subalgebras of products, which in turn allowed for a characterization of algebraic functions in members of every finitely generated variety of Łukasiewicz implication algebras.

Outline. In the following section we provide definitions and summarize basic facts on Łukasiewicz implication algebras and EFD-sentences. Section 3 is devoted to prove that every finite Łukasiewicz implication algebra is isomorphic with a global subdirect product with factors in a concise family of globally indecomposable algebras (see Corollary 3.2). In Section 4 we show that, under quite restrictive circumstances, the algebraic functions of an algebra $\mathbf{A} \leq \prod \mathbf{A}_i$ are determined by the algebraic functions of the $\mathbf{A}_i$'s (see Theorem 4.2). In Section 5 we apply the results of the previous sections to give a characterization of algebraic functions for every member of a finitely generated variety of Łukasiewicz implication algebras (see Theorem 5.3). The special case of algebras in the variety generated by the 3-element chain is worked out in detail in Subsection 5.1. Section 6 concludes the paper with a partial description of the lattice of AE classes in the variety of Łukasiewicz implication algebras.

2. Preliminaries

Following the usual convention in universal algebra we use boldface letters for algebraic structures, whereas the corresponding letters in italics stand for their universes. We also note with I, H, S, P, P_u, V the standard class operators for isomorphic images, homomorphic images, subalgebras, direct products, ultraproducts and generated variety, respectively.

2.1. **Łukasiewicz implication algebras.** Łukasiewicz implication algebras are the $\{\to, 1\}$-subreduct of MV-algebras and the algebraic semantics of the implicative fragment of the infinite-valued Łukasiewicz logic (see [11]). They were also studied by Berman and Blok under the name of Łukasiewicz residuation algebras (see [2]). They can be defined as algebras in the language $\{\to, 1\}$ that satisfy the following equations:

- $1 \to x \approx x$,
- $(x \to y) \to ((y \to z) \to (x \to z)) \approx 1$,
- $(x \to y) \to y \approx (y \to x) \to x$,
- $(x \to y) \to (y \to x) \approx y \to x$.

Thus the class $\mathbb{L}$ of Łukasiewicz implication algebras is a variety. The lattice of subvarieties was determined by Komori in [11] and it was shown by the authors in [5] that every subquasivariety is a variety. Given a positive integer n, we let $L_n := \{\frac{k}{n} : 0 \leq k \leq n, k \in \mathbb{Z}\}$. For $a, b \in L_n$, we define $a \to b := \min\{1, 1 - a + b\}$. Thus $\mathbf{L}_n := \langle L_n, \to, 1\rangle$ is a Łukasiewicz implication algebra. Observe that $\mathbf{L}_n$ is embeddable in $\mathbf{L}_m$ if and only if $n \leq m$. The lattice of sub(quasi)varieties of $\mathbb{L}$ is given by

$$\mathbb{T} \subsetneq \mathsf{V}(\mathbf{L}_1) \subsetneq \mathsf{V}(\mathbf{L}_2) \subsetneq \dots \subsetneq \mathsf{V}(\mathbf{L}_n) \subsetneq \dots \subsetneq \mathbb{L},$$

where $\mathbb{T}$ denotes the variety of trivial algebras. Note that $\mathsf{V}(\mathbf{L}_1)$ is the variety of implication algebras, also known as Tarski algebras, and it consists of the $\{\to, 1\}$-subreducts of Boolean algebras (see [1] for details about these algebras). It is worth mentioning that each $\mathbf{L}_n$ is simple, and that these are all the finite subdirectly irreducible members in $\mathbb{L}$.

The following lemma summarizes some basic properties of these algebras that will be used throughout the article.

Lemma 2.1. *Given a Łukasiewicz implication algebra* $\mathbf{A}$*:*
(1) *The relation defined by* $x \leq y$ *iff* $x \to y = 1$ *is a partial ordering on* A *with last element* 1.
(2) *Given* $a, b, c \in A$*, if* $a \leq b$*, then* $c \to a \leq c \to b$ *and* $b \to c \leq a \to c$.
(3) *The join of any two elements in* A *exists and is given by the term* $x \vee y := (x \to y) \to y$.
(4) *Given* $a, b, c \in A$*,* $c \to (a \vee b) = (c \to a) \vee (c \to b)$ *and* $(a \vee b) \to c = (a \to c) \wedge (b \to c)$ *(the meet of* $a \to c$ *and* $b \to c$*).*
(5) *Given* $a, b, c \in A$ *such that* $a \wedge b$ *exists,* $c \to (a \wedge b) = (c \to a) \wedge (c \to b)$ *and* $(a \wedge b) \to c = (a \to c) \vee (b \to c)$.

An *implicative filter* on a Łukasiewicz implication algebra $\mathbf{A}$ is a subset $F \subseteq A$ such that $1 \in F$ and, whenever $a, a \to b \in F$, it is also the case that $b \in F$. There is a one-to-one correspondence between congruences and implicative filters. Indeed, if θ is a congruence on $\mathbf{A}$, then $1/\theta$, the congruence class of 1 modulo θ, is an implicative filter. Conversely, if F is an implicative filter, then $\theta_F := \{(a, b) \in A^2 : a \to b, b \to a \in F\}$ is a congruence.

The algebras $\mathbf{L}_n$ are finite chains with the ordering given in the last lemma and, furthermore, it can be shown that these are, up to isomorphism, the only totally ordered finite Łukasiewicz implication algebras.

With the help of the properties in the last lemma, it is easy to show that, for any given $a, b \in A$, $b \leq a \to b$. This implies that any increasing subset of $\mathbf{A}$ is a subalgebra. In [5] we exploit this fact and give a nice representation theorem for finite algebras, which we state below. If a Łukasiewicz implication algebra $\mathbf{P}$ has a bottom, its complemented elements form a lattice under the induced ordering. We refer to this lattice as the *Boolean skeleton* of $\mathbf{P}$.

Lemma 2.2. *Given a finite algebra* $\mathbf{A} \in \mathbb{L}$*, there exists a product of finite chains* $\mathbf{P}$ *such that* A *is an increasing subset of* $\mathbf{P}$ *and contains the co-atoms of the Boolean skeleton of* $\mathbf{P}$.

In the same article we also prove that the congruences on $\mathbf{A}$ are in a one-to-one correspondence with the subsets of the set of co-atoms of the Boolean skeleton of $\mathbf{P}$, and, more importantly, that every homomorphic image of a finite algebra $\mathbf{A}$ is isomorphic to a subalgebra of $\mathbf{A}$. In fact, every homomorphic image of a finite algebra $\mathbf{A}$ is a retract. As a consequence, since free algebras in $\mathsf{V}(\mathbf{L}_n)$ are finite, every finite algebra in $\mathsf{V}(\mathbf{L}_n)$ is a retract of a free algebra, that is, every finite algebra in $\mathsf{V}(\mathbf{L}_n)$ is weakly projective in $\mathsf{V}(\mathbf{L}_n)$ (see [5] for details).

2.2. **Algebraic functions and EFD-sentences.** In this subsection we introduce the formal definitions of EFD-sentence and algebraic function, and some of their basic properties that are needed in the sequel.

Let $\mathcal{L}$ be a first order language without relation symbols. An *Equational Function Definition sentence* (EFD-sentence for brevity) in the language $\mathcal{L}$ is a sentence of the form

$$\forall x_1 ... x_n \, \exists! z_1 ... z_m \, \bigwedge_{i=1}^{k} s_i(\bar{x}, \bar{z}) \approx t_i(\bar{x}, \bar{z}),$$

where s_i, t_i are $\mathcal{L}$-terms, $n \geq 0$ and $m \geq 1$. Observe that conjunctions of identities can be cast as EFD-sentences; for example $\forall \bar{x} \ \bigwedge_{i=1}^{k} s_i(\bar{x}) \approx t_i(\bar{x})$ is equivalent to $\forall \bar{x}\, \exists! z\ z \approx x_1 \wedge \bigwedge_{i=1}^{k} s_i(\bar{x}) \approx t_i(\bar{x})$. If φ is the above EFD-sentence, we define:

$$E(\varphi) := \forall \bar{x}\, \exists \bar{z} \ \bigwedge_{i=1}^{k} s_i(\bar{x}, \bar{z}) \approx t_i(\bar{x}, \bar{z}),$$

and

$$U(\varphi) := \forall \bar{x}\, \bar{z}\, \bar{y} \ \left(\bigwedge_{i=1}^{k} s_i(\bar{x}, \bar{z}) \approx t_i(\bar{x}, \bar{z}) \wedge \bigwedge_{i=1}^{k} s_i(\bar{x}, \bar{y}) \approx t_i(\bar{x}, \bar{y}) \right) \to \bar{z} \approx \bar{y}.$$

Thus φ is logically equivalent to the conjunction $E(\varphi) \wedge U(\varphi)$. Given an $\mathcal{L}$-structure $\mathbf{A}$ such that $\mathbf{A} \vDash \varphi$ the *functions defined by* φ in $\mathbf{A}$ are $[\varphi]_1^{\mathbf{A}}, \ldots, [\varphi]_m^{\mathbf{A}} : A^n \to A$, given by

$$\left([\varphi]_1^{\mathbf{A}}(\bar{a}), \ldots, [\varphi]_m^{\mathbf{A}}(\bar{a})\right) := \text{the unique } \bar{b} \in A^m \text{ such that } \bigwedge_{i=1}^{k} s_i(\bar{a}, \bar{b}) \approx t_i(\bar{a}, \bar{b}),$$

for all $\bar{a} \in A^n$. An *algebraic function* on $\mathbf{A}$ is a function of the form $[\varphi]_j^{\mathbf{A}}$ for some EFD-sentence φ that holds in $\mathbf{A}$. A *monoalgebraic function* on $\mathbf{A}$ is an algebraic function on $\mathbf{A}$ defined by an EFD-sentence ψ with a single existential quantifier (in this case we may drop the subscript and just write $[\psi]^{\mathbf{A}}$ to denote it). The collection of all algebraic functions on $\mathbf{A}$ is denoted by $\mathrm{Clo}_{\mathrm{alg}}\, \mathbf{A}$.

We say that an algebra $\mathbf{B}$ is a *descendant* of $\mathbf{A}$, and write $\mathbf{B} \prec \mathbf{A}$, if every EFD-sentence true of $\mathbf{A}$ is also true of $\mathbf{B}$. We denote the class of all descendants of $\mathbf{A}$ by $\mathsf{D}(\mathbf{A})$. The lemma below collects some basic properties of algebraic functions and EFD-sentences. Recall that a *clone* on a set A is a set of operations of finite arity on A which contains all projections and is closed under composition. Of course, the term-functions on an algebra $\mathbf{A}$ form a clone on A, which is denoted by $\mathrm{Clo}\, \mathbf{A}$.

