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Homology of left non-degenerate set-theoretic solutions to the Yang–Baxter equation ☆

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ABSTRACT

This paper deals with left non-degenerate set-theoretic solutions to the Yang–Baxter equation (= LND solutions), a vast class of algebraic structures encompassing groups, racks, and cycle sets. To each such solution there is associated a shelf (i.e., a self-distributive structure) which captures its major properties. We consider two (co)homology theories for LND solutions, one of which was previously known, in a reduced form, for biracks only. An explicit isomorphism between these theories is described. For groups and racks we recover their classical (co)homology, whereas for cycle sets we get new constructions. For a certain type of LND solutions, including quandles and non-degenerate cycle sets, the (co)homologies split into the degenerate and the normalized parts. We express 2-cocycles of our theories in terms of group cohomology, and, in the case of cycle sets, establish connexions with extensions. This leads to a construction of cycle sets with interesting properties.

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Braided homology
Extension
Cubical homology

Contents

1. Introduction

The *Yang–Baxter equation* (= *YBE*) plays a fundamental role in such apparently distant fields as statistical mechanics, particle physics, quantum field theory, quantum group theory, and low-dimensional topology; see for instance [32] for a brief introduction. The study of its solutions has been a vivid research area for the last half of a century. Following Drinfel'd [11], set-theoretic solutions, or *braided sets*, received special attention. Concretely, these are sets X endowed with a *braiding*, i.e., a not necessarily invertible map $\sigma\colon X^{\times 2} \to X^{\times 2}$, often written as $\sigma(a,b) = ({}^a b, a^b)$, satisfying the YBE

$$(\sigma \times \mathrm{Id})(\mathrm{Id} \times \sigma)(\sigma \times \mathrm{Id}) = (\mathrm{Id} \times \sigma)(\sigma \times \mathrm{Id})(\mathrm{Id} \times \sigma)\colon X^{\times 3} \to X^{\times 3}. \tag{1.1}$$

Two families of braided sets are particularly well explored:

- The map

  $$\sigma(a,b) = (b, a^b)$$

  is a braiding if and only if the operation $a \triangleleft b := a^b$ is *self-distributive*, in the sense of

  $$(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c). \tag{1.2}$$

  Such datum $(X, \triangleleft)$ is called a *shelf*. The term *rack* is used if moreover the right translations $a \mapsto a \triangleleft b$ are bijections on X for all $b \in X$, which is equivalent to the invertibility of σ. A *quandle* is a rack satisfying $a \triangleleft a = a$ for all a, which means that σ is the identity on the diagonal of $X^{\times 2}$. A group with the conjugation operation $a \triangleleft b = b^{-1}ab$ yields an important example of quandles. A systematic study of self-distributivity dates back to Joyce [24] and Matveev [31].

- A *cycle set*, or *right-cyclic quasigroup*, is a set X with a binary operation $\cdot$ satisfying

$$(a \cdot b) \cdot (a \cdot c) = (b \cdot a) \cdot (b \cdot c) \tag{1.3}$$

and having all the left translations $a \mapsto b \cdot a$ bijective, the inverse operation being denoted by $a \mapsto b * a$. As pointed out by Rump [34], these give rise to involutive braidings

$$\sigma(a, b) = ((b * a) \cdot b, b * a),$$

and all braidings of a certain type can be obtained this way.

Racks and *non-degenerate* cycle sets (i.e., for which the squaring map $a \mapsto a \cdot a$ is bijective) can be included into the much more general—and hence less understood—family of *biracks*, introduced by Fenn, Rourke, and Sanderson in [19]. These are sets with invertible braidings which are *left* and *right non-degenerate*, i.e., their maps $a \mapsto a^b$ and $a \mapsto {}^b a$ are bijective.

Important advances in knot-theoretic and Hopf-algebraic applications of self-distributivity are due to the *homological approach*, initiated by Fenn–Rourke–Sanderson [20] and Carter–Jelsovsky–Kamada–Langford–Saito [6], and further developed by Andruskiewitsch–Graña [1]. Rack (co)homology theories were generalized to the case of arbitrary braided sets by Carter–Elhamdadi–Saito [5] and further developed by the first author [26]. For biracks, Fenn, Rourke, and Sanderson [19] constructed an alternative, and more manageable, (co)homology theory, recently revived by Ceniceros–Elhamdadi–Green–Nelson [9]. The knot invariant construction, which motivated the rack cohomology theory, survived in all these generalized settings. Another application of these cohomology theories is a construction of new examples of racks and braided sets via an extension procedure using cocycles of low degree. Recently, the extension techniques were adapted to cycle sets by the second author [37], resulting in counter-examples to several conjectures concerning involutive solutions to the Yang–Baxter equation.

The starting point of this paper is a study of the (co)homology of left non-degenerate (= LND) braided sets, including biracks. We introduce a coefficient version of the complex from [19] and extend it from biracks to all LND braided sets (Theorem 4.2). It is then related to the complex from [5,26] (recalled in Theorem 3.5) by an explicit isomorphism (Theorem 7.1). For a concrete braided set, one can thus choose between the two constructions the one more suitable for computations.

We also extend to LND braidings of a certain type the degenerate and normalized (co)homology constructions, given for quandles in [6] and for a more general class of biracks in [19]. These (co)homologies turn out to be related by a splitting theorem (Theorem 8.2), which generalizes the analogous result for quandles, obtained by Litherland

and Nelson in [29]. More precisely, we establish a splitting theorem in the abstract context of (a skew version of) cubical homology (Theorem 2.2), and then refine our concrete complexes into cubical structures.

On the way we show how to associate to any LND braiding a shelf operation that captures many of its properties: invertibility, involutivity, the structure (semi)group, the action of positive braid monoids on its tensor powers, etc. (Propositions 5.7 and 6.2, Theorem 6.3). This reduces the study of certain aspects of LND braided sets to that of shelves. An analogous construction for biracks was considered by Soloviev [36] and, in the more restricted case of braided groups, by Lu, Yan, and Zhu [30].

The last block of our results concerns cycle sets, whose (co)homological aspects remained unexplored until now. A cycle set is automatically an LND braided set (but not necessarily a birack!), which allows an application of our general (co)homology constructions described above. We provide a detailed analysis of cycle set extensions in terms of their cohomology groups (Theorem 9.15). Some explicit examples of extensions are given (Theorem 10.6), implying the estimate $N_m \leqslant 2N_{m-1}$ for the minimal size N_m of square-free multipermutation cycle sets of level m (see Section 10 for the definitions). We disprove the relation $N_m = 2^{m-1}+1$ conjectured by Cameron and Gateva-Ivanova [22], using a computer-aided computation of the N_m for small m.

Finally, we express the second cohomology of LND braided sets—in particular, cycle sets—in terms of the first group cohomology of their structure groups (Theorem 11.2), generalizing Etingof and Graña's result for racks [16].

Graphical tools play a central role in most of our constructions and proofs, making them more intuitive and concise.

2. Skew cubical structures and homology

The chain complexes we work with in this paper carry a much richer structure than a differential. This section reviews such enriched structures and establishes a homology splitting result for one of them.

Definition 2.1. A *pre-cubical structure* in a category $\mathcal{C}$ consists of a family of objects C_k, $k \geqslant 0$, and of two families of morphisms $d_i^+, d_i^- : C_k \to C_{k-1}$ for $k \geqslant 1$, $1 \leqslant i \leqslant k$, satisfying the compatibility conditions

$$d_i^\varepsilon d_j^\zeta = d_{j-1}^\zeta d_i^\varepsilon \qquad \text{for all } i<j \text{ and } \varepsilon,\zeta \in \{+,-\}. \tag{2.1}$$

These d_i are referred to as *boundaries*. Such a structure is called *weak skew cubical* if it also includes *degeneracies* $s_i : C_k \to C_{k+1}$ for $k \geqslant 1$, $1 \leqslant i \leqslant k$, subject to relations

$$d_i^\varepsilon s_j = s_{j-1} d_i^\varepsilon \qquad \text{for all } i<j \text{ and } \varepsilon \in \{+,-\}, \tag{2.2}$$

$$d_i^\varepsilon s_j = s_j d_{i-1}^\varepsilon \qquad \text{for all } i>j+1 \text{ and } \varepsilon \in \{+,-\}, \tag{2.3}$$

$$d_i^\varepsilon s_i = d_{i+1}^\varepsilon s_i \qquad \text{for all } i \text{ and } \varepsilon \in \{+,-\}. \tag{2.4}$$

A *skew* (respectively, *semi-strong skew*) *cubical structure* satisfies moreover the property

$$s_i s_j = s_{j+1} s_i \qquad \text{for all } i \leqslant j \tag{2.5}$$

and the upgraded version

$$d_i^{\varepsilon} s_i = d_{i+1}^{\varepsilon} s_i = \mathrm{Id} \tag{2.6}$$

of condition (2.4) for all $\varepsilon \in \{+, -\}$ (respectively, for $\varepsilon = +$ only).

In this paper we stick to a purely algebraic treatment of pre-cubical structures. For a topological interpretation, see the classical references [35,25,3]. We sketch it in Remark 2.5 only. That remark also compares our structures with the much more classical cubical and simplicial ones.

The importance of the different types of structures we introduced is illustrated by the following result. For simplicity, it is stated for the category $\mathbf{Mod}_R$ of modules over a unital commutative ring R.

Theorem 2.2. *Let $(C_k,\, d_i^+,\, d_i^-)$ be a pre-cubical structure in $\mathbf{Mod}_R$.*

(1) *The R-modules C_k endowed with the alternating sum maps*

$$\partial_k^{(\alpha,\beta)} = \alpha \sum\nolimits_{i=1}^{k} (-1)^{i-1} d_i^+ + \beta \sum\nolimits_{i=1}^{k} (-1)^{i-1} d_i^- \tag{2.7}$$

form a chain complex for any $\alpha, \beta \in R$.

(2) *If the boundaries $(d_i^+,\, d_i^-)$ can be completed with degeneracies s_i, then the images $C_k^{\mathrm{D}} = \sum_{i=1}^{k-1} \mathrm{Im}\, s_i$ form a sub-complex of $(C_k, \partial_k^{(\alpha,\beta)})$ for any choice of $\alpha, \beta \in R$.*

(3) *If the structure $(C_k,\, d_i^+,\, d_i^-,\, s_i)$ is moreover semi-strong skew cubical, then one has R-module decompositions*

$$C_k = C_k^{\mathrm{D}} \oplus C_k^{\mathrm{N}}, \qquad\qquad C_k^{\mathrm{N}} = \mathrm{Im}\, \eta_k, \tag{2.8}$$
$$\text{where } \; \eta_k = (\mathrm{Id} - s_1 d_2^+)(\mathrm{Id} - s_2 d_3^+) \cdots (\mathrm{Id} - s_{k-1} d_k^+).$$

It yields a chain complex splitting for $(C_k, \partial_k^{(\alpha,\beta)})$ for any $\alpha, \beta \in R$.

According to the theorem, decomposition (2.8) induces a decomposition in homology. We use the standard terminology for such splittings:

Definition 2.3. The D- and N-parts of the complexes and homology groups above are called *degenerate* and, respectively, *normalized*.

Proof. The first two points are classical and can be verified by a straightforward computation. We give a detailed proof for the last point, which to our knowledge is new. Fix an arbitrary choice of $\alpha, \beta \in R$, and put $\partial_k = \partial_k^{(\alpha,\beta)}$.

We first show that the η_k form an endomorphism of the complex (C_k, ∂_k). For this, it suffices to verify the relations

$$\sum\nolimits_{i=1}^{k}(-1)^{i-1}d_i^{\varepsilon}\eta_k = \eta_{k-1}\sum\nolimits_{i=1}^{k}(-1)^{i-1}d_i^{\varepsilon}, \qquad \varepsilon \in \{+,-\}. \tag{2.9}$$

Put $p_i = \mathrm{Id} - s_i d_{i+1}^{+}$, and rewrite η_k as $p_1 \cdots p_{k-1}$. The weak skew cubical axioms imply the following commutation rules for the p_i and the boundaries:

$$d_i^{\varepsilon}p_j = p_{j-1}d_i^{\varepsilon}, \quad i < j; \qquad\qquad d_i^{\varepsilon}p_j = p_j d_i^{\varepsilon}, \quad i > j+1; \tag{2.10}$$

$$(d_i^{\varepsilon} - d_{i+1}^{\varepsilon})p_i = d_i^{\varepsilon} - d_{i+1}^{\varepsilon}. \tag{2.11}$$

Further, semi-strong skew cubical axioms imply the simplification rule

$$p_i d_{i+1}^{\varepsilon} p_{i+1} = d_{i+1}^{\varepsilon} p_{i+1}. \tag{2.12}$$

Indeed, this property rewrites as

$$s_i d_{i+1}^{+} d_{i+1}^{\varepsilon}(\mathrm{Id} - s_{i+1}d_{i+2}^{+}) = 0,$$

which follows from the computation

$$s_i d_{i+1}^{+} d_{i+1}^{\varepsilon} s_{i+1} d_{i+2}^{+} \overset{(2.1)}{=} s_i d_{i+1}^{\varepsilon} d_{i+2}^{+} s_{i+1} d_{i+2}^{+} \overset{(2.6)}{=} s_i d_{i+1}^{\varepsilon} d_{i+2}^{+} \overset{(2.1)}{=} s_i d_{i+1}^{+} d_{i+1}^{\varepsilon}.$$

Now, relations (2.10) allow one to rewrite the right-hand side of (2.9) as

$$\begin{aligned} &p_1 \cdots p_{k-2}\sum\nolimits_{i=1}^{k}(-1)^{i-1}d_i^{\varepsilon} \\ &\quad = \sum\nolimits_{i=1}^{k-1}(-1)^{i-1}p_1 \cdots p_{i-1}d_i^{\varepsilon}p_{i+1}\cdots p_{k-1} + (-1)^{k-1}p_1 \cdots p_{k-2}d_k^{\varepsilon}. \end{aligned}$$

To conclude, we obtain the identical expression for the left-hand side of (2.9) by repeatedly using the following computation (with $s < k-1$):

$$\begin{aligned} &\sum\nolimits_{i=s}^{k}(-1)^{i-1}d_i^{\varepsilon}p_s \cdots p_{k-1} \\ &\overset{(2.10),(2.11)}{=} \sum\nolimits_{i=s+2}^{k}(-1)^{i-1}p_s d_i^{\varepsilon}p_{s+1}\cdots p_{k-1} \\ &\qquad\qquad + ((-1)^{s-1}d_s^{\varepsilon} + (-1)^{s}d_{s+1}^{\varepsilon})p_{s+1}\cdots p_{k-1} \\ &\overset{(2.12)}{=} (-1)^{s-1}d_s^{\varepsilon}p_{s+1}\cdots p_{k-1} + p_s(\sum\nolimits_{i=s+1}^{k}(-1)^{i-1}d_i^{\varepsilon})p_{s+1}\cdots p_{k-1}. \end{aligned}$$

Both $C_k^{\mathrm{D}} = \sum \mathrm{Im}\, s_i$ and $C_k^{\mathrm{N}} = \mathrm{Im}\,\eta_k$ are thus sub-complexes of $(C_k, \partial_k^{(\alpha,\beta)})$. It remains to establish the R-module decomposition $C_k = C_k^{\mathrm{D}} \oplus C_k^{\mathrm{N}}$. The definition of the map η_k

directly gives the property $\operatorname{Im}(\operatorname{Id} - \eta_k) \subseteq \sum_i \operatorname{Im} s_i$. Further, the semi-strong skew cubical axioms imply the following commutation rules for the p_i and the degeneracies:

$$s_i p_j = p_{j+1} s_i, \quad i \leqslant j; \qquad\qquad s_i p_j = p_j s_i, \quad i > j+1;$$
$$p_i s_i = 0; \qquad p_i s_{i+1} = s_{i+1} - s_i; \qquad (s_i - s_{i+1}) p_i = s_i - s_{i+1}.$$

These relations imply that the map η_k vanishes on $\operatorname{Im} s_i$ for all $1 \leqslant i \leqslant k-1$. We conclude by applying to the data $C_k^{\mathrm{D}} \overset{\iota_k}{\hookrightarrow} C_k \overset{\eta_k}{\to} C_k$ (where ι_k is the inclusion map) the following lemma.

