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Lyapunov-type inequalities for partial differential equations



Pablo L. de Nápoli, Juan P. Pinasco

Departamento de Matemática, IMAS - CONICET, FCEyN UBA, Ciudad Universitaria, Av. Cantilo s/n, 1428, Buenos Aires, Argentina

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ABSTRACT

In this work we present a Lyapunov inequality for linear and quasilinear elliptic differential operators in N -dimensional domains Ω . We also consider singular and degenerate elliptic problems with A_p coefficients involving the p -Laplace operator with zero Dirichlet boundary condition.

As an application of the inequalities obtained, we derive lower bounds for the first eigenvalue of the p -Laplacian, and compare them with the usual ones in the literature.

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1. Introduction

In his classical work [23], Lyapunov proved that, given a continuous periodic and positive function w with period L , the solution u of the ordinary differential equation $u'' + w(t)u = 0$, in $(-\infty, +\infty)$, was stable if

E-mail addresses: pdenapo@dm.uba.ar (P.L. de Nápoli), jpinasco@dm.uba.ar (J.P. Pinasco).

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$$L \int_0^L w(t) dt < 4.$$

Then, Borg in [4] introduced the Lyapunov inequality in his proof of the stability criteria for sign changing weights w . He showed that the inequality

$$\frac{4}{L} \leq \int_0^L |w(t)| dt \quad (1.1)$$

must be satisfied in order to have a nontrivial solution in $[0, L] \subset \mathbb{R}$ of the problem

$$\begin{cases} u'' + w(t)u = 0, \\ u(0) = 0 = u(L). \end{cases} \quad (1.2)$$

Since then, it was rediscovered and generalized many times. Inequality (1.1) was applied in stability problems, oscillation theory, a priori estimates, other inequalities, and eigenvalue bounds for ordinary differential equations. Different proofs of this inequality have appeared in the literature: the proof of Patula [28] by direct integration, or the one of Nehari [24] showing the relationship with Green's functions, among several others. See the survey [5] for other proofs.

In the nonlinear setting, the following inequality

$$\frac{2^p}{L^{p-1}} \leq \int_0^L w(t) dt \quad (1.3)$$

generalized Lyapunov inequality (1.1) to p -Laplacian problems,

$$\begin{cases} (|u'|^{p-2}u')' + w(t)|u|^{p-2}u = 0, \\ u(0) = 0 = u(L). \end{cases}$$

Here, $w \in L^1$ and $1 < p < \infty$, for $p = 2$ we recover the linear problem (1.2). Several proofs were given in the last years, see [21,27,29,33]; although it seems to be derived first by Elbert [14].

Later, we extended it in [10] to nonlinear operators in Orlicz spaces generalizing the p -Laplacian,

$$-(\varphi(u'))' = \lambda r(t)\varphi(u), \quad (1.4)$$

where $\varphi(s)$ is a convex nondecreasing function, such that $s\varphi(s)$ satisfies the Δ_2 condition. Moreover, we also extend it to systems of resonant type (see [3]) involving p - and

q -Laplacians in [11]. We refer the interested reader to the book [30] for a review of recent developments in these problems.

Beside the one dimensional case, there are few works devoted to similar inequalities for partial differential equations. An exception are the works of Cañada, Montero and Villegas [6,7], where the following problem was considered,

$$\begin{cases} \Delta u + w(x)u = 0, & x \in \Omega \\ \frac{\partial u}{\partial \eta} = 0, & x \in \partial\Omega \end{cases} \quad (1.5)$$

and a nonexistence result was obtained for general domains. The authors gives some bounds involving the second Neumann eigenvalue μ_2 . However, it is well known that μ_2 fails to reflect geometric properties of Ω , and can be made arbitrarily close to zero by adding a slight perturbation of the domain as in [8]. Also, some results of Egorov and Kondriatev, included in their book [13], contain Lyapunov type inequalities for higher order linear differential operators.

The aim of this work is to prove a Lyapunov inequality for N -dimensional (linear and quasilinear) elliptic operators with zero Dirichlet boundary conditions, reflecting more geometric information than the measure of the domain. Our toy model is the p -Laplace operator, and we consider here the following problem,

$$\begin{cases} \Delta_p u + w(x)|u|^{p-2}u = 0, & x \in \Omega \\ u = 0, & x \in \partial\Omega. \end{cases} \quad (1.6)$$

As usual, we denote $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ for any $1 < p < +\infty$, and the weight $w \in L^s$ for some s depending on p and N . We include a short appendix with some facts about the eigenvalues of the p -Laplace operator that we will need later.

Let us fix the following notations that will be used below: let us call r_Ω the inner radius of Ω ,

$$r_\Omega = \max_{x \in \Omega} d_\Omega(x)$$

where

$$d_\Omega(x) = d(x, \Omega^c) = \inf_{y \in \partial\Omega} |x - y|$$

is the distance from $x \in \Omega$ to the boundary.

Now, let us note that the length L of the interval in inequality (1.3) can be thought as the *measure* of the interval, but it can be understood also as twice the *inner radius* of the interval, by rewriting the inequality as

$$2 \left(\frac{2}{L} \right)^{p-1} \leq \int_0^L q(t) dt.$$

This is our main objective here: to derive some Lyapunov type inequalities involving the inner radius of the domain and norms of the weight w .

We divide the paper in two main parts, in the first we cover the case $p > N$, and we prove the existence of a Lyapunov inequality involving the L^1 norm of the weight and the inner radius of the domain. We also consider singular problems, and we need to prove a Morrey's theorem for A_p weights.

In the second one we analyze the case $p < N$, we show that there are Lyapunov type inequalities involving the L^s norm for $s > N/p$.

