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Global stability results for switched systems based on weak Lyapunov functions

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Abstract—**In this paper we study the stability of nonlinear and time-varying switched systems under restricted switching. We approach the problem by decomposing the system dynamics into a nominal-like part and a perturbation-like one. Most stability results for perturbed systems are based on the use of strong Lyapunov functions, i.e. functions of time and state whose total time derivative along the nominal system trajectories is bounded by a negative definite function of the state. However, switched systems under restricted switching may not admit strong Lyapunov functions, even when asymptotic stability is uniform over the set of switching signals considered. The main contribution of the current paper consists in providing stability results that are based on the stability of the nominal-like part of the system and require only a weak Lyapunov function. These results may have wider applicability than results based on strong Lyapunov functions. The results provided follow two lines. First, we give very general global uniform asymptotic stability results under reasonable boundedness conditions on the functions that define the dynamics of the nominal-like and the perturbation-like parts of the system. Second, we provide input-to-state stability (ISS) results for the case when the nominal-like part is switched linear-time-varying. We provide two types of ISS results: standard ISS that involves the essential supremum norm of the input and a modified ISS that involves a power-type norm.**

I. Introduction

Switched systems appear naturally in many engineering instances or as abstractions of more complicated systems [1]–[4]. The stability properties of switched systems have been extensively investigated in the last two decades (see [1], [2], [5]–[7] and references therein).

Lyapunov functions are central tools in the study of stability of nonautonomous (non-switched) nonlinear systems (see, e.g. [8]–[10]). Loosely speaking, Lyapunov functions can be classified as either strong or weak, depending on whether their mere existence is enough to ensure uniform asymptotic stability (strong) or just uniform Lyapunov stability (weak). In some cases, a strong Lyapunov function can be constructed if a weak one is available [11]–[14]. Although a weak Lyapunov function by itself gives no asymptotic stability guarantee, it can be supplemented with extensions of LaSalle's invariance principle [15] (see [16] and references therein) or other approaches (such as those based on persistence of excitation [17]–[19]), in order to yield asymptotic stability.

For switched systems, the existence of a strong Lyapunov function common to all of the subsystems implies that the switched system is globally uniformly asymptotically stable (GUAS) under arbitrary switching (the converse holds for time-invariant switched systems [20]). As a consequence, no strong Lyapunov function exists for a switched system that is GUAS for switching signals within some proper class but not GUAS under arbitrary switching. In many of these cases, however, it is indeed possible to find a common weak Lyapunov function (see the interesting discussion in Example 2 in Section VI of [21]). This fact motivated the development of several stability results for switched time-invariant systems: extensions of LaSalle's invariance principle [22]–[29] and other approaches [30]–[33]. To the best of our knowledge, results based on weak Lyapunov functions for switched time-varying nonlinear systems can only be found in [34]–[36], where the concept of persistence of excitation plays a fundamental role.

For systems with inputs/disturbances, one of the most useful formulations is given by the Input-to-State Stability (ISS) property [37], [38]. As for ISS of switched nonlinear systems under arbitrary switching, uniform (with respect to the switching signals) ISS is equivalent to the existence of a common ISS-Lyapunov function [39]. When no common ISS-Lyapunov function exists but the ISS property holds for each component subsystem, results for establishing ISS of the switched system were given in [40] for dwell-time switching and in [41] for average dwell-time switching. Recently, results for establishing ISS of switched systems where the ISS property does not necessarily hold for all subsystems have been given in [42], where the existence of an ISS-Lyapunov function for each ISS subsystem is assumed.

In the first part of this paper we address the following problem: assuming the existence of a weak common Lyapunov function V for a switched time-varying nonlinear system with switching signals belonging to some class $\mathcal{S}$, determine conditions under which the system is globally uniformly asymptotically stable with respect to $\mathcal{S}$ (GUAS w.r.t. $\mathcal{S}$, see Definition 2.1 in Section II). We will prove that under reasonable boundedness conditions, the switched system is GUAS w.r.t. $\mathcal{S}$ when the dynamics of each component subsystem can be decomposed into a switched time-varying part (denoted *the nominal switched system*) that is GUAS, and a switched time-varying part (denoted *the perturbation*) that satisfies a very mild vanishing property. Although such an approach resembles the classical one for perturbed systems (e.g. Chapter 9 of [8]),

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the techniques used in this paper differ greatly from standard ones, since we do not assume V to be a Lyapunov function for the nominal switched system.

In the second part of the paper we consider the following problem: determine conditions under which a switched time-varying nonlinear system with inputs/disturbances is ISS, uniformly with respect to switching signals in some class $\mathcal{S}$ (see Definition 4.1 in Section IV). We will give results for the case in which the zero-input switched system dynamics can be decomposed into a switched linear-time-varying (LTV) (the *nominal system*) part, which is GUAS (and hence globally uniformly exponentially stable) with respect to $\mathcal{S}$, and a switched time-varying nonlinear one (*the perturbation*) which satisfies specific bounds. Once again, our approach departs from the standard one since no common ISS-Lyapunov function is assumed to exist for the nominal switched system. In addition to the standard ISS concept, we also provide results employing a power-type norm instead of the supremum norm. This ISS variant is stronger and provides a better description of the behaviour of the system than the standard one.

The remainder of the paper is organized as follows. In Section II, we state the problems addressed and introduce some of the concepts and assumptions employed. Our main results are contained in Sections III (GUAS) and IV (ISS). The main body of the proofs of our main results are given in Section V and some concluding remarks in Section VI. The Appendix contains supplementary proofs for our ISS results.

Notation. $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}$, $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{>0}$ denote the natural numbers, nonnegative integers, reals, nonnegative reals, and positive reals, respectively. $|x|$ denotes the Euclidean norm of any $x \in \mathbb{R}^p$. $\|A\|$ and A' denote, respectively, the induced operator norm and the transpose of any matrix $A \in \mathbb{R}^{m\times p}$. If $g : \mathbb{R}_{\geq 0} \to \mathbb{R}^p$, $\tau \geq 0$ and $I \subset \mathbb{R}_{\geq 0}$ is an interval, then $g^\tau = g(\cdot + \tau)$ and $g_I : \mathbb{R}_{\geq 0} \to \mathbb{R}^p$ is such that $g_I(s) = g(s)$ if $s \in I$ and $g_I(s) = 0$ otherwise. For any interval $I \subset \mathbb{R}$, $L^1(I)$ is the set of Lebesgue integrable functions $f : I \to \mathbb{R}$. For any $n \in \mathbb{N}, \mathcal{U}_n$ and L_n^∞ denote the sets of all the locally essentially bounded functions and respectively the set of all the essentially bounded functions $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$. We note that $L_n^\infty \subset \mathcal{U}_n$. A Carathéodory function is a function $h : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$ such that $h(t, \cdot)$ is continuous for every $t \geq 0$ and $h(\cdot, \xi)$ is Lebesgue measurable for every $\xi \in \mathbb{R}^n$. We write $\alpha \in \mathcal{K}$ if $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is continuous, strictly increasing and $\alpha(0) = 0$, and $\alpha \in \mathcal{K}_\infty$ if, in addition, α is unbounded. Finally, $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a function of class $\mathcal{KL}$ if $\beta(\cdot, t) \in \mathcal{K}_\infty$ for any $t \geq 0$ and, for any fixed $r \geq 0$, $\beta(r, t)$ monotonically decreases to zero as $t \to \infty$.

II. Preliminaries and Problem Statement

Consider the switched time-varying nonlinear system

$$\dot{x} = f(t, x, \sigma) \tag{1}$$

where x takes values in $\mathbb{R}^n$, $\sigma : \mathbb{R}_{\geq 0} \to \Gamma$, with Γ an index set, is a *switching signal*, *i.e.*, σ is piecewise constant (it has at most a finite number of jumps in each compact interval) and continuous from the right (considering the discrete topology in Γ), $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \Gamma \to \mathbb{R}^n$, and $f_i(\cdot, \cdot) = f(\cdot, \cdot, i)$ is a Carathéodory function for every $i \in \Gamma$. Given a family $\mathcal{S}$ of switching signals, we say that (1) is forward complete w.r.t. $\mathcal{S}$ if for every $t_0 \geq 0$, $x_0 \in \mathbb{R}^n$ and $\sigma \in \mathcal{S}$, any maximal solution x of (1) corresponding to σ and such that $x(t_0) = x_0$ is defined for all $t \geq t_0$. Along the paper we will employ the following definition of global uniform asymptotic stability w.r.t. $\mathcal{S}$, where uniformity is understood in the strongest possible sense, *i.e.* w.r.t. initial times and w.r.t. the switching signals in $\mathcal{S}$.

Definition 2.1: System (1) is GUAS w.r.t. a family $\mathcal{S}$ of switching signals if it is forward complete w.r.t. $\mathcal{S}$ and there exists $\beta \in \mathcal{KL}$ such that for every $t_0 \geq 0$, $x_0 \in \mathbb{R}^n$ and $\sigma \in \mathcal{S}$, any maximal solution x of (1) corresponding to σ such that $x(t_0) = x_0$ satisfies

$$|x(t)| \leq \beta(|x_0|, t - t_0) \quad \forall t \geq t_0. \tag{2}$$

Remark 2.1: The GUAS w.r.t. $\mathcal{S}$ property can be defined equivalently in the classical $\varepsilon - \delta$ form, as it is done, for example, in [34, Defn. 1]. The equivalence between these definitions can be proved with the same technique used to prove Proposition 2.5 in [43]. ◦

Throughout the paper, we require the following standing assumptions.

Assumption 1: The function f in (1) can be written as

$$f(t, \xi, i) = \hat{f}(t, \xi, i) + g(t, \xi, i), \tag{3}$$

where $\hat{f}_i(\cdot, \cdot) = \hat{f}(\cdot, \cdot, i)$ is a Carathéodory function for every $i \in \Gamma$, $\hat{f}(t, \cdot, i)$ is locally Lipschitz, uniformly in t and in i, i.e. for all compact $B \subset \mathbb{R}^n$, there exists $L \geq 0$ such that $|\hat{f}(t, \xi, i) - \hat{f}(t, \xi', i)| \leq L|\xi - \xi'|$ for all $\xi, \xi' \in B$, all $t \geq 0$ and all $i \in \Gamma$, and the switched system (4) is GUAS w.r.t. $\mathcal{S}$.

$$\dot{x} = \hat{f}(t, x, \sigma) \tag{4}$$

For ease of reference, $\hat{f}$ will be called the *nominal* system function and g the *perturbation* term. The nominal system (4) is often a ‘simplified’ version of the system (1) for which the GUAS w.r.t. $\mathcal{S}$ property is easier to prove.

Assumption 2: There exists a common weak Lyapunov function V for (1), i.e. $V : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable, and

i) there exist $\phi_1, \phi_2 \in \mathcal{K}_\infty$ such that

$$\phi_1(|\xi|) \leq V(t, \xi) \leq \phi_2(|\xi|), \quad \forall \xi \in \mathbb{R}^n,\ \forall t \geq 0; \tag{5}$$

ii) for all $i \in \Gamma$, all $t \geq 0$ and all $\xi \in \mathbb{R}^n$,

$$\dot{V}_i(t, \xi) := \frac{\partial V(t, \xi)}{\partial t} + \frac{\partial V(t, \xi)}{\partial \xi} f_i(t, \xi) \leq -\eta_i(t, \xi), \tag{6}$$

where $\eta_i : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a Carathéodory function for every $i \in \Gamma$.