Lemma 2.4. *Let M, M' be two R-modules, and let $M' \overset{\alpha}{\to} M \overset{\beta}{\to} M$ be two R-linear maps satisfying the conditions $\beta\alpha = 0$ and $\operatorname{Im}(\operatorname{Id} - \beta) \subseteq \operatorname{Im} \alpha$. Then M decomposes as $\operatorname{Im} \alpha \oplus \operatorname{Im} \beta$.*

Proof. Condition $\operatorname{Im}(\operatorname{Id} - \beta) \subseteq \operatorname{Im} \alpha$ implies that $\operatorname{Im} \alpha + \operatorname{Im} \beta$ covers the whole R-module M. Let us show that this sum is direct. Relation $\beta\alpha = 0$ means $\operatorname{Im} \alpha \subseteq \operatorname{Ker} \beta$, implying $\operatorname{Im}(\operatorname{Id} - \beta) \subseteq \operatorname{Im} \alpha \subseteq \operatorname{Ker} \beta$, which translates as $\beta^2 = \beta$. But then the intersection $\operatorname{Ker} \beta \cap \operatorname{Im} \beta$ is zero, hence so is its sub-module $\operatorname{Im} \alpha \cap \operatorname{Im} \beta$. □ □

Remark 2.5. Eilenberg and MacLane [12] used the morphisms η_k to compare the full and the normalized versions of *simplicial homology* (recall that simplicial structures are similar to skew cubical ones, except that they include only one family of boundaries d_i^+). For *cubical homology*, they employed the morphisms $\eta'_k = (\operatorname{Id} - s_1 d_1^+) \cdots (\operatorname{Id} - s_k d_k^+)$ instead. Note that we use the classical notion of pre-cubical structure, but our skew cubical structures are different from the classical cubical ones. Concretely, a cubical structure bears $k+1$ degeneracies $s_1, \ldots, s_{k+1}$ on C_k while we stop at s_k, and it satisfies conditions $d_i^\varepsilon s_i = \operatorname{Id}$ and $d_{i+1}^\varepsilon s_i = s_i d_i^\varepsilon$ instead of (2.6). Topologically, cubical degeneracies correspond to compressing the unit cube in $\mathbb{R}^n$ in the direction of one of the axes via the maps

$$(x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k),$$

while our s_i can be thought of as squeezings onto the diagonal hyperplane $x_i = x_{i+1}$ via the map

$$\varsigma_i \colon (x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_{i-1}, x_i + x_{i+1} - x_i x_{i+1}, \ldots, x_k).$$

This explains the word *skew* in our terminology. More explicitly, together with the topological boundaries

$$\delta_i^+ \colon (x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_{i-1}, 0, x_i, \ldots, x_k),$$
$$\delta_i^- \colon (x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_{i-1}, 1, x_i, \ldots, x_k),$$

the ς_i satisfy the relations dual to our semi-strong skew cubical axioms. Observe that for a cubical structure, the C_k^{D} form a sub-complex of $(C_k, \partial_k^{(\alpha,\beta)})$ only for $\beta = -\alpha$; no splitting of type (2.8) is known in this case. The structures we will treat in Section 8 will be semi-strong skew cubical but neither skew cubical nor cubical.

3. Braided homology

Fix a braided set X, with a braiding $\sigma\colon X^{\times 2} \to X^{\times 2}$, $(a, b) \mapsto ({}^a b, a^b)$. In this section we recall and slightly extend the (co)homology theory for (X, σ), developed in [26]. It is referred to as *braided (co)homology* here.

We first comment on the *graphical calculus*, which renders our constructions more intuitive. Braided diagrams represent here maps between sets, a set being associated to each strand; horizontal glueing corresponds to Cartesian product, vertical glueing to composition (which should be read from bottom to top), straight vertical lines to identity maps, crossings to the braiding σ, and opposite crossings to its inverse (whenever it exists), as shown in Fig. 3.1Ⓐ. With these conventions, the Yang–Baxter equation (1.1) for σ becomes the diagram from Fig. 3.1Ⓑ, which is precisely the braid- and knot-theoretic RIII (= Reidemeister III) move. Associating *colors* (i.e., arbitrary elements of the corresponding sets) to the bottom free ends of a diagram and applying to them the map encoded by the diagram, one determines the colors of the top free ends; Fig. 3.1Ⓐ contains a simple case of this process, referred to as *color propagation*.

The role of coefficients in the braided homology will be played by the following structures:

Definition 3.1. A *right (braided) module* over a braided set (X, σ) is a pair (M, ρ), where M is a set and $\rho\colon M \times X \to M$, $(m, a) \mapsto m \cdot a$, is a map compatible with σ in the sense of

$$(m \cdot a) \cdot b = (m \cdot {}^a b) \cdot a^b$$

for all $m \in M$, $a, b \in X$. *Left modules* $(N, \lambda\colon X \times N \to N)$ over (X, σ) are defined similarly. See Fig. 3.2 for a diagrammatic version.

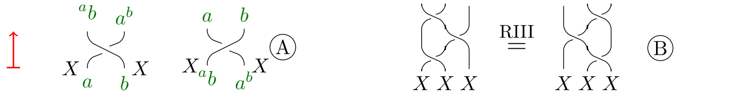

Fig. 3.1. Color propagation through a crossing and its opposite, and the RIII move representing the YBE.

Fig. 3.2. Right and left braided modules.

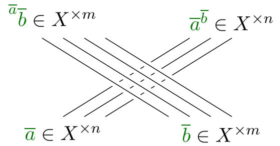

Fig. 3.3. The braiding of X extended to $T(X)$.

Example 3.2. The braided set (X, σ) is a right and a left module over itself, with the actions $\rho\colon (a,b) \mapsto a^b$ and $\lambda\colon (a,b) \mapsto {}^a b$. These modules are called *adjoint*. More generally, any of the powers $X^{\times n}$ is a right and a left module over $T(X) = \coprod_{i\geqslant 0} X^{\times i}$, with the module structure adjoint to the extension of the braiding σ to $T(X)$, see Fig. 3.3.

Example 3.3. The one-element set $I = \{*\}$ with the unique map $X \to I$ yields an example of a right and a left (X, σ)-module simultaneously. It is referred to as the *trivial* right/left (X, σ)-module.

Notation 3.4. Let (M, ρ) be a right module and (N, λ) be a left module over a braided set (X, σ). We write

$$\sigma_i = \mathrm{Id}_M \times \mathrm{Id}_X^{\times(i-1)} \times \sigma \times \mathrm{Id}_X^{\times(n-i-1)} \times \mathrm{Id}_N,$$
$$\rho_0 = \rho \times \mathrm{Id}_X^{\times(n-1)} \times \mathrm{Id}_N, \qquad \lambda_n = \mathrm{Id}_M \times \mathrm{Id}_X^{\times(n-1)} \times \lambda$$

(these are all maps from $M \times X^{\times n} \times N$ to $M \times X^{\times(n-1)} \times N$).

The following result extends a construction from [26]:

Theorem 3.5. *Let (M, ρ) be a right module and (N, λ) be a left module over a braided set (X, σ). Consider the sets $C_n = M \times X^{\times n} \times N$.*

(1) *The following maps form a pre-cubical structure on the C_n:*

$$d_i^{l,+} = \rho_0 \circ \sigma_1 \circ \cdots \circ \sigma_{i-1}, \qquad d_i^{r,-} = \lambda_n \circ \sigma_{n-1} \circ \cdots \circ \sigma_i.$$

(2) *Now suppose that the braiding σ is invertible, and consider the maps*

$$d_i^{l,-} = \rho_0 \circ \sigma_1^{-1} \circ \cdots \circ \sigma_{i-1}^{-1}, \qquad d_i^{r,+} = \lambda_n \circ \sigma_{n-1}^{-1} \circ \cdots \circ \sigma_i^{-1}$$

(Fig. 3.4). For any choice of $\varepsilon, \zeta \in \{l, r\}$, the families $(d_i^{\varepsilon,+}, d_i^{\zeta,-})$ with $n \geqslant 1$, $1 \leqslant i \leqslant n$, form a pre-cubical structure.

Proof. Conditions (2.1) are easily verified by diagram manipulations, using ambient isotopy, the third Reidemeister move, and the definition of braided modules (Figs. 3.1–3.2). $\square$

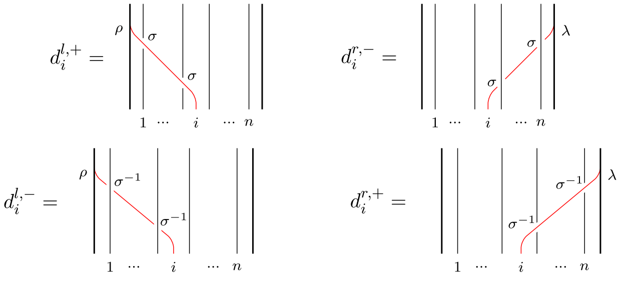

Fig. 3.4. Braided homology.

Theorem 2.2 now yields a collection of graphically defined differentials for a braided set, with coefficients in braided modules. Quite remarkably, they admit many alternative interpretations of completely different nature. For instance, choosing trivial modules (Example 3.3) as coefficients and the values $\alpha = 1$, $\beta = -1$ as parameters, one gets the complex of Carter–Elhamdadi–Saito [5]. It was described topologically in terms of certain n-dimensional cubes, inspired by the preferred squares approach to rack spaces, due to Fenn–Rourke–Sanderson [19,20]. The cochain version of that complex can also be regarded as the diagonal part of Eisermann's Yang–Baxter cohomology, which controls deformations of the braided set [14,15]. Recently the braided (co)homology received two complementary interpretations, boasting new applications: one based on Rosso's quantum shuffle machinery [26,28], and one in terms of a special differential graded bialgebra of Farinati and García-Galofre [18].

Remark 3.6. Theorem 3.5 is easily transportable from the category of sets to a general monoidal category. Moreover, the categorical duality yields a cohomological version of our constructions. Finally, degeneracies can be built out of a comultiplication on X compatible with the braiding σ. Details on these and other related points can be found in [26].

4. Birack homology

Recall that a *birack* is a braided set whose braiding $(a, b) \mapsto ({}^a b, a^b)$ is invertible and *non-degenerate*, i.e., the maps $a \mapsto a^b$ and $a \mapsto {}^b a$ are bijections $X \xrightarrow{\sim} X$ for all $b \in X$.

Notation 4.1. The inverses of the maps $a \mapsto a^b$ and $a \mapsto {}^b a$ are denoted by $a \mapsto a^{\tilde{b}}$ and $a \mapsto {}^{\tilde{b}} a$ respectively. We also use the notations

$$b \tilde{\cdot} a = a^{\tilde{b}}, \qquad a \cdot b = {}^{b \tilde{\cdot} a} b.$$

A homology theory for biracks was developed in [19,9] as follows:

Theorem 4.2. *Let* (X, σ) *be a birack.*

Fig. 4.1. Sideways maps.

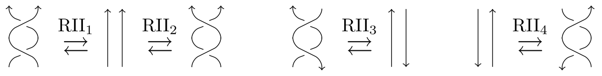

Fig. 4.2. Reidemeister II moves.

(1) *The maps* $X^{\times n} \to X^{\times (n-1)}$ *given by*

$$d_i \colon (x_1, \ldots, x_n) \mapsto (x_i \mathbin{\tilde{\cdot}} x_1, \ldots, x_i \mathbin{\tilde{\cdot}} x_{i-1}, x_i \cdot x_{i+1}, \ldots, x_i \cdot x_n),$$
$$d'_i \colon (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n),$$

form a pre-cubical structure.

(2) *The assertion remains true with the maps* d_i *replaced with*

$$d^{\star}_i \colon (x_1, \ldots, x_n) \mapsto (x_i \cdot x_1, \ldots, x_i \cdot x_{i-1}, x_i \mathbin{\tilde{\cdot}} x_{i+1}, \ldots, x_i \mathbin{\tilde{\cdot}} x_n).$$

If the relation $a \cdot a = a \mathbin{\tilde{\cdot}} a$ *holds for all* $a \in X$, *then the maps*

$$s_i \colon (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{i-1}, x_i, x_i, x_{i+1}, \ldots, x_n)$$

enrich any of the two pre-cubical structures above into a cubical one.

A proof by straightforward verifications is given in [9], whereas [19] treats only the chain complex $(X^{\times n}, \partial_n^{(1,-1)})$ from Theorem 2.2 using its topological realization. We propose here a diagrammatic interpretation of the boundary maps from the theorem, which will be instrumental in subsequent sections.

First, observe that the invertibility and the non-degeneracy of σ allow one to propagate colors through a crossing not only from bottom to top, but also from top to bottom (this corresponds to the map σ^{-1}), from right to left (this is the map $(a^b, b) \mapsto ({}^a b, a)$, or, in our notations, $(a, b) \mapsto (a \cdot b, b \mathbin{\tilde{\cdot}} a)$), and from left to right (this is the map $({}^a b, a) \mapsto (a^b, b)$). The right-to-left versions of σ and σ^{-1} are presented in Fig. 4.1. These maps are fundamental in birack theory, and are often called *sideways maps*. Note also that our treatment of crossings and their opposites validates the use of Reidemeister II moves (Fig. 4.2) in our diagrams. From now on strand orientations become relevant and are thus indicated in diagrams; all the strands in Figs. 3.1–3.4 should be considered as oriented upwards.

Now, consider the upper left diagram from Fig. 4.3. The colors $x_1, \ldots$ of its rightmost arcs (indicated in the diagram) can be propagated to the left, and uniquely determine the colors of all the remaining arcs. Probably the easiest way to see this is to start with

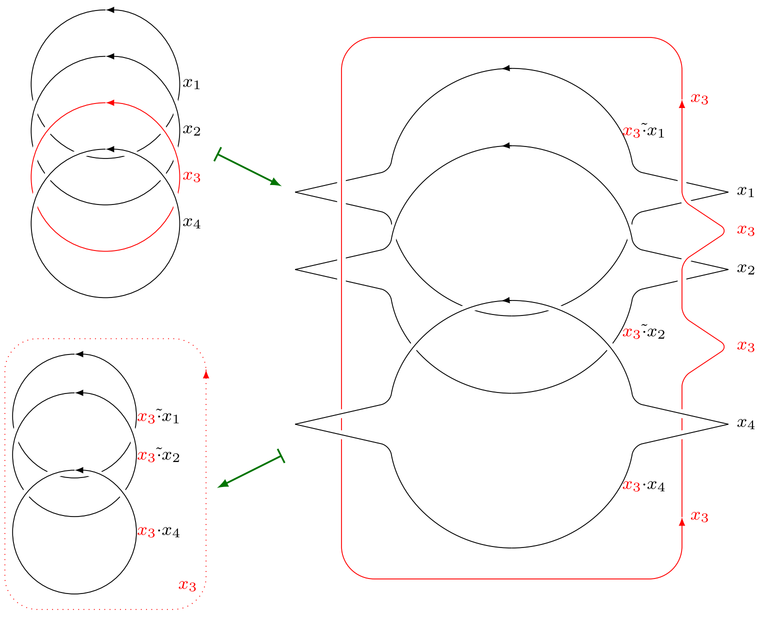

Fig. 4.3. A diagrammatic version of the boundary map d_i: the ith circle inflates and then disappears. Here $n = 4$, $i = 3$.

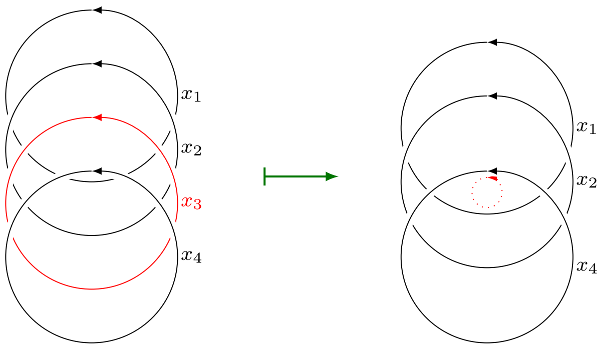

Fig. 4.4. A diagrammatic version of the map d'_i: the ith circle shrinks and then disappears.

n horizontally aligned disjoint circles of the same size, colored by $x_1, \ldots$, and then to continuously bring them closer in the vertical direction, until they are piled up as on this diagram; during the stacking procedure some local Reidemeister II moves occur, provoking local color changes but preserving the rightmost colors. Imagine then the ith circle inflating until it encloses the other ones, and then disappearing, as shown in the figure (where the circles are deformed for the sake of readability). RII and RIII moves with induced local color changes happen during the inflation. From the figure one sees that the new colors of the rightmost arcs of the remaining circles yield the value of $d_i(x_1, \ldots)$.