We do not consider here the case $p = N$. For $p = N = 2$, we mention two interesting results from Osserman [26]:

Theorem 1.1. (See Osserman, [26].) *Given a domain $\Omega \in \mathbb{R}^2$ of connectivity $k \geq 2$, the first Dirichlet eigenvalue of problem*

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

satisfy

$$\lambda_1 \geq \frac{1}{k^2 r_\Omega^2}.$$

Theorem 1.2. (See Osserman, [26].) *Let $\Omega \in \mathbb{R}^2$, and Ω_ε the domain obtained by removing from Ω a finite number of disjoint disks of radius ε centered at a fixed set E of points in Ω . Then,*

$$\lim_{\varepsilon \rightarrow 0} \lambda_1(\Omega_\varepsilon) = \lambda_1(\Omega).$$

Clearly, both results are enough to conclude that we cannot expect a general inequality involving the inner radius of the domain when $p = N$, although it would be very interesting to find a related inequality.

Finally, we show the optimality of the bounds, and we apply them to eigenvalue problems. We compare them with Sturmian and isoperimetric bounds. Also, we consider some related inequalities of Anane [1] and Cuesta [9], which involves the measure of the set Ω .

2. Statement of the results and organization of the paper

Let us state precisely our results in this section.

In Section 3, we consider the case $p > N$ and we prove first:

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^N$ be an open set, let $w \in L^1(\Omega)$ be a non-negative weight, and let $u \in W_0^{1,p}(\Omega)$ with $p > N$ be a nontrivial solution of*

$$\begin{cases} -\Delta_p u = w(x)|u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then,

$$\frac{C}{r_\Omega^{p-N}} \leq \|w\|_{L^1(\Omega)} \quad (2.7)$$

where C is an universal constant depending only on p and N .

Let us note that the constant C is the same for any $\Omega \subset \mathbb{R}^N$, since it is related to the constant given by Morrey's Theorem; we believe that it can be improved for particular domains. However, the power of the inner radius is optimal.

Then, we consider the following problem

$$-\operatorname{div}(v(x)|\nabla u|^{p-2}\nabla u) = w(x)|u|^{p-2}u$$

where now v is a singular or degenerate weight, typically a power of the distance to the boundary or powers of $|x|$ (as in Henon equations, and Caffarelli–Kohn–Nirenberg inequalities).

Here, the problem is more subtle since we need the density of continuous functions in the weighted Sobolev space

$$W_0^{1,p}(\mathbb{R}^N, v, w) := \{u \in L_{loc}^1(\mathbb{R}^N) : w^{1/p}u \in L^p(\mathbb{R}^N) \text{ and } v^{1/p}\nabla u \in [L^p(\mathbb{R}^N)]^N\}$$

where ∇u is a distributional gradient in the sense of Schwartz.

Following [20], this is true when $v = w$ belong to the Muckenhoupt class A_p , that is, v is a nonnegative function in $L_{loc}^1(\mathbb{R}^N)$, and there exists a constant $c_{p,v}$ such that

$$\left(\int_B v(x) dx \right) \left(\int_B v(x)^{-\frac{1}{p-1}} dx \right)^{p-1} \leq c_{p,v} |B| \quad (2.8)$$

for every ball $B \in \mathbb{R}^N$.

The same argument applies for different weights v, w in A_p , as we will show in Lemma 3.3 below. So, we will restrict ourselves to weights $v, w \in A_t$ with $t < p/N$, and in this case we prove the following Lyapunov type inequality:

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^N$, and let $v \in A_t(\mathbb{R}^N)$, with $t < p/N$, and $v \geq 0$. Let us define*

$$g(r_\Omega) = \sup_{x \in \Omega} \int_{B(x, r_\Omega)} v^{-\frac{1}{t-1}}(x) dx.$$

Let $u \in W_0^{1,p}(\Omega)$ be a nontrivial solution of

$$\begin{cases} -\operatorname{div}(v(x)|\nabla u|^{p-2}\nabla u) = w(x)|u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then, we have the following Lyapunov-type inequality

$$1 \leq C(p, t, N) r_\Omega^{p-tN} g(r_\Omega)^{t-1} \int_\Omega w(z) dz \quad (2.9)$$

where the constant $C(p, t, N)$ depends only on p , t , and N .

Theorem 2.2 is based on the fact that $A_t \subset A_p$ whenever $t < p$. Briefly, we will bound u by the fractional integral (or Riesz potential) of its gradient, and after adding the corresponding power of the coefficient, we wish to use Holder's inequality with exponents p in the gradient, and an exponent close to p' in $|\cdot|^{1-N}$.

Remark 2.3. This theorem can be thought as a Morrey's embedding with A_p weights. To our knowledge, no such result was proved before for the case $p > N$. For $p < N$, we refer the interested reader to the book of Turesson [32].

Although the terms in the Lyapunov inequality (2.9) seems difficult to compute, in certain interesting case are rather simple to compute. We choose as an example a coefficient which is a power of the distance to the boundary, $v(x) = d_\Omega^\gamma(x)$, and in this case we obtain a very clean bound,

$$1 \leq C r_\Omega^{p-N-\gamma} \int_\Omega w(z) dz,$$

where C depends only on N , p , and γ . Of course, γ is restricted by the A_t condition, let us recall that $d_\Omega^\gamma(x) \in A_t$ for $-1 < \gamma < t-1$, see [25].

For $1 < p < N$, a similar inequality cannot hold for arbitrary domains, as we mention in the Introduction. Perhaps the easiest way to understand why is to remove a discrete set of points with zero capacity from a ball, and the first eigenvalue remains the same.

So, in Section 4, we prove the following weaker inequality:

Theorem 2.4. Let $\Omega \subset \mathbb{R}^N$ be a smooth domain, $\frac{N}{p} < s$, and $w \in L^s(\Omega)$. Let $u \in W_0^{1,p}(\Omega)$ be a nontrivial solution of

$$\begin{cases} -\Delta_p u = w(x)|u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Then, we have the following Lyapunov inequality

$$\frac{C}{r^{\frac{sp-N}{s}}} \leq \|w\|_{L^s(\Omega)}. \quad (2.10)$$

The constant C depends on p , N , and the capacity of $\mathbb{R}^N \setminus \Omega$.