Remark 2.2: Without loss of generality, we may assume that the function $\eta : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \Gamma \to \mathbb{R}$ in (7) is bounded on $[0, T] \times B \times \Gamma$ for every $T \geq 0$ and every compact set $B \subset \mathbb{R}^n$. Otherwise, just replace η_i by $\min\{\eta_i(t, \xi), |\xi|^2\}$. ◦

$$\eta(t, \xi, i) := \eta_i(t, \xi) \tag{7}$$

Remark 2.3: It must be pointed out that the function V in Assumption 2 is not necessarily a Lyapunov function (be it weak or strong) for the nominal system (4). ◦

In Section III, we give GUAS results for the switched system (1), based on weak Lyapunov functions. In Section IV, we consider system (1) under the effect of disturbances:

$$\dot{x} = f(t, x, \sigma) + G(t, x, \sigma)u, \tag{8}$$

where $G : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \Gamma \to \mathbb{R}^{n \times m}$, every column of $G(\cdot, \cdot, i)$ is a Carathéodory function for every $i \in \Gamma$, and u represents the disturbance input. By 'input' we mean a function $u \in \mathcal{U}_m$. We will give conditions to establish ISS of (8) uniformly over switching signals in a given set $\mathcal{S}$ when only a weak Lyapunov function is available for the zero-input system (1).

III. Global Uniform Asymptotic Stability

This section focuses on GUAS results for switched nonlinear time-varying systems, based on weak Lyapunov functions. Results are given by imposing additional assumptions on the nominal system function $\hat{f}$ and on the perturbation term g appearing in Assumption 1. We will require the following boundedness condition.

Definition 3.1: A function $h : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \Gamma \to \mathbb{R}^p$ is uniformly bounded if h is bounded on $\mathbb{R}_{\geq 0} \times B \times \Gamma$ for every compact set $B \subset \mathbb{R}^n$.

Assumption 3: The nominal system function $\hat{f}$ and the perturbation term g in (3) are uniformly bounded.

We are now ready to formulate our main GUAS result.

Theorem 3.1: Consider (1) and a family $\mathcal{S}$ of switching signals, for which Assumptions 1 to 3 hold. In addition, suppose that the functions g in Assumption 1 and η in (7) satisfy the following condition:

(C) If $\{(t_k, \xi_k, i_k)\}$ is a sequence in $\mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \Gamma$ such that $t_k \to \infty$ and for some $0 < \varepsilon \leq 1$, $\varepsilon \leq |\xi_k| \leq 1/\varepsilon$ for all k then $\lim_{k\to\infty} \eta(t_k, \xi_k, i_k) = 0 \implies \lim_{k\to\infty} g(t_k, \xi_k, i_k) = 0$.

Then, (1) is GUAS w.r.t. $\mathcal{S}$.

The proof of Theorem 3.1 requires the concept of output-persistent excitation introduced in [34]. A particular case of this definition directly adapted to our problem is given next.

Definition 3.2: Let f be as in (1) and let $h : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \Gamma \to \mathbb{R}^p$ be such that for every continuous function $z : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ and for every switching signal $\sigma \in \mathcal{S}$, $h(\cdot, z(\cdot), \sigma(\cdot))$ is Lebesgue measurable and locally essentially bounded. The pair (h, f) is output-persistently exciting (output-PE) w.r.t. $\mathcal{S}$ if for every $0 < \varepsilon \leq 1$ there exist $T = T(\varepsilon) > 0$ and $r = r(\varepsilon) > 0$ such that for every solution x of (1) corresponding to a switching signal $\sigma \in \mathcal{S}$ and every $t \geq 0$ the following implication holds

$$\varepsilon \leq |x(\tau)| \leq \frac{1}{\varepsilon}, \quad \forall \tau \in [t, t+T] \implies \int_t^{t+T} |h(\tau, x(\tau), \sigma(\tau))|^2 \, d\tau \geq r. \tag{9}$$

Lemma 3.1: Under the assumptions of Theorem 3.1, the pair (h, f), with $h : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \Gamma \to \mathbb{R}$ defined via

$$h(t, \xi, i) = \sqrt{\eta(t, \xi, i)}$$

with η as in (7), is output-PE w.r.t. $\mathcal{S}$.

Note that h in Lemma 3.1 is well defined and satisfies the measurability and local essential boundedness conditions in the first part of Definition 3.2 due to Assumption 2, Remark 2.2, and the fact that a switching signal is piecewise constant. The proof of Lemma 3.1 is given in Section V-A.

Theorem 3.1 is a straightforward consequence of Lemma 3.1 and the following result, which is a corollary of Theorem 2 in [34].

Theorem 3.2: Consider (1) and a family $\mathcal{S}$ of switching signals. Let Assumption 2 hold and the pair (h, f), with h as in Lemma 3.1, be output-PE. Then (1) is GUAS w.r.t. $\mathcal{S}$.

Proof: The theorem readily follows from Theorem 2 in [34], since the hypotheses of that theorem are fulfilled with Φ the set of all the pairs (x, σ) with x a maximal solution of (1) corresponding to $\sigma \in \mathcal{S}$ and the covering $\chi = \{\chi_i\}_{i\in\Gamma}$, with $\chi_i = \mathbb{R}^n$ for all $i \in \Gamma$. In fact, Φ is invariant for χ, V is a piecewise Lyapunov function w.r.t. Φ which verifies (17) in [34] and the pair (h, f) is output-PE. ∎

Theorem 3.1 is established by applying an existing result, namely Theorem 2 of [34]. The main difficulty in the application of the latter result lies in showing that the output-PE assumption holds. Lemma 3.1 is thus the main technical tool. The proof of Lemma 3.1 requires the concept of limiting solutions of switched systems introduced in [34] and is inspired by the methods used in [16]. See Section V-A for details.

Example 3.1: Consider the switched system (1) with two modes $\dot{x} = f_i(t, x)$, $i = 1, 2$, where for all $t \geq 0$ and $\xi \in \mathbb{R}^2$

$$f_i(t, \xi) = \begin{bmatrix} h_{i,1}(t)\xi_1 + \xi_2 \\ -\xi_1^3 - h_{i,2}(t)\xi_2 \end{bmatrix} \quad i = 1, 2, \tag{10}$$

and for $i = 1, 2$, $h_{i,1} \in L^1(\mathbb{R}_{\geq 0})$, it is bounded and $\lim_{t\to\infty} h_{i,1}(t) = 0$, and $h_{i,2}$ is measurable, bounded and $\liminf_{t\to\infty} h_{i,2}(t) = a_i > 0$. We claim that (1) is GUAS for arbitrary switching, *i.e.* it is GUAS w.r.t. the family of all the switching signals. To show that, consider for $i = 1, 2$

$$\hat{f}_i(t, \xi) = \bar{f}(\xi) = \begin{bmatrix} \xi_2 \\ -\xi_1^3 - a\xi_2 \end{bmatrix}$$

where $a = \min\{a_1/2, a_2/2\}$, and

$$g_i(t, \xi) = \begin{bmatrix} h_{i,1}(t)\xi_1 \\ (a - h_{i,2}(t))\xi_2 \end{bmatrix}.$$

Note that for $i = 1, 2$ the functions $\hat{f}$ and g defined by $\hat{f}(t, \xi, i) = \hat{f}_i(t, \xi)$ and $g(t, \xi, i) = g_i(t, \xi)$ are uniformly bounded. Also, note that (4) is GUAS for arbitrary switching since $\dot{x} = \bar{f}(x)$ is a non-switched GUAS system. This statement follows readily from LaSalle's invariance principle by using the weak Lyapunov function $W(\xi) = \xi_1^4 + 2\xi_2^2$.

Let $T_a > 0$ be such that $h_{i,2}(t) \geq a$ for all $t \geq T_a$ and let $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be any uniformly continuous function, such that $\rho \in L^1(\mathbb{R}_{\geq 0})$ and

a) for all $0 \leq t \leq T_a$

$$\rho(t) \geq \max\{|h_{1,1}(t)|, |h_{2,1}(t)|, |a - h_{1,2}(t)|, |a - h_{2,2}(t)|\} + e^{-t}$$

b) $\rho(t) \geq \max\{|h_{1,1}(t)|, |h_{2,1}(t)|\} + e^{-t}$ for all $t \geq T_a$.

Define $\gamma(t) = e^{-\int_0^t 4\rho(s)\,ds}$ and $V(t,\xi) = \gamma(t)W(\xi)$. We have that γ is continuous, decreasing and $\lim_{t\to\infty}\gamma(t) = e^{-\int_0^\infty 4\rho(s)\,ds} = \bar{\gamma} > 0$ and that $\dot{\gamma}(t) = -4\rho(t)\gamma(t)$. Since ρ is uniformly continuous and belongs to $L^1(\mathbb{R}_{\geq 0})$, due to Barbalat's Lemma $\rho(t) \to 0$ as $t \to \infty$. Then $\lim_{t\to\infty}\dot{\gamma}(t) = 0$. It is easy to see that V satisfies (5) and that

$$\dot{V}_i(t,\xi) \leq -\eta_i(t,\xi) \leq 0, \quad \forall t \geq 0,\ i = 1,2,$$

with $\eta_1(t,\xi) = \eta_2(t,\xi) = 4\gamma(t)[e^{-t}\xi_1^4 + a\,\xi_2^2]$. Note that the function η defined by $\eta(t,\xi,i) = \eta_i(t,\xi)$ is uniformly bounded.

Finally, g_i satisfies condition (C) in Theorem 3.1. In fact, if $\eta(t_k,\xi_k,i_k) \to 0$, with $t_k \to \infty$ and $\varepsilon \leq |\xi_k| \leq 1/\varepsilon$, for some $\varepsilon \in (0,1]$, then, if $\xi_k = [\xi_k^1\ \xi_k^2]'$, we have $\xi_k^2 \to 0$. In this case,

$$g(t_k,\xi_k,i_k) = [h_{i_k,1}(t_k)\xi_k^1\ (a - h_{i_k,2}(t_k))\xi_k^2]^T \to 0,$$

since $h_{i_k,1}(t_k) \to 0$, $\xi_k^2 \to 0$ and $\{\xi_k^1\}$ and $\{h_{i_k,2}(t_k)\}$ are bounded and $\{|a - h_{i_k,2}(t_k)|\}$ is eventually bounded from below by $a > 0$.

Therefore the conditions of Theorem 3.1 are satisfied and hence the switched system (1) is GUAS for arbitrary switching. We note that in this example, we have that $\dot{V}_i(t,\xi) < 0$ for all $t \geq 0$ and $\xi \neq 0$. Nevertheless, V is not a strong Lyapunov function for the switched system since there do not exist positive definite functions μ_i such that $\dot{V}_i(t,\xi) \leq -\mu_i(\xi)$ for all $t \geq 0$, for all $\xi \in \mathbb{R}^2$ and for $i = 1,2$.