Lemma 2.3. *Let φ be an EFD-sentence with m existential quantifiers.*

(1) *The composition of algebraic functions on $\mathbf{A}$ is algebraic, and each term-operation on $\mathbf{A}$ is an algebraic function. So $\mathrm{Clo}_{\mathrm{alg}}\, \mathbf{A}$ is a clone, and it contains $\mathrm{Clo}\, \mathbf{A}$.*
(2) *If $\{\mathbf{A}_i : i \in I\}$ is a family of algebras such that $\mathbf{A}_i \vDash \varphi$ for all $i \in I$ and $\mathbf{P} = \prod_{i \in I} \mathbf{A}_i$, then $\mathbf{P} \vDash \varphi$ and*
$[\varphi]_j^{\mathbf{P}}(p_1, ..., p_n)(i) = [\varphi]_j^{\mathbf{A}_i}(p_1(i), ..., p_n(i))$ for every $i \in I$, $j = 1, ..., m$, and any $p_1, ..., p_n \in \mathbf{P}$.
(3) *Suppose $\mathbf{A} \vDash \varphi$, and let $\mathbf{B} \leq \mathbf{A}$. Then $\mathbf{B} \vDash \varphi$ iff $[\varphi]_j^{\mathbf{A}}(B^n) \subseteq B$ for $j = 1, ..., m$.*
(4) *If $\mathbf{A} \vDash \varphi$ and $\mathbf{B} \in \mathsf{S}(\mathbf{A}) \cap \mathsf{H}(\mathbf{A})$, then $\mathbf{B} \vDash \varphi$. I.e., $\mathsf{S}(\mathbf{A}) \cap \mathsf{H}(\mathbf{A}) \subseteq \mathsf{D}(\mathbf{A})$.*
(5) *If $f \in \mathrm{Clo}_{\mathrm{alg}}\, \mathbf{A}$ then $\mathrm{Clo}_{\mathrm{alg}}(\mathbf{A}, f) = \mathrm{Clo}_{\mathrm{alg}}\, \mathbf{A}$.*
(6) *Let γ be an automorphism of $\mathbf{A}$ with at least one fixed point, and let $\mathbf{Fix}\, \gamma$ denote the subalgebra of $\mathbf{A}$ whose universe is the fixed-point set of γ. Then $\mathbf{Fix}\, \gamma \prec \mathbf{A}$.*
(7) *If $\mathbf{B}, \mathbf{C} \leq \mathbf{A}$ and $\mathbf{A}, \mathbf{B}, \mathbf{C} \vDash \varphi$, then $\mathbf{B} \cap \mathbf{C} \vDash \varphi$.*

Proof. For (1-5) see [7], and (6) can be found in [8]. □

For a finite Łukasiewicz implication algebra $\mathbf{A}$, we have that $\mathsf{H}(\mathbf{A}) \subseteq \mathsf{IS}(\mathbf{A})$ (see the comments following Lemma 2.2). Hence, point (4) in the last lemma can be considerably sharpened.

Corollary 2.4. *If $\mathbf{A}$ is a finite algebra in $\mathbb{L}$, then $\mathsf{H}(\mathbf{A}) \subseteq \mathsf{D}(\mathbf{A})$.*

3. Global representation of finite Łukasiewicz implication algebras

Global subdirect products were introduced and developed by P. Krauss and D. Clark in [12] as a special type of representation in terms of sheaves of algebras. They preserve EFD-sentences and thus prove very useful in the study of axiomatizability by these sentences. We recall here a simplified yet equivalent definition. Given a subdirect product $\mathbf{A} \leq \prod_{i \in I} \mathbf{A}_i$, we say that it is *global* if there exists a topology τ on the index set I such that:

(1) $E(a, b) := \{i \in I : a(i) = b(i)\} \in \tau$ for every $a, b \in A$.
(2) $\mathbf{A}$ has the following patchwork property over τ: given any subset $\mathcal{S} \subseteq \tau$ such that $\bigcup \mathcal{S} = I$ and given elements $a_S \in A$ for every $S \in \mathcal{S}$ such that $S_1 \cap S_2 \subseteq E(a_{S_1}, a_{S_2})$ for every $S_1, S_2 \in \mathcal{S}$, there exists a (unique) $a \in A$ such that $S \subseteq E(a, a_S)$ for every $S \in \mathcal{S}$.

In the present section we find a family $\mathcal{G}$ of algebras in $\mathbb{L}$ with the property that every finite member of $\mathbb{L}$ is isomorphic to a global subdirect product with factors in $\mathcal{G}$. The results are based upon and generalize the ones in [4].

Put $\mathbf{F}_1 := \mathbf{L}_1$ and let $\mathbf{F}_n$, $n \geq 2$, be the subalgebra of $\mathbf{L}_1^n$ with universe $F_n := \{0,1\}^n - \{(0, \ldots, 0)\}$, that is, $\mathbf{F}_n$ is obtained by removing the least element of $\mathbf{L}_1^n$. Let $\mathcal{G}^*$ be the class of all finite algebras $\mathbf{A}$ in $\mathbb{L}$ satisfying:

- $\mathbf{A} \cong \mathbf{L}_n$ for some $n \geq 1$, or
- $\mathbf{A}$ has n co-atoms and has a subalgebra isomorphic to $\mathbf{F}_n$ that has no meet in $\mathbf{A}$.

Let $\mathcal{G}$ be a class of algebras that consists of one algebra from each isomorphism class of algebras in $\mathcal{G}^*$.

We shall arrive at the desired representation result through the application of Theorem 3.5 in [14], which links global subdirect products with systems of congruence equations. Given $\mathbf{A} \in \mathbb{L}$, we let $\Sigma_{\mathbf{A}}$ be the set of congruences θ on $\mathbf{A}$ such that $\mathbf{A}/\theta \in \mathcal{G}^*$. Given congruences $\theta_1, \ldots, \theta_n$ on $\mathbf{A}$ and elements $a_1, \ldots, a_n \in A$, a tuple $(\theta_1, \ldots, \theta_n; a_1, \ldots, a_n)$ is said to be *a congruence system on* $\mathbf{A}$ if $(a_i, a_j) \in \theta_i \vee \theta_j$ for $i, j = 1, \ldots, n$. An element $b \in A$ is a *solution* for the system $(\theta_1, \ldots, \theta_n; a_1, \ldots, a_n)$ if $(b, a_i) \in \theta_i$ for $i = 1, \ldots, n$. The following is a *Chinese remainder theorem* with respect to $\Sigma_{\mathbf{A}}$.

Theorem 3.1. *Let* $\mathbf{A}$ *be a finite algebra in* $\mathbb{L}$*. Assume* $(\theta_1, \ldots, \theta_n; a_1, \ldots, a_n)$ *is a congruence system on* $\mathbf{A}$ *with the property that for every* $\theta \in \Sigma_{\mathbf{A}}$ *there exists* $i \in I$ *such that* $\theta_i \subseteq \theta$*. Then this system has a solution in* $\mathbf{A}$*.*

Proof. By Lemma 2.2 we may assume $\mathbf{A} \leq \mathbf{P}$, where $\mathbf{P}$ is a finite product of finite chains and $\mathbf{A}$ contains the set co-atoms C of the Boolean skeleton of $\mathbf{P}$. Let D be the set of meet-irreducible elements of $\mathbf{A}$. It is clear that $C \subseteq D$ and D consists of mutually incomparable chains, the elements of C being the least elements of these chains. For each $a \in A$ we set $D_a := \{d \in D : a \leq d\}$ and for each implicative filter F on $\mathbf{A}$ we set $D_F := F \cap D$.

Let F_i be the implicative filter corresponding to θ_i, $i = 1, \ldots, n$. Then the condition $(a_i, a_j) \in \theta_i \vee \theta_j$ is equivalent to $D_{a_i} \setminus (D_{F_i} \cup D_{F_j}) = D_{a_j} \setminus (D_{F_i} \cup D_{F_j})$ (see [5] for details).

Let $B := \bigcup_{i=1}^{n} (D_{a_i} \setminus D_{F_i})$. We claim that $b := \bigwedge B$ exists in $\mathbf{A}$ and is the solution to the congruence system. Suppose on the contrary that B has no meet in $\mathbf{A}$ and let $M \subseteq B$ be minimal with respect to the property of not having meet in $\mathbf{A}$; in other words, M has no meet in $\mathbf{A}$, but every proper subset of M does. Note that M contains at most one element of each chain of meet-irreducible elements. Let $M := \{d_1, \dots, d_k\}$ and let M' consist of the meets of proper subsets of M. Here we consider $\bigwedge \emptyset = 1 \in M'$. Since

$$d_i \to d_j = \begin{cases} 1 & \text{if } i = j, \\ d_j & \text{if } i \neq j, \end{cases}$$

it follows that

$$\bigwedge_{s=1}^{u} d_{i_s} \to \bigwedge_{t=1}^{v} d_{j_t} = \bigwedge_{t=1}^{v} \underbrace{\bigvee_{s=1}^{u} (d_{i_s} \to d_{j_t})}_{\in M \cup \{1\}} \in M'.$$

Thus M' is a subuniverse of $\mathbf{A}$ and it is clear that $\mathbf{M}' \cong \mathbf{F}_k$.

Let $U := \{c \in C : \text{for every } d \in M, c \not\leq d\}$ and let F be the implicative filter corresponding to U. Then, $\mathbf{A}/F$ has k co-atoms and contains a subalgebra isomorphic to $\mathbf{F}_k$ without meet, that is, the congruence that corresponds to F belongs to $\Sigma_{\mathbf{A}}$ and hence $F_\ell \subseteq F$ for some index ℓ. Then

$$M \subseteq B \setminus D_F \subseteq B \setminus D_{F_\ell} = \left(\bigcup_{i=1}^{n} (D_{a_i} \setminus D_{F_i}) \right) \setminus D_{F_\ell} = \bigcup_{i=1}^{n} (D_{a_i} \setminus (D_{F_i} \cup D_{F_\ell}))$$
$$= \bigcup_{i=1}^{n} (D_{a_\ell} \setminus (D_{F_i} \cup D_{F_\ell})) \subseteq D_{a_\ell}.$$

Thus $M \subseteq [a_\ell)$, which contradicts the assumption that M has no meet in $\mathbf{A}$.

So far we have proved that B has meet in $\mathbf{A}$. It remains to show that $b = \bigwedge B$ is in fact the solution to the congruence system. As B is an increasing subset of D, $B = D_b$ and, consequently, it suffices to show that $B \setminus D_{F_i} = D_{a_i} \setminus D_{F_i}$, $1 \leq i \leq n$. Indeed,

$$B \setminus D_{F_j} = \left(\bigcup_{i=1}^{n} D_{a_i} \setminus D_{F_i} \right) \setminus D_{F_j} = \bigcup_{i=1}^{n} \big(D_{a_i} \setminus (D_{F_i} \cup D_{F_j})\big) = \bigcup_{i=1}^{n} \big(D_{a_j} \setminus (D_{F_i} \cup D_{F_j})\big)$$
$$= D_{a_j} \cap D_{F_j}^c \cap \bigcup_{i=1}^{n} D_{F_i}^c = D_{a_j} \setminus D_{F_j}.$$

□

Corollary 3.2. *Every finite Łukasiewicz implication algebra is a global subdirect product of algebras in $\mathcal{G}$.*

Proof. This follows from Theorem 3.1 and [14, Theorem 3.5]. □

To prove that this is an optimal result, we claim that the algebras in $\mathcal{G}$ are globally indecomposable, that is, they cannot be represented as a global subdirect product of their proper homomorphic images. For each $n \geq 2$ put

$$s_i^n(x_1, \dots, x_n) := \bigvee_{j \neq i} x_j,$$

and define the following EFD-sentence

$$\varphi_n := \forall x_1, \dots, x_n \, \exists! z \left(z \to s_1^n(\bar{x}) \approx 1 \,\&\, \dots \,\&\, z \to s_n^n(\bar{x}) \approx 1 \,\&\, \bigvee_{i=1}^{n} (s_i^n(\bar{x}) \to z) \approx 1 \right).$$

These sentences were introduced in [4], where it is shown that every EFD-sentence in the language $\{\to, 1\}$ is equivalent over $V(\mathbf{L}_1)$ to one of these. Notice that for $\mathbf{A} \in \mathbb{L}$ we have that $\mathbf{A} \vDash \varphi_n$ if and only the meet

$$\bigwedge_{i=1}^{n} s_i^n(\bar{a})$$

exists in $\mathbf{A}$ for all $\bar{a}$. Observe as well that $[\varphi_n]^{\mathbf{A}} = \bigwedge_{i=1}^{n} s_i^n(\bar{a})$. In particular, φ_2 defines the meet in every algebra satisfying it.

Lemma 3.3. *If* $\mathbf{A}$ *is a finite algebra in* $\mathbb{L}$ *with* n *co-atoms, then* $\mathbf{A} \vDash \varphi_m$ *for every* $m > n$.