The proof of this theorem is based on the Sobolev immersion with critical exponent and Hardy's inequality, and for this reason the p -capacity of $\mathbb{R}^N \setminus \Omega$ appears on the constant. Although the constant is domain-dependent, for certain classes of sets we can give an uniform constant, i.e., for Lipschitz or convex domains, we have an explicit constant depending only on p and N (see the details below at the end of Section 4).

Remark 2.5. We do not consider singular problems when $p < N$. Similar results as in Section 3 can be obtained by combining the results in [32] with Hardy-type inequalities involving A_p weights, see the book of Opic and Kufner [25], following the proof of Theorem 2.4.

Let us note that we have the following lower bounds for the first eigenvalue of the p -Laplacian with zero Dirichlet boundary conditions:

Corollary 2.6. Let λ_1 be the first eigenvalue of

$$-\Delta_p u = \lambda w(x)|u|^{p-2}u,$$

in Ω with zero Dirichlet boundary conditions in $\partial\Omega$. Then,

- for $p > N$ and w as in Theorem 2.1, we have

$$\frac{C}{r_\Omega^{p-N} \|w(x)\|_1} \leq \lambda_1,$$

- for $p < N$ and w as in Theorem 2.4,

$$\frac{C}{r_\Omega^{\frac{sp-N}{s}} \|w(x)\|_s} \leq \lambda_1.$$

This corollary follows directly from Theorems 2.1 and 2.4, by replacing w with $\lambda_1 w$. In Section 5, we apply the bounds of Corollary 2.6 to eigenvalue problems.

First, we show that the powers of the inner radius appearing in Theorems 2.1 and 2.4 are optimal:

Proposition 2.7. *Let $B(0, R)$ be the ball of radius R centered at the origin, and let*

$$\gamma = \begin{cases} p - N & \text{if } p > N \\ \frac{sp - N}{s} & \text{if } p < N. \end{cases}$$

- *Let $R > 1$. For any $\beta < \gamma$, and C fixed, there exists a non-negative weight w , and a solution $u_\beta \in W_0^{1,p}(B(0, R))$ of*

$$\begin{cases} -\Delta_p u = w(x)|u|^{p-2}u & \text{in } B(0, R) \\ u = 0 & \text{on } \partial B(0, R) \end{cases}$$

such that the inequality

$$\frac{C}{R^\beta} \leq \|w\|_{L^1(B(0, R))}$$

does not hold.

- *Let $R < 1$. For any $\beta > \gamma$, and C fixed, there exists a non-negative weight w , and $u_\beta \in W_0^{1,p}(B(0, R))$ a solution of*

$$\begin{cases} -\Delta_p u = w(x)|u|^{p-2}u & \text{in } B(0, R) \\ u = 0 & \text{on } \partial B(0, R) \end{cases}$$

such that the inequality

$$\frac{C}{R^\beta} \leq \|w\|_{L^1(B(0, R))}$$

does not hold.

The result follows by computing a bound of the first eigenvalue of the p -Laplacian on a ball with a radial weight restricted to a small ball of radius ε for a suitable ε .

Finally, we compare the lower bounds for the first eigenvalue of the p -Laplacian in [Corollary 2.6](#) with the ones obtained with other techniques.

A classical tool for problems without weights is the Faber–Krahn inequality,

$$\lambda_1(B) \leq \lambda_1(\Omega),$$

where B is the ball with Lebesgue measure $|B| = |\Omega|$. Several proofs of this inequality for the p -Laplacian appeared in the literature, and they are based on the ideas of Talenti. Some improvements involving measures of the asymmetry of the domain Ω are known, see [\[2,17\]](#).

For bounded weights, a Sturmian comparison argument combined with the variational characterization of the first eigenvalue (see equation (A.2) in the Appendix), enable us to replace w with the norm $\|w\|_{L^\infty}$, obtaining now lower bounds for λ_1 .

For arbitrary weights, there are few inequalities involving their norms and the measure of the domain, namely the works of Anane [1] and Cuesta [9].

We show that for certain domains and weights, the bounds given by Lyapunov inequality are better.

We close the paper with a short Appendix where we include some basic facts about p -Laplacian eigenvalues.

3. Lyapunov's inequality for $p > N$

Let us recall first Morrey inequality:

Theorem 3.1 (Morrey). *If $p > n$, there exists a constant $C(N, p)$ such that for all $u \in W_0^{1,p}(\Omega)$,*

$$|u(x) - u(y)| \leq C(n, p) \|\nabla u\|_{L^p} |x - y|^\alpha$$

for all $x, y \in \overline{\Omega}$, and $\alpha = 1 - \frac{N}{p}$.

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. Let $u \in W_0^{1,p}(\Omega)$ be a nontrivial solution of

$$-\Delta_p u = w(x)|u|^{p-2}u$$

with Dirichlet boundary conditions. Multiplying by u and integrating by parts, we obtain

$$\int_{\Omega} |\nabla u|^p = \int_{\Omega} w(x)|u|^p.$$

Since $p > N$, u is continuous and let us choose $c \in \Omega$ a the point of $\overline{\Omega}$ where $|u(x)|$ achieves its maximum. Then, for $y = c$ and $x \in \partial\Omega$ we have that

$$|u(c)| \leq C(N, p) \left(\int_{\Omega} |\nabla u|^p dx \right)^{\frac{1}{p}} |x - c|^\alpha.$$

By using that $|x - c| \leq r_\Omega$, the inner radius of Ω , we get

$$|u(c)| \leq C(N, p) \left(\int_{\Omega} w(x)|u|^p dx \right)^{\frac{1}{p}} r_\Omega^\alpha.$$

Hence,

$$|u(c)| \leq C(N, p) |u(c)| \left(\int_{\Omega} w(x) dx \right)^{\frac{1}{p}} r_{\Omega}^{\alpha}$$

and canceling out $|u(c)|$ we have the Lyapunov inequality

$$\frac{1}{r_{\Omega}^{\alpha}} \leq C(N, p) \left(\int_{\Omega} w(x) dx \right)^{\frac{1}{p}},$$

with $\alpha = 1 - \frac{N}{p}$.