Example 3.2: Consider the ideal switched model of the semi-quasi-Z-source inverter [44], connected to a cubic-law time-varying resistive load and under zero input voltage:

$$\begin{aligned}
&\dot{x} = f(t,x,\sigma) = \tilde{A}_\sigma x - e_4\tilde{g}_\sigma(t, e_4'x),\\
&e_4 = [0\ 0\ 0\ 1]', \quad P = \operatorname{diag}(L_1, L_2, C_1, C_2)\\
&\tilde{A}_1 = P^{-1}\begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 1\\ 0 & -1 & 0 & 0\\ 0 & -1 & 0 & 0\end{bmatrix}, \quad \tilde{A}_2 = P^{-1}\begin{bmatrix} 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\end{bmatrix},\\
&\tilde{g}_i(t,v) = \frac{G_i(t)}{C_2}v^3, \quad G_i(t) = |\cos(t^2 + a_i)| + \epsilon_i,
\end{aligned}$$

for some $a_i \in \mathbb{R}$ and $\epsilon_i > 0$, for $i = 1,2$. The positive constants L_1, L_2, C_1, C_2 represent the inverter inductance and capacitance values. Irrespective of the load function $\tilde{g}_i$, stability of this inverter model can only be ensured by constantly switching between modes 1 and 2, and imposing additional restrictions on the time spent in mode 2 [45]. Let $\mathcal{S}$ denote the set of switching signals $\sigma : \mathbb{R}_{\geq 0} \to \{1,2\}$ where each mode has minimum ($d_{\min}$) and maximum ($d_{\max}$) dwell-times satisfying $0 < d_{\min} < d_{\max} < \pi\sqrt{L_1C_1}$. The time-invariant positive definite quadratic function $V(t,x) = \bar{V}(x) = \frac{1}{2}x'Px$ satisfies (5) with $\phi_1(s) = \lambda_{\min}s^2$ and $\phi_2(s) = \lambda_{\max}s^2$ with $\lambda_{\min}, \lambda_{\max}$ the minimum and maximum eigenvalues of $P/2$. The function V is a common weak Lyapunov function for this system and Assumption 2 is satisfied, since

$$\begin{aligned}
\dot{V}_i(t,\xi) &= \xi' P\tilde{A}_i\xi - \xi' C_2 e_4\tilde{g}_i(t, e_4'\xi)\\
&= -G_i(t)(e_4'\xi)^4 =: -\eta_i(t,\xi) \leq 0.
\end{aligned}$$

Note that $P\tilde{A}_i$ is skew-symmetric for $i = 1,2$ and hence $\xi' P\tilde{A}_i\xi = 0$ for all $\xi \in \mathbb{R}^n$. The switched-linear system $\dot{x} = \tilde{A}_\sigma x$ is thus lossless, since $\frac{\partial}{\partial\xi}\bar{V}(\xi)\tilde{A}_i\xi \equiv 0$. As a consequence, the evident decomposition of the system into the form (3) with nominal system function $\hat{f}(t,\xi,i) = \tilde{A}_i\xi$ is not useful because the GUAS w.r.t. $\mathcal{S}$ requirement of Assumption 1 is not satisfied. Let $K > 0$ and put the system into the form $\dot{x} = A_\sigma x + g_\sigma(t,x)$, with, for $i = 1,2$,

$$A_i = \tilde{A}_i - Ke_4e_4', \quad g_i(t,\xi) = -e_4[\tilde{g}_i(t, e_4'\xi) - Ke_4'\xi].$$

Following the same lines as in Lemma 1 and Theorem 2 of [44], we can show that the switched-linear system $\dot{x} = A_\sigma x$ is GUAS w.r.t. $\mathcal{S}$. Assumption 1 is now satisfied by decomposing the system into the form (3) with $\hat{f}(t,\xi,i) = A_i\xi$ and $g(t,\xi,i) = g_i(t,\xi)$. Note however that V is not a strong Lyapunov function for the nominal system and no such function exists in this case. We have $|g_i(t,\xi)| \leq |\tilde{g}_i(t, e_4'\xi)| + K|e_4'\xi|$, $|\tilde{g}_i(t, e_4'\xi)| \leq (1+\epsilon_i)|e_4'\xi|^3/C_2$, and $G_i(t) \geq \epsilon_i > 0$. Then, Assumption 3 is satisfied. In addition, $\epsilon_i(e_4'\xi)^4 \leq \eta_i(t,\xi) \leq (1+\epsilon_i)(e_4'\xi)^4$ for all t and hence Condition (C) of Theorem 3.1 is satisfied. By Theorem 3.1, the system is GUAS w.r.t. $\mathcal{S}$. ◦

Remark 3.1: Given a switched system (1) which satisfies Assumption 2 and a family of switching signals $\mathcal{S}$, two issues arise in the application of Theorem 3.1: a) how to decompose the switched system into suitable nominal and perturbation parts and b) how to prove that the nominal part is GUAS w.r.t. $\mathcal{S}$. Usually, the nominal system is obtained by replacing f by some function $\hat{f}$ which is simpler than f and such that, roughly speaking, $f - \hat{f} \to 0$ when the virtual output $h = \sqrt{\eta} \to 0$. This step should be carefully performed, since if $\hat{f}$ is too simple it could result in a non GUAS nominal part. For example, if in Example 3.1 we take $\hat{f}_i = [0\ -\xi_1^3]'$ for $i = 1,2$, then the perturbed terms satisfy (C) in Theorem 3.1 but the nominal system is not GUAS w.r.t. any family of switching signals. The same occurs in Example 3.2 if we take as the nominal terms: $\hat{f}_i(t,\xi) = \tilde{A}_i\xi$, for $i = 1,2$.

Once the decomposition of the system is performed, the GUAS of the nominal part w.r.t. $\mathcal{S}$ may be proved by means of existing results or *ad hoc* methods as, e.g., in [45]. There are systems for which no useful decomposition is possible, save for the trivial one $\hat{f} = f$ and $g = 0$. The latter happens, e.g., for the switched system (29) in Section V of [36], which is GUAS for arbitrary switching. ◦

IV. Input-to-state Stability

In this section, we consider system (8) and provide conditions for the ISS w.r.t. the disturbance input u, uniform over switching signals in a given set $\mathcal{S}$. We say that (8) is forward complete w.r.t. $\mathcal{S}$ if for every initial time $t_0 \geq 0$, initial state $x_0 \in \mathbb{R}$, switching signal $\sigma \in \mathcal{S}$ and input $u \in \mathcal{U}_m$, every maximal solution x of (8) corresponding to σ and u that satisfies $x(t_0) = x_0$, is defined for all $t \geq t_0$. Besides standard ISS involving the essential supremum norm $\|u\|_\infty = \operatorname{ess\,sup}_{t\geq 0}|u(t)|$, we will also consider the following family of norms, which we will collectively name the *power* norms. Given $p \in [1,\infty)$ and $\tau > 0$, we define for $u \in \mathcal{U}_m$:

$$\|u\|_{p,\tau} := \sup_{t\geq 0}\left(\int_t^{t+\tau}|u(s)|^p\,ds\right)^{1/p}. \tag{11}$$

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We employ the following definitions.

Definition 4.1: Let $\mathcal{S}$ be a family of switching signals. System (8) is input-to-state stable (ISS) uniformly w.r.t. $\mathcal{S}$ if it is forward complete w.r.t. $\mathcal{S}$ and there exist $\beta \in \mathcal{KL}$ and $\nu \in \mathcal{K}$ such that for every $t_0 \geq 0$, $x_0 \in \mathbb{R}^n$ and maximal solution x of (8) corresponding to any switching signal $\sigma \in \mathcal{S}$, any input $u \in \mathcal{U}_m$, and such that $x(t_0) = x_0$, the following holds:

$$|x(t)| \leq \beta(|x_0|, t - t_0) + \nu(\|u\|_\infty) \quad \forall t \geq t_0. \tag{12}$$

Definition 4.2: Let $\mathcal{S}$ be a family of switching signals. System (8) is power ISS (pISS) uniformly w.r.t. $\mathcal{S}$ if it is forward complete w.r.t. $\mathcal{S}$ and there exist $\tau > 0$, $\beta \in \mathcal{KL}$ and $\nu \in \mathcal{K}$ such that for every $t_0 \geq 0$, $x_0 \in \mathbb{R}^n$ and maximal solution x of (8) corresponding to any switching signal $\sigma \in \mathcal{S}$, any input $u \in \mathcal{U}_m$, and such that $x(t_0) = x_0$, then

$$|x(t)| \leq \beta(|x_0|, t - t_0) + \nu(\|u\|_{1,\tau}) \quad \forall t \geq t_0. \tag{13}$$

Remark 4.1: By the Markov and causality properties, equivalent definitions of ISS and pISS are obtained if u in (12) and (13) is replaced by $u_{[t_0,t]}$. ◦

Remark 4.2: By means of techniques analogous to those used for proving Lemma 2.7 of [46], one can prove that system (8) is ISS (pISS) uniformly w.r.t. $\mathcal{S}$ if and only if the following conditions hold with $\|\cdot\| = \|\cdot\|_\infty$ ($\|\cdot\| = \|\cdot\|_{1,\tau}$):

i) For every $T > 0$, $r > 0$ and $s > 0$ there exists $C > 0$ such that every maximal solution x of (8) corresponding to any switching signal $\sigma \in \mathcal{S}$, any input u such that $\|u\| \leq s$, and any $t_0 \geq 0$ for which $|x(t_0)| \leq r$, satisfies $|x(t)| \leq C$ for all $t \in [t_0, t_0 + T]$.

ii) For each $\epsilon > 0$ there exists $\delta > 0$ such that every maximal solution x of (8) corresponding to any switching signal $\sigma \in \mathcal{S}$, any input u such that $\|u\| \leq \delta$, and any $t_0 \geq 0$ for which $|x(t_0)| \leq \delta$, satisfies $|x(t)| \leq \epsilon$ for all $t \geq t_0$.

iii) There exists $\nu \in \mathcal{K}$ such that, for any $r \geq \epsilon > 0$, there is a $T > 0$ so that for every maximal solution x of (8) corresponding to any switching signal $\sigma \in \mathcal{S}$, any input u, and any $t_0 \geq 0$ for which $|x(t_0)| \leq r$, then

$$|x(t)| \leq \epsilon + \nu(\|u\|) \quad \forall t \geq t_0 + T. \quad ◦$$

Remark 4.3: The following assertions hold.

i) Given positive real numbers τ and τ', there exists a constant $k = k(\tau, \tau')$ such that $\|u\|_{1,\tau} \leq k\|u\|_{1,\tau'}$. In consequence, if $\|u\|_{1,\tau}$ is finite for some $\tau > 0$, then $\|u\|_{1,\tau'}$ is finite for every $\tau' > 0$.

ii) Taking into account that for $p > 1$, $\tau > 0$ and every $t \geq 0$, $\int_t^{t+\tau} |u(s)|\, ds \leq \tau^{1/q}(\int_t^{t+\tau} |u(s)|^p\, ds)^{1/p}$, where q satisfies $1/p + 1/q = 1$, we have that $\|u\|_{1,\tau} \leq \tau^{1/q}\|u\|_{p,\tau}$.

iii) Since for every $\tau > 0$ and every $t \geq 0$, $\int_t^{t+\tau} |u(s)|\, ds \leq \tau\|u_{[t,t+\tau]}\|_\infty \leq \tau\|u\|_\infty$, it follows that $\|u\|_{1,\tau} \leq \tau\|u\|_\infty$.

iv) For $t \geq t_0 \geq 0$, we have that $\|u_{[t_0,t]}\|_{p,\tau}$ is finite for all $p \geq 1$ and all $\tau > 0$. In addition, the norm inequalities in the previous items hold when replacing u by $u_{[t_0,t]}$. ◦

Remark 4.4:

i) From Remark 4.3i) we have that if (13) holds for some norm $\|\cdot\|_{1,\tau}$, then it holds for any norm $\|\cdot\|_{1,\tau'}$ if we replace ν by $\tilde{\nu}(r) := \nu(k(\tau, \tau')r)$.

ii) It follows from Remark 4.3ii) that if system (8) is pISS uniformly w.r.t. $\mathcal{S}$, then (13) holds with $\|u\|_{p,\tau}$ instead $\|u\|_{1,\tau}$ and with $\tilde{\nu}(r) := \nu(\tau^{1/q}r)$ instead of ν. ◦

Remark 4.5: Due to Remark 4.3iii), if (8) is pISS uniformly w.r.t. $\mathcal{S}$ then it is ISS uniformly w.r.t. $\mathcal{S}$. In addition, the bound (13) gives a better description of the behavior of the switched system than the bound (12). For example, consider an input u unbounded on $[0, \infty)$ and whose integrals on any interval of length τ are uniformly bounded. Then (12) does not give useful information while (13) implies that the states remain bounded and converge to a ball centered at the origin. ◦

Assumption 4 below is somewhat weaker than requiring Assumption 1 and that the nominal part of the zero-input system (1) be switched LTV.