The value of $d'_i(x_1, \ldots)$ is obtained by a similar procedure, except that the ith circle shrinks into the area which is interior to all the circles (Fig. 4.4); the colors of the rightmost arcs are not affected by this procedure. Observe that changing the order of shrinking and/or inflation of different circles does not modify the colors in the resulting diagram. This implies relations (2.1), and hence Theorem 4.2. To switch from the d_i to the $d_i^\star$, one should replace all the crossings in our diagrams with their opposites.

5. Associated shelves

This section contains a reminder from the zoology of braided sets. We recall/introduce several algebraic structures (a shelf, a semigroup, and a group) associated to every braiding, and capturing its properties. They will be instrumental in further sections.

Definition 5.1. A braiding $\sigma\colon X^{\times 2} \to X^{\times 2}$, $(a,b) \mapsto ({}^a b, a^b)$ and the corresponding braided set are called

- *left (or right) non-degenerate* if the map $a \mapsto a^b$ (respectively, $a \mapsto {}^b a$) is a bijection $X \xrightarrow{\sim} X$;
- *non-degenerate* if both left and right non-degenerate;
- *involutive* if $\sigma\sigma = \mathrm{Id}$;
- *weakly RI-compatible* if there exists a map $t\colon X \to X$ such that $\sigma(t(a),a) = (t(a),a)$ for all $a \in X$;
- *RI-compatible* if weakly RI-compatible with a bijective map t.

Observe that the left (or right) non-degeneracy is equivalent to the sideways map (or its right version) being well defined. Also note that the RI-compatibility is related to the color propagation through the kink of a Reidemeister I move (Fig. 5.1), hence the name.

Example 5.2. Recall that a *shelf* is a set X with a self-distributive operation $\triangleleft$, in the sense of (1.2). It gives rise to a braiding $\sigma_\triangleleft(a,b) = (b, a \triangleleft b)$, which is

- invertible if and only if the right translations $a \mapsto a \triangleleft b$ are bijective, i.e., $(X, \triangleleft)$ is a *rack*;
- left non-degenerate if and only if $(X, \triangleleft)$ is a rack;
- always right non-degenerate;
- involutive if and only if the shelf is *trivial*, i.e., $a \triangleleft b = a$ for all a, b;
- (weakly) RI-compatible if and only if one has $a \triangleleft a = a$ for all a, implying $t = \mathrm{Id}_X$; in this case $(X, \triangleleft)$ is called a *spindle*.

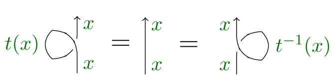

Fig. 5.1. Reidemeister I move and the map t.

The notion of braided module over $(X, \sigma_\triangleleft)$ recovers the classical notion of module over the shelf $(X, \triangleleft)$. Another braiding on X is defined by $\sigma'_\triangleleft(a, b) = (b \triangleleft a, a)$. It should be thought of as the braiding $\sigma_\triangleleft$ with the entries read from right to left. It has the same properties as $\sigma_\triangleleft$, except that the right and left non-degeneracies change places.

Example 5.3. A *birack* can be seen as an invertible, left and right non-degenerate braided set. A birack—or, more generally, any left non-degenerate braided set—is weakly RI-compatible if and only if one has $a \cdot a = a \,\tilde{\cdot}\, a$ for all $a \in X$ (Notation 4.1), and RI-compatible if and only if the map $t\colon a \mapsto a \cdot a = a \,\tilde{\cdot}\, a$ is bijective (note that its injectivity follows from the left non-degeneracy, so for finite biracks weak and usual RI-compatibility properties are equivalent).

Example 5.4. Recall that a *cycle set* is a set X with an operation $\cdot$ satisfying the cycle property (1.3), such that all the translations $a \mapsto b \cdot a$ admit inverses $a \mapsto b * a$. Its associated braiding $\sigma_\cdot(a, b) = ((b * a) \cdot b, b * a)$ is

- always involutive, left non-degenerate, and weakly RI-compatible, with $t(a) = a \cdot a$;
- right non-degenerate if and only if $(X, \cdot)$ is *non-degenerate*, i.e., the squaring map $a \mapsto a \cdot a$ is bijective;
- RI-compatible if and only if $(X, \cdot)$ is non-degenerate.

Example 5.5. As noticed in [26], any *monoid* X, with an associative operation $(a, b) \mapsto a \star b$ and a unit element $e \in X$, carries the following braiding:

$$\sigma_\star(a, b) = (e, a \star b).$$

Even better: the YBE for $\sigma_\star$ is equivalent to the associativity of $\star$, if one admits the unit property of e. This braiding is

- almost never invertible, nor right non-degenerate, nor involutive, nor RI-compatible;
- left non-degenerate if and only if right translations are bijective, which holds for instance when X is a group;
- weakly RI-compatible, with $t(a) = e$;
- idempotent, in the sense of $\sigma_\star \sigma_\star = \sigma_\star$.

A module over our monoid is automatically a braided module over $(X, \sigma_\star)$, with the same action.

We now show how to associate a shelf to any left non-degenerate braided set. Our result generalizes that of Soloviev [36], see also [1, Prop. 5.4]. Recall the notations $b \,\tilde{\cdot}\, a$, $a \cdot b$ (Notation 4.1) and the sideways map (Fig. 4.1) for biracks, which still make sense in our more general context. Moreover, observe that the left non-degeneracy is sufficient for performing

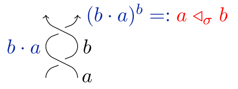

Fig. 5.2. The colors a, b are propagated to the left, and then upwards; $a \triangleleft_\sigma b$ is defined as the induced upper right color.

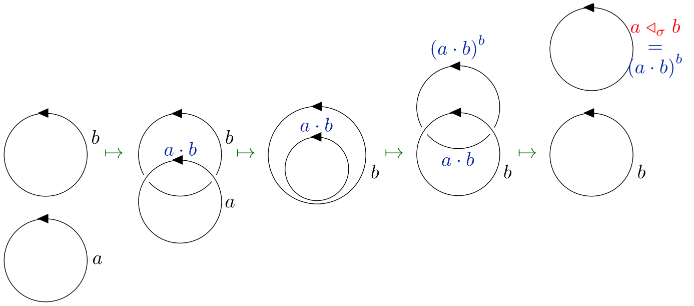

Fig. 5.3. The a-colored circle passes through the b-colored one; its color changes to $a \triangleleft_\sigma b$.

- all the oriented versions of the RII move (Fig. 4.2) in the direction $\leftarrow$ of the equivalences $\rightleftarrows$, and
- the disentangling $\rightarrow$-directed moves RII_1 and RII_3 provided that for each strand, the colors of its lower and upper ends coincide.

Definition 5.6. These directed RII moves are called *allowed.*

Proposition 5.7.

(1) *A left non-degenerate braided set (X, σ) carries the following self-distributive operation (Fig. 5.2):*

$$a \triangleleft_\sigma b = (b \cdot a)^b.$$

(2) *It defines a rack structure if and only if σ is invertible.*
(3) *The resulting shelf is trivial if and only if σ is involutive.*
(4) *The following assertions are equivalent:*
  (a) *The resulting shelf is a spindle.*
  (b) *The braiding σ is weakly* RI*-compatible.*
  (c) *The braiding σ is weakly* RI*-compatible with $t(a) = a \cdot a = a \mathbin{\tilde{\cdot}} a$.*

Definition 5.8. The structure from the proposition is called the *associated shelf/rack structure* for (X, σ).

The associated shelf operation can also be interpreted in terms of colored circles (in the spirit of Figs. 4.3–4.4), as shown in Fig. 5.3. Note that this passing-through procedure involves only allowed RII moves.

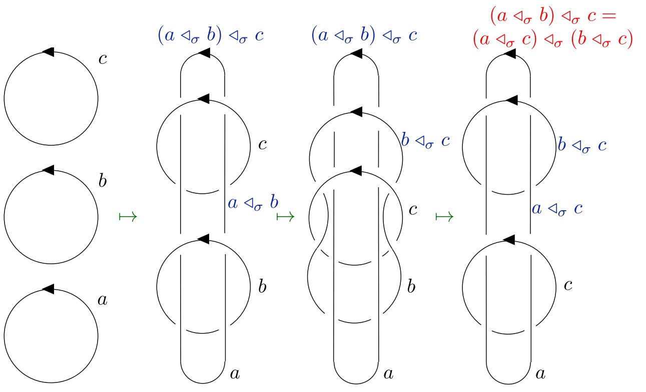

Fig. 5.4. A proof of $(a \lhd_\sigma b) \lhd_\sigma c = (a \lhd_\sigma c) \lhd_\sigma (b \lhd_\sigma c)$. The b-colored circle passes through the c-colored one, while both are encircling the elongated a-colored circle.

Proof of Proposition 5.7.

(1) Fig. 5.4 contains a diagrammatic proof of the self-distributivity of $\lhd_\sigma$; only allowed RII moves and nicely oriented RIII moves are used. Certainly, algebraic manipulations also do the trick.

(2) Suppose that σ is invertible. This implies that, for a given $b \in X$, the map $a \mapsto b \cdot a$ is bijective. The map $a \mapsto a^b$ being bijective by left non-degeneracy, so is $a \mapsto a \lhd_\sigma b = (b \cdot a)^b$. In the other direction, the bijectivity of $a \mapsto a \lhd_\sigma b$ and of $a \mapsto a^b$ yields the bijectivity of $a \mapsto b \cdot a$. Then σ^{-1} is defined by $\sigma^{-1}(c, b) = (a \,\tilde{\cdot}\, b, a)$ where a is the unique element of X satisfying $b \cdot a = c$.

(3) Suppose that $a \lhd_\sigma b = a$ for all a, b. Then the color of the upper left arc in Fig. 5.2 has to be $a \cdot b$. Further, the color of the middle left arc is $b \cdot a$ when determined from the lower crossing, and $b \,\tilde{\cdot}\, a$ when determined from the upper one. Hence the operations $\cdot$ and $\tilde{\cdot}$ coincide, and from the figure one reads $\sigma^2(a \,\tilde{\cdot}\, b, a) = (a \,\tilde{\cdot}\, b, a)$, for all a, b. Since the map $b \mapsto a \,\tilde{\cdot}\, b = b^{\tilde{a}}$ is bijective, one obtains $\sigma^2 = \mathrm{Id}$. The opposite direction is obvious.

(4) Implications 4c $\Rightarrow$ 4b $\Rightarrow$ 4a are clear. Let us prove 4a $\Rightarrow$ 4c. Relation $a \lhd_\sigma a = a$ means $(a \cdot a)^a = a$, or equivalently, using the left non-degeneracy, $a \cdot a = a^{\tilde{a}} = a \,\tilde{\cdot}\, a$, which implies $\sigma(a \cdot a, a) = (a \cdot a, a)$ as desired. $\square$

Example 5.9. The self-distributive operation associated to the braiding $\sigma_\lhd$ or $\sigma'_\lhd$ for a shelf $(X, \lhd)$ is simply its original operation $\lhd$.

Example 5.10. For the braiding coming from a right-invertible monoid (Example 5.5), the associated operation is $a \lhd_\sigma b = b$ for all a, b.

We finish this section by recalling how to associate a (semi)group to a braiding. This construction appeared in [17] and [23], and since then became a key tool in the study of the YBE.

Definition 5.11. The *structure (semi)group of a braided set* (X, σ) is the (semi)group $G_{(X,\sigma)}$ (respectively, $SG_{(X,\sigma)}$), defined by its generators $a \in X$ and relations $ab = {}^a b a^b$, for all $a, b \in X$. The *structure (semi)group of a shelf* $(X, \triangleleft)$, denoted by $(S)G_{(X,\triangleleft)}$, is simply the structure (semi)group of the associated braided set $(X, \sigma'_\triangleleft)$. Similarly, for a cycle set $(X, \cdot)$, one puts $(S)G_{(X,\cdot)} = (S)G_{(X,\sigma_\cdot)}$.

The importance of these constructions comes, among others, from the following elementary property:

Lemma 5.12. *For a right (X, σ)-module $(M, \cdot)$, the assignment $(m, a) \mapsto m \cdot a$, $a \in X$, $m \in M$ extends to a unique $SG_{(X,\sigma)}$-module structure on M. If X acts on M by bijections (such modules are called* solid*), then this assignment also defines a unique $G_{(X,\sigma)}$-module structure. This yields a bijection between (solid) (X, σ)-module and $SG_{(X,\sigma)}$- (respectively, $G_{(X,\sigma)}$-) module structures on M. Analogous properties hold true for right modules.*

Example 5.13. In rack theory, the *associated group* of a rack is a widely used notion. In our language, it is the group $G_{(X,\sigma_\triangleleft)}$, isomorphic to $G_{(X,\triangleleft)}$ via the order-reversion map $(x_1, x_2, \ldots, x_n) \mapsto (x_n, \ldots, x_2, x_1)$.

Example 5.14. For the braided set associated to a monoid X (Example 5.5), the structure semigroup is isomorphic to the monoid itself, via the map sending $x_1 \cdots x_n \in SG_X$ to the product $x_1 \star \cdots \star x_n \in X$.

6. Guitar map

This section is devoted to the remarkable guitar map and its applications to the study of braided sets and their structure (semi)groups. Applications to homology will be treated in the next section.

Definition 6.1. The *n-guitar map* for a braided set (X, σ) is the map $J\colon X^{\times n} \to X^{\times n}$, defined by

$$J(\overline{x}) = (J_1(\overline{x}), \ldots, J_n(\overline{x})), \qquad \overline{x} = (x_1, \ldots, x_n),$$
$$J_i(\overline{x}) = x_i^{x_{i+1}\cdots x_n},$$

with the notation $x_i^{x_{i+1}\cdots x_n} = (\ldots(x_i^{x_{i+1}})\ldots)^{x_n}$. We will often abusively talk about the *guitar map* J meaning the family of n-guitar maps for all $n \geqslant 1$.

The name comes from the resemblance of a diagrammatic version of the map J (Fig. 6.1) with the position of guitar strings when a chord is played.

Recall the notation σ_i (Notation 3.4), which we will use here with trivial coefficients. Also recall the extension of σ to $T(X) = \coprod_{i \geqslant 0} X^{\times i}$, and the right braided action

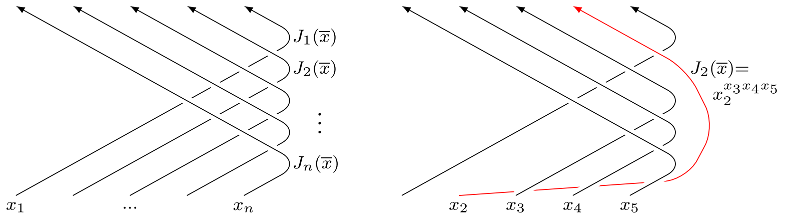

Fig. 6.1. The guitar map (left) and a formula for calculating its components (right); the two diagrams are related by a sequence of RIII moves.

$(\overline{a}, \overline{b}) \mapsto \overline{a}^{\overline{b}}$ of $T(X)$ on itself (Example 3.2). One easily checks that both the extended braiding and its adjoint action descend from $T(X)$ to the semigroup $SG_{(X,\sigma)}$. For $\overline{a}, \overline{b} \in T(X)$, their concatenation product will be denoted by $\overline{a}\overline{b}$.

Proposition 6.2. *Let (X, σ) be a left non-degenerate braided set, $\lhd_\sigma$ its associated shelf operation, and $\sigma' = \sigma'_{\lhd_\sigma} : (a, b) \mapsto (b \lhd_\sigma a, a)$ the braiding extracted from $\lhd_\sigma$.*

(1) *The n-guitar map J for (X, σ) is bijective for all $n \geqslant 1$.*
(2) *J satisfies the cocycle property $J(\overline{a}\overline{b}) = (J(\overline{a}) \leftthreetimes \overline{b})J(\overline{b})$, $\overline{a}, \overline{b} \in T(X)$, where the action $\leftthreetimes$ of $T(X)$ on itself is defined by*

$$(a_1, \dots, a_n) \leftthreetimes \overline{b} = (a_1^{\overline{b}}, \dots, a_n^{\overline{b}}). \tag{6.1}$$

(3) *J entwines the adjoint action and the action $\leftthreetimes$ above, in the sense of $J(\overline{a}^{\overline{b}}) = J(\overline{a}) \leftthreetimes \overline{b}$.*
(4) *J entwines σ and σ', in the sense of $J\sigma_i = \sigma'_i J$.*
(5) *The operation $\leftthreetimes$ induces an action of $SG_{(X,\sigma)}$ on $SG_{(X,\lhd_\sigma)}$.*
(6) *J induces a bijective cocycle $J^{SG} : SG_{(X,\sigma)} \xrightarrow{\sim} SG_{(X,\lhd_\sigma)}$, with the $SG_{(X,\sigma)}$-action on $SG_{(X,\lhd_\sigma)}$ from the previous point.*

In practice, the braiding σ' and the semigroup $SG_{(X,\lhd_\sigma)}$ turn out to be much simpler than σ and $SG_{(X,\sigma)}$. For instance, for an involutive σ, one obtains the *flip* $\sigma' : (a, b) \mapsto (b, a)$ (cf. Proposition 5.7), and $SG_{(X,\lhd_\sigma)}$ becomes the free abelian semigroup on X. This reduction to shelves simplifies the study of certain aspects of braided sets. For example, the proposition implies that the action on $X^{\times n}$ of the positive braid monoid B_n^+ (or, for invertible σ, of the whole braid group B_n) induced by σ is conjugated to the action induced by $\lhd_\sigma$. Thus, as far as $B_n^{(+)}$-actions are concerned, left non-degenerate braided sets yield nothing new compared to shelves.