The proof is finished. \square

Remark 3.2. In particular, let λ_1 be the first eigenvalue of

$$-\Delta_p u = \lambda w(x) |u|^{p-2} u$$

in Ω with zero Dirichlet boundary conditions in $\partial\Omega$. We have

$$\frac{C(N, p)^{-p}}{r_{\Omega}^{p-N} \|w(x)\|_1} \leq \lambda_1, \quad (3.11)$$

which gives the lower bound for λ_1 in [Corollary 2.6](#).

3.1. Singular and degenerate weights

The following lemma extend the results in [\[20\]](#) for different weights in the function and its distributional gradient:

Lemma 3.3. *For $v, w \in A_p$, the space $W_0^{1,p}(\mathbb{R}^N, v, w)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ with the norm*

$$\|\cdot\|_{p,v,w} := (\|\nabla \cdot\|_{[L^p(\mathbb{R}^N, v)]^N}^p + \|\cdot\|_{L^p(\mathbb{R}^N, w)}^p)^{1/p}.$$

Proof. The proof follows by taking $u \in W_0^{1,p}(\mathbb{R}^N, v, w)$ and regularizing it by convolution with a mollifier η_j . Now, from Lemma 1.5 in [\[20\]](#),

$$\begin{aligned} \eta_j * u &\rightarrow u && \text{in } L^p(\mathbb{R}^N, w) \\ \nabla(\eta_j * u) &= \eta_j * \nabla u \rightarrow \nabla u && \text{in } [L^p(\mathbb{R}^N, v)]^N \end{aligned}$$

that is, $\eta_j * u \rightarrow u$ in $W_0^{1,p}(\mathbb{R}^N, v, w)$. \square

We are ready to prove [Theorem 2.2](#).

Proof. Thanks to [Lemma 3.3](#), we can choose a smooth function u . Now, given $x, y \in \bar{\Omega}$, such that $r = |x - y| \leq r_\Omega$, let us call $A = B(x, r) \cap B(y, r)$. Hence,

$$\begin{aligned} |u(x) - u(y)| &\leq \frac{1}{|A|} \int_A |u(x) - u(z)| dz + \frac{1}{|A|} \int_A |u(y) - u(z)| dz \\ &\leq C \int_{B(x, r)} \frac{|\nabla u(z)|}{|x - z|^{N-1}} dz + C \int_{B(y, r)} \frac{|\nabla u(z)|}{|y - z|^{N-1}} dz \\ &= I_1 + I_2 \end{aligned}$$

where the constant C depends only on N , see for instance, Evans [\[15\]](#).

Let us bound now I_1 . We need to include the coefficient v appearing in the equation, and let us call $B = B(x, r)$. By using Holder's inequality:

$$\begin{aligned} I_1 &= C \int_B \frac{|\nabla u(z)|}{|x - z|^{N-1}} v^{\frac{1}{p}} v^{-\frac{1}{p}} dz \\ &\leq C \left(\int_B v |\nabla u(z)|^p dz \right)^{\frac{1}{p}} \left(\int_B \frac{1}{|x - z|^{q(N-1)}} dz \right)^{\frac{1}{q}} \left(\int_B v^{-\frac{s}{p}} dz \right)^{\frac{1}{s}} \end{aligned}$$

where

$$\begin{aligned} \frac{1}{p} + \frac{1}{q} + \frac{1}{s} &= 1, \\ s &= \frac{p}{t-1}. \end{aligned}$$

Now, we have following bounds:

$$\int_B v(z) |\nabla u(z)|^p dz \leq \int_\Omega w(z) |u(z)|^p dz, \quad (3.12)$$

$$\int_B \frac{1}{|x - z|^{q(N-1)}} dz \leq c r_\Omega^{q-qN+N}, \quad (3.13)$$

$$\int_B v^{-\frac{s}{p}}(z) dz \leq g(r_\Omega). \quad (3.14)$$

We have used that v is positive, and by integrating by parts the equation multiplied by u in Ω , we get the first inequality. The second one follows by integrating in polar coordinates in a bigger ball of radius r_Ω , the constant c can be computed explicitly and

depends only on N , p and q . The last one was defined in this way in the hypotheses of the theorem.

The bound for I_2 is almost identical, although we need first to impose some extra condition on u . Since we are working in $W_0^{1,p}$, we can extend any function by zero outside Ω , and we can take a smooth function u supported in Ω . So, we can integrate only over $B(y, r) \cap \Omega$ in the first inequality (3.12), and we get

$$|u(x) - u(y)| \leq C r_\Omega^{1-N+\frac{N}{q}} g(r_\Omega)^{\frac{1}{s}} \left(\int_\Omega w(z) |u(z)|^p dz \right)^{\frac{1}{p}}$$

where C is a universal constant depending only on N , p and q .

We are able to choose yet the points x and y , and this is the last step of the proof. Let x be the point where $|u|$ is maximized, and y one of the points in $\partial\Omega$ which minimizes $|x - y|$. So, $u(y) = 0$ and $|x - y| < r_\Omega$.