Assumption 4 (Switched LTV GUAS nominal system): The function f in (8) can be written as in (3), where the nominal system function $\hat{f} : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \Gamma \to \mathbb{R}^n$ satisfies

$$\hat{f}(t, \xi, i) = A(t, i)\xi, \tag{14}$$

with $A(t, i) \in \mathbb{R}^{n\times n}$ measurable in $t \geq 0$ for every $i \in \Gamma$, and for every $T \geq 0$ there exists $M > 0$ such that $\|A(t, i)\| \leq M$ for all $t \in [0, T]$ and all $i \in \Gamma$. In addition, the nominal system $\dot{x} = A(t, \sigma)x$ is GUAS w.r.t. $\mathcal{S}$.

Remark 4.6: Consider Assumption 4, and let $\Phi(t, s, \sigma)$ denote the state transition matrix of the GUAS w.r.t. $\mathcal{S}$ LTV system $\dot{x} = A(t, \sigma)x$, satisfying $\Phi(s, s, \sigma) = I$. Following similar lines as in, e.g., [8, Theorem 4.11], it follows that there exist positive constants Ψ and λ such that

$$\|\Phi(t, s, \sigma)\| \leq \Psi e^{-\lambda(t-s)}, \quad \forall t \geq s \geq 0, \forall\sigma \in \mathcal{S}. \tag{15}$$

We require the following strengthened version of Assumption 2 and Condition (C) of Theorem 3.1.

Assumption 5: Let Assumption 2 hold. Suppose that there exist a nondecreasing and Borel measurable function $\omega : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, such that $\omega(s) > 0$ if $s > 0$, a continuous function $\phi_3 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\phi_3(0) = 0$ and a continuous and nondecreasing function $\phi_4 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\phi_4(s) > 0$ if $s > 0$ and

i) for all $t \geq 0$, all $\xi \in \mathbb{R}^n$ and all $i \in \Gamma$,

$$|g_i(t, \xi)| \leq \phi_3(\eta_i(t, \xi)), \tag{16}$$

$$\|G(t, \xi, i)\| \leq \omega(V(t, \xi)), \tag{17}$$

$$\left|\frac{\partial V}{\partial \xi}(t, \xi)\, G(t, \xi, i)\right| \leq \phi_4(V(t, \xi)), \quad \text{and} \tag{18}$$

ii) The function ϕ_5 defined in (19) satisfies $\phi_5 \in \mathcal{K}_\infty$.

$$\phi_5(s) = \int_0^s \frac{1}{\phi_4(\tau)} d\tau, \tag{19}$$

where the integral is considered in the Lebesgue sense.

We require the following lemma, whose proof is given in the Appendix.

Lemma 4.1: Let $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a continuous function such that $\phi(0) = 0$. The following are equivalent

i) There exists a continuous and nondecreasing function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\gamma(0) = 0$ and

$$\int_0^1 \phi(|h(s)|)ds \leq \gamma\left(\int_0^1 |h(s)|ds\right), \quad \forall h \in L^1([0, 1]);$$

ii) $\limsup_{r\to\infty} \frac{\phi(r)}{r} < \infty$.

The following is our main ISS result based on weak Lyapunov functions.

Theorem 4.1: Consider (8) and a family $\mathcal{S}$ of switching signals, for which Assumptions 4 and 5 hold, as well as item a) below. Consider Ψ as in (15) in Remark 4.6 and $\kappa_2(s) := \phi_2(2\Psi s)$. If item b) holds with

$$\tilde{\ell}(a,b,r) := \phi_5^{-1}\big(\phi_5(a) + rb\big) \tag{20}$$

and item c) holds, then (8) is ISS uniformly w.r.t. $\mathcal{S}$.

If item b) holds with

$$\tilde{\ell}(a,b,r) := \phi_5^{-1}\big(\phi_5(a) + b\big) \tag{21}$$

and item d) holds, then (8) is pISS uniformly w.r.t. $\mathcal{S}$.

a) There exists a continuous and nondecreasing function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $\gamma(0) = 0$, such that for all $h \in L^1([0,1])$,
$$\int_0^1 \phi_3(|h(s)|)ds \leq \gamma\left(\int_0^1 |h(s)|ds\right); \tag{22}$$

b) $\limsup_{a\to\infty} \frac{\rho(a,b,M)}{a} < \frac{1}{2}$ for all $b \geq 0$ and $M > 0$, where
$$\rho(a,b,M) := \sup_{0<r\leq M} h_5(a,b,r), \tag{23}$$
$$h_5(a,b,r) := \kappa_2\left\{r\gamma\left(\tfrac{\tilde{\ell}(a,b,r)-a}{r}\right) + \int_a^{\tilde{\ell}(a,b,r)} \frac{\omega(s)}{\phi_4(s)}ds\right\};$$

c) $\limsup_{a\to\infty} \frac{\phi_1^{-1}(a)}{\phi_2^{-1}(a/2)} < \infty$;

d) $\sup_{a>0} \frac{\phi_1^{-1}(a)}{\phi_2^{-1}(a/2)} < \infty$.

The proof of Theorem 4.1 is given in Section V-B. We next provide some comments on the required assumptions.

According to Lemma 4.1 and since $\phi_3 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is continuous and satisfies $\phi_3(0) = 0$ by Assumption 5, the existence of γ as required by item a) is equivalent to $\limsup_{r\to\infty} \frac{\phi_3(r)}{r} < \infty$. Once the function ϕ_3 is known, the latter inequality can be used to easily decide whether such ϕ_3 is suitable. The function γ is then required in order to build ρ for item b). A suitable tool for establishing a) directly is Hölder inequality, which can be employed when ϕ_3 is of the form $\phi_3(s) = \sum_{i=1}^k c_i s^{1/p_i}$ for some $k \in \mathbb{N}$, $c_i > 0$ and $p_i \geq 1$, yielding $\gamma(s) = \phi_3(s)$. In case ϕ_3 does not have such form, the proof of Lemma 4.1 gives some indication on how the required function γ can be constructed. The evaluation of items c) or d) is straightforward, as is that of item b) once γ is known.

Remark 4.7: The assumptions required in Theorem 4.1 in order to ascertain the pISS property imply those required for the ISS property. This is because item d) implies item c) and if item b) holds with $\tilde{\ell}$ as in (21) then item b) also holds with $\tilde{\ell}$ as in (20). To see the latter fact, let ρ_{ISS} and ρ_{pISS} denote (23) with $\tilde{\ell}$ as in (20) or (21), respectively. Note that $\rho_{\text{ISS}}(a,b,M) \leq \rho_{\text{pISS}}(a,bM,M)$ for all $a > 0$, $b \geq 0$, and $M > 0$, and then $\limsup_{a\to\infty} \frac{\rho_{\text{ISS}}(a,b,M)}{a} \leq \limsup_{a\to\infty} \frac{\rho_{\text{pISS}}(a,bM,M)}{a} < 1/2$. ∘

The proof of Theorem 4.1 takes advantage of the linear + perturbation (+ input) form of the system dynamics. The ISS property is established via the equivalent formulation of Remark 4.2. A very interesting strategy that is employed in the proof is that solutions are 'sampled' at specific time instants. These time instants are selected so that either the 'nominal part' of the solution is ensured to cause a substantial decrease in the value of the (weak) Lyapunov function or the increase in this value caused by the input u does not exceed a specific threshold. See Section V-B for details. Existing results that employ a related type of sampling in order to ensure the decrease of a weak Lyapunov function or of the magnitude of the state can be found in [47], [48].

Corollary 4.1 below gives simpler to check but more restrictive assumptions that also ensure the uniform ISS and pISS properties considered.

Corollary 4.1: Consider (8) and a family $\mathcal{S}$ of switching signals. Let Assumption 2 hold, with ϕ_1, ϕ_2 satisfying item c) of Theorem 4.1 and $\phi_2(s) = \sum_{j=1}^{k_2} b_j s^{l_j}$ for some $k_2 \in \mathbb{N}$, $b_j > 0$ and $l_j > 0$. Let Assumptions 3 and 4 hold. Let (16) be satisfied with $\phi_3(s) = \sum_{j=1}^{k_3} c_j s^{1/p_j}$ for some $k_3 \in \mathbb{N}$, $c_j > 0$ and $p_j \geq 1$. Let (17) hold for some $\omega : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ nondecreasing, Borel measurable, and satisfying $\omega(s) > 0$ if $s > 0$ and $\omega(s) = \varrho s^{n_\omega}$ for all $s \geq \bar{s}$, for some $\varrho > 0$, $n_\omega \geq 0$ and $\bar{s} > 0$. Let (18) be satisfied with $\phi_4(s) = ds^{1/m}$ for some $d > 0$ and $m > 1$. If $\max_{1\leq j\leq k_2} l_j < m \min_{1\leq j\leq k_3} p_j$ and $n_\omega < \frac{1}{\max_{1\leq j\leq k_2} l_j}$, then (8) is ISS uniformly w.r.t. $\mathcal{S}$. If, in addition, ϕ_1, ϕ_2 are such that item d) of Theorem 4.1 is satisfied, then (8) is pISS uniformly w.r.t. $\mathcal{S}$.

Proof: As explained above, the fact that $p_j \geq 1$ allows the application of Hölder inequality and shows that item a) of Theorem 4.1 is satisfied with $\gamma = \phi_3$. By direct computation from the definition (19), we have $\phi_5(s) = \frac{q}{d}s^{1/q}$, where we defined $q := m/(m-1)$. Note that $\phi_5 \in \mathcal{K}_\infty$ because $m > 1$ gives $q > 0$. Then, Assumption 5 is satisfied. Let $\kappa_2(s) = \phi_2(2\Psi s) = \sum_{j=1}^{k_2} b_j (2\Psi)^{l_j} s^{l_j}$. Define $h(a,b,r) := r\gamma\left(\frac{\phi_5^{-1}(\phi_5(a)+b)-a}{r}\right) + \int_a^{\phi_5^{-1}(\phi_5(a)+b)} \frac{\omega(s)}{\phi_4(s)}ds$. For $a \geq \bar{s}$, we can compute

$$h(a,b,r) = r\sum_{j=1}^{k_3} c_j \left[\frac{\left(a^{1/q} + db/q\right)^q - a}{r}\right]^{1/p_j} + \varrho\Xi,$$
$$\text{with } \Xi = \frac{q/d}{qn_\omega + 1}\left[\left(a^{\frac{1}{q}} + \frac{db}{q}\right)^{qn_\omega+1} - a^{n_\omega + \frac{1}{q}}\right].$$

Define also

$$h'(a,b,M) := \begin{cases} \sup_{0<r\leq M} \kappa_2\{h(a,br,r)\} & \text{(ISS)} \\ \sup_{0<r\leq M} \kappa_2\left\{h(a,b,r)\right\} & \text{(pISS)} \end{cases}$$

Since $p_j \geq 1$ for all $j = 1,\ldots,k_3$, then both $h(a,br,r)$ and $h(a,b,r)$ are nondecreasing in r. In consequence, we have

$$h'(a,b,M) := \begin{cases} \kappa_2\{h(a,bM,M)\} & \text{(ISS)} \\ \kappa_2\left\{h(a,b,M)\right\} & \text{(pISS)} \end{cases}$$