Proof. Let $\mathbf{A}$ be a finite algebra in $\mathbb{L}$ with n co-atoms, and fix $m > n$. Note that Lemma 2.2 implies that there is a natural number k such that $\mathbf{A}$ is an increasing subset of $\mathbf{L}_k^n$. Since $\mathbf{L}_k$ satisfies φ_m it follows from (2) in Lemma 2.3 that $\mathbf{L}_k^n$ satisfies it as well, and that

$$[\varphi_m]^{\mathbf{L}_k^n}(u_1, \dots, u_m)(i) = [\varphi_m]^{\mathbf{L}_k}(u_1(i), \dots, u_m(i)),$$

for all $\bar{u}$ from $\mathbf{L}_k^n$ and $i = 1, \dots, n$. By (3) in Lemma 2.3 we are done if we can show that A is closed under $[\varphi_m]^{\mathbf{L}_k^n}$. Let us take a look at how $[\varphi_m]^{\mathbf{L}_k}$ behaves. Given $a_1, \dots, a_m \in L_k$ and $\tilde{a} := \max\{a_1, \dots, a_m\}$ we have:

- $[\varphi_m]^{\mathbf{L}_k}(a_1, \dots, a_m) = \tilde{a}$, if $\tilde{a}$ appears more than once among the elements $a_1, \dots, a_m$,
- $[\varphi_m]^{\mathbf{L}_k}(a_1, \dots, a_m) = \max(\{a_1, \dots, a_m\} \setminus \{\tilde{a}\})$, otherwise.

Thus $[\varphi_m]^{\mathbf{L}_k}(a_1, \dots, a_m) \geq a_j$ for all but possibly one index j. Now given $u_1, \dots, u_m \in L_k^n$ and $i \in \{1, \dots, n\}$ we have that

$$[\varphi_m]^{\mathbf{L}_k^n}(u_1, \dots, u_m)(i) \geq u_j(i),$$

for every j except possibly for $j = j_i$. Since $m > n$ there is an index j_0 such that

$$[\varphi_m]^{\mathbf{L}_k^n}(u_1, \dots, u_m)(i) \geq u_{j_0}(i),$$

for $i = 1, \dots, n$; so $[\varphi_m]^{\mathbf{L}_k^n}(u_1, \dots, u_m) \geq u_{j_0}$. Thus, as A is an increasing subset of $\mathbf{L}_k^n$ it must be closed under $[\varphi_m]^{\mathbf{L}_k^n}$. ☐

Corollary 3.4. *Every algebra in* $\mathcal{G}$ *is globally indecomposable.*

Proof. Let $\mathbf{A} \in \mathcal{G}$. If $\mathbf{A} \cong \mathbf{L}_n$ for some n, the conclusion is clear since $\mathbf{A}$ is subdirectly irreducible. Assume $\mathbf{A}$ has n co-atoms and a subalgebra isomorphic to $\mathbf{F}_n$ that has no meet in $\mathbf{A}$. It follows that $\mathbf{A} \nvDash \varphi_n$. Suppose now that $\mathbf{A}$ were a global subdirect product of some of its proper homomorphic images. Since every proper homomorphic image of $\mathbf{A}$ has less co-atoms than $\mathbf{A}$, we conclude that every factor satisfies φ_n, which in turn implies that $\mathbf{A} \vDash \varphi_n$, a contradiction. ☐

4. Algebraic function factorization in subalgebras of products

In this section we look at the following problem: given a subalgebra $\mathbf{A}$ of a product $\mathbf{P} := \prod_{i \in I} \mathbf{A}_i$, what is the connection (if any) between the algebraic functions of the $\mathbf{A}_i$'s and the algebraic functions of $\mathbf{A}$? As it happens not much can be said in the general case, however under some fairly special circumstances the algebraic functions of $\mathbf{A}$ are exactly the restrictions of the algebraic functions of $\mathbf{P}$ that preserve A. In turn, the algebraic functions of $\mathbf{P}$ are not hard to determine in terms of the algebraic functions of the $\mathbf{A}_i$'s.

We start out with a basic lemma concerning existential formulas.

Lemma 4.1. *Let $\mathbb{K}$ be a class of structures and $\phi(\bar{x},\bar{y})$ an existential formula such that $\exists\bar{y}\ \phi(\bar{x},\bar{y})$ is preserved by substructures of structures in $\mathbb{K}$ (i.e., if $\mathbf{B}\in\mathbb{K}$ and $\mathbf{A}\leq\mathbf{B}$, then for all $\bar{a}$ from A we have that $\mathbf{B}\vDash\exists\bar{y}\ \phi(\bar{a},\bar{y})$ implies $\mathbf{A}\vDash\exists\bar{y}\ \phi(\bar{a},\bar{y})$). Then,*

$$\mathbb{K}\vDash\exists\bar{y}\ \phi(\bar{x},\bar{y})\leftrightarrow\bigvee\nolimits_{\bar{t}\in T(\bar{x})^m}\phi(\bar{x},\bar{t}(\bar{x})),$$

where $T(\bar{x})$ denotes the set of all terms in the variables $\bar{x}$.

Furthermore, if the free algebra $\mathbf{F}_{\mathsf{V}(\mathbb{K})}(\bar{x})$ is finite or $\mathbb{K}$ is closed under ultraproducts, then the disjunction above can be taken over a finite set of tuples in $T(\bar{x})^m$.

Proof. We only prove the part concerning ultraproducts. Suppose for the sake of contradiction that for every finite $F\subseteq T(\bar{x})^m$ there are $\mathbf{A}_F\in\mathbb{K}$ and $\overline{a_F}$ from A_F such that

$$\mathbf{A}_F\vDash\exists\bar{y}\ \phi(\overline{a_F},\bar{y})\wedge\neg\bigvee\nolimits_{\bar{t}\in F}\phi(\overline{a_F},\bar{t}(\overline{a_F})).$$

Let u be an ultrafilter over $I:=\mathcal{P}_{fin}(T(\bar{x})^m)$ such that for each $\bar{t}\in T(\bar{x})^m$ we have $\{F\in I:\bar{t}\in F\}\in u$. Then, if we take $\mathbf{U}:=\prod_{F\in I}\mathbf{A}_F/u$ and $\bar{a}$ from $\prod_{F\in I}A_F$ given by $\bar{a}(F)=\overline{a_F}$, we have

$$\mathbf{U}\vDash\exists\bar{y}\ \phi(\bar{a}/u,\bar{y})\wedge\neg\bigvee\nolimits_{\bar{t}\in T(\bar{x})^m}\phi(\bar{a}/u,\bar{t}(\bar{a}/u)),$$

clearly a contradiction. □

Call a class of algebras $\mathbb{K}$ *disjunctive* if there is a finite conjunction of equations $\tau(x,y,z,w)$ such that

$$\mathbb{K}\vDash\tau(x,y,z,w)\leftrightarrow x\approx y\vee z\approx w.$$

In the article [9] the reader can find several equivalent conditions for a class to be disjunctive. Observe that if $\mathbb{K}$ is disjunctive and $\rho(\bar{x})$ is a positive open formula, then there is a conjunction of equations $\delta(\bar{x})$ such that $\mathbb{K}\vDash\rho(\bar{x})\leftrightarrow\delta(\bar{x})$.

If C is a clone over a set B and $A\subseteq B$, we define

$$\mathrm{C}|_A:=\{f|_A:f\in\mathrm{C}\text{ and }A\text{ is closed under }f\}.$$

It is easy to check that $\mathrm{C}|_A$ is a clone on A.

For a class of algebras $\mathbb{K}$ let

$$\mathrm{Clo}_{\mathbb{K}\text{-alg}}\,\mathbf{A}:=\{[\varphi]_j^{\mathbf{A}}:\varphi\text{ is an EFD-sentence and }\mathbb{K}\cup\{\mathbf{A}\}\vDash\varphi\}.$$

So $\mathrm{Clo}_{\mathbb{K}\text{-alg}}\,\mathbf{A}$ consists of the algebraic functions of $\mathbf{A}$ that can be defined by an EFD-sentence valid in $\mathbb{K}$. The proof of [7, Proposition 2] showing that algebraic functions on a structure form a clone can be adapted with little work to see that $\mathrm{Clo}_{\mathbb{K}\text{-alg}}\,\mathbf{A}$ is a clone as well. In fact it is a subclone of $\mathrm{Clo}_{\mathrm{alg}}\,\mathbf{A}$ and it contains $\mathrm{Clo}\,\mathbf{A}$.

Here is the main result of this section.

Theorem 4.2. *Suppose $\mathbf{A}\leq\prod_{i\in I}\mathbf{A}_i$ and assume $\{\mathbf{A}_i:i\in I\}$ is disjunctive and it is contained in $\mathsf{D}(\mathbf{A})$. Let $\mathbb{K}:=\mathsf{SP_u}(\{\mathbf{A}_i:i\in I\})\cap\mathsf{D}(\mathbf{A})$. Then,*

$$\mathrm{Clo}_{\mathrm{alg}}\,\mathbf{A}=\left(\mathrm{Clo}_{\mathbb{K}\text{-alg}}\prod_{i\in I}\mathbf{A}_i\right)\Bigg|_A.$$

Proof. The proof of ($\subseteq$) is routine. We prove ($\supseteq$). Set $\mathbf{P} := \prod_{i\in I} \mathbf{A}_i$ and take $f \in \mathrm{Clo}_{\mathbb{K}\text{-alg}}\, \mathbf{P}$ such that f preserves A. There is an EFD-sentence $\varphi := \forall \bar{x}\, \exists! \bar{z}\ \varepsilon(\bar{x}, \bar{z})$ such that $\mathbb{K} \cup \{\mathbf{P}\} \vDash \varphi$ and $[\varphi]_1^{\mathbf{P}} = f$. Next, observe that $\mathbb{K}' := \mathsf{SP_u}(\{\mathbf{A}_i : i \in I\})$ is disjunctive. Let Σ be the set of EFD-sentences valid in $\mathbf{A}$. Clearly $\mathbf{A} \vDash \Sigma$ and $\mathrm{Mod}\,\Sigma \cap \mathbb{K}' = \mathbb{K}$. Let $\mathbb{K}_\Sigma$ be the algebraic expansion of $\mathbb{K}$ with respect to Σ (see [7] for the definition), and note that $\mathbb{K}_\Sigma$ is a universal and disjunctive class satisfying φ. By Lemma 4.1 there are $\overline{t_1}(\bar{x}), \dots, \bar{t_k}(\bar{x}) \in T(\bar{x})^{m-1}$ such that

$$\mathbb{K}_\Sigma \vDash \exists \bar{z}\ \varepsilon(\bar{x}, z, z_2, \dots, z_m) \leftrightarrow \exists z \bigvee_{j=1}^{k} \varepsilon(\bar{x}, z, \overline{t_j}(\bar{x})).$$

Now let $\delta(\bar{x}, z)$ be a conjunction of equations such that

$$\mathbb{K}_\Sigma \vDash \bigvee_{j=1}^{k} \varepsilon(\bar{x}, z, \overline{t_j}(\bar{x})) \leftrightarrow \delta(\bar{x}, z),$$

and define $\psi := \forall \bar{x}\, \exists! z\ \delta(\bar{x}, z)$. As $\mathbb{K}_\Sigma \vDash \psi$ we have that $\mathbf{P}_\Sigma \vDash \psi$, and it is not hard to check that $f = [\psi]^{\mathbf{P}_\Sigma}$. We know that f preserves A, thus $\mathbf{A}_\Sigma \vDash \psi$ and $f|_A \in \mathrm{Clo}_{\mathrm{alg}}\, \mathbf{A}_\Sigma$. Finally, (5) in Lemma 2.3, implies $f|_A \in \mathrm{Clo}_{\mathrm{alg}}\, \mathbf{A}$. ☐

5. Algebraic functions in Łukasiewicz implication algebras

We now put to work the results in the previous section. In order to apply Theorem 4.2 to Łukasiewicz implication algebras we first need the following fact.

Lemma 5.1. *The class of* $\mathbb{L}$*-chains is disjunctive.*

Proof. Let $\tau(x, y, z, w)$ be the conjunction of the following four equations: $(x \to y) \vee (z \to w) \approx 1$, $(x \to y) \vee (w \to z) \approx 1$, $(y \to x) \vee (z \to w) \approx 1$, $(y \to x) \vee (w \to z) \approx 1$. Then $\tau(x, y, z, w)$ is equivalent to $x \approx y \vee z \approx w$ over the class of $\mathbb{L}$-chains. ☐

Next we look at $\mathbb{K}$-algebraic functions on a finite chain.