Historically, the map J seems to be first considered by Etingof, Schedler, and Soloviev [17] for involutive braidings (and thus with σ' being the flip). In their setting, they showed Point 4 of the proposition, without assuming the left non-degeneracy of σ. Their construction was extended to non-involutive braidings by Soloviev [36] and by Lu, Yan, and Zhu [30]; the latter used a mirror version of the guitar map and de-

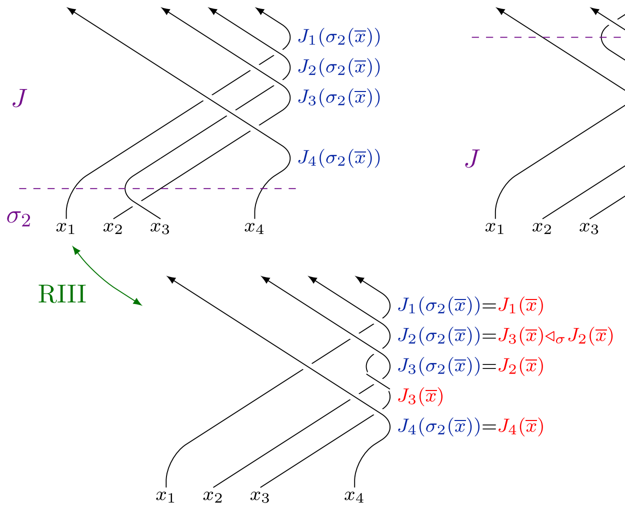

Fig. 6.2. The entwining relation $J\sigma_i = \sigma'_i J$ (here $n = 4$, $i = 2$) is established by comparing the colors of the rightmost arcs in the bottom diagram as calculated from the upper left (blue labels) and the upper right diagrams (red labels). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

noted it by T_n. For cycle sets, Dehornoy [10] developed a right-cyclic calculus, used to obtain short proofs for the existence of the Garside structure, of the I-structure, and of Coxeter-like groups for structure groups $G_{(X,\cdot)}$. Certain maps Ω_i, crucial in his calculus, can in fact be expressed as $\Omega(x_1, \dots, x_n) = J^{-1}(x_n, \dots, x_1)$, and their properties follow from those of the guitar map. In yet another particular case, that of braidings associated to shelves, the guitar map was reintroduced by Przytycki [33] under the name “the remarkable map f”, and used in a study of multi-term distributive homology.

Proof of Proposition 6.2. In Point 1, the bijectivity of J is equivalent to the left non-degeneracy of σ. Points 2 and 3 can be checked by playing with the diagrammatic definition of J. Point 4 is proved in Fig. 6.2. In 5, the only non-trivial property to check is $(\sigma'_i(\overline{c})) \leftthreetimes \overline{b} = \sigma'_i(\overline{c} \leftthreetimes \overline{b})$. Since J is bijective, we will verify it for the tuples $\overline{c}$ of the form $J(\overline{a})$ only:

$$(\sigma'_i J(\overline{a})) \leftthreetimes \overline{b} = (J\sigma_i(\overline{a})) \leftthreetimes \overline{b} = J((\sigma_i(\overline{a}))^{\overline{b}}) = J\sigma_i(\overline{a}^{\overline{b}}) = \sigma'_i J(\overline{a}^{\overline{b}})$$
$$= \sigma'_i(J(\overline{a}) \leftthreetimes \overline{b}).$$

The last point is a consequence of the preceding ones. $\square$

Our next aim is to upgrade the guitar map so that it induces a bijective cocycle $G_{(X,\sigma)} \overset{\sim}{\to} G_{(X,\triangleleft_\sigma)}$. This will be done for the case of a non-degenerate, invertible, and RI-compatible braided set (X, σ). In particular, one is now allowed to use all Reidemeister moves with all possible orientations.

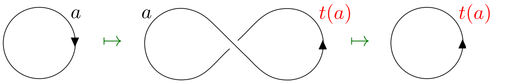

Fig. 6.3. The toss map.

Glue together two copies of X into $\overline{X} = \{ a^{+1}, a^{-1} \mid a \in X \}$, called the *double* of X. The braiding σ extends to $\overline{X}$ via

- $\overline{\sigma}(a^{+1}, b^{+1}) = (c^{+1}, d^{+1})$, where $\sigma(a, b) = (c, d)$;
- $\overline{\sigma}(a^{-1}, b^{-1}) = (c^{-1}, d^{-1})$, where $\sigma(d, c) = (b, a)$;
- $\overline{\sigma}(a^{+1}, b^{-1}) = (c^{-1}, d^{+1})$, where $\sigma(d, b) = (c, a)$;
- $\overline{\sigma}(a^{-1}, b^{+1}) = (c^{+1}, d^{-1})$, where $\sigma(a, c) = (b, d)$.

This definition is best seen graphically: the diagram for $\overline{\sigma}$ on $(a_1^{\varepsilon_1}, a_2^{\varepsilon_2})$ is the one for σ, with the orientation of the ith strand reversed whenever $\varepsilon_i = -1$. The new braiding $\overline{\sigma}$ inherits the non-degeneracy, invertibility, and RI-compatibility properties of σ (with $t(a^{+1}) = t(a)^{+1}$, and $t(a^{-1}) = (t^{-1}(a))^{-1}$).

We also need the *toss map* $K\colon \overline{X}^{\times n} \to \overline{X}^{\times n}$ defined by $K(a^{+1}) = a^{+1}$, $K(a^{-1}) = t(a)^{-1}$, and $K(x_1^{\varepsilon_1}, \ldots, x_n^{\varepsilon_n}) = (K(x_1^{\varepsilon_1}), \ldots, K(x_n^{\varepsilon_n}))$. The change from a to $t(a)$ for "negatively oriented" elements can be regarded as the color change happening to a negatively oriented circle when it flips and gets positive orientation (Fig. 6.3), hence the name.

Theorem 6.3. *Let (X, σ) be a non-degenerate invertible RI-compatible braided set, $\lhd_\sigma$ its associated shelf operation, $(\overline{X}, \overline{\sigma})$ its double, J the guitar map of $(\overline{X}, \overline{\sigma})$, and K the toss map. The map $\overline{J} = KJ\colon \overline{X}^{\times n} \to \overline{X}^{\times n}$ induces a bijective cocycle $G_{(X,\sigma)} \overset{\sim}{\to} G_{(X,\lhd_\sigma)}$, where the group $G_{(X,\sigma)}$ act on $G_{(X,\lhd_\sigma)}$ via*

$$(a_1^{\varepsilon_1} \cdots a_n^{\varepsilon_n}) \,\overline{\leftthreetimes}\, \overline{b} = (a_1^{\overline{b}})^{\varepsilon_1} \cdots (a_n^{\overline{b}})^{\varepsilon_n}, \tag{6.2}$$

for $a_i \in X$, $\varepsilon_i \in \{1, -1\}$, $\overline{b} \in G_{(X,\sigma)}$.

This result first appeared in [17] for involutive braidings, in [30] for braided groups, and in [36] in the general case. Its known proofs are rather technical and do not use the guitar map.

Note that the operation $\overline{\leftthreetimes}$ above is in general different from the operation $\leftthreetimes$ from (6.1), considered here for the braided set $(\overline{X}, \overline{\sigma})$.

Proof. Fig. 6.4 shows that the map $\overline{J}$ behaves well on the "inverse pairs", in the sense of

$$\overline{J}(a^{+1}, a^{-1}) = (t(a)^{+1}, t(a)^{-1}), \qquad \overline{J}(a^{-1}, a^{+1}) = (a^{-1}, a^{+1}).$$

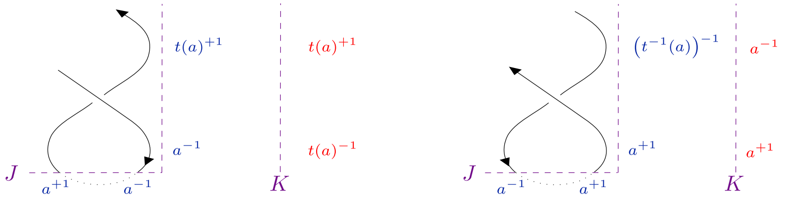

Fig. 6.4. Computation of $\overline{J}(a^{+1}, a^{-1})$ and $\overline{J}(a^{-1}, a^{+1})$ in two steps: first applying the graphical guitar map J, next modifying the result by the toss map K.

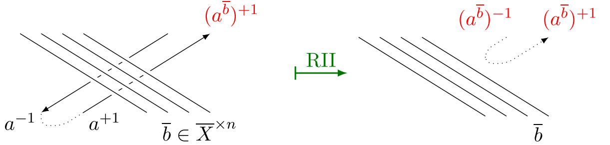

Fig. 6.5. Computation of the adjoint action $(a^{-1}, a^{+1})^{\overline{b}}$ via the capping trick.

The adjoint right action also preserves the "inverse pairs": one has

$$(a^{+1}, a^{-1})^{\overline{b}} = (c^{+1}, c^{-1}), \qquad (a^{-1}, a^{+1})^{\overline{b}} = ((a^{\overline{b}})^{-1}, (a^{\overline{b}})^{+1}),$$

where $\overline{b}$ lives in $\overline{X}^{\times n}$, and $c \in X$ is defined via $c^{-1} = (a^{-1})^{\overline{b}}$ in $\overline{X}$. The second identity is proved in Fig. 6.5; the first one is similar.

The same graphical capping trick yields the properties

$$\overline{a}^{(b^{+1}, b^{-1})} = \overline{a} = \overline{a}^{(b^{-1}, b^{+1})}. \tag{6.3}$$

Now, using Proposition 6.2, one has

$$\begin{aligned}\overline{J}(\overline{a}\overline{b}) &= KJ(\overline{a}\overline{b}) = K((J(\overline{a}) \leftthreetimes \overline{b})J(\overline{b})) = K(J(\overline{a}^{\overline{b}})J(\overline{b})) = KJ(\overline{a}^{\overline{b}})KJ(\overline{b}) \\ &= \overline{J}(\overline{a}^{\overline{b}})\overline{J}(\overline{b}),\end{aligned}$$

and thus

$$\begin{aligned}\overline{J}(\overline{a}b^{+1}b^{-1}\overline{d}) &= \overline{J}((\overline{a}b^{+1}b^{-1})^{\overline{d}})\overline{J}(\overline{d}) = \overline{J}(\overline{a}^{b^{+1}b^{-1}\overline{d}})\overline{J}((b^{+1}b^{-1})^{\overline{d}})\overline{J}(\overline{d}) \\ &= \overline{J}(\overline{a}^{\overline{d}})\overline{J}(c^{+1}c^{-1})\overline{J}(\overline{d}) = \overline{J}(\overline{a}^{\overline{d}})t(c)^{+1}t(c)^{-1}\overline{J}(\overline{d}),\end{aligned}$$

where $c \in X$ is defined via $c^{-1} = (b^{-1})^{\overline{d}}$. Together with the invertibility of the maps J, K, and t and of the adjoint action on $T(\overline{X})$, this implies that the map $\overline{J}\colon T(\overline{X}) \to T(\overline{X})$ survives when on both sides one mods out the relations $\overline{a}b^{+1}b^{-1}\overline{d} = \overline{a}\overline{d}$, $b \in X$, $\overline{a}, \overline{d} \in \overline{X}$. Relations $\overline{a}b^{-1}b^{+1}\overline{d} = \overline{a}\overline{d}$ can be treated analogously. Moreover, J entwines σ and $\sigma' = \sigma'_{\triangleleft_\sigma}$ (Proposition 6.2), and K does not alter the elements $a^{+1} \in \overline{X}$, so $\overline{J} = KJ$ still survives when one mods out the relations $\overline{a}b^{+1}c^{+1}\overline{d} = \overline{a}\sigma(b^{+1}, c^{+1})\overline{d}$ on the left and $\overline{a}b^{+1}c^{+1}\overline{d} = \overline{a}\sigma'(b^{+1}, c^{+1})\overline{d}$ on the right. Hence $\overline{J}$ induces a bijection $G_{(X,\sigma)} \xrightarrow{\sim} G_{(X,\triangleleft_\sigma)}$.

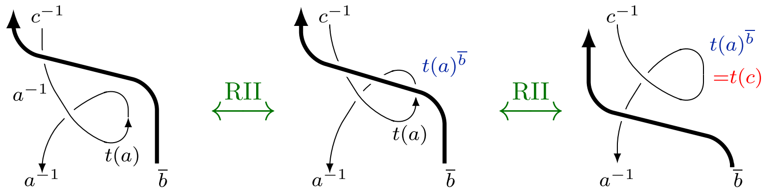

Fig. 6.6. Comparing $t(a)^{\overline{b}}$ and $t(c)$ under the assumption $c^{-1} = (a^{-1})^{\overline{b}}$. Thick strands stand here for bundles of parallel strands.

Next, formula (6.2) defines an action of the semigroup $T(\overline{X})$ on itself. It behaves well with respect to the inverse pairs: the property

$$(a_1^{\varepsilon_1}, \ldots, a_i^{\varepsilon_i}, a_i^{-\varepsilon_i}, \ldots, a_n^{\varepsilon_n}) \mathbin{\overline{\leftthreetimes}} \overline{b} = ((a_1^{\overline{b}})^{\varepsilon_1}, \ldots, (a_i^{\overline{b}})^{\varepsilon_i}, (a_i^{\overline{b}})^{-\varepsilon_i}, \ldots (a_n^{\overline{b}})^{\varepsilon_n})$$

is clear from the definition, and the property

$$\overline{a} \mathbin{\overline{\leftthreetimes}} (\overline{b} c^{\varepsilon} c^{-\varepsilon} \overline{d}) = \overline{a} \mathbin{\overline{\leftthreetimes}} (\overline{b}\overline{d})$$

follows from (6.3). Nice behavior with respect to the braidings σ and σ' is proved in the same way as for the action $\leftthreetimes$ in Proposition 6.2. Altogether, this shows that $\overline{\leftthreetimes}$ induces an action of $G_{(X,\sigma)}$ on $G_{(X,\triangleleft_\sigma)}$. It remains to check for $\overline{J}$ the cocycle property with respect to this action. It will follow from the relation $\overline{J}(\overline{a}\overline{b}) = \overline{J}(\overline{a}^{\overline{b}})\overline{J}(\overline{b})$ from the previous paragraph if we manage to prove the identity

$$\overline{J}(\overline{a}^{\overline{b}}) = \overline{J}(\overline{a}) \mathbin{\overline{\leftthreetimes}} \overline{b}. \tag{6.4}$$

The relation $J(\overline{a}^{\overline{b}}) = J(\overline{a}) \leftthreetimes \overline{b}$ from Proposition 6.2 reduces (6.4) to

$$K(J(\overline{a}) \leftthreetimes \overline{b}) = (KJ(\overline{a})) \mathbin{\overline{\leftthreetimes}} \overline{b}.$$

The toss map K and the operations $\leftthreetimes$ and $\overline{\leftthreetimes}$ acting component-wise, it suffices to consider the relation $K(a^{+1} \leftthreetimes \overline{b}) = K(a^{+1}) \mathbin{\overline{\leftthreetimes}} \overline{b}$, $a \in X$, in which case both sides equal $(a^{\overline{b}})^{+1}$; and $K(a^{-1} \leftthreetimes \overline{b}) = K(a^{-1}) \mathbin{\overline{\leftthreetimes}} \overline{b}$, which translates as $t(a)^{\overline{b}} = t(c)$, where $c \in X$ is defined via $c^{-1} = (a^{-1})^{\overline{b}}$. This latter property is verified graphically in Fig. 6.6. $\square$

7. The two homology theories coincide

Recall that for biracks we have seen two homology constructions: the general braided homology (Section 3), and a specific theory (Section 4). We now establish the equivalence of these theories using the guitar map J. Moreover, we extend the birack homology (and our equivalence of complexes) to the more general left non-degenerate (= LND) braided sets, and add coefficients to the complexes involved.