Note that h' is continuous, $h'(a,0,M) = 0$, and that $h'(a,\cdot,M)$ is nondecreasing. Since $p_j \geq 1$, $m > 1$, $0 <$

$\max_{1\leq j\leq k_2} l_j < m\min_{1\leq j\leq k_3} p_j$ and $n_\omega < \frac{1}{\max_{1\leq j\leq k_2} l_j}$, simple computations show that

$$\lim_{a\to\infty} \frac{h'(a,b,M)}{a} = 0. \quad (24)$$

This follows after taking into account that, for all $K\in\mathbb{R}$,

$$\lim_{a\to\infty} \frac{(a^\ell+K)^{1/\ell}-a}{a^t} = 0 \quad \text{if } \ell+t>1.$$

Then, item b) of Theorem 4.1 is satisfied because $\rho = h'$ in either case. By Theorem 4.1, the result follows. ■

Example 4.1: Consider a switched system of the form (8), where $f(t,x,\sigma)$ coincides with that of Example 3.2 and $G(t,\xi,i) = I$ (identity matrix) for all $t\geq 0, \xi\in\mathbb{R}^n, i\in\Gamma$. The system equations may represent, for example, the semi-quasi-Z-source inverter under time-varying input voltage [45]. Consider the same set of switching signals $\mathcal{S}$ and weak Lyapunov function V of Example 3.2. Since V is quadratic, then V satisfies (5) with $\phi_1(s) = \lambda_{\min}s^2$, $\phi_2(s) = \lambda_{\max}s^2$ and $\lambda_{\min},\lambda_{\max}$ the minimum and maximum eigenvalues of $P/2$. Note that ϕ_1,ϕ_2 satisfy item d) and hence item c) of Theorem 4.1. Consider the same decomposition (3) for the zero-input system as that in Example 3.2. We have $|g_i(t,\xi)| \leq \frac{G_i(t)}{C_2}|e_4'\xi|^3 + K|e_4'\xi|$ and $\eta_i(t,\xi) := G_i(t)(e_4'\xi)^4$, whence $|e_4'\xi| = [\eta_i(t,\xi)/G_i(t)]^{1/4}$ with $\epsilon_i\leq G_i(t)\leq 1+\epsilon_i$ for all $t\geq 0$. Consequently, (16) is satisfied with $\phi_3(s) = c_1 s^{1/p_1} + c_2 s^{1/p_2}$, with $c_1 = (1+\max\{\epsilon_1,\epsilon_2\})^{1/4}/C_2$, $c_2 = K/(\min\{\epsilon_1^{1/4},\epsilon_2^{1/4}\})$, $p_1 = 4/3$, and $p_2 = 4$. Also, (17) is satisfied with $\omega(s)\equiv 1 = 1\cdot s^0$. Since V is quadratic, then (18) is satisfied with $\phi_4(s) = ds^{1/m}$, with $d = 2\lambda_{\max}/\sqrt{\lambda_{\min}}$ and $m = 2$. Since $2 < 2\cdot 4/3$ and $0 < 1/2$, by Corollary 4.1, the considered system is pISS uniformly w.r.t. $\mathcal{S}$ and in consequence also ISS w.r.t. $\mathcal{S}$. As regards the semi-quasi-Z-source inverter connected to the specific nonlinear load considered, this establishes a very important property: if the input voltage u is bounded or, more generally, the integrals of $|u|$ or of $|u|^2$ on intervals of length τ are uniformly bounded, then irrespective of its time evolution (even if it is discontinuous and changes sign) the system variables will be bounded when switching is performed so that the corresponding switching signals are in $\mathcal{S}$. ◦

Remark 4.8: Examples 3.2 and 4.1 deal with a real application: the semi-quasi-Z-source inverter switched model. Being a physical system, a 'natural' Lyapunov function is given by the circuit's energy function. Neither subsystem of this switched system is ISS since neither zero-input subsystem is asymptotically stable. As a consequence, neither strong common nor strong individual Lyapunov functions (and in consequence neither ISS common nor individual ISS Lyapunov functions) exist for the zero-input system. In addition, not even standard multiple Lyapunov functions exist for this circuit. This happens because the usual requirement that the value of the Lyapunov function at every onset time of a same subsystem must be lower than the previous one cannot be satisfed, since the constraints on the switching times are time-dependent but not state-dependent. As a consequence, Example 4.1 gives a practical example where, to the best of our knowledge, no other existing results can be applied in order to establish the ISS uniformly w.r.t. $\mathcal{S}$ property, let alone by means of the natural energy function. ◦

V. Proofs

A. Proof of Lemma 3.1

For proving Lemma 3.1 we will employ the concept of limiting solution of a switched system introduced in [34]. We next give a definition of this concept adapted to our purpose.

Definition 5.1: A continuous function $\bar{x}:\mathbb{R}_{\geq 0}\to\mathbb{R}^n$ is a limiting solution of the switched system (1) with switching signals in $\mathcal{S}$ if there exist an unbounded sequence $\{t_k\}$ in $\mathbb{R}_{\geq 0}$, a sequence $\{(x_k,\sigma_k)\}$, with x_k a maximal solution of (1) corresponding to the switching signal $\sigma_k\in\mathcal{S}$, and a compact set $K\subset\mathbb{R}^n$ such that $x_k(t)\in K$ for all $t\in[t_k,t_k+k]$ and all k, and $\{x_k(\cdot+t_k)\}$ converges to $\bar{x}$ uniformly on $[0,T]$ for all $T>0$.

Remark 5.1: Note that the sequence of switching signals $\{\sigma_k(\cdot+t_k)\}$ in Definition 5.1 is not required to converge in any sense. ◦

The following lemma is a consequence of Lemma 3 in [34].

Lemma 5.1: Consider the switched system (1) with switching signals in $\mathcal{S}$. Suppose that f in (1) is uniformly bounded. Let $\{t_k\}$ be an unbounded sequence in $\mathbb{R}_{\geq 0}$, $K\subset\mathbb{R}^n$ be compact and $\{(x_k,\sigma_k)\}$ be a sequence such that for every k:

1) x_k is a maximal solution of (1) corresponding to the switching signal $\sigma_k\in\mathcal{S}$; and
2) $x_k(t)\in K$ for every $t\in[t_k,t_k+k]$.

Then there exists a subsequence $\{k_l\}$ of $\{k\}$ and a limiting solution $\bar{x}:\mathbb{R}_{\geq 0}\to\mathbb{R}^n$ of (1) with switching signals in $\mathcal{S}$ such that $\{x_{k_l}(\cdot+t_{k_l})\}$ converges to $\bar{x}$ uniformly on $[0,T]$ for all $T>0$.

Now, we are in position to prove Lemma 3.1.

Proof of Lemma 3.1: We will prove the lemma by contradiction. Suppose that the pair (h,f) is not output-PE. Then there exist $0<\varepsilon_0\leq 1$ and sequences $\{t_k'\}$ in $\mathbb{R}_{\geq 0}$, $\{\sigma_k\}$ in $\mathcal{S}$ and $\{z_k\}$, where for every $k\in\mathbb{N}$ z_k is a solution of (1) corresponding to $\sigma_k\in\mathcal{S}$, such that for each $k\in\mathbb{N}$:

1) $\varepsilon_0\leq|z_k(t)|\leq 1/\varepsilon_0$ for all $t\in[t_k',t_k'+2k]$;
2) $\int_{t_k'}^{t_k'+2k}|h(s,z_k(s),\sigma_k(s)|^2\,ds\leq 1/k$.

Let $t_k = t_k'+k$. Then $t_k\to\infty$. If we consider the compact set $K = \{\xi\in\mathbb{R}^n : \varepsilon\leq|\xi|\leq 1/\varepsilon_0\}$, we have that $z_k(t)\in K$ for all $t\in[t_k,t_k+k]$. Since the functions $\hat{f}$ and g are uniformly bounded, it follows that f is also uniformly bounded. By applying Lemma 5.1 there exists a subsequence of $\{t_k\}$, which we still denote by $\{t_k\}$, and a limiting solution $\overline{x}$ of (1) with switching signals in $\mathcal{S}$ such that the sequence $\{x_k\}$, with $x_k = z_k(\cdot+t_k)$, converges to $\bar{x}$ uniformly on $[0,T]$ for all $T>0$. Note that from 1) above, we have that $|\overline{x}(t)|\geq\varepsilon_0$ for all $t\geq 0$.

Now, taking 2) into account, for every k we have

$$0\leq\int_0^k \eta(s+t_k,x_k(s),\sigma_k^{t_k}(s))\,ds\leq 1/k.$$

For any k let $\rho_k(s) := \eta(s+t_k,x_k(s),\sigma_k^{t_k}(s))$ for $s\in[0,k]$ and $\rho_k(s) := 0$ for $s>k$. Then

$$\lim_{k\to\infty}\int_0^\infty \rho_k(s)\,ds = 0.$$

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From the latter, the nonnegativeness of the integrands and well-known results of real analysis it follows that there exists a subsequence of $\{\rho_k\}$, which we still denote by $\{\rho_k\}$ such that for almost all $s \in \mathbb{R}_{\geq 0}$

$$\lim_{k\to\infty} \rho_k(s) = 0.$$

Since $\rho_k(s) = \eta(s + t_k, x_k(s), \sigma_k^{t_k}(s))$ for all $s \leq k$, then

$$\lim_{k\to\infty} \eta(s + t_k, x_k(s), \sigma_k^{t_k}(s)) = 0 \tag{25}$$

for almost all $s \in \mathbb{R}_{\geq 0}$. Let $s \geq 0$ be such that (25) holds. Then, due to condition (C),

$$\lim_{k\to\infty} g(s + t_k, x_k(s), \sigma_k^{t_k}(s)) = 0.$$

Since (25) is true for almost all $s \in \mathbb{R}_{\geq 0}$, it follows that

$$\lim_{k\to\infty} g(s + t_k, x_k(s), \sigma_k^{t_k}(s)) = 0 \text{ a.e. on } \mathbb{R}_{\geq 0}. \tag{26}$$

Next we prove that $\bar{x}$ is a limiting solution of the switched system (4) with switching signals in $\mathcal{S}$.