After bounding $|u(z)| \leq |u(x)|$ at the right hand side, and canceling out with the one in the left hand side, we get

$$1 \leq C(p, t, N) r_\Omega^{p-pN+\frac{pN}{q}} g(r_\Omega)^{\frac{p}{s}} \int_\Omega w(z) dz.$$

Finally, let us observe that the relationship between Holder's exponent implies that

$$\frac{p}{q} = p - t, \quad \frac{p}{s} = t - 1.$$

The proof is finished. \square

Remark 3.4. Let us note that inequality (3.13) holds when $q - qN + N > 0$, and $q \geq p'$ in Holder's inequality. That is,

$$\frac{p}{p-1} < q < \frac{N}{N-1}$$

which makes sense because $p > N > 1$.

On the other hand, the bigger is q , the bigger is s . When $q \rightarrow \frac{N}{N-1}$, we have that $s \rightarrow \frac{pN}{p-N}$, and the integral in inequality (3.14) is well defined when $v \in A_t$ with

$$t < p/N.$$

As an application of Theorem 2.2 we have the following result for quasilinear problems involving the distance to the boundary.

Proposition 3.5. *Let $\Omega \in \mathbb{R}^N$ a bounded open set, $p > N$, and $u \in W_0^{1,p}(\Omega, d^\gamma, w)$ a nontrivial solution of*

$$-div(d_\Omega^\gamma(x) |\nabla u|^{p-2} \nabla u) = w(x) |u|^{p-2} u$$

in Ω with zero Dirichlet boundary conditions in $\partial\Omega$, where $d_\Omega(x)$ is the distance to the boundary. Then,

$$1 \leq C r_\Omega^{p-N-\gamma} \int_{\Omega} w(z) dz,$$

where C depends only on N , p , and γ .

In order to prove this proposition, we can repeat the previous proof, although only inequality (3.14) depends on d'_Ω . So, we will improve this bound by integrating in $B(x, d_\Omega(x))$ instead of $B(x, r_\Omega)$.

Proof. We divide the proof in two cases, depending on the sign of γ .

First, we consider $\gamma < 0$.

Given $z \in \Omega$, we choose $y \in \partial\Omega$ with $r = |x - y| = d_\Omega(x)$, clearly we have $r \leq r_\Omega$. After a translation if necessary, we can suppose that $y = 0$, and we have $d_\Omega(z) \leq |z|$. Then,

$$d_\Omega^{-\frac{s\gamma}{p}}(z) \leq |z|^{-\frac{s\gamma}{p}}.$$

Hence, we can estimate $g(r_\Omega)$ by computing

$$\int_{B(x,r)} d_\Omega^{-\frac{s\gamma}{p}}(z) \leq \int_{B(x,r)} |z|^{-\frac{s\gamma}{p}} dz = r^{N-\frac{s\gamma}{p}} \int_{B(x/r,1)} |\eta|^{-\frac{s\gamma}{p}} d\eta \leq C r_\Omega^{N-\frac{s\gamma}{p}},$$

where in the last step we changed variables, $\eta = z/r$.

So, we can bound

$$\int_{B(x,r)} d_\Omega^{-\frac{s\gamma}{p}}(z) \leq C r_\Omega^{N-\frac{s\gamma}{p}}.$$

Let us consider now $\gamma > 0$.

Given $z \in \Omega$ and $y \in \partial\Omega$ with $r = |x - y| = d_\Omega(x) \leq r_\Omega$ as before, clearly we have $r \leq r_\Omega$. After a translation if necessary, we can suppose that $x = 0$, and we have $d_\Omega(z) \geq d_{\partial B(0,r)}(z)$, the distance to the boundary of the ball.

Then, since $\gamma > 0$,

$$d_\Omega^{-\frac{s\gamma}{p}}(z) \leq d_{\partial B(0,r)}^{-\frac{s\gamma}{p}}(z),$$

and

$$\int_{B(0,r)} d_\Omega^{-\frac{s\gamma}{p}}(z) \leq \int_{B(0,r)} (r - |z|)^{-\frac{s\gamma}{p}} dz$$

$$\begin{aligned}
&= c_N \int_0^r (r - \rho)^{-\frac{s\gamma}{p}} \rho^{N-1} d\rho \\
&= c_N r^{N-\frac{s\gamma}{p}} \int_0^1 (1 - \hat{\rho})^{-\frac{s\gamma}{p}} \hat{\rho}^{N-1} d\hat{\rho} \\
&= C r^{N-\frac{s\gamma}{p}}.
\end{aligned}$$

Again, we have the bound

$$\int_{B(0,r)} d_{\Omega}^{-\frac{s\gamma}{p}}(z) \leq C r_{\Omega}^{N-\frac{s\gamma}{p}}.$$

The last step is to replace this bound instead of the power of $g(r_{\Omega})$ in Lyapunov's inequality given by [Theorem 2.1](#). By using that $p/s = t - 1$, we have

$$1 \leq C r_{\Omega}^{p-N-\gamma} \int_{\Omega} w(z) dz$$

and the proof is finished. \square

4. Lyapunov-type inequality for $p < N$

Let us prove now [Theorem 2.4](#).

Proof. Let us define

$$q = \alpha p + (1 - \alpha)p^*,$$

where p^* is the Sobolev conjugate exponent, and $\alpha \in (0, 1)$ which will be chosen later.