Lemma 5.2. *Let* ψ *be an EFD-sentence valid in* $\mathbf{L}_n$*, and set* $\mathbb{K} := \mathrm{Mod}\,\psi \cap \mathsf{S}(\mathbf{L}_n)$*. Then, a function* f *is in* $\mathrm{Clo}_{\mathbb{K}\text{-alg}}\, \mathbf{L}_n$ *if and only if*
(1) *every* $\mathbf{L} \in \mathbb{K}$ *is closed under* f*, and*
(2) *if* $\mathbf{L}, \mathbf{L}' \in \mathbb{K}$ *and* $\gamma : \mathbf{L} \to \mathbf{L}'$ *is an isomorphism, then* γ *preserves* f*.*

Proof. We prove the nontrivial direction. Suppose f satisfies (1) and (2). Consider the expansion

$$\mathbf{L}_n^* := \left\langle \mathbf{L}_n, \wedge, [\psi]_1^{\mathbf{L}_n}, \dots, [\psi]_m^{\mathbf{L}_n} \right\rangle,$$

where all the functions defined by ψ are listed. Note that every function added, including the meet, is in $\mathrm{Clo}_{\mathbb{K}\text{-alg}}\, \mathbf{L}_n$. Also note that the term

$$t := ((x \leftrightarrow y) \overset{n}{\to} z) \wedge (((x \leftrightarrow y) \overset{n}{\to} x) \to x))$$

interprets as the ternary discriminator on $\mathbf{L}_n^*$; here $x \leftrightarrow y$ stands for $(x \to y) \wedge (y \to x)$, and $x \overset{n}{\to} y$ is defined recursively by $x \overset{0}{\to} y := y$, $x \overset{n+1}{\to} y := x \to (x \overset{n}{\to} y)$. Hence, by (1), (2) and the fact that $\mathsf{S}(\mathbf{L}_n^*) = \mathbb{K}$, we can apply Pixley's Theorem characterizing term-functions in quasiprimal algebras (see Theorem 3.4.4 in [10]) to conclude that $f \in \mathrm{Clo}\, \mathbf{L}_n^*$. So, as $\mathrm{Clo}\, \mathbf{L}_n^* \subseteq \mathrm{Clo}_{\mathbb{K}\text{-alg}}\, \mathbf{L}_n$, we have $f \in \mathrm{Clo}_{\mathbb{K}\text{-alg}}\, \mathbf{L}_n$. ☐

We are ready to present our main result characterizing algebraic functions in algebras from every proper subvariety of $\mathbb{L}$.

Theorem 5.3. *Let* $\mathbf{A} \leq \mathbf{L}_n^I$*, suppose* $\mathbf{L}_n \in \mathsf{H}(\mathbf{A})$ *and let* $\mathbb{K} := \mathsf{S}(\mathbf{L}_n) \cap \mathsf{D}(\mathbf{A})$*. For a function* $f : A^k \to A$ *the following are equivalent:*

(1) *f is algebraic on* $\mathbf{A}$*.*
(2) $f \in \left(\mathrm{Clo}_{\mathbb{K}\text{-alg}}\, \mathbf{L}_n^I\right)\big|_A$*.*
(3) *There is a function* $f^* : L_n^k \to L_n$ *such that:*
  (*a*) $f(\bar{a})(i) = f^*(\bar{a}(i))$*, for all* $\bar{a} \in A^k$ *and* $i \in I$*.*
  (*b*) *If* $\mathbf{L} \in \mathbb{K}$*, then* L *is closed under* f^**.*
  (*c*) *If* $\mathbf{L}, \mathbf{L}' \in \mathbb{K}$ *and* $\gamma : \mathbf{L} \to \mathbf{L}'$ *is an isomorphism, then* γ *preserves* f^**.*
(4) *There is* $f^* \in \mathrm{Clo}_{\mathbb{K}\text{-alg}}\, \mathbf{L}_n$ *such that* $f(\bar{a})(i) = f^*(\bar{a}(i))$*, for all* $\bar{a} \in A^k$ *and* $i \in I$*.*

Proof. We know that $\mathbf{L}_n$ is disjunctive (Lemma 5.1) and it is a descendant of $\mathbf{A}$ (Corollary 2.4). Thus (1)⇔(2) is immediate from Theorem 4.2. We prove (2)⇒(3) next. If $f \in \left(\mathrm{Clo}_{\mathbb{K}\text{-alg}}\, \mathbf{L}_n^I\right)\big|_A$ there is an EFD-sentence φ such that $\mathbb{K} \vDash \varphi$ and $f = [\varphi]_1^{\mathbf{A}} = [\varphi]_1^{\mathbf{L}_n^I}|_A$. Taking $f^* := [\varphi]_1^{\mathbf{L}_n}$ it is routine to check that (3) holds.

Observe that (3)⇒(4) easily follows from Lemma 5.2. It only remains to show now that (4)⇒(2). Let f^* be as in (4). Fix an EFD-sentence φ such that $\mathbb{K} \vDash \varphi$ and $[\varphi]_1^{\mathbf{L}_n} = f^*$. Then (2) in Lemma 2.3 implies that $\mathbf{L}_n^I \vDash \varphi$ and we have that

$$[\varphi]_1^{\mathbf{L}_n^I}(\bar{a})(i) = [\varphi]_1^{\mathbf{L}_n}(\bar{a}(i)) = f^*(\bar{a}(i)) = f(\bar{a})(i),$$

for all $\bar{a} \in A^k$ and $i \in I$. Thus A is closed under $[\varphi]_1^{\mathbf{L}_n^I}$ and

$$f = [\varphi]_1^{\mathbf{L}_n^I}|_A \in \left(\mathrm{Clo}_{\mathbb{K}\text{-alg}}\, \mathbf{L}_n^I\right)\big|_A.$$

□

For $\mathbf{A}$ in a proper subvariety of $\mathbb{L}$, there is a largest finite chain $\mathbf{L}_n$ that is a homomorphic image of $\mathbf{A}$. Theorem 5.3 says that $\mathrm{Clo}_{\mathrm{alg}}\, \mathbf{A}$ can be determined in two steps:

(1) find the class $\mathbb{K}$ of subchains of $\mathbf{L}_n$ that are descendants of $\mathbf{A}$, and then
(2) compute $\mathrm{Clo}_{\mathbb{K}\text{-alg}}\, \mathbf{L}_n$.

The hardest part here is the first step; once it is accomplished there are tools that make the second step manageable (for example through Lemma 5.2 above). Even though we were not able to produce a satisfactory description of $\mathbb{K}$ in terms of $\mathbf{A}$ in the general case, we illustrate in the upcoming subsection how this issue can be handled in a particular case.

5.1. **Case study: Algebraic functions in $\mathsf{V}(\mathbf{L}_2)$.** Next we apply the results of the previous sections to characterize the algebraic functions on every algebra in $\mathsf{V}(\mathbf{L}_2)$. We will need the following two auxiliary results first.

Given a term $t(\bar{x})$ in the language $\{\to, \wedge, 1\}$ and an algebra $\mathbf{A} \in \mathbb{L}$ we write $t^{\mathbf{A}}$ to denote the obvious partial function defined by $t(\bar{x})$ in $\mathbf{A}$. If s and t are $\{\to, \wedge, 1\}$-terms, the expression $\mathbf{A} \vDash t \approx s$ means that $t^{\mathbf{A}} = s^{\mathbf{A}}$.

Lemma 5.4. *Given a term t in the language* $\{\to, \wedge, 1\}$*, there exist terms* $t_1, \ldots, t_n$ *in the language* $\{\to, 1\}$ *such that* $\mathbb{L} \vDash t \approx t_1 \wedge t_2 \wedge \ldots \wedge t_n$*.*

Proof. By induction on t. The statement is clear when t is a variable or a constant. Let t be a term of size n and let us assume the result holds for terms of size less than n. The are two possibilities:

- $t = t_1 \to t_2$: There exist terms t_{ik} in the language $\{\to, 1\}$ such that $\mathbb{L} \vDash t_i \approx \bigwedge_{k=1}^{m_i} t_{ik}$, $i = 1, 2$. Hence $\mathbb{L}$ satisfies

$$t \approx t_1 \to t_2 \approx \bigwedge_{k=1}^{m_1} t_{1k} \to \bigwedge_{r=1}^{m_2} t_{2r} \approx \bigwedge_{r=1}^{m_2} \left(\bigwedge_{k=1}^{m_1} t_{1k} \to t_{2r} \right) \approx \bigwedge_{r=1}^{m_2} \bigvee_{k=1}^{m_1} (t_{1k} \to t_{2r}).$$

- $t = t_1 \wedge t_2$: This case is immediate.

□

Corollary 5.5. *Let* $\mathbf{A} \in \mathbb{L}$ *and let* t *be a* $\{\to, \wedge, 1\}$*-term such that* $t^{\mathbf{A}}$ *is a total function. Then* $t^{\mathbf{A}}$ *is monoalgebraic on* $\mathbf{A}$*.*

Proof. By Lemma 5.4, there is a term $\bigwedge_{k=1}^{m} t_k$, with each t_k a $\{\to, 1\}$-term, such that $\mathbf{A} \vDash t \approx \bigwedge_{k=1}^{m} t_k$. Let

$$\varphi := \forall \bar{x}\, \exists! z \left(z \to t_1(\bar{x}) \approx 1 \,\&\, z \to t_2(\bar{x}) \approx 1 \,\&\, \ldots \,\&\, z \to t_m(\bar{x}) \approx 1 \,\&\, \bigvee_{k=1}^{m} (t_k(\bar{x}) \to z) \approx 1 \right).$$

Then $\mathbf{A} \vDash \varphi$ and $[\varphi]^{\mathbf{A}} = t^{\mathbf{A}}$. □

5.1.1. *Algebraic functions in* $\mathsf{V}(\mathbf{L}_1)$. Recall the definition of φ_n for each $n \geq 2$ from Section 3. To make the following statement more compact let us define $\varphi_\infty := \forall x\, \exists! z\; x \approx z$.

Theorem 5.6. *Let* $\mathbf{A} \in \mathsf{V}(\mathbf{L}_1)$*. For a function* $f : A^k \to A$ *the following are equivalent:*

(1) f *is algebraic on* $\mathbf{A}$*.*
(2) f *is monoalgebraic on* $\mathbf{A}$*.*
(3) *There is a term* t *in the language* $\{\wedge, \to, 1\}$ *such that* $f = t^{\mathbf{A}}$*.*
(4) $f \in \mathrm{Clo}\left(\mathbf{A}, [\varphi_m]^{\mathbf{A}}\right)$*, where* m *is the least element of* $\{n \in \omega : 2 \leq n\} \cup \{\infty\}$ *such that* $\mathbf{A} \vDash \varphi_m$*.*

Proof. Note that (4)⇒(3) and (2)⇒(1) are both trivial, and (3)⇒(2) follows from Corollary 5.5. We prove (1)⇒(4). Let $f \in \mathrm{Clo}_{\mathrm{alg}}\, \mathbf{A}$, and take an EFD-sentence φ such that $f = [\varphi]_1^{\mathbf{A}}$. By [4] there is $t \in \{n \in \omega : 2 \leq n\} \cup \{\infty\}$ such that φ is equivalent over $\mathsf{V}(\mathbf{L}_1)$ to φ_t. So, by [7, Proposition 8] $f \in \mathrm{Clo}\left(\mathbf{A}, [\varphi_t]^{\mathbf{A}}\right)$. It is easy to see that $t \geq m$ implies $[\varphi_t]^{\mathbf{A}} \in \mathrm{Clo}\left(\mathbf{A}, [\varphi_m]^{\mathbf{A}}\right)$; thus $f \in \mathrm{Clo}\left(\mathbf{A}, [\varphi_m]^{\mathbf{A}}\right)$. □

As we had very detailed information on the EFD-sentences in $\mathsf{V}(\mathbf{L}_1)$, it was not necessary to invoke Theorem 5.3 in the proof above. It is not hard to see though, that (1-3) are equivalent using Theorem 5.3.