As usual, for LND braidings we make use of the notations $b \mathbin{\tilde{\cdot}} a$ and $a \cdot b$ (Notation 4.1), and of the graphical calculus involving allowed RII moves (Definition 5.6) and nicely oriented RIII moves. See Fig. 4.1 for the coloring rules expressed in terms of the operations $\cdot$ and $\tilde{\cdot}$. We will also need the maps

$$\chi_i(y_1,\ldots,y_k) = J_1^{-1}((y_i \triangleleft_\sigma y_{i-1})\cdots \triangleleft_\sigma y_1, y_1,\ldots,y_{i-1},y_{i+1},\ldots,y_k) \tag{7.1}$$

from $X^{\times k}$ to X, where we write the inverse of the k-guitar map J as $J^{-1} = (J_1^{-1},\ldots,J_k^{-1})$.

Theorem 7.1. *Let (X, σ) be a left non-degenerate braided set. Let $(M, \cdot)$ be a right module and $(N, \cdot)$ be a left module over (X, σ).*

(1) *A pre-cubical structure on $C_k = M \times X^{\times k} \times N$ can be given by the maps*

$$d_i\colon (m, y_1,\ldots,y_k,n) \mapsto (m, y_i \tilde{\cdot}\, y_1,\ldots,y_i \tilde{\cdot}\, y_{i-1}, y_i\cdot y_{i+1},\ldots,y_i\cdot y_k, y_i\cdot n),$$
$$d'_i\colon (m, y_1,\ldots,y_k,n) \mapsto (m\cdot \chi_i(\overline{y}), y_1,\ldots,y_{i-1},y_{i+1},\ldots,y_k,n).$$

(2) *The extended guitar map $J := \mathrm{Id}_M \times J \times \mathrm{Id}_N$ yields an isomorphism between the pre-cubical structure $(d_i^{r,-}, d_i^{l,+})$ from Theorem 3.5 and the structure (d_i, d'_i) above.*

As a consequence, the chain complexes obtained from these pre-cubical structures via Theorem 2.2 are isomorphic.

Note that when the braided set is a birack and the coefficients M, N are trivial (Example 3.3), the pre-cubical structure from our theorem specializes to that from Theorem 4.2. For braided sets associated to cycle sets, the exotic map χ_i takes the simpler form $J_1^{-1}(y_i, y_1,\ldots,y_{i-1},y_{i+1},\ldots,y_k)$, and appears in Dehornoy's right-cyclic calculus [10].

As usual, the theorem remains true when the operations $\cdot$ and $\tilde{\cdot}$ exchange places; this yields a generalization of the structure $(d_i^\star, d'_i)$ from Theorem 4.2.

Proof. Since the guitar map is bijective, it suffices to prove that it entwines the structures in question, i.e. that one has

$$J\circ d_i^{r,-} = d_i\circ J, \qquad J\circ d_i^{l,+} = d'_i\circ J$$

for all $n \geqslant 1$, $1 \leqslant i \leqslant n$. This would imply in particular that (d_i, d'_i) is indeed a pre-cubical structure. A graphical proof is presented in Fig. 7.1.

In this proof we worked with a slightly modified definition of the maps χ_i, in the sense of the following lemma.

Lemma 7.2. *The families of maps χ_i and $\chi'_i\colon X^{\times k} \to X$ defined by*

$$\chi_i(\overline{y}) = J_1^{-1}((y_i \triangleleft_\sigma y_{i-1})\cdots \triangleleft_\sigma y_1, y_1,\ldots,y_{i-1},y_{i+1},\ldots,y_k),$$
$$\chi'_i(\overline{y}) = {}^{x_1\ldots x_{i-1}}x_i, \qquad \overline{x} = J^{-1}(\overline{y}) \tag{7.2}$$

(with notations from Fig. 3.3), coincide.

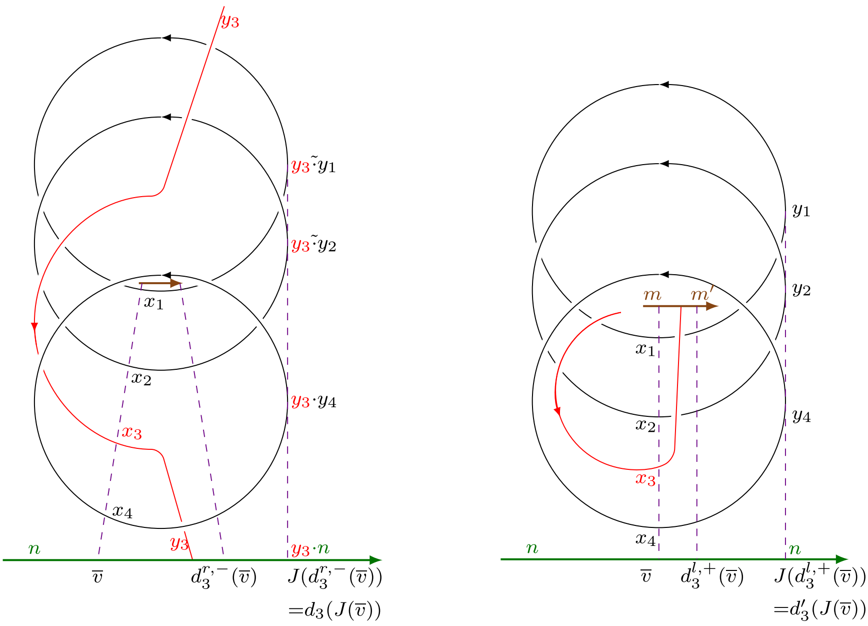

Fig. 7.1. Comparing $d_i^{r,-}$ with d_i, and $d_i^{l,+}$ with d'_i, $n = 4$, $i = 3$. Here $\overline{v} = (m, x_1, \ldots, x_4, n)$, and $J(\overline{v}) = (m, y_1, \ldots, y_4, n)$. On the left the brown M-colored strand has a constant color m, and on the right its color changes from m to $m' = m \cdot ({}^{x_1 x_2} x_3) = m \cdot \chi_3(\overline{y})$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Proof. The properties of the guitar map (Proposition 6.2) legitimize the following calculation:

$$\begin{aligned}\sigma_1 \cdots \sigma_{i-1}(\overline{x}) &= J^{-1} J \sigma_1 \cdots \sigma_{i-1}(\overline{x}) = J^{-1} \sigma'_1 \cdots \sigma'_{i-1} J(\overline{x}) = J^{-1} \sigma'_1 \cdots \sigma'_{i-1}(\overline{y}) \\ &= J^{-1}((y_i \triangleleft_\sigma y_{i-1}) \cdots \triangleleft_\sigma y_1, y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_k).\end{aligned}$$

The desired relation is obtained by comparing the first components of the resulting k-tuples. □ □

Remark 7.3. Note that the symbol $\cdot$ in the expressions $y_i \cdot y_k$ and $y_i \cdot n$ from the theorem has different meaning. In order to motivate this abuse of notation, we remark that the operations $y_i \cdot y_k$ and $y_i \tilde{\cdot} y_k$ both define a left action of (X, σ) on itself; this is an easy consequence of the Yang–Baxter relation.

8. Degeneracies and a homology splitting

For a weakly RI-compatible (Definition 5.1) left non-degenerate braided set, we now enrich the pre-cubical structure from Theorem 7.1 into a semi-strong skew cubical structure. Theorem 2.2 then yields a decomposition of the corresponding chain complexes into the degenerate and the normalized parts, generalizing the homology decomposition for quandles from [29].

Recall that for LND braided sets, weak RI-compatibility is equivalent to the condition $a \cdot a = a \,\tilde{\cdot}\, a$ for all $a \in X$; in this case the map t from the definition is necessarily the squaring map $t(a) = a \cdot a$ (Example 5.3). Recall also the maps χ_i defined by (7.1) or, equivalently, by (7.2).

Definition 8.1. A right module $(M, \cdot)$ over a braided set (X, σ) is called *solid* if every $a \in X$ acts on M bijectively. In this case, the inverse of the bijection $m \mapsto m \cdot a$ is denoted by $m \mapsto m \cdot a^{-1}$. Solid left modules are defined similarly.

Theorem 8.2. *Let (X, σ) be a weakly RI-compatible left non-degenerate braided set. Let $(M, \cdot)$ be a solid right module and $(N, \cdot)$ a left module over (X, σ). Then the pre-cubical structure $(M \times X^{\times k} \times N, d'_i, d_i)$ from Theorem 7.1 can be enriched into a semi-strong skew cubical one by the degeneracies*

$$s_i \colon (m, y_1, \ldots, y_k, n) \mapsto (m \cdot t(\chi_i(\overline{y}))^{-1}, y_1, \ldots, y_{i-1}, y_i, y_i, y_{i+1}, \ldots, y_k, n).$$

Further, given an abelian group A, the abelian groups $C_k(X, M, N; A) = AM \times X^{\times k} \times N$, $k \geqslant 0$, decompose as

$$C_k(X, M, N; A) = C^{\mathrm{D}}_k(X, M, N; A) \oplus C^{\mathrm{N}}_k(X, M, N; A), \tag{8.1}$$
$$C^{\mathrm{D}}_k(X, M, N; A) = \sum\nolimits_{i=1}^{k-1} A \operatorname{Im} s_i, \qquad C^{\mathrm{N}}_k(X, M, N; A) = \operatorname{Im} \eta_k,$$

where η_k is the A-linearization of the map

$$\eta_k = (\mathrm{Id} - s_1 d'_2)(\mathrm{Id} - s_2 d'_3) \cdots (\mathrm{Id} - s_{k-1} d'_k).$$

For any $\alpha, \beta \in \mathbb{Z}$, this decomposition is preserved by the differentials

$$\partial^{(\alpha,\beta)}_k = \alpha \sum\nolimits_i (-1)^{i-1} d'_i + \beta \sum\nolimits_i (-1)^{i-1} d_i.$$

As usual, decomposition (8.1) induces homology splittings.

Recall that the guitar map sends the pre-cubical structure $(d^{l,+}_i, d^{r,-}_i)$ isomorphically onto the structure (d'_i, d_i) (Theorem 7.1). Hence the maps $J^{-1} s_i J$ yield degeneracies for $(d^{l,+}_i, d^{r,-}_i)$, explicitly written as

$$s^{\mathrm{B}}_i \colon (m, \overline{x}_-, x, \overline{x}_+, n) \mapsto (m \cdot t({}^{(\overline{x}_-)}x)^{-1}, t(x) \,\tilde{\cdot}\, \overline{x}_-, t(x), x, \overline{x}_+, n),$$

where $\overline{x}_- \in X^{\times(i-1)}$, $\overline{x}_+ \in X^{\times(k-i)}$, and the operation $\tilde{\cdot}$ is extended from X to $T(X)$ using the extension to $T(X)$ of the braiding σ (Example 3.2). Here and afterwards we use the subscript B or the prefix B when referring to the braided homology. The modification of the M-component, which seemed surprising in the definition of s_i, becomes more intuitive on the level of s^{B}_i: indeed, it can be read off from Fig. 8.1. This figure is presented here for giving intuition only; to make thing rigorous, one should explain the

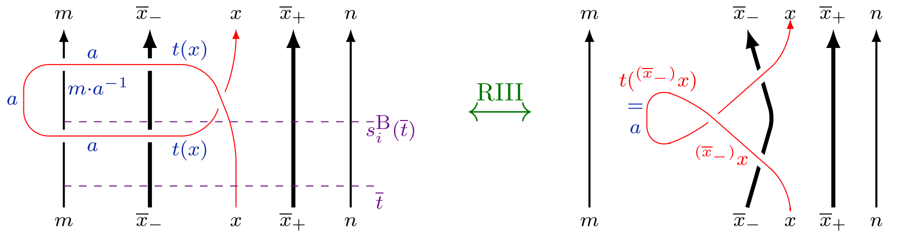

Fig. 8.1. A graphical definition of s_i^{B} (left) and a computation of the element $a \in X$ acting on $m \in M$ (right).

use of badly oriented RIII moves and the coloring rules around a crossing between an X-strand and an M-strand (which make possible appropriate RII and RIII moves).

Further, decomposition (8.1) implies the decomposition

$$C_k(X, M, N; A) = J^{-1}(C_k^{\mathrm{D}}(X, M, N; A)) \oplus J^{-1}(C_k^{\mathrm{N}}(X, M, N; A))$$

(the maps J and J^{-1} are extended by linearity), preserved by the differentials

$$\partial_k^{\mathrm{B},(\alpha,\beta)} = \alpha \sum\nolimits_i (-1)^{i-1} d_i^{l,+} + \beta \sum\nolimits_i (-1)^{i-1} d_i^{r,-}.$$

Explicitly, one has the following result:

Corollary 8.3. *In the context of Theorem 8.2, for every $\alpha, \beta \in \mathbb{Z}$ the chain complex $(C_k(X, M, N; A), \partial_k^{\mathrm{B},(\alpha,\beta)})$ admits the following direct summand:*

$$BC_k^{\mathrm{D}}(X, M, N; A) = \sum AM \times X^{\times(i-1)} \times (x \cdot x, x) \times X^{\times(k-i-1)} \times N,$$

where the sum is over all $1 \leqslant i \leqslant k-1$ and $x \in X$.

For the proof of Theorem 8.2 we shall need the following lemma.

Lemma 8.4. *Let (X, σ) be a weakly RI-compatible LND braided set. Consider the relation $(a_1, a_2)^b = (c_1, c_2)$ in $T(X)$. Then condition $a_1 = t(a_2)$ is equivalent to $c_1 = t(c_2)$. The same equivalence holds for the relation ${}^b(a_1, a_2) = (c_1, c_2)$.*

Proof. The first equivalence is established in Fig. 8.2, the second one is similar. $\square$

Proof of Theorem 8.2. The verification of the semi-strong skew cubical relations (2.2)–(2.6) is easy on the X- and N-components of $M \times T(X) \times N$. One should be more careful with how the left- and the right-hand sides of these relations modify the M-component. For example, on the level of the M-components, relation $d'_i s_j = s_j d'_{i-1}$ for $i > j + 1$ reads

$$(m \cdot t(\chi_j(\overline{y}))^{-1}) \cdot \chi_i(s_j(\overline{y})) = (m \cdot \chi_{i-1}(\overline{y})) \cdot t(\chi_j(d'_{i-1}(\overline{y})))^{-1}$$

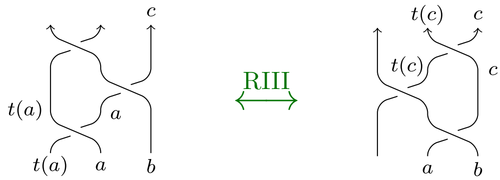

Fig. 8.2. The colors on the left-hand side have the indicated pattern if and only if they do so on the right. This shows that the neighboring colors $(t(x), x)$ remain of the same type when passing (in any direction) under another strand.

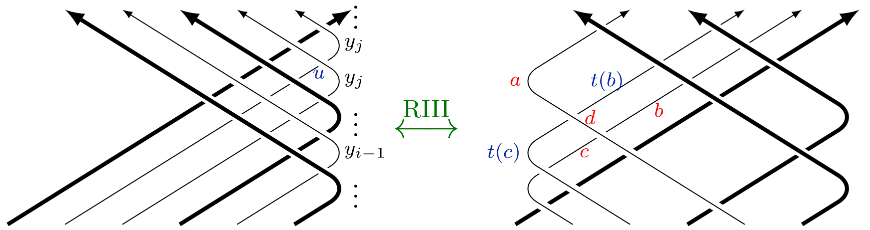

Fig. 8.3. A proof of relation (8.2). Here the thick lines stand for half-twisted bundles of strands. On the left, the jth strand is doubled (under the action of s_j). The color identifications $a = \chi_i(s_j(\overline{y}))$, $b = \chi_j(d'_{i-1}(\overline{y}))$, $c = \chi_j(\overline{y})$, $d = \chi_{i-1}(\overline{y})$ are established using the expression (7.2) for the maps χ. The colors $t(b)$ and $t(c)$ are obtained by a repeated application of Lemma 8.4, starting from the colors $u = t(y_j)$ and y_j on the left.

for all $\overline{y} \in X^{\times k}$, $m \in M$ (here the maps s_j and d'_{i-1} are used with trivial coefficients). This is equivalent to the relation

$$(m \cdot \chi_i(s_j(\overline{y}))) \cdot t(\chi_j(d'_{i-1}(\overline{y}))) = (m \cdot t(\chi_j(\overline{y}))) \cdot \chi_{i-1}(\overline{y}).$$

Since M is a braided module, it is sufficient to show the property

$$\sigma(t(\chi_j(\overline{y})), \chi_{i-1}(\overline{y})) = (\chi_i(s_j(\overline{y})), t(\chi_j(d'_{i-1}(\overline{y})))), \tag{8.2}$$

which we establish in Fig. 8.3.