Then, we have

$$\frac{1}{r_{\Omega}^{\alpha p}} \int_{\Omega} |u|^q dx \leq \int_{\Omega} \frac{|u|^q}{d(x)^{\alpha p}} dx,$$

where $d(x)$ is the distance from x to the boundary. Now, Holder's inequality with exponents $1/\alpha$ and $(1/\alpha)' = 1/(\alpha - 1)$ gives

$$\int_{\Omega} \frac{|u|^{\alpha p} |u|^{(1-\alpha)p^*}}{d(x)^{\alpha p}} dx \leq \left(\int_{\Omega} \frac{|u|^p}{d(x)^p} dx \right)^{\alpha} \left(\int_{\Omega} |u|^{p^*} dx \right)^{1-\alpha}. \quad (4.15)$$

Let us recall Hardy and Sobolev inequalities,

$$\int_{\Omega} \frac{|u|^p}{d(x)^p} dx \leq C_h \int_{\Omega} |\nabla u|^p dx,$$

$$\int_{\Omega} |u|^{p^*} dx \leq C_s \left(\int_{\Omega} |\nabla u|^p dx \right)^{p^*/p}$$

and by using them in equation (4.15), we get

$$\left(\int_{\Omega} \frac{|u|^p}{d(x)^p} dx \right)^{\alpha} \left(\int_{\Omega} |u|^{p^*} dx \right)^{1-\alpha} \leq C_{hs} \left(\int_{\Omega} |\nabla u|^p dx \right)^{\alpha+(1-\alpha)p^*/p}$$

where C_{hs} is a constant depending only on C_h and C_s , the constants involved in Hardy and Sobolev inequalities.

Hence, by using the weak formulation for equation $-\Delta_p u = w(x)|u|^{p-2}u$, and applying again Holder's inequality with exponents s and s' we obtain

$$\begin{aligned} \left(\int_{\Omega} |\nabla u|^p dx \right)^{\frac{\alpha p + (1-\alpha)p^*}{p}} &= \left(\int_{\Omega} w(x)|u|^p dx \right)^{\frac{\alpha p + (1-\alpha)p^*}{p}} \\ &\leq \left(\int_{\Omega} w(x)^s dx \right)^{\frac{\alpha p + (1-\alpha)p^*}{ps}} \left(\int_{\Omega} |u|^{ps'} dx \right)^{\frac{\alpha p + (1-\alpha)p^*}{ps'}}. \end{aligned}$$

We choose now α such that $ps' = q$. Let us observe that

$$\frac{\alpha p + (1-\alpha)p^*}{ps'} = 1,$$

$$\frac{\alpha p + (1-\alpha)p^*}{ps} = \frac{s'}{s},$$

and

$$\alpha = \frac{p^* - ps'}{p^* - p}.$$

Finally, we get

$$\frac{1}{r_{\Omega}^{\alpha p}} \int_{\Omega} |u|^q dx \leq \|w\|_{L^s}^{s'} \int_{\Omega} |u|^q dx,$$

and the theorem is proved. \square

Remark 4.1. A tedious computation shows that

$$\frac{\alpha p}{s'} = \frac{p}{s'} \frac{p^* - ps'}{p^* - p} = \frac{sp - N}{s}.$$

Since $s > N/p$, the exponent is positive.

Remark 4.2. The constant C depends on the constant H appearing on the Hardy inequality. When Ω is convex, we have $H = \left(\frac{p}{N-p}\right)^p$; for other domains, the constant depends on the capacity of $\mathbb{R}^N \setminus \Omega$; for Lipschitz domains the constant is close to 1/2, see [19,22] for details.

Remark 4.3. In particular, let λ_1 be the first eigenvalue of

$$-\Delta_p u = \lambda w(x) |u|^{p-2} u$$

in Ω with zero Dirichlet boundary conditions in $\partial\Omega$. We have

$$\frac{C}{r_\Omega^{\frac{sp-N}{s}} \|w(x)\|_s} \leq \lambda_1, \quad (4.16)$$

which gives the lower bound for λ_1 in [Corollary 2.6](#).

5. Some applications to eigenvalue problems

We close the paper with a discussion about the optimality of the lower bounds and its application to eigenvalue problems. We show that in certain cases the new bounds are better than the known ones.

5.1. Optimality of the bounds

Let us show the optimality of the power of the inner radius appearing in the inequality.

Proof of Proposition 2.7. For brevity, we will consider only the case $p > N$, $R > 1$ since the remaining ones follow exactly in the same way.

Fix $R > 1$, and let us show that the bound (3.11) from [Remark 3.2](#) cannot hold for some power $\beta < p - N$ and

$$w(r) = \chi_{[0,\varepsilon]}(r) r^{1-N},$$

where $\chi_{[0,\varepsilon]}(r)$ is the characteristic function of $[0, \varepsilon]$.

Clearly, $\|w\|_1 = \omega_{N-1}\varepsilon$, where ω_{N-1} is the surface measure of the unit ball, since

$$\int_{B(0,R)} \chi_{[0,\varepsilon]}(|x|)|x|^{1-N}dx = \int_{\omega_{N-1}} \int_0^\varepsilon r^{1-N}r^{N-1}drd\theta.$$

Let $\lambda_1^{(R)}$ and $\lambda_1^{(\varepsilon)}$ be the first eigenvalues of the p -Laplacian problem

$$-\Delta_p u = \lambda w(x)|u|^{p-2}u$$

with Dirichlet boundary conditions in $B(0, R)$ and $B(0, \varepsilon)$ respectively. We have $\lambda_1^{(R)} < \lambda_1^{(\varepsilon)}$, since extending the functions by zero, we have $W_0^{1,p}(B(0, \varepsilon)) \subset W_0^{1,p}(B(0, R))$, and the inequality follows by using the variational characterization,

$$\begin{aligned} \lambda_1^{(R)} &= \inf_{\{u \in W_0^{1,p}(B(0,R)): u \not\equiv 0\}} \frac{\int_{B(0,R)} |\nabla u|^p dx}{\int_{B(0,R)} \chi_{[0,\varepsilon]}(|x|)|x|^{1-N} dx} \\ \lambda_1^{(\varepsilon)} &= \inf_{\{u \in W_0^{1,p}(B(0,\varepsilon)): u \not\equiv 0\}} \frac{\int_{B(0,\varepsilon)} |\nabla u|^p dx}{\int_{B(0,\varepsilon)} |x|^{1-N} dx}. \end{aligned}$$