It is worth to point out that Theorem 5.6 characterizes the algebras in $\mathsf{V}(\mathbf{L}_1)$ in which every algebraic function is a term-function precisely as those that do not satisfy φ_n for $1 \leq n < \infty$. An example of this case is given by the algebra obtained by removing the bottom element of $\mathbf{L}_1^{\omega}$.

5.1.2. *Algebraic functions in* $\mathsf{V}(\mathbf{L}_2) \setminus \mathsf{V}(\mathbf{L}_1)$. Given $\mathbf{A} \in \mathsf{V}(\mathbf{L}_2) \setminus \mathsf{V}(\mathbf{L}_1)$ we have that $\mathbf{L}_2$ is necessarily a quotient of $\mathbf{A}$, and thus a descendant of $\mathbf{A}$. In order to have Theorem 5.3 yield a characterization of the algebraic functions on $\mathbf{A}$ we need to determine whether $\mathbf{L}_1$ is a descendant of $\mathbf{A}$ or not.

Define

$$\varphi_b := \forall x \, \exists! z \, (x \to z \approx p(x,z) \to (p(x,z) \to x)),$$

where $p(x,z) := (z \to x) \vee z$. We introduce a new function symbol b, and write $b^{\mathbf{A}}$ to denote $[\varphi_b]^{\mathbf{A}}$ in the algebras satisfying φ_b, (we may omit the superscript in $b^{\mathbf{A}}$ if there is no risk of confusion). It is easily verified that $\mathbf{L}_2 \vDash \varphi_b$ and that $b^{\mathbf{L}_2}(1) = 1$, $b^{\mathbf{L}_2}(\frac{1}{2}) = 0$, $b^{\mathbf{L}_2}(0) = \frac{1}{2}$. Note also that $\mathbf{L}_1 \nvDash \varphi_b$.

Given $n \geq 2$, we denote by $\mathbf{T}_n$ the subalgebra of $\mathbf{L}_2^n$ with universe

$$T_n := \{(a_1, \ldots, a_n) \in L_2^n : a_i = 1 \text{ for some } i\}.$$

We also set $\mathbf{T}_1 := \mathbf{L}_2$.

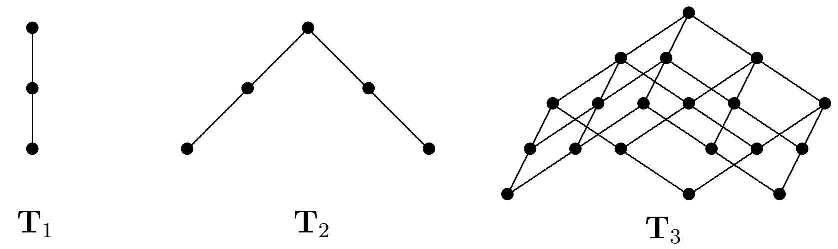

FIGURE 5.1. $\mathbf{T}_1$, $\mathbf{T}_2$ and $\mathbf{T}_3$

Theorem 5.7. *Let* $\mathbf{A} \in \mathsf{V}(\mathbf{L}_2)$. *The following statements are equivalent:*

(1) $\mathbf{A} \vDash \varphi_b$.

(2) $\mathbf{L}_1$ *is not a descendant of* $\mathbf{A}$.

Furthermore, if $\mathbf{A}$ *is finite, the following statement is also equivalent:*

(3) $\mathbf{A}$ *is a global subdirect product of algebras in* $\{\mathbf{T}_n : n \geq 1\}$.

We have some work to do to prove this theorem. Recall that any algebra in $\mathsf{V}(\mathbf{L}_2)$ is a global subdirect product of algebras in the family $\mathcal{G}_2 := \mathcal{G} \cap \mathsf{V}(\mathbf{L}_2)$. Note that $\mathbf{T}_n \in \mathcal{G}_2$ for every $n \geq 1$.

Lemma 5.8. $\mathbf{T}_n \prec \mathbf{T}_{n+1}$ *for every* $n \geq 1$. *Furthermore,* $\mathbf{T}_{n+1} \not\prec \mathbf{T}_n$ *for all* $n \geq 1$.

Proof. For the case $n = 1$, just note that $\mathbf{T}_1 \in \mathsf{H}(\mathbf{T}_2)$. For $n > 1$, let $\gamma : \mathbf{T}_{n+1} \to \mathbf{T}_{n+1}$ be the automorphism that permutes the first two coordinates. Then $S := \{\bar{x} \in T_{n+1} : \gamma(\bar{x}) = \bar{x}\}$ is a subuniverse of $\mathbf{T}_{n+1}$ and $\mathbf{S} \cong \mathbf{T}_n$. By item (6) in Lemma 2.3, $\mathbf{T}_n \prec \mathbf{T}_{n+1}$. □

Our next step is to show that $\{\mathbf{T}_n : n \geq 1\}$ is axiomatized by φ_b relative to $\mathcal{G}_2$.

Lemma 5.9. *If* $\mathbf{F} \leq \mathbf{L}_2^n$ *and* $\mathbf{F} \cong \mathbf{L}_1^n$, *then there is a partition* $\{I, J\}$ *of* $\{1, 2, \ldots, n\}$ *such that*

$$F = \{(a_1, \ldots, a_n) \in L_2^n : a_i \in \{0,1\} \textit{ for } i \in I, a_i \in \{\tfrac{1}{2}, 1\} \textit{ for } i \in J\}.$$

Proof. Fix $i \in I$ and consider the ith projection $\pi_i : L_2^n \to L_2$. Since $\pi_i(\mathbf{F})$ cannot be isomorphic to $\mathbf{L}_2$, $\pi_i(F)$ must be $\{0,1\}$, $\{\frac{1}{2},1\}$ or $\{1\}$. We have thus shown that there exist $I, J, K \subseteq \{1,2,\ldots,n\}$, pairwise disjoint, such that $I \cup J \cup K = \{1,2,\ldots,n\}$ and

$$F \subseteq \{(a_1,\ldots,a_n) \in L_2^n : a_i \in \{0,1\} \text{ for } i \in I, a_i \in \{\tfrac{1}{2},1\} \text{ for } i \in J, a_i = 1 \text{ for } i \in K\}.$$

Since F has 2^n elements, it is clear that $K = \emptyset$ and that the equality holds. □

Lemma 5.10. *For all* $\mathbf{A} \in \mathcal{G}_2$ *we have* $\mathbf{A} \models \varphi_b$ *iff* $\mathbf{A} = \mathbf{T}_n$ *for some* $n \geq 1$.

Proof. By item 2 in Lemma 2.3, $\mathbf{L}_2^n \models \varphi_b$. It is clear that T_n is closed under $b^{\mathbf{L}_2^n}$ and thus $\mathbf{T}_n \models \varphi_b$.

Conversely, assume $\mathbf{A} \in \mathcal{G}_2$ is such that $\mathbf{A} \models \varphi_b$. Since $\mathbf{L}_1 \not\models \varphi_b$, the algebra $\mathbf{L}_1$ cannot be a homomorphic image of $\mathbf{A}$ (recall Corollary 2.4). Consequently we may consider $\mathbf{A}$ as an increasing subset of $\mathbf{L}_2^n$ that contains the co-atoms of the Boolean skeleton of $\mathbf{L}_2^n$. Observe that $\mathbf{A}$ must be closed under $b^{\mathbf{L}_2^n}$.

Assume A has an element in $\mathbf{L}_2^n$ all of whose coordinates differ from 1. Since A is increasing, $(\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2}) \in A$. Thus, $b^{\mathbf{L}_2^n}(\frac{1}{2},\ldots,\frac{1}{2}) = (0,\ldots,0) \in A$ and, therefore, $\mathbf{A} = \mathbf{L}_2^n$. This is a contradiction since $\mathbf{L}_2^n \notin \mathcal{G}_2$. We conclude that $A \subseteq T_n$. In order to see that the equality holds, it is sufficient to show that A contains the atoms of the Boolean skeleton of $\mathbf{L}_2^n$.

Since $\mathbf{A} \in \mathcal{G}_2$, it has a subalgebra isomorphic to $\mathbf{F}_n$ without meet. Lemma 5.9 says that there is a partition $\{I, J\}$ of $\{1,2,\ldots,n\}$ such that

$$\{(a_1,\ldots,a_n) \in L_2^n : a_i \in \{0,1\} \text{ for } i \in I, a_i \in \{\tfrac{1}{2},1\} \text{ for } i \in J\} \cap T_n \subseteq A.$$

From this and the fact that A is increasing, it follows that $(1,\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2}) \in A$. Thus

$$b^{\mathbf{L}_2^n}(1,\tfrac{1}{2},\tfrac{1}{2},\ldots,\tfrac{1}{2}) = (1,0,\ldots,0) \in A.$$

Likewise the rest of the atoms of the Boolean skeleton of $\mathbf{L}_2^n$ belong to A as well. Therefore, $A = T_n$. □

Lemma 5.11. *If* $\mathbf{A}$ *is an algebra in* $\mathcal{G}_2$ *and* $\mathbf{L}_1$ *is not a descendant of* $\mathbf{A}$*, then* $\mathbf{A} = \mathbf{T}_n$ *for some* $n \geq 1$.

Proof. Let $\mathbf{A} \in \mathcal{G}_2$ such that $\mathbf{A} \neq \mathbf{T}_n$ for every n. We will show that $\mathbf{L}_1 \prec \mathbf{A}$. By Corollary 2.4, we may assume that $\mathbf{L}_1 \notin \mathsf{H}(\mathbf{A})$. Hence, we assume that $\mathbf{A}$ is an increasing subset of $\mathbf{L}_2^n$ that contains the co-atoms of the Boolean skeleton of $\mathbf{L}_2^n$. In particular we have that $\mathbf{L}_2 \prec \mathbf{A}$ and, by Lemma 2.3 (2), $\mathbf{L}_2^n \prec \mathbf{A}$. Let $\sigma_1, \sigma_2, \ldots, \sigma_{n!}$ be the automorphisms of $\mathbf{L}_2^n$ that permute the coordinates and let

$$B := \bigcap_{i=1}^{n!} \sigma_i(A).$$

Since one of the automorphisms is the identity, B is a subuniverse of $\mathbf{A}$ and, furthermore, using Lemma 2.3 (7), it follows that $\mathbf{B} \prec \mathbf{A}$. Moreover, since $\sigma_j(B) = B$ for each j, $\sigma_j|_B : B \to B$ is an automorphism of B. Now put

$$\mathbf{C} := \bigcap_{j=1}^{n!} \mathbf{Fix}\, \sigma_j|_B,$$

and note that $C \subseteq \{(\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2}),(1,1,\ldots,1)\}$. It follows that $\mathbf{C} \prec \mathbf{B}$ and, consequently, $\mathbf{C} \prec \mathbf{A}$.

Suppose $A \not\subseteq T_n$. Thus there is $(a_1, \ldots, a_n) \in A$ such that $a_i \neq 1$ for every i. Since A is increasing, it follows that $m := (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) \in A$. It is easy to see that $m \in C$, so $\mathbf{C} \cong \mathbf{L}_1$ and we have proved that $\mathbf{L}_1 \prec \mathbf{A}$.

It remains to show that the same result follows assuming $A \subsetneq T_n$. We prove this part using induction on n. For $n = 1$ the statement is trivial since $\mathbf{A} \cong \mathbf{L}_1$. Assume that the statement is true for $k < n$.