The remaining semi-strong cubical relations follow from the properties

$$\begin{aligned}
\sigma(\chi_i(s_j(\overline{y})), t(\chi_{j-1}(d'_i(\overline{y})))) &= (t(\chi_j(\overline{y})), \chi_i(\overline{y})), && i < j\\
\sigma(t(\chi_i(s_j(\overline{y}))), t(\chi_j(\overline{y}))) &= (t(\chi_{j+1}(s_i(\overline{y}))), t(\chi_i(\overline{y}))), && i \leqslant j\\
\chi_i(s_i(\overline{y})) = \chi_{i+1}(s_i(\overline{y})) &= t(\chi_i(\overline{y})),\\
\chi_j(\overline{y}) &= \chi_j(d_{i-1}(\overline{y})), && i > j+1\\
\chi_j(\overline{y}) &= \chi_{j-1}(d_i(\overline{y})), && i < j.
\end{aligned}$$

These are established by a similar graphical procedure: in the guitar map diagram, one pulls to the left the strings responsible for the degeneracies and/or boundaries involved, and determines the induced colors.

Alternatively, one could show the semi-strong skew cubical relations for the data $(d_i^{l,+}, d_i^{r,-}, s_i^{\mathrm{B}})$ using the graphical calculus, based on the diagrammatic interpretations of these maps from Figs. 3.4 and 8.1.

By linearization, one obtains a semi-strong skew cubical structure on $C_k(X, M, N; A)$, which we regard as $\mathbb{Z}$-modules. The desired decomposition and its compatibility with the differentials now follow from Theorem 2.2. $\square$

Let us now explore the applications of Theorems 7.1 and 8.2 to two particular cases of braided sets.

Example 8.5. A rack $(X, \triangleleft)$ can be seen as a LND braided set, with the braiding $\sigma_\triangleleft(a, b) = (b, a \triangleleft b)$. The operations $\cdot$ and $\tilde{\cdot}$ become here $a \cdot b = b$, $a \,\tilde{\cdot}\, b = b \,\widetilde{\triangleleft}\, a$, where the operation $b \mapsto b \,\widetilde{\triangleleft}\, a$ is defined as the inverse of $b \mapsto b \triangleleft a$. The maps χ_i from (7.1) simplify as

$$\chi_i(y_1, \ldots, y_k) = (\cdots (y_i \,\widetilde{\triangleleft}\, y_{i+1}) \,\widetilde{\triangleleft} \cdots) \,\widetilde{\triangleleft}\, y_k,$$

denote here by $y_i \,\widetilde{\triangleleft}\, y_{i+1} \cdots y_k$ for the sake of readability. For a right module M and a left module N over $(X, \triangleleft)$ (which are thus braided modules, cf. Example 5.2), the pre-cubical structure from Theorem 7.1 writes

$$d_i \colon (m, y_1, \ldots, y_k, n) \mapsto (m, y_1 \,\widetilde{\triangleleft}\, y_i, \ldots, y_{i-1} \,\widetilde{\triangleleft}\, y_i, y_{i+1}, \ldots, y_k, y_i \cdot n),$$

$$d_i' \colon (m, y_1, \ldots, y_k, n) \mapsto (m \cdot (y_i \,\widetilde{\triangleleft}\, y_{i+1} \cdots y_k), y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_k, n).$$

The inverse

$$J^{-1}(m, y_1, \ldots, y_{k-1}, y_k, n) = (m, y_1 \,\widetilde{\triangleleft}\, y_2 \cdots y_k, \ldots, y_{k-1} \,\widetilde{\triangleleft}\, y_k, y_k, n)$$

of the guitar map sends these boundary maps to

$$d_i^{r,-} \colon (m, x_1, \ldots, x_k, n) \mapsto (m, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k, (x_i \triangleleft x_{i+1} \cdots x_k) \cdot n),$$

$$d_i^{l,+} \colon (m, x_1, \ldots, x_k, n) \mapsto (m \cdot x_i, x_1 \triangleleft x_i, \ldots, x_{i-1} \triangleleft x_i, x_{i+1}, \ldots, x_k, n).$$

For trivial coefficients, this result was established by Przytycki [33]; see his work for the meaning of the corresponding isomorphisms of complexes in the self-distributive homology theory.

Suppose now that X acts on M by bijections (which is a standard assumption in rack theory). If our rack is a quandle, then the braiding $\sigma_\triangleleft$ is RI-compatible, with $t(a) = a$. Theorem 8.2 then yields the degeneracies

$$s_i \colon (m, y_1, \ldots, y_k, n) \mapsto \\ (m \cdot (y_i \,\widetilde{\triangleleft}\, y_{i+1} \ldots y_k)^{-1}, y_1, \ldots, y_{i-1}, y_i, y_i, y_{i+1}, \ldots, y_k, n)$$

for (d_i', d_i). The decomposition (8.1) then generalizes the splitting known in the case of trivial coefficients since the work of Litherland and Nelson [29].

Example 8.6. A group $(X, \star, e)$ is also an RI-compatible LND braided set, with the braiding $\sigma_\star(a,b) = (e, a \star b)$, and the constant map $t(a) = e$ (Example 5.5). For this structure, one calculates

$$\begin{aligned} a \cdot b &= e, & \chi_i(y_1, \ldots, y_k) &= e, \quad i \geqslant 2, \\ a \mathbin{\tilde{\cdot}} b &= b \star a^{-1}, & \chi_1(y_1, \ldots, y_k) &= y_1 \star y_2^{-1} \end{aligned}$$

(we declare $y_2 = e$ if $k = 1$). Take also a right module M and a left module N over the group X (which are thus solid braided modules). Theorem 7.1 then says that the pre-cubical structures

$$\begin{aligned} d_i &\colon (m, y_1, \ldots, y_k, n) \mapsto (m, y_1 \star y_i^{-1}, \ldots, y_{i-1} \star y_i^{-1}, e, \ldots, e, y_i \cdot n), \\ d'_i &\colon (m, y_1, \ldots, y_k, n) \mapsto (m, y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_k, n), \ i \geqslant 2, \\ d'_1 &\colon (m, y_1, \ldots, y_k, n) \mapsto ((m \cdot y_1) \cdot y_2^{-1}, y_2, \ldots, y_k, n); \\ \text{and } d_i^{r,-} &\colon (m, x_1, \ldots, x_k, n) \mapsto (m, x_1, \ldots, x_{i-1}, e, \ldots, e, x_i \cdot \ldots \cdot (x_k \cdot n)), \\ d_i^{l,+} &\colon (m, x_1, \ldots, x_k, n) \mapsto (m, x_1, \ldots, x_{i-2}, x_{i-1} \star x_i, x_{i+1}, \ldots, x_k, n), \ i \geqslant 2, \\ d_1^{l,+} &\colon (m, x_1, \ldots, x_k, n) \mapsto (m \cdot x_1, x_2 \ldots, x_k, n) \end{aligned}$$

are connected by the isomorphisms

$$\begin{aligned} J(m, x_1, \ldots, x_{k-1}, x_k, n) &= (m, x_1 \star \cdots \star x_k, \ldots, x_{k-1} \star x_k, x_k), \\ J^{-1}(m, y_1, \ldots, y_{k-1}, y_k, n) &= (m, y_1 \star y_2^{-1}, \ldots, y_{k-1} \star y_k^{-1}, y_k, n). \end{aligned}$$

One recognizes the two equivalent forms $\sum_i (-1)^{i-1} d'_i$ and $\sum_i (-1)^{i-1} d_i^{l,+}$ of the bar differential for groups. Przytycki [33] noticed the resemblance between this equivalence of differentials and the corresponding phenomenon in the self-distributive situation. Our unified braided interpretation of the two homology theories offers a conceptual explanation of these parallels.

According to Theorem 8.2, the partial diagonal maps

$$s_i \colon (m, y_1, \ldots, y_k, n) \mapsto (m, y_1, \ldots, y_{i-1}, y_i, y_i, y_{i+1}, \ldots, y_k, n)$$

are degeneracies for (d'_i, d_i), and the sum of their images forms a direct summand of any of the complexes constructed in Theorem 2.2.

9. Cycle sets: cohomology and extensions

In this section we specialize our (co)homology study above to cycle sets (Example 5.4), and apply it to an analysis of cycle set extensions. In particular, we interpret the latter in terms of certain 2-cocycles.

Recall that a cycle set is a set X with a binary operation $\cdot$ satisfying $(a \cdot b) \cdot (a \cdot c) = (b \cdot a) \cdot (b \cdot c)$ and having all the translations $a \mapsto b \cdot a$ bijective, with the inverses $a \mapsto b * a$. A cycle set carries the involutive left non-degenerate braiding $\sigma_{\cdot}(a, b) = ((b * a) \cdot b, b * a)$. The operations $\cdot$ and $\tilde{\cdot}$ (Notation 4.1) for this braiding both coincide with the original operation $\cdot$, making the sideways map (Fig. 4.1) symmetric: it takes the form $(a, b) \mapsto (a \cdot b, b \cdot a)$.

We now apply Theorem 7.1 to the braided set $(X, \sigma_{\cdot})$, with trivial coefficients (Example 3.3) on the left, and adjoint coefficients $(X, \cdot)$ (Remark 7.3) on the right. More precisely, for our data we consider the chain complex obtained from the pre-cubical structure (d_i, d'_i) via Theorem 2.2 with $\alpha = 1$, $\beta = -1$, and its cohomological counterpart:

Definition 9.1. The *cycles / boundaries / homology groups of a cycle set* $(X, \cdot)$ with coefficients in an abelian group A are the cycles / boundaries / homology groups of the chain complex $C_n(X, A) = AX^{\times n}$, $n \geqslant 0$, with

$$\begin{aligned}\partial_n(x_1, \ldots, x_n) = \sum\nolimits_{i=1}^{n-1} (-1)^i ((x_1, \ldots, \widehat{x_i}, \ldots, x_n) \\ - (x_i \cdot x_1, \ldots, x_i \cdot x_{i-1}, x_i \cdot x_{i+1}, \ldots, x_i \cdot x_n)),\end{aligned}$$

where $(x_1, \ldots, \widehat{x_i}, \ldots, x_n) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$, and $\partial_1 = 0$. The *cocycles / coboundaries / cohomology groups of* $(X, \cdot)$ are defined by the differentials $\partial^n \colon f \mapsto f \circ \partial_{n+1}$ on $C^n(X, A) = \mathrm{Fun}(X^n, A)$. The constructed cycle / boundary / homology groups are denoted by $Z_n(X, A)$, $B_n(X, A)$, and $H_n(X, A)$ respectively, with the analogous notations $\ldots^n (X, A)$ in the co-case.

Note the subscript shift in C_n with respect to previous sections.

Example 9.2. For a trivial cycle set $(X, x \cdot y = y)$ all the differentials ∂_n and ∂^n vanish, hence one has $H_n(X, A) = AX^{\times n}$, $H^n(X, A) = \mathrm{Fun}(X^n, A)$.

Example 9.3. The first differentials read $\partial_1 = 0$ and $\partial_2(x, y) = y - x \cdot y$. Thus the homology group $H_1(X, A)$ is the A-module freely generated by the *orbits* of our cycle set $(X, \cdot)$, i.e. by the classes of the equivalence relation on X generated by $x \sim y \cdot x$, $x, y \in X$. The cohomology group $H^1(X, A)$ is the group of those maps $X \to A$ which are constant on every orbit.

We now turn to a study of the 2-cocycles of $(X, \cdot)$, i.e., maps $f \colon X \times X \to A$ such that for all $x, y, z \in X$ one has

$$f(x, z) + f(x \cdot y, x \cdot z) = f(y, z) + f(y \cdot x, y \cdot z). \tag{9.1}$$

Example 9.4. Let f, g be two commuting endomorphisms of an abelian group X such that g is invertible and f squares to zero. Then X is a cycle set with

$$x \cdot y = -f(g(x)) + g(y), \quad x, y \in X.$$

For $h \in \mathrm{End}(X)$, the map $X \times X \to X$, $(x, y) \mapsto h(y - x)$, is a 2-cocycle if and only if h satisfies $h = hg$.

Example 9.5. Fix two distinct elements $\alpha_0 \neq \alpha_1$ in an abelian group A. Then the map $f(x, y) = \begin{cases} \alpha_1 & \text{if } x = y, \\ \alpha_0 & \text{if } x \neq y, \end{cases}$ is a 2-cocycle of the cycle set $(X, \cdot)$. Indeed, relations $x = z$ and $y \cdot x = y \cdot z$ are equivalent in X (since the left translations are invertible), which yields the desired property (9.1).

In the remaining part of this section we will show that 2-cocycles are closely related to cycle set extensions. This was one of the motivations behind our definition of cycle set cohomology.

Lemma 9.6. *Let $(X, \cdot)$ be a cycle set, A an abelian group, and $f\colon X \times X \to A$ be a map. Then $A \times X$ with $(\alpha, x) \cdot (\beta, y) = (\beta + f(x, y), x \cdot y)$ for $\alpha, \beta \in A$ and $x, y \in X$ is a cycle set if and only if $f \in Z^2(X, A)$.*

Notation 9.7. The cycle set from the lemma is denoted by $A \times_f X$.

Proof. The left translation invertibility for $A \times X$ follows from the same property for X. Indeed, one can define inverses as $(\alpha, x) * (\beta, y) = (\beta - f(x, x * y), x * y)$. Further, the cycle property

$$((\alpha, x) \cdot (\beta, y)) \cdot ((\alpha, x) \cdot (\gamma, z)) = ((\beta, y) \cdot (\alpha, x)) \cdot ((\beta, y) \cdot (\gamma, z)),$$

for $A \times X$ reads

$$\gamma + f(x, z) + f(x \cdot y, x \cdot z) = \gamma + f(y, z) + f(y \cdot x, y \cdot z),$$

which is equivalent to f being a 2-cocycle. $\square$

Remark 9.8. One can mimic Definition 9.1 (with the preceding argument) for a general left non-degenerate braided set (X, σ). In this situation, 2-cocycles are defined by the property

$$f(x, z) + f(x \cdot y, x \cdot z) = f(y, z) + f(y \mathbin{\tilde{\cdot}} x, y \cdot z).$$

Changing the pre-cubical structure (d_i, d'_i) to $(d^\star_i, d'_i)$ (Theorem 4.2), one gets an alternative (co)homology theory, with the 2-cocycles, called *star 2-cocycles* here, defined by

$$f^\star(x,z)+f^\star(x \,\tilde{\cdot}\, y, x \,\tilde{\cdot}\, z) = f^\star(y,z)+f^\star(y\cdot x, y \,\tilde{\cdot}\, z).$$

Further, observe that a left non-degenerate map $(a,b) \mapsto ({}^a b, a^b)$ satisfies the Yang–Baxter equation if and only if the associated maps $\cdot, \tilde{\cdot}$ (Notation 4.1) obey the following three properties:

$$(a \,\tilde{\cdot}\, b)\cdot(a\cdot c) = (b\cdot a)\cdot(b\cdot c),$$
$$(a\cdot b) \,\tilde{\cdot}\, (a \,\tilde{\cdot}\, c) = (b \,\tilde{\cdot}\, a) \,\tilde{\cdot}\, (b \,\tilde{\cdot}\, c),$$
$$(a \,\tilde{\cdot}\, b)\cdot(a \,\tilde{\cdot}\, c) = (b\cdot a) \,\tilde{\cdot}\, (b\cdot c)$$

(this is classical for biracks, and the proof extends directly to general left non-degenerate braided sets). Now, developing the argument from the lemma above, one shows that the formulas

$$(\alpha,x)\cdot(\beta,y) = (\beta+f(x,y), x\cdot y), \qquad (\alpha,x) \,\tilde{\cdot}\, (\beta,y) = (\beta+f^\star(x,y), x \,\tilde{\cdot}\, y)$$

are associated to a left non-degenerate braiding on $A\times X$ if and only if f is a 2-cocycle, $f^\star$ is a star 2-cocycle, and the two are compatible in the sense of

$$f(x,z)+f^\star(x\cdot y, x\cdot z) = f^\star(y,z)+f(y \,\tilde{\cdot}\, x, y \,\tilde{\cdot}\, z).$$

Inspired by the theory of abelian extensions of quandles [4,13], we define extensions of cycle sets by abelian groups, of which the structure from Lemma 9.6 will be a fundamental example.