Since the first eigenfunction in a ball is radial,

$$\begin{aligned} \lambda_1^{(R)} \leq \lambda_1^{(\varepsilon)} &= \inf_{\{u \in W^{1,p}(0,\varepsilon): u(\varepsilon)=0, u \not\equiv 0\}} \frac{\int_0^\varepsilon r^{N-1}|u'|^p dr}{\int_0^\varepsilon |u|^p dr} \\ &\leq \varepsilon^{N-1} \frac{\pi_p^p}{\varepsilon^p}. \end{aligned}$$

Then,

$$\frac{C}{R^\beta} \leq \lambda_1 \omega_{N-1} \varepsilon.$$

Let $\varepsilon = R^\alpha$, and if we can choose $\alpha < 1$ such that $\beta - \alpha(p - N) < 0$, we reach a contradiction:

$$R^{\alpha(p-N)} \leq cR^\beta.$$

However, this is equivalent to find α satisfying

$$0 < \frac{\beta}{p-N} < \alpha < 1,$$

and we can find it if

$$\frac{\beta}{p-N} < 1,$$

which holds exactly when $\beta < p - N$. \square

Remark 5.1. Clearly, $\beta > \gamma$ is of no interest when the inner radius is greater than 1, since we get a worse bound instead of an improvement. Similar observations hold for the remaining cases.

5.2. Comparison with other estimates

Let us consider the following eigenvalue problem:

$$\begin{cases} -\Delta_p u = \lambda w(x)|u|^{p-2}u, & x \in \Omega \\ u = 0, & x \in \partial\Omega \end{cases} \quad (5.17)$$

There are few ways to obtain lower bounds for the eigenvalues of the p -Laplacian. In the constant coefficient case, we can use symmetrization and then compare with the first eigenvalue of a ball with the same measure as Ω , since the Faber–Krahn inequality implies

$$\lambda_1(B) \leq \lambda_1(\Omega).$$

For weighted problems, a Sturmian-type comparison theorem is available, that is, if $w_1(x) \leq w_2(x)$, then

$$\lambda_k(w_2) \leq \lambda_k(w_1),$$

since the eigenvalues are computed with the Rayleigh quotient. Also, Anane and Cuesta obtained some inequalities that we will review below.

In the rest of the section we compare those bounds with the one obtained from [Corollary 2.6](#) when $p > N$ and, $N = 2$. Similar results hold for $p < N$, and higher dimensions.

Faber–Krahn. In order to compare Faber–Krahn inequality and Lyapunov inequality [\(2.7\)](#), we can expect that the former will be worse in thin domains. So, let us take the following family of domains in \mathbb{R}^2

$$\Omega_R = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq R, 0 \leq y \leq 1/R\}$$

with $0 < R \leq 1$.

Since $|\Omega_R| = 1$, Faber–Krahn gives a fixed lower bound for any Ω_R . However, Lyapunov inequality (with $w \equiv 1$) implies

$$\frac{C(2, p)^{-p}}{r_{\Omega_R}^{p-2} \|w(x)\|_1} \leq \frac{C(2, p)^{-p}}{(R/2)^{p-2}} = \frac{C}{R^{p-2}} \leq \lambda_1.$$

Now, from equations [\(A.3\)](#), when $R \rightarrow 0$,

$$\hat{\lambda}_1 = \frac{\pi_p^p}{R^p} + \pi_p^p R^p = O\left(\frac{\pi_p^p}{R^p}\right),$$

and by using (A.6) from Appendix,

$$\lambda_1 = O\left(\frac{\pi_p^p}{R^p}\right).$$

Lyapunov inequality is better for R small, although it is not optimal in this family of sets.

Faber–Krahn inequality can be improved as in [2,17]. Following Fusco, Maggi and Pratelli,

$$\lambda_1(\Omega) \geq \lambda_1(B) \left\{ 1 + \frac{A(\Omega)^{2+p}}{C(N,p)} \right\},$$

where $C(N,p)$ is a fixed constant, and $A(E)$ is the Fraenkel asymmetry of a set E with finite measure,

$$A(E) := \inf \left\{ \frac{|E\Delta(x_0 + rB(0,1))|}{|E|} : x_0 \in \mathbb{R}^N, r^N |B(0,1)| = |E| \right\}.$$

Since A is bounded above by 2, the maximum constant that can be involved in the lower bound is independent of R for the previous family of sets.

Sturm type bounds. Intuitively, this kind of bounds can be improved because by adding a highly concentrated spike with very low mass in a given weight we can change slightly the eigenvalue, and the supremum norm of the weight can be made arbitrarily big. The proof follows easily by using the eigenfunction of the unperturbed weight as a test function.

However, the improvement can be better, even for domains with an inner radius of the same order than the diameter of the domain. Suppose that $0 \leq w \leq M$, $\Omega = [0, R] \times [0, R]$, and $R \gg 1$, with $\int_{\Omega} w(x) = 1$. The variational characterization of the first eigenvalue, together with (A.3) and (A.6) implies

$$\frac{2\pi_p^p}{MR^p} \leq \lambda_1.$$

Now, Lyapunov inequality gives the bound

$$\frac{C}{R^{p-2}} \leq \lambda_1.$$

Let us observe that the difference between them not depend only on M , but on a factor MR^2 . Indeed, we always have

$$R^{p-2} \int_{\Omega} w dx \leq R^p M.$$

Bounds involving norms of the weights. For arbitrary weights, there are few estimates involving their norms and the measure of the domain.