Since $A \subsetneq T_n$, there is an atom of the Boolean skeleton of $\mathbf{L}_2^n$ that does not belong to A. Without loss of generality we can assume that $(1, 0, \ldots, 0) \notin A$. Let $\mathbf{D}$ be the projection of $\mathbf{A}$ on coordinates $2, 3, \ldots, n$. Note that:

- D is increasing in $\mathbf{L}_2^{n-1}$.
- $(0, 0, \ldots, 0) \notin D$.
  Indeed, if $(0, 0, \ldots, 0) \in D$, then there exists $a \in \{0, \frac{1}{2}, 1\}$ such that $(a, 0, 0, \ldots, 0) \in A$. Since $A \subseteq T_n$, it follows that $a = 1$, so $(1, 0, \ldots, 0) \in A$, a contradiction.
- $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) \in D$.
  Indeed, as $\mathbf{A} \in \mathcal{G}_2$, $\mathbf{A}$ contains a subalgebra isomorphic to $\mathbf{F}_n$ without meet in $\mathbf{A}$. Using Lemma 5.9 and the fact that A is increasing, it follows that $(1, \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) \in A$. Hence $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) \in D$.
- $\mathbf{D} \not\models \varphi_b$.
  It follows immediately from the two previous remarks.

Now $\mathbf{D}$ may be represented as a global subdirect product of algebras in $\mathcal{G}_2$ with at most $n-1$ co-atoms. All these factor algebras cannot be isomorphic to algebras in $\{\mathbf{T}_n : n \geq 1\}$, for, if this were the case, φ_b would be valid in $\mathbf{D}$. Thus $\mathbf{D}$ has a homomorphic image $\mathbf{E}$ in $\mathcal{G}_2$ with at most $n - 1$ co-atoms that is not isomorphic to any $\mathbf{T}_n$, $n \geq 1$. Using the inductive hypothesis, $\mathbf{L}_1 \prec \mathbf{E}$. Hence $\mathbf{L}_1 \prec \mathbf{E} \prec \mathbf{D} \prec \mathbf{A}$, as was to be proved. □

Proof of Theorem 5.7 . We first prove the theorem assuming A finite. Implication (1)⇒(2) follows immediately from the fact that $\mathbf{L}_1 \not\models \varphi_b$. Implication (3)⇒(1) is a consequence of Proposition 5.10. In order to prove that (2) implies (3), assume $\mathbf{L}_1 \not\prec \mathbf{A}$. Suppose $\mathbf{A}$ is a global subdirect product of $\{\mathbf{A}_1, \ldots, \mathbf{A}_k\} \subseteq \mathcal{G}_2$. For each i, $\mathbf{A}_i \in \mathsf{H}(\mathbf{A})$, so $\mathbf{A}_i \prec \mathbf{A}$ and, consequently, $\mathbf{L}_1 \not\prec \mathbf{A}_i$. Thus, by Lemma 5.11, $\{\mathbf{A}_1, \ldots, \mathbf{A}_k\} \subseteq \{\mathbf{T}_n : n \geq 1\}$, which completes the proof for the finite case.

It remains to show that implication (2)⇒(1) holds for infinite algebras as well. Assume $\mathbf{A} \not\models \varphi_b$. We may suppose $\mathbf{L}_1 \notin \mathsf{H}(\mathbf{A})$, since otherwise $\mathbf{L}_1 \prec \mathbf{A}$. As $\mathbf{L}_2$ must be a quotient of $\mathbf{A}$, it follows by Corollary 2.4 that $\mathbf{L}_2 \prec \mathbf{A}$. We know that $\mathbf{A} \leq \mathbf{L}_2^I$ for some set I. Since $\mathbf{L}_2 \models U(\varphi_b)$, it follows that $\mathsf{V}(\mathbf{L}_2) = Q(\mathbf{L}_2) \models U(\varphi_b)$ and, in particular, $\mathbf{A} \models U(\varphi_b)$. Hence $\mathbf{A} \not\models E(\varphi_b)$. However, as $\mathbf{L}_2^I \models \varphi_b$, we conclude that there is $a \in A$ such that $b^{\mathbf{L}_2^I}(a) \notin A$.

We construct a finite algebra $\mathbf{B}$ such that $\mathbf{B} \prec \mathbf{A}$ and $\mathbf{B} \not\models \varphi_b$. Let Σ be the set of EFD-sentences that are valid in $\mathbf{A}$. Since $\mathbf{L}_2 \models \Sigma$, it is clear that A is closed under every algebraic function determined by an EFD-sentence $\varphi \in \Sigma$ on $\mathbf{L}_2^I$. Let $\mathbf{L}_2^*$ and $\mathbf{A}^*$ be the algebras that result by expanding $\mathbf{L}_2$ and $\mathbf{A}$ with the algebraic functions determined by sentences in Σ. Clearly $\mathbf{A}^* \leq (\mathbf{L}_2^*)^I$. In particular, $\mathbf{A}^* \in \mathsf{V}(\mathbf{L}_2^*)$, so $\mathbf{A}^*$ is a locally finite algebra. Hence $\mathbf{B}^* = \mathbf{Sg}^{\mathbf{A}^*}(a)$ is a finite algebra. Moreover, if $\mathbf{B}$ denotes the $\{\rightarrow, 1\}$-reduct of $\mathbf{B}^*$, $\mathbf{B} \prec \mathbf{A}$ since B is closed under every algebraic function of $\mathbf{A}$. Furthermore, since $b^{\mathbf{L}_2^I}(a) \notin B$, we have $\mathbf{B} \not\models \varphi_b$. Thus $\mathbf{B}$ is a finite algebra in $\mathsf{V}(\mathbf{L}_2)$ such that $\mathbf{B} \not\models \varphi_b$. Consequently, $\mathbf{L}_1 \prec \mathbf{B}$, and finally, since $\mathbf{B} \prec \mathbf{A}$, we get that $\mathbf{L}_1 \prec \mathbf{A}$. □

A final piece of information we need to apply Theorem 5.3 is a description of the clones $\mathrm{Clo}_{\mathrm{alg}}\,\mathbf{L}_2$ and $\mathrm{Clo}_{\{\mathbf{L}_1,\mathbf{L}_2\}\text{-alg}}\,\mathbf{L}_2$.

Proposition 5.12. *Given $f : L_2^k \to L_2$, the following are equivalent:*

(1) $f \in \mathrm{Clo}_{\mathrm{alg}}\,\mathbf{L}_2$,
(2) $f \in \mathrm{Clo}\langle L_2, \to, \wedge, b, 1\rangle$,
(3) $f(1, 1, \dots, 1) = 1$.

Proof. We prove (1)⇒(3)⇒(2)⇒(1). Note that (1) implies (3) because trivial algebras satisfy all EFD-sentences. To prove (3)⇒(2) observe that $\langle L_2, \to, \wedge, b, 1\rangle$ is quasiprimal (recall the ternary discriminator term given in the proof of Lemma 5.2). So an easy application of Pixley's Theorem (see Theorem 3.4.4 in [10]) yields the desired implication. Finally, (2)⇒(1) follows from the fact that composition of algebraic functions is again algebraic. □

Proposition 5.13. *Given $f : L_2^k \to L_2$, the following are equivalent:*

(1) $f \in \mathrm{Clo}_{\{\mathbf{L}_1,\mathbf{L}_2\}\text{-alg}}\,\mathbf{L}_2$, *that is, $f = [\varphi]_i^{\mathbf{L}_2}$ for some index i and some EFD-sentence φ such that $\mathbf{L}_1, \mathbf{L}_2 \vDash \varphi$,*
(2) $f \in \mathrm{Clo}\langle L_2, \to, \wedge, 1\rangle$,
(3) *f satisfies:*
  - $f(1, 1, \dots, 1) = 1$.
  - $f(\{0, 1\}^k) \subseteq \{0, 1\}$.
  - $f(b(a_1), \dots, b(a_k)) = b(f(a_1, \dots, a_k))$ *for every* $a_1, \dots, a_k \in \{0, 1\}$.

Proof. We show (1)⇒(3)⇒(2)⇒(1).

Let φ be an EFD-sentence such that $\mathbf{L}_1, \mathbf{L}_2 \vDash \varphi$, $\varphi = \forall x_1, \dots, x_k\, \exists! z_1, \dots, z_m\ \varepsilon(\bar{x}, \bar{z})$ and let $f := [\varphi]_i^{\mathbf{L}_2}$. The fact that $\varepsilon(1, 1, \dots, 1)$ holds in $\mathbf{L}_2$ yields that $f(1, 1, \dots, 1) = 1$. Since $\mathbf{L}_1 \vDash \varphi$ and $\{0, 1\}$ is a subuniverse of $\mathbf{L}_2$ isomorphic to $\mathbf{L}_1$, it is also immediate that $f(\{0, 1\}^k) \subseteq \{0, 1\}$. Furthermore, b is an isomorphism between the subuniverses $\{0, 1\}$ and $\{\frac{1}{2}, 1\}$. Thus, the third condition on f follows.

The implication (3)⇒(2) follows from Pixley's Theorem (see Theorem 3.4.4 in [10]). And (2)⇒(1) is a consequence of the fact that $\mathrm{Clo}_{\{\mathbf{L}_1,\mathbf{L}_2\}\text{-alg}}$ is a clone containing $\wedge$. □

Remark 5.14. Note that the functions in the previous proposition are monoalgebraic. This follows from Corollary 5.5.

Given a term $t(\bar{x})$ in the language $\{b, \wedge, \to, 1\}$ and $\mathbf{A} \in \mathbb{L}$ such that $\mathbf{A} \vDash \varphi_b$, let $t^{\mathbf{A}}$ denote the (partial) function defined by t in $\mathbf{A}$. Note that for any set I, the function $t^{\mathbf{L}_2^I}$ is always total, and $t^{\mathbf{L}_2^I}(\bar{u})(i) = t^{\mathbf{L}_2}(\bar{u}(i))$, for all $\bar{u}$ from $\mathbf{L}_2^I$ and $i \in I$. Furthermore, if $\mathbf{A} \leq \mathbf{L}_2^I$, then $\bar{a}$ is in the domain of $t^{\mathbf{A}}$ iff $t^{\mathbf{L}_2^I}(\bar{a}) \in A$; and in that case $t^{\mathbf{L}_2^I}(\bar{a}) = t^{\mathbf{A}}(\bar{a})$.

Theorem 5.15. *Let $\mathbf{A} \in \mathsf{V}(\mathbf{L}_2) \setminus \mathsf{V}(\mathbf{L}_1)$, and let $f : A^k \to A$.*

(1) *Suppose $\mathbf{A} \vDash \varphi_b$ or equivalently that $\mathbf{L}_1 \not\prec \mathbf{A}$. The following are equivalent:*
  (*a*) *f is algebraic on $\mathbf{A}$.*
  (*b*) *There is a term t in the language $\{b, \wedge, \to, 1\}$ such that $t^{\mathbf{A}} = f$.*
(2) *Suppose $\mathbf{A} \nvDash \varphi_b$ or equivalently that $\mathbf{L}_1 \prec \mathbf{A}$. The following are equivalent:*
  (*a*) *f is algebraic on $\mathbf{A}$.*
  (*b*) *f is monoalgebraic on $\mathbf{A}$.*
  (*c*) *There is a term t in the language $\{\wedge, \to, 1\}$ such that $t^{\mathbf{A}} = f$.*

Proof. (1). Assume $\mathbf{A} \leq \mathbf{L}_2^I$. We prove (a)⇒(b). Suppose $f \in \mathrm{Clo}_{\mathrm{alg}}\, \mathbf{A}$. By Theorem 5.3 there is $f^* \in \mathrm{Clo}_{\mathrm{alg}}\, \mathbf{L}_2$ such that $f(\bar{a})(i) = f^*(\bar{a}(i))$, for all $\bar{a} \in A^k$ and $i \in I$. Proposition 5.12 implies that there is a $\{b, \wedge, \to, 1\}$-term t such that $t^{\mathbf{L}_2} = f^*$. Note that

$$t^{\mathbf{A}}(\bar{a})(i) = t^{\mathbf{L}_2^I}(\bar{a})(i) = t^{\mathbf{L}_2}(\bar{a}(i)) = f^*(\bar{a}(i)) = f(\bar{a})(i),$$

for all $\bar{a} \in A^k$ and $i \in I$. Thus $t^{\mathbf{A}} = f$.