Taking into account that for all $t \geq 0$

$$x_k(t) = x_k(0) + \int_0^t \hat{f}(s + t_k, x_k(s), \sigma_k^{t_k}(s))\, ds + \int_0^t g(s + t_k, x_k(s), \sigma_k^{t_k}(s))\, ds, \tag{27}$$

that $x_k \to \bar{x}$ uniformly on $[0, t]$, that $\hat{f}$ is locally Lipschitz uniformly in t and in i, the boundedness condition on g, (26) and the Lebesgue Convergence Theorem we arrive to

$$\bar{x}(t) = \bar{x}(0) + \lim_{k\to\infty} \int_0^t \hat{f}(s + t_k, \bar{x}(s), \sigma_k^{t_k}(s))\, ds. \tag{28}$$

For any k, let w_k be the maximal solution of (4) corresponding to the switching signal σ_k that satisfies $w_k(t_k) = \bar{x}(0)$. Since (4) is GUAS w.r.t. $\mathcal{S}$, the solution w_k is defined for every $t \geq t_k$ and $|w_k(t)| \leq \beta(|\bar{x}(0)|, t - t_k)$ for all $t \geq t_k$, where $\beta \in \mathcal{KL}$. So $w_k(t) \in K^*$ for all $t \geq t_k$, where K^* is the compact set $K^* = \{\xi \in \mathbb{R}^n : |\xi| \leq \beta(|\bar{x}(0)|, 0)\}$. If in addition we take into account the boundedness condition on $\hat{f}$, Lemma 5.1 asserts that there exist a subsequence of $\{t_k\}$, which we still denote by $\{t_k\}$, and a limiting solution $\bar{w}$ of (4) with switching signals in $\mathcal{S}$ such that $\{w_k(\cdot + t_k)\}$ converges to $\bar{w}$ uniformly on any interval $[0, T]$, with $T > 0$. If for any k we denote $\omega_k(\cdot) = w_k(\cdot + t_k)$ then

$$\omega_k(t) = \bar{x}(0) + \int_0^t \hat{f}(s + t_k, \omega_k(s), \sigma_k^{t_k}(s))\, ds.$$

The uniform convergence of $\{\omega_k\}$ to $\bar{w}$ on $[0, t]$ and the Lipschitz condition on $\hat{f}$ yield

$$\bar{w}(t) = \bar{x}(0) + \lim_{k\to\infty} \int_0^t \hat{f}(s + t_k, \bar{w}(s), \sigma_k^{t_k}(s))\, ds. \tag{29}$$

From (28) and (29) it follows that for every $t \geq 0$

$$\bar{w}(t) - \bar{x}(t) = \lim_{k\to\infty} \int_0^t \theta_k(s)\, ds,$$

where $\theta_k(s) = \hat{f}(s+t_k, \bar{w}(s), \sigma_k^{t_k}(s)) - \hat{f}(s+t_k, \bar{x}(s), \sigma_k^{t_k}(s))$. The fact that $\bar{x}(s)$ and $\bar{w}(s)$ belong to some compact set for all $s \geq 0$ and the Lipschitz condition on $\hat{f}$ imply the existence of a constant $L \geq 0$ such that $|\theta_k(s)| \leq L|\bar{w}(s) - \bar{x}(s)|$ for all $s \geq 0$. In consequence, for all $t \geq 0$

$$\begin{aligned} |\bar{w}(t) - \bar{x}(t)| &= \lim_{k\to\infty} \left| \int_0^t \theta_k(s)\, ds \right| \\ &\leq \limsup_{k\to\infty} \int_0^t |\theta_k(s)|\, ds \\ &\leq \int_0^t L|\bar{w}(s) - \bar{x}(s)|\, ds. \end{aligned}$$

By applying Gronwall's Lemma, it follows that $|\bar{w}(t) - \bar{x}(t)| \leq e^{Lt}|\bar{w}(0) - \bar{x}(0)|$. Since $\bar{w}(t) = \lim_{k\to\infty} w_k(t + t_k)$, then $\bar{w}(0) = \bar{x}(0)$ and hence $\bar{w}(t) = \bar{x}(t)$ for all $t \in \mathbb{R}_{\geq 0}$. Also, since $|w_k(t + t_k)| \leq \beta(|\bar{x}(0)|, t)$ for all $t \in \mathbb{R}_{\geq 0}$, it follows that $|\bar{x}(t)| \leq \beta(|\bar{x}(0)|, t)$ for all $t \geq 0$, and *a posteriori* that $\bar{x}(t) \to 0$ as $t \to \infty$, which contradicts the fact that $|\bar{x}(t)| \geq \varepsilon_0 > 0$ for all $t \geq 0$. ■

B. Proof of Theorem 4.1

The proofs of the ISS and pISS cases are very similar and hence we will establish both cases simultaneously. We first prove that item i) of Remark 4.2 holds for the norm $\|\cdot\|_\infty$ and for the norm $\|\cdot\|_{1,\tau}$, with any $\tau > 0$.

Let $t_0 \geq 0$, let $x_0 \in \mathbb{R}^n$, let $u \in \mathcal{U}_m$ and let x denote any maximal solution of (8) corresponding to some $\sigma \in \mathcal{S}$ and satisfying $x(t_0) = x_0$. By Assumption 5, we have

$$\begin{aligned} \frac{d}{dt}[V(t, x(t))] &\leq \frac{\partial V}{\partial \xi}(t, x(t)) G(t, x(t), \sigma(t)) u(t) \\ &\leq \phi_4 \circ V(t, x(t))\, |u(t)|, \end{aligned} \tag{30}$$

for almost all $t \geq t_0$ for which x is defined. Applying the Comparison Lemma (see, e.g. [8, Lemma 3.4]), then for all $t \geq t_0$ for which x is defined,

$$V(t, x(t)) \leq \phi_5^{-1}\left(\phi_5(V(t_0, x(t_0)) + \int_{t_0}^t |u(s)| ds\right). \tag{31}$$

Due to (5), (31) and standard results for differential equations we have that (8) is forward complete w.r.t. $\mathcal{S}$.

Let $T > 0$, $r > 0$ and $s > 0$. Then, if $|x(t_0)| \leq r$ and the input u satisfies $\|u\|_\infty \leq s$ or $\|u\|_{1,\tau} \leq s$, from (5), (31), Remark 4.3i) and the fact that $u \in \mathcal{U}_m$, it follows that $|x(t)| \leq C$ for all $t \in [t_0, t_0 + T]$, where $C = \phi_1^{-1}(\phi_5^{-1} \circ \phi_2(r) + Ts)$ in the case of the supremum norm and where $C = \phi_1^{-1}(\phi_5^{-1} \circ \phi_2(r) + k(T, \tau)s)$ in the case of the norm $\|\cdot\|_{1,\tau}$.

We next introduce auxiliary functions. Define $R : (0, \infty) \to (0, \infty)$ and $\bar{R} : (0, \infty) \to (0, \infty)$ as

$$\begin{aligned} R(a) &:= \frac{1}{\lambda} \log \frac{2\Psi\phi_1^{-1}(a)}{\phi_2^{-1}(a/2)}. \\ \bar{R}(b) &:= \sup_{a \geq b} R(a). \end{aligned} \tag{32}$$

Note that item d) implies item c), and that $\bar{R}(b)$ is finite for all $b > 0$ due to continuity and item c). For $a > 0$ and $b \geq 0$, define

$$\ell(a, b) := \tilde{\ell}(a, b, R(a))$$

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and consider the expression

$$h_1(a,b,c) := a/2 + \kappa_2 \left\{ R(a)\gamma\left(\frac{\ell(a,b)-c}{R(a)}\right) + \int_a^{\ell(a,b)} \frac{\omega(s)}{\phi_4(s)} ds \right\},$$

valid for $0 \leq c \leq \ell(a,b)$. Define the function $h_2(\cdot,\cdot)$ via

$$h_2(a,b) = \sup\left\{c \in [0, \ell(a,b)] : c \leq h_1(a,b,c)\right\}.$$

Note that h_2 satisfies $h_2(a,b) \leq \ell(a,b)$. The following claims, whose proofs are given in the Appendix, will be used in the proof of the theorem.

Claim 1: There exist $\alpha, \tilde{\alpha} \in \mathcal{K}_\infty$ such that $\tilde{\alpha}(b) \geq \alpha(b)$ for all $b \geq 0$ and (33) holds. In addition, (34) holds for the ISS proof and (35) holds for the pISS proof.

$$0 < h_2(a,b) < a \qquad \text{for all } a \geq \alpha(b), b > 0. \tag{33}$$

$$\phi_5^{-1}[\phi_5(\alpha(b)) + \bar{R}(\alpha(b))b] \leq \tilde{\alpha}(b) \quad \text{for all } b > 0. \tag{34}$$

$$\phi_5^{-1}(\phi_5(\alpha(b)) + b) \leq \tilde{\alpha}(b) \quad \text{for all } b > 0. \tag{35}$$

Claim 2: Consider sequences $\{y_k\}$ and $\{w_k\}$ of nonnegative real numbers satisfying $\sup_{k\geq 0} w_k \leq U$ and $y_{k+1} \leq h_2(y_k, w_k)$ for all $k \in \mathbb{N}_0$ for which $y_k \geq \alpha(U)$. Then, for all $\tilde{r} > 0$ and $0 < U_0 \leq U \leq U_1$, there exists $K = K(\tilde{r}, U_0, U_1) \in \mathbb{N}$ such that if $\alpha(U) \leq y_0 \leq \tilde{r}$ then $y_k < \alpha(U)$ for some $k \leq K$.

We next proceed to prove that items ii) and iii) of Remark 4.2 hold for the norms $\|\cdot\|_\infty$ and $\|\cdot\|_{1,\tau}$, where

$$\tau := \lim_{b\to 0^+} \bar{R}(b).$$

Note that item d) implies that $\tau < \infty$.

Let $U > 0$. Let x denote any solution to (8) corresponding to a switching signal $\sigma \in \mathcal{S}$ and to an input $u \in \mathcal{U}_m$ such that $\|u\| \leq U$, where in the sequel $\|u\|$ denotes $\|u\|_\infty$ for the ISS proof (in which case $u \in L_m^\infty$) and $\|u\|_{1,\tau}$ for the pISS proof.

Let $t_0 \geq 0$ be such that $x(t_0)$ is defined. In correspondence with solution x, U and t_0, we define recursively the sequences $\{t_k\}$, $\{x_k\}$ and $\{V_k\}$ as follows

$$r_0 := 0, \quad x_0 := x(t_0), \quad V_0 := V(t_0, x_0),$$

and, while $V_k \geq \alpha(U)$

$$r_{k+1} := R(V_k) \qquad t_{k+1} := t_k + r_{k+1}$$
$$x_{k+1} := x(t_{k+1}) \qquad V_{k+1} := V(t_{k+1}, x_{k+1}).$$

For any k such that t_k and t_{k+1} are defined, we also define

$$u_k := \text{ess sup}_{t_k \leq t \leq t_{k+1}} |u(t)|. \tag{ISS}$$

$$u_k := \int_{t_k}^{t_{k+1}} |u(s)| ds. \tag{pISS}$$

Note that the sequences are finite if for some $k \in \mathbb{N}_0$, $V_k < \alpha(U)$. In particular, if $V_0 < \alpha(U)$ the sequences $\{t_k\}$, $\{x_k\}$ and $\{V_k\}$ are only defined for $k = 0$, while the sequence $\{u_k\}$ is undefined. Note also that $u_k \leq U$ for every k for which u_k is defined. The latter fact follows straightforwardly for the ISS proof and from $t_{k+1} = t_k + R(V_k) \leq t_k + \tau$ for the pISS proof.

The following fact about the sequences defined above is proved in the Appendix.

Claim 3: The sequences $\{V_k\}$ and $\{u_k\}$ satisfy

$$V_{k+1} \leq h_2(V_k, u_k) \leq h_2(V_k, U) < V_k.$$

In consequence, combining Claims 2 and 3 we can establish the following fact about the sequence $\{V_k\}$:

Fact 1: For every $\tilde{r} > 0$ and positive numbers $U_0 \leq U_1$ there exists $K = K(\tilde{r}, U_0, U_1)$ such that: if $V_0 \leq \tilde{r}$ and $U \in [U_0, U_1]$ then there exists $k^* \leq K$ such that $V_{k^*} < \alpha(U)$.

From Fact 1, the definition of $\{t_k\}$ and the monotony of $\{V_k\}$, we derive the following result.