First, Anane obtained in [1] the following estimate:

$$\frac{C}{|\Omega|^\sigma \|w\|_\infty} \leq \lambda,$$

where

$$\begin{aligned} \sigma &= p/N & \text{if } 1 < p \leq N, \\ \sigma &= 1/2 & \text{if } N < p. \end{aligned}$$

Also, Cuesta proved in [9] the following inequality:

$$\frac{C}{|\Omega|^{\frac{sp-N}{sN}} \|w\|_s} \leq \lambda,$$

where

$$\begin{aligned} s &> N/p & \text{if } 1 < p \leq N, \\ s &= 1 & \text{if } N < p. \end{aligned}$$

Clearly, they are Lyapunov type inequalities, involving the measure of the domain instead of the inner radius. Those inequalities were widely used to show that the first eigenvalue is isolated, since any other eigenfunction has at least two nodal domains, and one of them must shrink, but the inequality implies that the first eigenvalue of the shrinking domain cannot converge to the first eigenvalue of the full domain.

Let us observe that

$$|\Omega|^{1/N} \geq Cr_\Omega$$

with equality only when Ω is a ball, so [Corollary 2.6](#) gives better bounds, except in Anane's bound for $p > N$, which is better when $w \simeq cte$, $|\Omega| \simeq r_\Omega^N$, and the measure of Ω is small enough.

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Appendix A. Eigenvalues of the p -Laplacian

We say that a function u is an eigenfunction of problem

$$\begin{cases} -\Delta_p u = \lambda w(x)|u|^{p-2}u, & x \in \Omega \\ u = 0, & x \in \partial\Omega \end{cases} \quad (\text{A.1})$$

corresponding to the eigenvalue λ if

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx = \lambda \int_{\Omega} w(x) |u|^{p-2} u \varphi \, dx$$

for any test-function $\varphi \in W_0^{1,p}(\Omega)$. The existence of infinitely many eigenvalues was proved by Garcia Azorero and Peral Alonso in [18] by using the critical point theory of Ljusternik–Schnirelmann, and the variational characterization given by the Rayleigh quotient,

$$\lambda_k = \inf_{C \in \mathcal{C}_k} \sup_{u \in C} \frac{\int_{\Omega} |\nabla u|^p \, dx}{\int_{\Omega} w(x) |u|^p \, dx}, \quad (\text{A.2})$$

where \mathcal{C}_k is the class of compact symmetric ($C = -C$) subsets of $W_0^{1,p}(\Omega)$ of Krasnoselskii genus greater or equal that k , see [31] for details.

It is well known that the first eigenfunction is positive and simple, see for instance [1]. Indeed, this result holds for more general operators, including the so-called pseudo- p -Laplacian operator,

$$-\hat{\Delta}_p v := - \sum_{i=1}^N \frac{\partial}{\partial x_i} \left(\left| \frac{\partial v}{\partial x_i} \right|^{p-2} \frac{\partial v}{\partial x_i} \right),$$

and the proof is exactly the same, the simplicity follows by a Picone type identity, and the positivity by considering $|u_1|$ as a test function, where u_1 is the first eigenfunction.

We will use the pseudo- p -Laplacian in order to control the eigenvalues of the p -Laplacian. The equivalence of norms in \mathbb{R}^N , $|x|_q \leq C_{p,q} |x|_p$ enable us to compare the first eigenvalue of each problem, since both can be defined

$$\hat{\lambda}_1 = \inf_{u \in B} \| |\nabla u|_p \|_p^p; \quad \lambda_1 = \inf_{u \in B} \| |\nabla u|_2 \|_2^p,$$

where

$$B = \left\{ u \in W_0^{1,p}(\Omega) : \int_{\Omega} w(x) |u|^p \, dx \right\}$$

Clearly,

$$\begin{aligned} \hat{\lambda}_1 &\leq \lambda_1 \leq N^{(p-2)/2} \hat{\lambda}_1 && \text{if } 2 < p, \\ N^{(p-2)/2} \hat{\lambda}_1 &\leq \lambda_1 \leq \hat{\lambda}_1 && \text{if } p < 2. \end{aligned} \quad (\text{A.3})$$

The first eigenvalue of the one dimensional problem with $w \equiv 1$

$$\begin{cases} -(|u'|^{p-2}u')' = \lambda|u|^{p-2}u & \text{in } (0, L) \\ u(0) = u(L) = 0 \end{cases} \quad (\text{A.4})$$

can be computed explicitly with the help of the function $\sin_p(x)$, defined implicitly as

$$x = \int_0^{\sin_p(x)} \left(\frac{p-1}{1-t^p} \right)^{1/p} dt,$$

and its first zero π_p

$$\pi_p = 2 \int_0^1 \left(\frac{p-1}{1-t^p} \right)^{1/p} dt.$$

We have

$$\lambda_1 = \frac{\pi_p^p}{L^p}.$$

Also, for the mixed boundary condition $u'(0) = u(L) = 0$, the first eigenvalue is given by

$$\lambda_1 = \frac{2^p \pi_p^p}{L^p}.$$

We refer the interested reader to the work of Del Pino, Drabek and Manasevich, [12] for more details about the one dimensional case.

Finally, for $w \equiv 1$ the first eigenvalue $\hat{\lambda}_1$ and the corresponding eigenfunction \hat{u}_1 of the pseudo- p -Laplacian in a cube $Q = [0, L]^N \subset \mathbb{R}^N$ can be computed explicitly. Following [16], we have

$$\hat{\lambda}_1 = \frac{\pi_p^p N}{L^p}, \quad \hat{u}_1(x) = \prod_{j=1}^N \sin_p \left(\frac{\pi_p x_j}{L} \right), \quad (\text{A.5})$$

which combined with inequalities (A.3) gives upper and lower bounds for the first eigenvalue of the p -Laplacian in Q with $w(x) \equiv 1$.

A similar computation gives, for $\Omega = \prod_{j=1}^N [0, L_j]$,

$$\hat{\lambda}_1 = \sum_{j=1}^N \frac{\pi_p^p}{L_j^p}, \quad \hat{u}_1(x) = \prod_{j=1}^N \sin_p \left(\frac{\pi_p x_j}{L_j} \right) \quad (\text{A.6})$$

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