Definition 9.9. An *(abelian) extension* of a cycle set $(X,\cdot)$ by an abelian group A is the data $(Y \xrightarrow{p} X, A)$, where $(Y,\cdot)$ is a cycle set endowed with a left A-action (denoted by $(\alpha,y)\mapsto \alpha y$), and $p\colon Y\to X$ is a surjective cycle set homomorphism, such that the following hold:

(1) A acts regularly on each fiber $p^{-1}(x)$ (i.e., for all y, z from the same fiber there is a unique $\alpha\in A$ such that $\alpha z = y$), and
(2) for all $\alpha\in A$ and $y,z\in Y$, one has $(\alpha y)\cdot z = y\cdot z$ and $y\cdot(\alpha z) = \alpha(y\cdot z)$.

Example 9.10. Let A be an abelian group, $(X,\cdot)$ a cycle set, and f a cocycle from $Z^2(X,A)$. Form the cycle set $A\times_f X$ (Lemma 9.6), and consider the canonical surjection $p_X\colon A\times_f X\to X$, $(\alpha,x)\mapsto x$. Let A act on $A\times_f X$ by $\alpha(\beta,x) = (\alpha+\beta,x)$. One readily sees that $(p_X\colon A\times_f X\to X, A)$ is an extension of X by A.

Definition 9.11. Extensions $(Y \xrightarrow{p} X, A)$ and $(Y' \xrightarrow{p'} X, A)$ are called *equivalent* if there exists a cycle set isomorphism $F\colon Y \xrightarrow{\sim} Y'$ satisfying $p = p'\circ F$ and $F(\alpha y) = \alpha F(y)$ for all $\alpha\in A$, $y\in Y$.

Lemma 9.12. *Let $(X, \cdot)$ be a cycle set and $(Y \xrightarrow{p} X, A)$ be its extension. Every set-theoretic section $s\colon X \to Y$ induces a 2-cocycle $f \in Z^2(X, A)$ such that*

$$f(x_1, x_2)s(x_1 \cdot x_2) = s(x_1) \cdot s(x_2) \tag{9.2}$$

for all $x_1, x_2 \in X$. Furthermore, if $s'\colon X \to Y$ is another section and $f' \in Z^2(X, A)$ is its associated 2-cocycle, then f and f' are cohomologous.

Proof. Take a section $s\colon X \to Y$ to $p\colon Y \to X$ and two elements $x_1, x_2 \in X$. Since p is a cycle set homomorphism, both $s(x_1 \cdot x_2)$ and $s(x_1) \cdot s(x_2)$ belong to $p^{-1}(x_1 \cdot x_2)$. By the regularity of the A-action on fibers, there exists a unique $f(x_1 \cdot x_2) \in A$ verifying (9.2). This defines a map $f\colon X \times X \to A$, which we claim to be a cocycle. Indeed, for $x_1, x_2, x_3 \in X$ one calculates

$$\begin{aligned}
&(f(x_1, x_3) + f(x_1 \cdot x_2, x_1 \cdot x_3))\, s((x_1 \cdot x_2) \cdot (x_1 \cdot x_3)) \\
&\quad = f(x_1, x_3)\, (s(x_1 \cdot x_2) \cdot s(x_1 \cdot x_3)) \\
&\quad = s(x_1 \cdot x_2) \cdot (f(x_1, x_3)s(x_1 \cdot x_3)) \\
&\quad = s(x_1 \cdot x_2) \cdot (s(x_1) \cdot s(x_3)) \\
&\quad = (-f(x_1, x_2)(s(x_1) \cdot s(x_2)) \cdot (s(x_1) \cdot s(x_3)) \\
&\quad = (s(x_1) \cdot s(x_2)) \cdot (s(x_1) \cdot s(x_3))\,.
\end{aligned}$$

Permuting the arguments, one obtains

$$\begin{aligned}
&(f(x_2, x_3) + f(x_2 \cdot x_1, x_2 \cdot x_3)\, s((x_2 \cdot x_1) \cdot (x_2 \cdot x_3)) \\
&\quad = (s(x_2) \cdot s(x_1)) \cdot (s(x_2) \cdot s(x_3))\,.
\end{aligned}$$

Now, the cycle property for X and Y and the regularity of the A-action on fibers imply $f(x_1, x_3) + f(x_1 \cdot x_2, x_1 \cdot x_3) = f(x_2, x_3) + f(x_2 \cdot x_1, x_2 \cdot x_3)$ as desired.

Suppose now that s' is another section, and take an $x \in X$. Since $s(x)$ and $s'(x)$ both belong to the fiber $p^{-1}(x)$, there exists a unique $\gamma(x) \in A$ such that $s'(x) = \gamma(x)s(x)$. Let us prove that $f(x_1, x_2) - f'(x_1, x_2) = \gamma(x_1 \cdot x_2) - \gamma(x_2)$ for all $x_1, x_2 \in X$, which means that f and f' are cohomologous. One has:

$$\begin{aligned}
\gamma(x_2)(s(x_1) \cdot s(x_2)) &= s(x_1) \cdot (\gamma(x_2)s(x_2)) \\
&= (\gamma(x_1)s(x_1)) \cdot (\gamma(x_2)s(x_2)) \\
&= s'(x_1) \cdot s'(x_2) \\
&= f'(x_1, x_2)s'(x_1 \cdot x_2) \\
&= f'(x_1, x_2)(\gamma(x_1 \cdot x_2)s(x_1 \cdot x_2))
\end{aligned}$$

$$\begin{aligned}
&= (f'(x_1, x_2) + \gamma(x_1 \cdot x_2))s(x_1 \cdot x_2) \\
&= (f'(x_1, x_2) + \gamma(x_1 \cdot x_2))(-f(x_1, x_2)(s(x_1) \cdot s(x_2))) \\
&= (f'(x_1, x_2) + \gamma(x_1 \cdot x_2) - f(x_1, x_2))(s(x_1) \cdot s(x_2)).
\end{aligned}$$

As usual, the regularity of the A-action on fibers allows one to conclude. □

Lemma 9.13. *Let $(X, \cdot)$ be a cycle set and A an abelian group. Every extension $(Y \xrightarrow{p} X, A)$ of X by A is equivalent to an extension $(A \times_f X \xrightarrow{p_X} X, A)$ for some $f \in Z^2(X, A)$.*

Proof. Let $s\colon X \to Y$ be any set-theoretic section to $p\colon Y \to X$. By Lemma 9.12, it induces a cocycle $f \in Z^2(X, A)$. Consider the map

$$F\colon A \times_f X \to Y, \quad (\alpha, x) \mapsto \alpha s(x).$$

It is a homomorphism of cycle sets, since one has

$$\begin{aligned}
F((\alpha, x)(\beta, y)) &= F((\beta + f(x, y), x \cdot y)) \\
&= (\beta + f(x, y))s(x \cdot y) = \beta((s(x) \cdot s(y)) \\
&= (\alpha s(x)) \cdot (\beta s(y)) = F(\alpha, x) \cdot F(\beta, y).
\end{aligned}$$

Let us prove that F is bijective. It is injective since relation $\alpha s(x) = \beta s(y)$ implies $x = p(s(x)) = p(\alpha s(x)) = p(\beta s(y)) = p(s(y)) = y$, and $\alpha = \beta$ follows by the regularity of the A-action. It is surjective since for every $y \in Y$ there exists a unique $\alpha \in A$ such that $\alpha sp(y) = y$, implying $F(\alpha, p(y)) = y$.

It remains to prove that F is a map of extensions. One has

$$\begin{aligned}
&(p \circ F)(\alpha, x) = p(\alpha s(x)) = ps(x) = x = p_X(\alpha, x), \\
&\alpha F(\beta, x) = \alpha(\beta s(x)) = (\alpha + \beta)s(x) = F(\alpha + \beta, x). \qquad \square
\end{aligned}$$

Lemma 9.14. *Let A be an abelian group, $(X, \cdot)$ a cycle set, and f, g cocycles in $Z^2(X, A)$. The extensions $(A \times_f X \xrightarrow{p_X} X, A)$ and $(A \times_g X \xrightarrow{p_X} X, A)$ are equivalent if and only if f and g are cohomologous.*

Proof. Suppose that F is an equivalence between $(A \times_f X \to X, A)$ and $(A \times_g X \to X, A)$, i.e. $F\colon A \times_f X \to A \times_g X$ is an isomorphism of cycle sets such that $p_X \circ F = p_X$ and $\alpha F(\beta, x) = F(\alpha + \beta, x)$ for all $\alpha, \beta \in A$ and $x \in X$. Let $\gamma\colon X \to A$ be defined by $x \mapsto p_A(F(0, x))$, where $p_A\colon A \times X \to A$, $(\alpha, x) \mapsto \alpha$, is the canonical surjection. Then one has

$$F(\alpha, x) = F(\alpha(0, x)) = \alpha F(0, x) = \alpha(\gamma(x), x) = (\alpha + \gamma(x), x).$$

This implies

$$F((\alpha,x)\cdot(\beta,y)) = (\beta + f(x,y) + \gamma(x\cdot y), x\cdot y), \tag{9.3}$$

$$F(\alpha,x)\cdot F(\beta,y) = (\beta+\gamma(y)+g(x,y), x\cdot y). \tag{9.4}$$

Since F is a cycle set morphism, one obtains $g(x,y) - f(x,y) = \gamma(x\cdot y) - \gamma(y)$ for all $x,y \in X$, thus f and g are cohomologous.

Conversely, if f and g are cohomologous, there exists $\gamma\colon X \to A$ such that $g(x,y) - f(x,y) = \gamma(x\cdot y) - \gamma(y)$ for all $x,y \in X$. Consider the map

$$F\colon A\times_f X \to A\times_g X, \quad (\alpha,x)\mapsto(\alpha+\gamma(x),x).$$

Computations (9.3)–(9.4) remain valid and show that F is a cycle set morphism. It is bijective with the inverse $F^{-1}(\alpha,x) = (\alpha-\gamma(x),x)$, and clearly satisfies $p_X \circ F = p_X$ and $\alpha F(\beta,x) = F(\alpha+\beta,x)$ for all $\alpha,\beta\in A$, $x\in X$. □

Put together, the preceding lemmas yield:

Theorem 9.15. *Let $(X,\cdot)$ be a cycle set and A an abelian group. The construction from Lemma 9.12 yields a bijective correspondence between the set $\mathcal{E}(X,A)$ of equivalence classes of extensions of X by A, and the cohomology group $H^2(X,A)$.*

Remark 9.16. The extension procedure allows the construction of new cycle sets, and thus new left non-degenerate involutive braided sets, out of simpler ones. Another enhancement procedure for braidings is their algebraic deformation, in the spirit of Gerstenhaber. It was extensively studied by Eisermann [14,15]. Except for the diagonal case, a deformation transforms a set-theoretic solution to the YBE into an intrinsically linear one, and thus forces one outside the realm of cycle sets. For instance, deformations of the flip $(a,b)\mapsto(b,a)$ include all the braidings coming from quantum groups. The interaction of these two enhancements reserves many open questions:

(1) How can one relate the deformation theories of the braidings corresponding to a cycle set and its extension?
(2) Do cycle set extensions form a class of deformations of the corresponding braiding?
(3) Can the cycle set cohomology, responsible for extensions, be recovered inside Eisermann's Yang–Baxter cohomology, which controls deformations?

The last two phenomena do hold for the braidings associated to racks [14,15].

We conclude this section with an estimation of the *Betti numbers* $\beta_n(X)$ of a cycle set $(X,\cdot)$—that is, the ranks of the free part of its integral homology groups $H_n(X,\mathbb{Z})$. Recall the notion of *orbits* of $(X,\cdot)$ from Example 9.3.

Proposition 9.17. *Let $(X, \cdot)$ be a finite cycle set with m orbits. Then the inequality $\beta_n(X) \geqslant m^n$ holds for all $n \geqslant 0$.*

Proof. Consider the set $\mathrm{Orb}(X)$ of orbits of X, endowed with the trivial cycle set operation $\mathcal{O} \cdot \mathcal{O}' = \mathcal{O}'$. The quotient map $X \twoheadrightarrow \mathrm{Orb}(X)$ is a cycle set morphism, and thus induces a chain complex surjection $C_n(X, \mathbb{Z}) \twoheadrightarrow C_n(\mathrm{Orb}(X), \mathbb{Z})$ and a map in homology $\phi\colon H_n(X, \mathbb{Z}) \to H_n(\mathrm{Orb}(X), \mathbb{Z})$. For a trivial cycle set the differentials ∂_n are all zero. So the abelian groups $H_n(\mathrm{Orb}(X), \mathbb{Z})$ are free, with m^n generators $[(\mathcal{O}_1, \dots, \mathcal{O}_n)]$, $\mathcal{O}_i \in \mathrm{Orb}(X)$. For such an n-tuple of orbits, put $e_{\mathcal{O}_1,\dots,\mathcal{O}_n} = \sum_{x_i \in \mathcal{O}_i} (x_1, \dots, x_n)$. The differential ∂_n vanishes on this element of $C_n(X, \mathbb{Z})$, since all its $n-1$ terms do. One gets a class $[e_{\mathcal{O}_1,\dots,\mathcal{O}_n}] \in H_n(X, \mathbb{Z})$, with $\phi([e_{\mathcal{O}_1,\dots,\mathcal{O}_n}]) = |\mathcal{O}_1| \cdots |\mathcal{O}_n| [(\mathcal{O}_1, \dots, \mathcal{O}_n)]$. The linear independence of the $[(\mathcal{O}_1, \dots, \mathcal{O}_n)]$ now implies that of the m^n elements $[e_{\mathcal{O}_1,\dots,\mathcal{O}_n}]$ of $H_n(X, \mathbb{Z})$. $\square$

This proposition and its proof are inspired by the analogous result for racks, due to Carter–Jelsovsky–Kamada–Saito [7]. Following them, one can extend the proposition to a certain class of infinite cycle sets. But this analogy does not go much further. For instance, for a wide class of shelves including all finite racks one actually has the equality $\beta_n(X) = |\,\mathrm{Orb}(X)|^n$ [16,27], which fails for many small cycle sets. Indeed, while preparing the paper [17], Etingof, Schedler, and Soloviev computed a complete list of non-degenerate involutive braidings of size $\leqslant 8$. Thanks to Schedler we could access this list and convert it into a readable database for Magma [2] and GAP [21]. This database (available from the authors immediately on request), and Rump's identification between such braidings and cycle sets in the finite setting, allowed us to write a computer program for calculating the homologies $H_n(X, \mathbb{Z})$ for small n and X. The results motivated

Question 9.18. *What information about a cycle set is contained in its Betti numbers?*

10. Applications to multipermutation braided sets

In this section we will apply the extension techniques developed above for constructing cycle sets with prescribed properties—namely, the multipermutation level. We will freely use notations from the previous section.

Let us first recall some notions and results from [17,34].

Definition 10.1. A cycle set $(X, \cdot)$ is called

- *non-degenerate* if its squaring map $a \mapsto a \cdot a$ is bijective;
- *square-free* if it satisfies $a = a \cdot a$ for all $a \in X$.

Of course, square-free cycle sets are automatically non-degenerate.

Proposition 10.2. *For a non-degenerate cycle set* $(X, \cdot)$, *consider the equivalence relation*

$$a \approx a' \quad \Longleftrightarrow \quad a \cdot b = a' \cdot b \ \text{ for all } b \in X.$$

The operation $\cdot$ *then induces a non-degenerate cycle set structure on the quotient set* $\overline{X} = X/\approx$.

Proof. To show that the induced operation is well defined, one should prove $a \cdot b \approx a' \cdot b'$ under the assumptions $a \approx a'$, $b \approx b'$. For any $c \in X$, one has

$$(a \cdot b) \cdot (a \cdot c) \overset{a \approx a'}{=} (a' \cdot b) \cdot (a' \cdot c) = (b \cdot a') \cdot (b \cdot c) \overset{b \approx b'}{=} (b' \cdot a') \cdot (b' \cdot c)$$
$$= (a' \cdot b') \cdot (a' \cdot c) \overset{a \approx a'}{=} (a' \cdot b') \cdot (a \cdot c).$$

Since every element of X can be written in the form $a \cdot c$, we are done.