Fact 2: Let $\tilde{r} > 0$ and $0 < U_0 \leq U_1$. Then there exists $T = T(\tilde{r}, U_0, U_1)$ such that for every solution x of (8) corresponding to a switching signal $\sigma \in \mathcal{S}$, an input u such that $\|u\| \leq U$, with $U \in [U_0, U_1]$, and such that $V(t_0, x(t_0)) \leq \tilde{r}$ for some $t_0 \geq 0$, there exists $t_x \in [t_0, t_0 + T]$ such that

$$V(t_x, x(t_x)) < \alpha(U).$$

If x is in the conditions of Fact 2, and $\{t_k\}$, $\{V_k\}$ are the sequences defined above, from Fact 1 we have that for some $0 \leq k^* \leq K = K(\tilde{r}, U_0, U_1)$, $V_{k^*} < \alpha(U)$. If $k^* = 0$, then $t_x = t_0$. If $k^* > 0$, then $V(t_{k^*}, x(t_{k^*})) < \alpha(U)$ and $t_{k^*} = t_0 + \sum_{j=0}^{k^*-1} R(V_j) \leq t_0 + k^* \bar{R}(\alpha(U)) \leq t_0 + K\bar{R}(\alpha(U_0))$. In both cases Fact 2 holds with $T = K\bar{R}(\alpha(U_0))$.

Fact 3: Let $U > 0$ and let x be a solution of (8) corresponding to a switching signal $\sigma \in \mathcal{S}$ and input u such that $\|u\| \leq U$. If $V(t_x, x(t_x)) \leq \alpha(U)$ for some $t_x \geq 0$, then

$$V(t, x(t)) \leq \tilde{\alpha}(U) \quad \forall t \geq t_x. \tag{36}$$

Suppose that (36) does not hold for some solution x in the conditions of Fact 3. Then, from the continuity of $V(\cdot, x(\cdot))$, there exists $t^* > t_0' \geq t_x$ such that $V(t_0', x(t_0')) = \alpha(U)$, $V(t, x(t)) > \alpha(U)$ for all $t \in (t_0', t^*]$ and $V(t^*, x(t^*)) > \tilde{\alpha}(U)$. If we consider the sequences $\{t_k\}$, $\{x_k\}$, $\{V_k\}$ and $\{u_k\}$ defined above, but with the initial data $t_0 := t_0'$, $x_0 := x(t_0')$ and $V_0 := V(t_0', x(t_0'))$, from Claim 3 we have that $V_1 = V(t_1, x(t_1)) < \alpha(U)$, where $t_1 = t_0' + R(\alpha(U))$. So, necessarily $t^* \in (t_0', t_1)$. Using (31) with t_0 replaced by t_0', it follows that for all $t \in [t_0', t_1]$

$$V(t, x(t)) \leq \phi_5^{-1}\left(\phi_5\big(V(t_0', x(t_0'))\big) + \int_{t_0'}^{t_1} |u(s)|\, ds\right).$$

In consequence,

$$V(t^*, x(t^*)) \leq \phi_5^{-1}\big(\phi_5(\alpha(U)) + R(\alpha(U))U\big) \leq \tilde{\alpha}(U),$$

for the ISS case, and

$$V(t^*, x(t^*)) \leq \phi_5^{-1}\big(\phi_5(\alpha(U)) + U\big) \leq \tilde{\alpha}(U),$$

for the pISS case. In both cases we arrive to a contradiction. So, Fact 3 holds.

Next, we prove that item ii) in Remark 4.2 (recall Remark 4.2) is satisfied. Let $\epsilon > 0$, let $U = \tilde{\alpha}^{-1} \circ \phi_1(\epsilon)$, and $\delta = \min\{U, \phi_2^{-1} \circ \alpha(U)\} > 0$. Let x be a solution of (8) corresponding to a switching signal $\sigma \in \mathcal{S}$ and an input u such that $\|u\| \leq \delta$, and $|x(t_0)| \leq \delta$. Then, $\|u\| \leq U$ and $|x(t_0)| \leq \phi_2^{-1} \circ \alpha(U)$. The latter inequality implies that $V(t_0, x(t_0)) \leq \alpha(U)$. From Fact 3, we have that

$V(t, x(t)) \leq \tilde{\alpha}(U) = \phi_1(\epsilon)$ for all $t \geq t_0$. From (5), then $|x(t)| \leq \epsilon$ for all $t \geq t_0$.

Next, we show that item iii) in Remark 4.2 is also satisfied. Fix $0 < \epsilon \leq r$. Then, $\phi_1(\epsilon) \leq \phi_2(r)$. Suppose that x is a solution of (8) corresponding to a switching signal $\sigma \in \mathcal{S}$ and to an input u, such that for the time $t_0 \geq 0$, $|x_0| = |x(t_0)| \leq r$. The latter implies that $V_0 = V(t_0, x(t_0)) \leq \phi_2(r) =: \tilde{r}$. We will consider two cases:

a) $V_0 \leq \alpha(\|u\|)$.
b) $V_0 > \alpha(\|u\|)$.

In case a), from Fact 3 with $U = \|u\|$ we have that $V(t, x(t)) \leq \tilde{\alpha}(U)$ and hence $|x(t)| \leq \epsilon + \phi_1^{-1} \circ \tilde{\alpha}(\|u\|)$ for all $t \geq t_0$.

In case b), and since $V_0 \leq \tilde{r}$, then $\alpha(\|u\|) < \tilde{r}$. Let $\tilde{\epsilon} := \alpha \circ \tilde{\alpha}^{-1}(\phi_1(\epsilon))$, $U_0 := \tilde{\alpha}^{-1}(\phi_1(\epsilon)) = \alpha^{-1}(\tilde{\epsilon})$ and $U_1 := \max\{\alpha^{-1}(\tilde{r}), U_0\}$. Take $U = \max\{\|u\|, U_0\}$ and note that $0 < U_0 \leq U \leq U_1$. Let T be as in Fact 2. Then, there exists $t_x \in [t_0, t_0 + T]$ such that $V(t_x, x(t_x)) \leq \alpha(U)$. From the latter and Fact 3 it follows that $V(t, x(t)) \leq \tilde{\alpha}(U)$ for all $t \geq t_0 + T$. If $\|u\| \geq U_0$, then $U = \|u\|$ and we have that $V(t, x(t)) \leq \tilde{\alpha}(\|u\|)$ and hence $|x(t)| \leq \phi_1^{-1} \circ \tilde{\alpha}(\|u\|) + \epsilon$ for all $t \geq t_0 + T$. If $\|u\| < U_0$, then $U = U_0$ and $V(t, x(t)) \leq \tilde{\alpha}(U_0) = \phi_1(\epsilon)$ and hence $|x(t)| \leq \phi_1^{-1} \circ \tilde{\alpha}(\|u\|) + \epsilon$ for all $t \geq t_0 + T$.

In either case, item iii) in Remark 4.2 is satisfied with $\nu = \phi_1^{-1} \circ \tilde{\alpha}$. ■

VI. CONCLUSIONS

We have provided a set of novel stability results for switched nonlinear and time-varying systems under restricted switching. These results are based on weak Lyapunov functions. The approach employed consists in decomposing the system dynamics into a nominal part that is GUAS with respect to the set $\mathcal{S}$ of switching signals considered, and a perturbation term that should satisfy specific bounds. Both GUAS w.r.t. $\mathcal{S}$ results and uniform w.r.t. $\mathcal{S}$ ISS results are provided. Our GUAS results require the perturbation term to satisfy a mild vanishing condition. The uniform ISS results provided require the nominal part of the system to be switched linear-time-varying, and more stringent assumptions are placed on the perturbation and the weak Lyapunov function. To the best of the authors' knowledge, no other GUAS or ISS results exist for switched nonlinear and time-varying systems under restricted switching based on weak Lyapunov functions and employing a perturbation approach. In addition to standard ISS involving the essential supremum norm of the input, we also provide another type of ISS result that employs a power-type norm of the input. An interesting approach employed in the ISS proofs consists in 'sampling' the system solutions at specific time instants related to a specific decrease in the magnitude of the solution.

APPENDIX
ISS SUPPLEMENTARY PROOFS

A. Proof of Claim 1

This proof requires the following result.

Lemma A.1: Let $\rho : \mathbb{R}_{>0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ be continuous and satisfy, for all $a > 0$ and $M > 0$,

a) $\rho(a, 0, M) = 0$;
b) $\rho(a, \cdot, M)$ is nondecreasing.

If $\limsup_{a\to\infty} \dfrac{\rho(a, b, M)}{a} < \dfrac{1}{2}$ for all $b \geq 0$ and $M > 0$, then for every $M > 0$ there exists $\tilde{\gamma} \in \mathcal{K}$ such that $\rho(a, b, M) < a/2$ for all $a > \tilde{\gamma}(b)$ and $b \geq 0$.

Proof: Given $M > 0$, for each $b \geq 0$ consider

$$C(b) := \{c \geq 0 : \forall a > c, \rho(a, b, M) < a/2\},$$
$$\varphi(b) := \inf C(b).$$

Note that $\varphi(b) \geq 0$ for all $b \geq 0$, $C(0) = [0, \infty)$ by a) and hence $\varphi(0) = 0$, and also $\varphi(b) < \infty$ for all $b \geq 0$. By definition, we have $\rho(a, b, M) < a/2$ for all $a > \varphi(b)$ and by continuity of ρ, $\rho(\varphi(b), b, M) = \varphi(b)/2$ if $b > 0$. By b), then $\varphi(\cdot)$ is nondecreasing. Hence, there exists $L \geq 0$ such that $\lim_{b\to 0^+} \varphi(b) = L$. For a contradiction, suppose that $L > 0$. Consider a sequence $\{d_j\}$ of positive numbers satisfying $\lim_{j\to\infty} d_j = 0$. We have $\rho(\varphi(d_j), d_j, M) = \varphi(d_j)/2$. But $\lim_{j\to\infty} \rho(\varphi(d_j), d_j, M) = \rho(L, 0, M) = 0$ and $\lim_{j\to\infty} \varphi(d_j)/2 = L/2 > 0$, which is a contradiction. Then, $L = 0$. The function φ thus satisfies $\varphi(0) = 0$, is nondecreasing, and continuous at 0, and hence can be bounded by a function $\tilde{\gamma} \in \mathcal{K}$ so that $\varphi(b) \leq \tilde{\gamma}(b)$ for all $b \geq 0$. Then, $\rho(a, b, M) < a/2$ for all $a > \tilde{\gamma}(b)$, for all $b \geq 0$. ■

For $a > 0$, $b \geq 0$ and $0 \leq c \leq \ell(a, b)$, $h_1(a, b, c)$ is continuous and nonincreasing in c. So $h_1(a, b, c) - c$ is strictly decreasing. Since $h_1(a, b, 0) > 0$, then the set of values $c \in [0, \ell(a, b)]$ satisfying $c \leq h_1(a, b, c)$ is an interval of the form $[0, h_2(a, b)]$, and $h_2(a, b) > 0$. For $a > 0$ and $b \geq 0$, define $h_3(a, b) := h_1(a, b, a) - a/2$. Note that h_3 is well defined because $a \leq \ell(a, b)$. The inequality $h_3(a, b) < a/2$ implies $a > h_1(a, b, a)$. The latter means that $c = a$ does not satisfy $c \leq h_1(a, b, c)$ and hence $h_3(a, b) < a/2$ implies that $h_2(a, b) < a$. Consider

$$A(b) := \{c \geq 0 : \forall a > c, h_3(a, b) < a/2\}$$
$$h_4(b) := \inf A(b).$$

We have $A(0) = [0, \infty)$ because $h_3(a, 0) = 0$ for all $a > 0$. Then, $h_4(0) = 0$. Since $h_3(a, b)$ is nondecreasing in b (because h_1 is), then $b_1 \leq b_2 \Rightarrow A(b_1) \supseteq A(b_2)$ and hence $h_4(b_1) \leq h_4(b_2)$. This shows that h_4 is nondecreasing. We next show that $A(b) \neq \emptyset$ for all $b > 0$. Note that $h_3(a, b) = h_5(a, b, R(a))$. Let $L > 0$ and $M = \bar{R}(L)$. Then $R(a) \leq M$ whenever $a \geq L$. For all a such that $R(a) \leq M$, it follows that $h_3(a, b) \leq \sup_{0<r\leq M} h_5(a, b, r) = \rho(a, b, M)$. Note that ρ is continuous, $\rho(a, 0, M) = 0$, and that $\rho(a, \cdot, M)$ and $\rho(a, b, \cdot)$ are nondecreasing. Hence, let $\tilde{\gamma} \in \mathcal{K}$ correspond to M as per Lemma A.1. Then, for all $a > \tilde{\gamma}(b)$ such that $R(a) \leq M$ it happens that $h_3(a, b) \leq \rho(a, b, M) < a/2$. Hence, $h_3(a, b) < a/2$ for all $a > \max\{L, \tilde{\gamma}(b)\}$ and $A(b)$ is thus nonempty. As a consequence, $h_4(b) < \infty$ for all $b \geq 0$.