To show that (b)⇒(a) just apply (4)⇒(1) of Theorem 5.3 with $f^* := t^{\mathbf{L}_2}$.
(2). The proof of (a)⇔(c) is analogous to the proof of (1) above. Clearly (b)⇒(a); and (c)⇒(b) follows from Corollary 5.5. □

6. Algebraically expandable classes

In this section we show how to apply the knowledge of the algebraic clones to give a partial description of the lattice of algebraically expandable classes of $\mathsf{V}(\mathbf{L}_2)$.

We start with some general results that hold for any of the varieties $\mathsf{V}(\mathbf{L}_n)$. Let $\mathcal{G}_n := \mathcal{G} \cap \mathsf{V}(\mathbf{L}_n)$ and let $Dec(\mathcal{G}_n)$ denote the collection of subclasses $\mathcal{C}$ of $\mathcal{G}_n$ such that for every $\mathbf{A}, \mathbf{B} \in \mathcal{G}_n$, if $\mathbf{A} \prec \mathbf{B}$ and $\mathbf{B} \in \mathcal{C}$, then $\mathbf{A} \in \mathcal{C}$. We also denote by $AE(\mathsf{V}(\mathbf{L}_n))$ the lattice of algebraically expandable classes that are contained in $\mathsf{V}(\mathbf{L}_n)$.

Proposition 6.1. *The map $\alpha : AE(\mathsf{V}(\mathbf{L}_n)) \to Dec(\mathcal{G}_n)$, given by $\alpha(\mathbb{K}) = \mathbb{K} \cap \mathcal{G}_n$, is one-to-one. Moreover, if $\mathbb{K}_1, \mathbb{K}_2 \in AE(\mathsf{V}(\mathbf{L}_n))$, then $\alpha(\mathbb{K}_1 \cap \mathbb{K}_2) = \alpha(\mathbb{K}_1) \cap \alpha(\mathbb{K}_2)$.*

Proof. It is straightforward to verify that $\mathbb{K} \cap \mathcal{G}_n \in Dec(\mathcal{G}_n)$ for every $\mathbb{K} \in AE(\mathsf{V}(\mathbf{L}_n))$.

Suppose $\mathbb{K}_1 \cap \mathcal{G}_n = \mathbb{K}_2 \cap \mathcal{G}_n$. For $i = 1, 2$, let $\mathbb{K}_i := \mathrm{Mod}\, \Sigma_i$, where Σ_i is a set of EFD-sentences. In [5] it is shown that $\mathsf{V}(\mathbf{L}_n)$ and all of its sub(quasi)varieties are finitely generated as quasivarieties. Therefore, we may apply Proposition 1 in [6], which states that, in order to show that $\mathbb{K}_1 = \mathbb{K}_2$, it is enough to show that $\mathbb{K}_1 \cap \mathsf{V}(\mathbf{L}_n)_{fin} = \mathbb{K}_2 \cap \mathsf{V}(\mathbf{L}_n)_{fin}$. Indeed, if $\mathbf{A} \in \mathbb{K}_1 \cap \mathsf{V}(\mathbf{L}_n)_{fin}$, $\mathbf{A} \vDash \Sigma_1$. Now, since $\mathbf{A}$ is finite, it is a global subdirect product of a subfamily $\mathcal{G}_{\mathbf{A}} \subseteq \mathcal{G}_n$. Using Corollary 2.4, it follows that $\mathcal{G}_{\mathbf{A}} \prec \mathbf{A}$. Hence $\mathcal{G}_{\mathbf{A}} \vDash \Sigma_1$, that is, $\mathcal{G}_{\mathbf{A}} \subseteq \mathbb{K}_1 \cap \mathcal{G}_n$. Therefore $\mathcal{G}_{\mathbf{A}} \subseteq \mathbb{K}_2 \cap \mathcal{G}_n$ and, hence, $\mathbf{A} \vDash \Sigma_2$, that is, $\mathbf{A} \in \mathbb{K}_2 \cap \mathsf{V}(\mathbf{L}_n)_{fin}$. The converse is analogous.

Lastly, it is immediate that α preserves meets since in both lattices, $AE(\mathsf{V}(\mathbf{L}_n))$ and $Dec(\mathcal{G}_n)$, the meet operation is the intersection. □

It is thus important to know the structure of $Dec(\mathcal{G}_n)$. In the case $n = 1$, we know that $\mathcal{G}_1$ consists of the algebras $\mathbf{F}_n$ for $n \geq 1$. In a similar way to Proposition 5.8, it may be shown that $\mathbf{F}_n \prec \mathbf{F}_{n+1}$, but $\mathbf{F}_{n+1} \not\prec \mathbf{F}_n$. It follows that $\mathcal{G}_1$ consists of an ω-chain:

$$\mathbf{F}_1 \prec \mathbf{F}_2 \prec \ldots \prec \mathbf{F}_n \prec \ldots$$

which, in turn, implies that the structure of the algebraically expandable classes contained in $\mathsf{V}(\mathbf{L}_1)$ is an $(\omega + 1)$-chain. These are the results found in [4].

In the case $n = 2$, we know of two ω-chains in $\mathcal{G}_2$, namely, the algebras $\mathbf{F}_n$, $n \geq 1$, and the algebras $\mathbf{T}_n$, $n \geq 1$. In this case the basic structure of $\mathcal{G}_2$ may be depicted as in Figure 6.1. The dotted lines represent the layers of algebras determined by the sentences φ_n, $n \geq 2$.

The following proposition shows that our understanding of the algebraic functions can help us determine the details of the structure of $Dec(\mathcal{G}_n)$.

Proposition 6.2. *Assume $\mathbf{A}, \mathbf{B} \leq \mathbf{C} = \prod_{i \in I} \mathbf{C}_i$ such that $\{\mathbf{C}_i : i \in I\}$ is disjunctive and it is contained in $\mathsf{D}(\mathbf{A}) \cap \mathsf{D}(\mathbf{B})$. Let $\mathbb{K} := \mathsf{SP}_u(\{\mathbf{C}_i : i \in I\}) \cap \mathsf{D}(\mathbf{B})$. Then, $\mathbf{A} \prec \mathbf{B}$ if and*

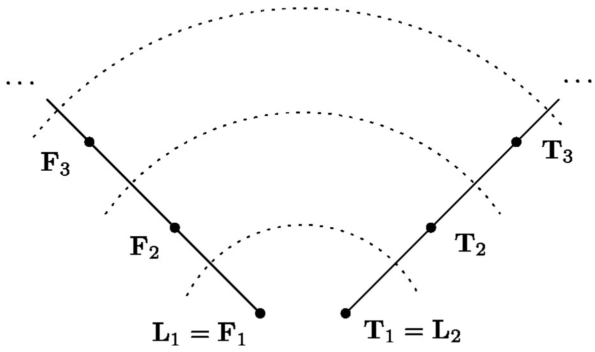

FIGURE 6.1. Basic structure of $\mathcal{G}_2$

only if A is closed under every $f \in \mathrm{Clo}_{\mathbb{K}\text{-alg}}\,\mathbf{C}$ such that B is closed under f. Moreover, if $\mathbf{A}$ is k-generated, $\mathbf{A} \prec \mathbf{B}$ if and only if A is closed under every k-ary $f \in \mathrm{Clo}_{\mathbb{K}\text{-alg}}\,\mathbf{C}$ such that B is closed under f.

Proof. The left-to-right implication follows immediately from Theorem 4.2. We now prove the converse implication. If $\mathbf{B} \vDash \varphi$, then $\mathbf{C} \vDash \varphi$ and, hence, $\mathbf{A} \vDash U(\varphi)$. Let $f_i := [\varphi]_i^{\mathbf{C}}$, $1 \leq i \leq n$. Since $\mathbb{K} \vDash \varphi$ and B is closed under each f_i, it follows, by the assumptions, that A is also closed under each f_i, that is, $\mathbf{A} \vDash E(\varphi)$. This shows that $\mathbf{A} \prec \mathbf{B}$.

For the "moreover" part, observe that if $\mathbf{A} \nvDash E(\varphi)$ and $\varphi = \forall x_1, \ldots, x_n\, \exists! z_1, \ldots, z_m\, \varepsilon(\bar{x}, \bar{z})$, there exist elements $a_1, \ldots, a_n \in A$ such that $\mathbf{A} \nvDash \exists z_1, \ldots, z_m\, \varepsilon(\bar{a}, \bar{z})$. If $\{g_1, \ldots, g_k\}$ is a set of generators for $\mathbf{A}$, there exist terms $t_1, \ldots, t_n$ in the variables $x_1, \ldots, x_k$ such that $t_i^{\mathbf{A}}(\bar{g}) = a_i$, $1 \leq i \leq n$. If we define $\psi := \forall x_1, \ldots, x_k\, \exists! z_1, \ldots, z_m\, \varepsilon(\bar{t}(\bar{x}), \bar{z})$, then $\mathbf{B} \vDash \psi$ and $\mathbf{A} \nvDash \psi$, so there must exist a k-variable function $f \in \mathrm{Clo}_{\mathbb{K}\text{-alg}}\,\mathbf{C}$ such that B is closed under f, but A is not. □

We illustrate how to apply the previous proposition by determining the structure of a the subfamily of $\mathcal{G}_2$ axiomatized by the EFD-sentence φ_3. This sentence limits the number of co-atoms to two, thus there are a finitely many algebras in $\mathcal{G}_2$ satisfying it. They are shown in Figure 6.2.

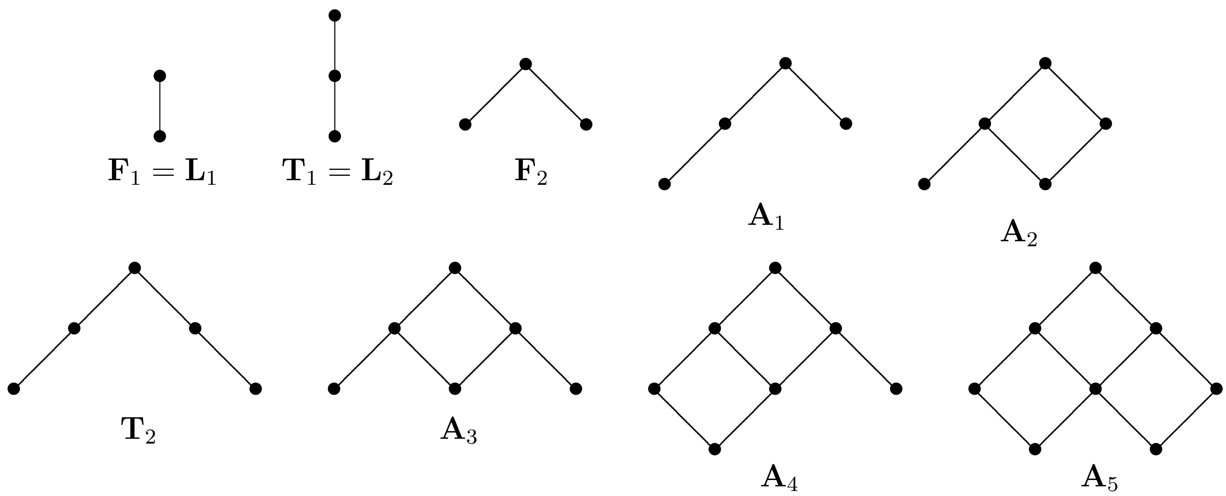

FIGURE 6.2. $\mathcal{G}_2 \cap \mathrm{Mod}\,\varphi_3$

Example 6.3. We show that $\mathbf{F}_2 \prec \mathbf{A}_1$. Consider $\mathbf{F}_2, \mathbf{A}_1 \leq \mathbf{L}_2^2$,

$$F_2 = \{(1, \tfrac{1}{2}), (\tfrac{1}{2}, 1), (1, 1)\}, \quad \text{and} \quad A_1 = \{(1, 0), (1, \tfrac{1}{2}), (\tfrac{1}{2}, 1), (1, 1)\}.$$