The cycle set property (1.3) for $\overline{X}$ and the surjectivity of the left translations and of the squaring map follow from the analogous properties for X. Let us now prove that the left translations on $\overline{X}$ are injective, i.e. that the relation $a \cdot b \approx a \cdot b'$ implies $b \approx b'$. Indeed, for any $c \in X$ one has

$$(b \cdot a) \cdot (b \cdot c) = (a \cdot b) \cdot (a \cdot c) \overset{a \cdot b \approx a \cdot b'}{=} (a \cdot b') \cdot (a \cdot c) = (b' \cdot a) \cdot (b' \cdot c).$$

For $c = a$ this yields, using the injectivity of the squaring map, the equality $b \cdot a = b' \cdot a$. The injectivity of the left translations on X then extracts from the computation above the desired property $b \cdot c = b' \cdot c$ for all $c \in X$.

The injectivity of the induced squaring map demands more work, and is proved in [34]. Note that it is automatic in two important cases: the square-free case ($\overline{X}$ is square-free since X is so), and the finite case (where surjectivity implies injectivity). $\square$

Definition 10.3.

- The induced structure from the proposition is called the *retraction* of $(X, \cdot)$, denoted by $\operatorname{Ret}(X, \cdot)$.
- A non-degenerate cycle set $(X, \cdot)$ is called *multipermutation (= MP) of level* $n \geqslant 0$ if n is the minimal number of retractions necessary to turn it into a one-element set (in the sense of $|\operatorname{Ret}^n(X, \cdot)| = 1$).
- For an integer $m \geqslant 0$, the number N_m denotes the minimal size of square-free MP cycle sets of level m.

Remark 10.4. The non-degeneracy is essential for the retraction construction to work: Rump [34] exhibited an example of a degenerate cycle set such that the left translations for the induced operation are not injective.

Example 10.5. The only possibility for a MP cycle set of level 0 is a one-element set with its unique binary operation. Level 1 consists of the structures $(X,\, a\cdot b=\theta(b))$, where θ is an arbitrary bijection $X \overset{\sim}{\to} X$ and X has at least two elements; they are naturally called *permutation cycle sets*. Such a cycle set is square-free if and only if θ is the identity map. These descriptions imply $N_0 = 1$ and $N_1 = 2$.

See [17] for more examples of and details on MP cycle sets.

In [8, Thm. 4] Cedó, Jespers, and Okniński constructed finite square-free MP solutions of arbitrary level. Our extension theory yields a similar result.

Theorem 10.6. *Any square-free multipermutation cycle set of level m and size N admits an extension of size $2N$ which is square-free and multipermutation of level $m+1$.*

This theorem comes with an important corollary:

Corollary 10.7. *For any $m \geqslant 0$,*

(1) *there exists a square-free MP cycle set of level m and size 2^m;*
(2) *one has $N_{m+1} \leqslant 2N_m$.*

Estimation $N_{m+1} \leqslant 2N_m + 1$ was obtained earlier by Cameron and Gateva-Ivanova [22].

Proof of Theorem 10.6. Given a square-free MP cycle set $(X,\, \cdot)$ of level m and size N, consider its 2-cocycle

$$f\colon X\times X \to \mathbb{Z}/2\mathbb{Z}, \qquad f(x,y) = \begin{cases} 0 & \text{if } x=y,\\ 1 & \text{if } x\neq y,\end{cases}$$

from Example 9.5. Form the extension $(\mathbb{Z}/2\mathbb{Z}\times_f X \xrightarrow{p_X} X,\, \mathbb{Z}/2\mathbb{Z})$ (Example 9.10). Explicitly, the cycle set operation on $\mathbb{Z}/2\mathbb{Z}\times_f X$ reads

$$\begin{aligned}(\alpha,x)\cdot(\beta,x) &= (\beta+f(x,x), x\cdot x) = (\beta,x),\\ (\alpha,x)\cdot(\beta,y) &= (\beta+f(x,y), x\cdot y) = (\beta+1, x\cdot y) \qquad \text{for } x\neq y.\end{aligned}$$

This clearly defines a square-free cycle set $(\mathbb{Z}/2\mathbb{Z}\times_f X,\, \cdot)$ of size $2N$. We will now show that the map $(\alpha,x)\mapsto x$ induces a cycle set isomorphism

$$\mathrm{Ret}(\mathbb{Z}/2\mathbb{Z}\times_f X,\, \cdot) \overset{\sim}{\to} (X,\, \cdot),$$

which implies that the cycle set $(\mathbb{Z}/2\mathbb{Z}\times_f X,\, \cdot)$ is MP of level $m+1$.

Concretely, we have to prove that the equality $(\alpha,x)\cdot(\beta,y) = (\alpha',x')\cdot(\beta,y)$ for all $(\beta,y)\in\mathbb{Z}/2\mathbb{Z}\times_f X$ is equivalent to $x=x'$. Indeed, for distinct x and x' one has

Table 10.1
Some values of N_m.

m	0	1	2	3	4	5
N_m	1	2	3	5	6	8

$(\alpha, x) \cdot (\beta, x) = (\beta, x)$ while $(\alpha', x') \cdot (\beta, x) = (\beta + 1, x' \cdot x)$, and for $x = x'$ one compares $(\alpha, x) \cdot (\beta, y) = (\beta + f(x, y), x \cdot y) = (\alpha', x) \cdot (\beta, y)$. $\square$

In fact, the retraction–extension interplay in our proof is more than a mere coincidence. Below is a more conceptual example of a connexion between the two constructions:

Proposition 10.8. *Let $(X, \cdot)$ be a non-degenerate cycle set. Then the natural projection $(X, \cdot) \twoheadrightarrow \mathrm{Ret}(X, \cdot)$ from X to its retraction factors through any extension $(X \xrightarrow{p} Y, A)$ (Definition 9.9) with the total cycle set X.*

Proof. It suffices to check that any x_1, x_2 from the same fiber $p^{-1}(y)$ have identical left translations, i.e., satisfy $x_1 \approx x_2$. Indeed, by the definition of an extension, for any $x_1, x_2 \in p^{-1}(y)$ there exists a $\alpha \in A$ with $x_1 = \alpha x_2$. But then one has $x_1 \cdot x = (\alpha x_2) \cdot x = x_2 \cdot x$, which was to be proved. $\square$

In Rump's identification between finite cycle sets and finite non-degenerate involutive braidings [34], square-freeness is equivalent to the diagonal preservation property $\sigma(x, x) = (x, x)$. Thus an inspection of the non-degenerate involutive braiding list from [17] allows one to compute the first values of the sequence N_m. The results are given in Table 10.1. This computation answers in the negative [22, Open Question 6.13I(3)], which asks if the relation $N_m = 2^{m-1} + 1$, valid for the first values of m, in fact holds for all m. It also shows that our estimations $N_{m+1} \leqslant 2N_m$ are not optimal.

11. Relation with group cohomology

This section proposes an interpretation of the second cohomology group $H^2(X, A)$ of a left non-degenerate (= LND) braided set (X, σ) (in the sense of Remark 9.8, the star version) in terms of group cohomology. This generalizes the analogous result for racks established by Etingof and Graña [16]. For our second favorite example—that of cycle sets—this can be helpful in studying extensions, in the light of Theorem 9.15.

Recall that the *group cohomology* $H^*(G, M)$ of a group G with coefficients in a right G-module M is the cohomology of the complex $(C^n(G, M), \partial^n_{\mathrm{G}})$ with $C^n(G, M) = \mathrm{Fun}(G^n, M)$ and

$$\begin{aligned} \partial^n_{\mathrm{G}} f(g_1, \ldots, g_{n+1}) &= f(g_2, \ldots, g_{n+1}) + (-1)^{n+1} f(g_1, \ldots, g_n) g_{n+1} \\ &\quad + \sum\nolimits_{i=1}^{n} (-1)^i f(g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_{n+1}) \end{aligned}$$

(here the multiplication and action symbols are omitted for simplicity). The group we are interested in here is the structure group $G_{(X,\sigma)}$ (Definition 5.11). Using Notation 4.1, it can be defined as the free group on the set X modulo the relation $(a \cdot b)a = (b \mathbin{\tilde{\cdot}} a)b$. The G-module we will consider comes from

Lemma 11.1. *Let (X, σ) be a LND braided set, and A an abelian group. The map*

$$\begin{aligned}\operatorname{Fun}(X, A) \times X &\to \operatorname{Fun}(X, A),\\ (\gamma, a) &\mapsto (\gamma \cdot a \colon b \mapsto \gamma(a \mathbin{\tilde{\cdot}} b))\end{aligned}$$

extends to a $G_{(X,\sigma)}$-module structure on the abelian group $\operatorname{Fun}(X, A)$*.*

Proof. The map $\lambda(a, b) = a \mathbin{\tilde{\cdot}} b$ defines a left (X, σ)-module structure on itself (Remark 7.3). By the left non-degeneracy, all the maps $b \mapsto a \mathbin{\tilde{\cdot}} b$ are bijective, hence λ extends to a unique $G_{(X,\sigma)}$-module structure on X (Lemma 5.12). This induces a right $G_{(X,\sigma)}$-module structure on $\operatorname{Fun}(X, A)$, which satisfies the desired property. □

The group cohomology with these choices turns out to be useful in studying the cohomology of our braided set:

Theorem 11.2. *Let (X, σ) be a left non-degenerate braided set, and A an abelian group. Then one has the following abelian group isomorphism:*

$$H^2(X, A) \simeq H^1(G_{(X,\sigma)}, \operatorname{Fun}(X, A)),$$

where on the left H^2 stands for the cohomology theory from Remark 9.8 (the star version), and on the right group cohomology is used (the module structure on $\operatorname{Fun}(X, A)$ *is described above).*

Proof. Consider two maps

$$\begin{aligned}\omega \colon Z^1(G_{(X,\sigma)}, \operatorname{Fun}(X, A)) &\to Z^2(X, A),\\ \theta &\mapsto ((x, y) \mapsto \theta(x)(y));\\ \nu \colon Z^1(G_{(X,\sigma)}, \operatorname{Fun}(X, A)) &\leftarrow Z^2(X, A),\\ (x \mapsto f(x, -)) &\leftarrow\!\shortmid f.\end{aligned}$$

We have to show that

(1) for a 1-cocycle θ, $\omega(\theta)$ is indeed a 2-cocycle;
(2) the map ν is well defined;
(3) ω sends 1-coboundaries to 2-coboundaries;
(4) ν sends 2-coboundaries to 1-coboundaries.

Since ω and ν are clearly mutually inverse, this will imply that they induce isomorphism in cohomology.

(1) Let θ be a map in $Z^1(G_{(X,\sigma)}, \mathrm{Fun}(X, A))$. It means that it satisfies the relation

$$\theta(g_1 g_2) = \theta(g_2) + \theta(g_1) \cdot g_2, \qquad g_1, g_2 \in G_{(X,\sigma)}.$$

For $f = \omega(\theta)$, one needs to check the relation

$$f(x, z) + f(x \mathbin{\tilde{\cdot}} y, x \mathbin{\tilde{\cdot}} z) = f(y, z) + f(y \cdot x, y \mathbin{\tilde{\cdot}} z)$$

for all $x, y, z \in X$, which rewrites as

$$\theta(x)(z) + \theta(x \mathbin{\tilde{\cdot}} y)(x \mathbin{\tilde{\cdot}} z) = \theta(y)(z) + \theta(y \cdot x)(y \mathbin{\tilde{\cdot}} z).$$

Recalling the definition of the $G_{(X,\sigma)}$-action on $\mathrm{Fun}(X, A)$, one transforms this into

$$\theta(x)(z) + (\theta(x \mathbin{\tilde{\cdot}} y) \cdot x)(z) = \theta(y)(z) + (\theta(y \cdot x) \cdot y)(z).$$

The 1-cocycle property for θ simplifies it to

$$\theta((x \mathbin{\tilde{\cdot}} y)x) = \theta((y \cdot x)y),$$

which follows from the relation $(x \mathbin{\tilde{\cdot}} y)x = (y \cdot x)y$ valid in the structure group $G_{(X,\sigma)}$.

(2) Recall that a map $\theta\colon G_{(X,\sigma)} \to \mathrm{Fun}(X, A)$ is a 1-cocycle if and only if the map $G_{(X,\sigma)} \to G_{(X,\sigma)} \ltimes \mathrm{Fun}(X, A)$ given by $g \mapsto (g, \theta(g))$ is a group morphism; here $G_{(X,\sigma)} \ltimes \mathrm{Fun}(X, A)$ is the set $G_{(X,\sigma)} \times \mathrm{Fun}(X, A)$ endowed with the group multiplication $(g, \gamma)(g', \gamma') = (gg', \gamma' + \gamma \cdot g')$. The verification of this well-known property is elementary. We will now show that, for $f \in Z^2(X, A)$, the assignment $\iota_f\colon X \to G_{(X,\sigma)} \ltimes \mathrm{Fun}(X, A)$, $x \mapsto (x, f(x, -))$, extends to a unique group morphism $G_{(X,\sigma)} \to G_{(X,\sigma)} \ltimes \mathrm{Fun}(X, A)$. For this it suffices to check the property $\iota_f(x \mathbin{\tilde{\cdot}} y)\iota_f(x) = \iota_f(y \cdot x)\iota_f(y)$ for all $x, y \in X$. Explicitly, it reads

$$(x \mathbin{\tilde{\cdot}} y, f(x \mathbin{\tilde{\cdot}} y, -))(x, f(x, -)) = (y \cdot x, f(y \cdot x, -))(y, f(y, -)),$$

which simplifies as

$$\begin{aligned} &((x \mathbin{\tilde{\cdot}} y)x, f(x \mathbin{\tilde{\cdot}} y, -) \cdot x + f(x, -)) = \\ &((y \cdot x)y, f(y \cdot x, -) \cdot y + f(y, -)). \end{aligned}$$

Since the relation $(x \mathbin{\tilde{\cdot}} y)x = (y \cdot x)y$ always holds in $G_{(X,\sigma)}$, it remains to show the equation

$$f(x \mathbin{\tilde{\cdot}} y, x \mathbin{\tilde{\cdot}} -) + f(x, -) = f(y \cdot x, y \mathbin{\tilde{\cdot}} -) + f(y, -),$$

which is precisely the definition of a 2-cocycle.

(3) A 1-coboundary in $C^1(G_{(X,\sigma)}, \mathrm{Fun}(X, A))$ is a map of the form

$$\partial^0_{\mathrm{G}}\gamma\colon g \mapsto \gamma - \gamma \cdot g \qquad \text{for some } \gamma \in \mathrm{Fun}(X, A).$$

Its image $\omega(\partial^0_{\mathrm{G}}\gamma)$ is then the map sending (x, y) to

$$(\gamma - \gamma \cdot x)(y) = \gamma(y) - \gamma(x \,\tilde{\cdot}\, y) = -\partial^1\gamma(x, y),$$

yielding $\omega(\partial^0_{\mathrm{G}}\gamma) = \partial^1(-\gamma)$.

(4) A 2-coboundary in $C^2(X, A)$ is a map of the form $\partial^1\gamma$ for some $\gamma \in \mathrm{Fun}(X, A)$. Since ω and ν are mutually inverse, the computation above implies the relation $\nu(\partial^1\gamma) = \partial^0_{\mathrm{G}}(-\gamma)$. $\square$

One could wonder if a similar result holds true for higher cohomology groups. A first step in this direction is the identification

$$H^n(X, A) \simeq H^{n-1}_{tr}(X, \mathrm{Fun}(X, A)),$$

which follows from the obvious isomorphism of cochain complexes. Here H^n stands for the cohomology theory from Remark 9.8 (the star version), and H^{n-1}_{tr} is the cohomology of (X, σ) with trivial coefficients, acting on $\mathrm{Fun}(X, A)$ on the right as in Lemma 11.1. This latter cohomology, described for instance in [28], mimics group cohomology. Our identification generalizes Etingof and Graña's result for racks [16, Proposition 5.1]. Note that they used $H^n_{tr}(X, A)$ instead of our $H^n(X, A)$; the two cohomology theories coincide for racks, but differ for more general braided sets. Unfortunately, precise relations between $H^{n-1}_{tr}(X, \mathrm{Fun}(X, A))$ and the group cohomology $H^{n-1}(G_{(X,\sigma)}, \mathrm{Fun}(X, A))$ are as for now very poorly understood. To our knowledge, the only connection between the two is the quantum symmetrizer map, known to be an isomorphism in some very particular cases [18,28].

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