Note that $F_2 \subseteq A_1$ and that $\mathbf{F}_2$ is generated by $\{(1, \frac{1}{2}), (\frac{1}{2}, 1)\}$. It is easy to check that the assumptions of Proposition 6.2 are met. If we assume, by contradiction, that $\mathbf{F}_2 \not\prec \mathbf{A}_1$, following the proof of Proposition 6.2, it follows that there exists $f \in \mathrm{Clo}_{\{\mathbf{L}_1,\mathbf{L}_2\}\text{-alg}}\, \mathbf{L}_2^2$ such that A_1 is closed under f but $f((1, \frac{1}{2}), (\frac{1}{2}, 1)) \notin F_2$. Thus, $f(x, y) = (g(x(1), y(1)), g(x(2), y(2)))$, where $g \in \mathrm{Clo}_{\{\mathbf{L}_1,\mathbf{L}_2\}\text{-alg}}\, \mathbf{L}_2$. Since $A_1 \backslash F_2 = \{(1, 0)\}$, it follows that $f((1, \frac{1}{2}), (\frac{1}{2}, 1)) = (1, 0)$, so $g(1, \frac{1}{2}) = 1$ and $g(\frac{1}{2}, 1) = 0$. This contradicts Proposition 5.13. □

Example 6.4. We show that $\mathbf{A}_1 \not\prec \mathbf{A}_2$. Consider $\mathbf{A}_1, \mathbf{A}_2 \leq \mathbf{L}_2^2$ given by

$$A_1 = \{(1, 0), (1, \tfrac{1}{2}), (\tfrac{1}{2}, 1), (1, 1)\}, \text{ and } A_2 = \{(1, 0), (1, \tfrac{1}{2}), (\tfrac{1}{2}, \tfrac{1}{2}), (\tfrac{1}{2}, 1), (1, 1)\},$$

and note that $\mathbf{A}_1$ is generated by $\{(1, 0), (1, \frac{1}{2}), (\frac{1}{2}, 1)\}$. We look for a function $f \in \mathrm{Clo}_{\{\mathbf{L}_1,\mathbf{L}_2\}\text{-alg}}\, \mathbf{L}_2^2$ given by $f(x, y, z) = (g(x(1), y(1), z(1)), g(x(2), y(2), z(2)))$, where $g : L_2^3 \to L_2$, such that

$$f((1, 0), (1, \tfrac{1}{2}), (\tfrac{1}{2}, 1)) \notin A_1, \qquad \text{but} \qquad f(A_2^2) \subseteq A_2.$$

Since $A_2 \setminus A_1 = \{(\frac{1}{2}, \frac{1}{2})\}$, it follows that $f((1, 0), (1, \frac{1}{2}), (\frac{1}{2}, 1)) = (\frac{1}{2}, \frac{1}{2})$, so $g(1, 1, \frac{1}{2}) = \frac{1}{2}$ and $g(0, \frac{1}{2}, 1) = \frac{1}{2}$. This is no contradiction with the fact that $g \in \mathrm{Clo}_{\{\mathbf{L}_1,\mathbf{L}_2\}\text{-alg}}\, \mathbf{L}_2$. In fact, if we define

$$g(x, y, z) := \begin{cases} \frac{1}{2} & \text{if } (x, y, z) = (1, 1, \frac{1}{2}), \\ \frac{1}{2} & \text{if } (x, y, z) = (0, \frac{1}{2}, 1), \\ 0 & \text{if } (x, y, z) = (1, 1, 0), \\ 1 & \text{otherwise}, \end{cases}$$

it may be checked that f satisfies all conditions. Hence $\mathbf{A}_1 \not\prec \mathbf{A}_2$.

From the definition of g, we may produce an EFD-sentence φ that is valid in $\mathbf{L}_1$ and $\mathbf{L}_2$ such that $g = [\varphi]^{\mathbf{L}_2}$. Recall that in Proposition 5.13 we showed that these functions are monoalgebraic and belong to $\mathrm{Clo}\langle L_2, \to, \wedge, 1\rangle$. We can write $g(x, y, z) = g_1(x, y, z) \wedge g_2(x, y, z)$, where

$$g_1(x, y, z) := \begin{cases} \frac{1}{2} & \text{if } (x, y, z) = (1, 1, \frac{1}{2}), \\ 0 & \text{if } (x, y, z) = (1, 1, 0), \\ 1 & \text{otherwise}, \end{cases} \quad \text{and} \quad g_2(x, y, z) := \begin{cases} \frac{1}{2} & \text{if } (x, y, z) = (0, \frac{1}{2}, 1), \\ 1 & \text{otherwise}. \end{cases}$$

Then it is not difficult to provide $\{\to, \wedge, 1\}$-terms that represent these functions:

$$t_1(x, y, z) := (x \wedge y) \to ((x \wedge y) \to z),$$
$$t_2(x, y, z) := z \to ((y \to x) \vee y),$$

respectively. Thus we get a term $t(x, y, z) := t_1(x, y, z) \wedge t_2(x, y, z)$ that represents g on $\langle L_2, \to, \wedge, 1\rangle$. Based on the proof of Lemma 5.4, we can write t as a meet of implicative terms. We thus get $t(x, y, z) = s_1(x, y, z) \wedge s_2(x, y, z)$, where

$$s_1(x, y, z) := ((x \to ((x \to z) \vee (y \to z))) \vee (y \to ((x \to z) \vee (y \to z)))),$$
$$s_2(x, y, z) := (z \to ((y \to x) \vee y)).$$

We can then build an EFD-sentence:

$$\varphi := \forall x_1, x_2, x_3\, \exists! z\ (z \to s_1(x_1, x_2, x_3) \approx 1\ \&\ z \to s_2(x_1, x_2, x_3) \approx 1\ \&\\ \&\ (s_1(x_1, x_2, x_3) \to z) \vee (s_2(x_1, x_2, x_3) \to z) \approx 1)$$

that defines g on $\mathbf{L}_2$. This way we have provided an EFD-sentence φ such that $\mathbf{A}_2 \vDash \varphi$, but $\mathbf{A}_1 \nvDash \varphi$. □

If we study all possible combinations of the algebras in Figure 6.2, we find that the relation $\prec$ is a partial ordering on these algebras and we obtain the Hasse diagram shown in Figure 6.3.

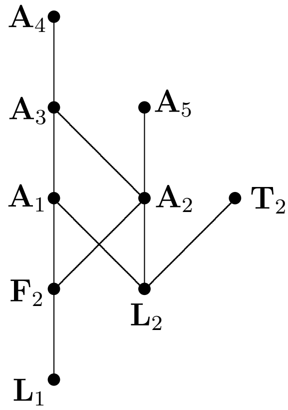

FIGURE 6.3. Ordering of $\mathcal{G}_2 \cap \mathrm{Mod}\,\varphi_3$

We can then compute $Dec(\mathcal{G}_2 \cap \mathrm{Mod}\,\varphi_3)$, which is shown in Figure 6.4. Observe that, upon distinguishing algebras by means of EFD-sentences as in Example 6.4, we are able to provide several EFD-sentences that, together with φ_3, determine certain algebraically expandable classes which correspond to certain elements in $Dec(\mathcal{G}_2 \cap \mathrm{Mod}\,\varphi_3)$. If we follow this procedure we find corresponding EFD-sentences to account for the encircled nodes in Figure 6.4. As an immediate consequence, any meet of these encircled nodes may also be equipped with a corresponding set of EFD-sentences. However, there are four nodes (the ones not marked by a bullet) that do not correspond to algebraically expandable classes. To show this in detail we need the following lemma.

Lemma 6.5. *If φ is an EFD-sentence such that $\mathbf{T}_2 \vDash \varphi$ and $\mathbf{L}_1 \vDash \varphi$, then $\mathbf{A}_1 \vDash \varphi$.*

Proof. Recall that $\mathbf{T}_2 \leq \mathbf{L}_2^2$ such that $T_2 = \{(1,0), (1, \frac{1}{2}), (1,1), (\frac{1}{2}, 1), (0,1)\}$, as may be seen in Figure 6.2. Consider the following two subuniverses of $\mathbf{L}_2^3$:

$$S_1 := \{(x, y, z) \in L_2^3 : (x, y) \in B_1, z = 1\},$$
$$S_2 := \{(x, y, z) \in L_2^3 : (x, z) \in B_1, y \in \{0, 1\}\}.$$

It is clear that $\mathbf{S}_1 \cong \mathbf{T}_2$ and $\mathbf{S}_2 \cong \mathbf{T}_2 \times \mathbf{L}_1$. By the assumptions, it follows that $\mathbf{L}_2^3, \mathbf{S}_1, \mathbf{S}_2 \vDash \varphi$. Using Lemma 2.3 (7), we obtain that $\mathbf{S}_1 \cap \mathbf{S}_2 \vDash \varphi$. Since $S_1 \cap S_2 = \{(x, y, z) \in L_2^3 : (x, y) \in B_1, y \in \{0, 1\}, z = 1\}$, it is clear that $\mathbf{S}_1 \cap \mathbf{S}_2 \cong \mathbf{A}_1$. Thus $\mathbf{A}_1 \vDash \varphi$. □

This lemma shows that any decreasing set of $\mathcal{G}_2 \cap \mathrm{Mod}\,\varphi_3$ that contains both $\mathbf{L}_1$ and $\mathbf{T}_2$ but does not contain $\mathbf{A}_1$ cannot be of the form $\mathbb{K} \cap \mathcal{G}_2$ for any algebraically expandable class

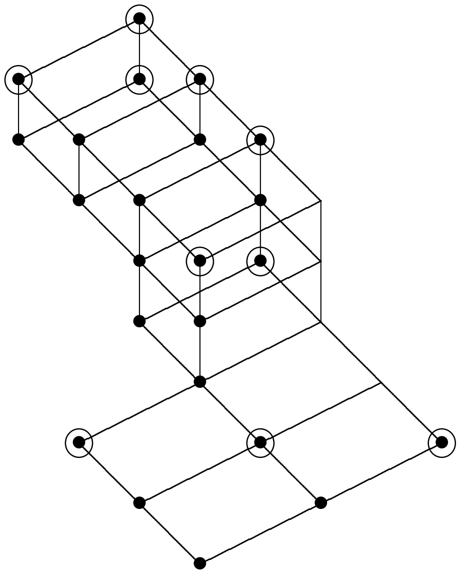

FIGURE 6.4. $Dec(\mathcal{G}_2 \cap \mathrm{Mod}\,\varphi_3)$

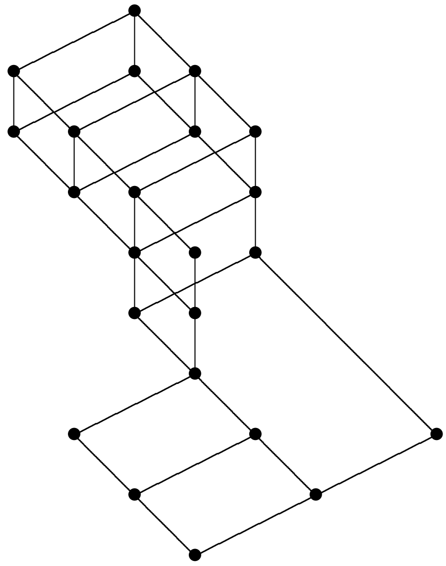

FIGURE 6.5. Lattice of algebraically expandable classes contained in $V(\mathbf{L}_2) \cap \mathrm{Mod}\,\varphi_3$

$\mathbb{K}$. The structure of the lattice of algebraically expandable classes within $\mathsf{V}(\mathbf{L}_2) \cap \mathrm{Mod}\,\varphi_3$ thus follows easily and is given in Figure 6.5.

Even though it seems feasible to do similar case studies in the variety $V(\mathbf{L}_n)$ for small n, a general solution for step 1 in the procedure described in the paragraph following Theorem 5.3 has eluded us. One of the reasons for this is that there does not appear to be a simple description of the algebraically expandable subclasses of $S(\mathbf{L}_n)$.

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