We next show that $\lim_{b\to 0^+} h_4(b) = 0$. Since h_4 is nondecreasing, there exists $L = \lim_{b\to 0^+} h_4(b) \geq 0$. For a contradiction, suppose that $L > 0$. Let $0 < \epsilon < L$, let $\tilde{\gamma} \in \mathcal{K}$ correspond to $M = \bar{R}(L - \epsilon)$ according to Lemma A.1 and

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let $b > 0$ be sufficiently small so that $\tilde{\gamma}(b) < L - \epsilon$. Note that $a \geq L - \epsilon$ implies that $R(a) \leq M$ and hence $h_3(a, b) < a/2$ for all $a \geq L - \epsilon$. Then, $h_4(b) \leq L - \epsilon$, contradicting the fact that h_4 is nondecreasing. The function h_4 is thus nondecreasing, continuous at 0, and satisfies $h_4(0) = 0$ and $h_4(b) < \infty$ for all $b > 0$. Therefore, we can find $\alpha \in \mathcal{K}_\infty$ so that $h_4(b) < \alpha(b)$ for all $b > 0$. We then have $h_3(a, b) < a/2$ for all $a \geq \alpha(b)$, $a > 0$, which implies that $h_2(a, b) < a$ for all $a \geq \alpha(b)$, $a > 0$. We have now established (33).

We next proceed with the ISS proof in order to establish (34). We have $0 \leq h_3(a, b) < a/2$ for all $a \geq \alpha(b)$, $a > 0$. In particular, $h_3(\alpha(b), b) < \alpha(b)/2$ and then $\lim_{b\to 0^+} h_3(\alpha(b), b) = 0$. From the definition of h_1, h_3 and ℓ, it follows that $h_3(a, b) \geq \kappa_2 \left(\int_a^{\ell(a,b)} \frac{\omega(s)}{\phi_4(s)} ds \right) \geq 0$. Define F via $F(r) := \int_0^r \frac{\omega(s)}{\phi_4(s)} ds$ for $r \geq 0$. Since ω is positive for $s > 0$, Borel measurable and nondecreasing, ϕ_4 is continuous and positive on $(0, \infty)$ and $\phi_5 \in \mathcal{K}_\infty$, then $F(0) = 0$, F is continuous and strictly increasing and $h_3(a, b) \geq \kappa_2(F(\ell(a, b)) - F(a)) \geq 0$. Since $\ell(a, b) = \phi_5^{-1}(\phi_5(a) + R(a)b)$, $\kappa_2 \in \mathcal{K}_\infty$, and $\lim_{b\to 0^+} h_3(\alpha(b), b) = 0$, it follows that $\lim_{b\to 0^+} F(\ell(\alpha(b), b)) = \lim_{b\to 0^+} F(\alpha(b)) = 0$. Then $\lim_{b\to 0^+} \ell(\alpha(b), b) = 0$ and, *a posteriori* $\lim_{b\to 0^+}[R(\alpha(b))b] = 0$. Next, we show that $\lim_{b\to 0^+}[\bar{R}(\alpha(b))b] = 0$. Since $\bar{R}(\alpha(\cdot))$ is nonincreasing, then $\lim_{b\to 0^+} \bar{R}(\alpha(b))$ exists (but may equal ∞). If $\lim_{b\to 0^+} \bar{R}(\alpha(b)) < \infty$, then $\lim_{b\to 0^+}[\bar{R}(\alpha(b))b] = 0$. If $\lim_{b\to 0^+} \bar{R}(\alpha(b)) = \infty$, consider a decreasing sequence $\{b_k\}$ of positive real numbers, satisfying $\lim_{k\to\infty} b_k = 0$. Due to the continuity of $R(\cdot)$ and hypothesis c), then $\bar{R}(a) < \infty$ for all $a > 0$, and for each b_k there exists a corresponding a_k satisfying $0 < b_k \leq a_k < \infty$ and $\bar{R}(\alpha(b_k)) = R(\alpha(a_k))$. The sequence $\{a_k\}$ necessarily satisfies $\lim_{k\to\infty} a_k = 0$. Therefore, $0 \leq \bar{R}(\alpha(b_k))b_k \leq R(\alpha(a_k))a_k$ and $\lim_{k\to\infty}[\bar{R}(\alpha(b_k))b_k] = 0$. This shows that $\lim_{b\to 0^+}[\bar{R}(\alpha(b))b] = 0$. The function $\bar{\ell}(b) := \phi_5^{-1}\big(\phi_5(\alpha(b)) + \bar{R}(\alpha(b))b\big)$ thus satisfies $\lim_{b\to 0^+} \bar{\ell}(b) = 0$ and is continuous for all $b > 0$. The existence of $\tilde{\alpha} \in \mathcal{K}_\infty$ satisfying (34) is thus ensured. The fact that $\alpha(b) \leq \tilde{\alpha}(b)$ for all $b \geq 0$ is a consequence of $\phi_5 \in \mathcal{K}_\infty$ and $\bar{R}(\alpha(b))b > 0$ for all $b > 0$.

Finally, we establish (35) for the pISS case. Since $\alpha, \phi_5 \in \mathcal{K}_\infty$, then $\tilde{\alpha}$ defined as $\tilde{\alpha}(b) = \phi_5^{-1}(\phi_5(\alpha(b)) + b)$ satisfies $\tilde{\alpha} \in \mathcal{K}_\infty$, (35) and $\alpha(b) \leq \tilde{\alpha}(b)$ for all $b \geq 0$. ∎

B. *Proof of Claim 2*

By the analysis performed at the beginning of the proof of Claim 1, we know that for $a > 0$ and $b \geq 0$, the set of values $c \in [0, \ell(a, b)]$ satisfying $c \leq h_1(a, b, c)$ is an interval of the form $[0, h_2(a, b)]$, with $h_2(a, b) > 0$. If $a \geq \alpha(b)$, by (33) and the fact that $a \leq \ell(a, b)$, then $h_2(a, b) < \ell(a, b)$. As a consequence, if $a \geq \alpha(b)$, then $h_2(a, b)$ is the unique value of $c \in [0, \ell(a, b)]$ satisfying $h_1(a, b, c) = c$. Consider the set $C = \{(a, b) \in \mathbb{R}^2_{\geq 0} : a \geq \alpha(b), b > 0\}$ and a sequence $\{(a_k, b_k)\}$ in C that converges to $(a, b) \in C$. We have $0 < h_2(a_k, b_k) < a_k$ for all k, and there exists a subsequence $\{(a_{k_l}, b_{k_l})\}$ so that $h_2(a_{k_l}, b_{k_l}) \to c$, where $0 \leq c \leq a \leq \ell(a, b)$. By the continuity of h_1, then $h_2(a_{k_l}, b_{k_l}) = h_1(a_{k_l}, b_{k_l}, h_2(a_{k_l}, b_{k_l})) \to c = h_1(a, b, c)$, and hence $c = h_2(a, b)$. This shows that $h_2(a_k, b_k) \to h_2(a, b)$ and hence h_2 is continuous in C.

Let $D = \{(a, b) : \alpha(b) \leq a \leq \tilde{r}, U_0 \leq b \leq U_1\}$. Define $m = m(\tilde{r}, U_0, U_1)$ via $m = \min_{(a,b)\in D}[a - h_2(a, b)]$. The function $a - h_2(a, b)$ is continuous and positive in the compact set $D \subset C$, and hence $m > 0$. Since $\sup_{k\geq 0} w_k \leq U$, then $y_k \geq \alpha(U)$ implies that $y_k \geq \alpha(w_k)$. By (33), then for each k for which $y_k \geq \alpha(U)$, we have $y_k - y_{k+1} \geq y_k - h_2(y_k, w_k) \geq y_k - h_2(y_k, U)$, where the last inequality follows because $h_2(a, \cdot)$ is nondecreasing. Let $l = \inf\{k \in \mathbb{N}_0 : y_k < \alpha(U)\}$. Since $y_0 \geq \alpha(U)$, then $l \geq 1$ and $y_k \geq \alpha(U)$ for $k = 0, 1, \ldots, l - 1$. Note that $(y_k, U) \in D$ and hence $y_k - y_{k+1} \geq m$ whenever $k = 0, 1, \ldots, l - 1$. This shows that $l < \infty$, and the result follows by taking K equal to the lowest integer not less that $\tilde{r}/m$. ∎

C. *Proof of Claim 3*

Suppose that V_k and V_{k+1} are defined. Then $V_k \geq \alpha(U) > 0$ and $t_{k+1} = t_k + R(V_k) < \infty$. Let $\Phi(t, s, \sigma)$ denote the state transition matrix of the linear time-varying system $\dot{x}(t) = A(t, \sigma(t))x(t)$, satisfying $\Phi(s, s, \sigma) = I$. From (3) and (14), we have

$$x_{k+1} = \Phi(t_{k+1}, t_k, \sigma)x_k + \int_{t_k}^{t_{k+1}} \Phi(t_{k+1}, s, \sigma)\big[g(s, x(s), \sigma(s)) + G(s, x(s), \sigma(s))u(s)\big] ds$$

Using (5) and the fact that for all $a, b \geq 0$ and $\phi \in \mathcal{K}$, then $\phi(a + b) \leq \phi(2a) + \phi(2b)$, it follows that $V_{k+1} = V(t_{k+1}, x_{k+1})$ satisfies, using (15),

$$V_{k+1} \leq \phi_2\big(2|\Phi(t_{k+1}, t_k, \sigma)x_k|\big) + \kappa_2 \left(\int_{t_k}^{t_{k+1}} \Big[\big|g(s, x(s), \sigma(s))\big| + \big|G(s, x(s), \sigma(s))u(s)\big| \Big] ds \right).$$

The first summand above satisfies

$$\phi_2\left(2\left|\Phi(t_{k+1}, t_k, \sigma)x_k\right|\right) \leq \phi_2(2\Psi e^{-\lambda(t_{k+1}-t_k)} \phi_1^{-1}(V_k))$$

where we employed (15). Given that $t_{k+1} - t_k = R(V_k)$,

$$e^{-\lambda(t_{k+1}-t_k)} = e^{-\lambda R(V_k)} = \frac{\phi_2^{-1}(V_k/2)}{2\Psi\phi_1^{-1}(V_k)},$$

where we have used (32). It thus follows that

$$\phi_2(2|\Phi(t_{k+1}, t_k, \sigma)x_k|) \leq V_k/2, \quad \forall \sigma \in \mathcal{S}.$$

By means of (22), we can write for every $h \in L^1([t_k, t_{k+1}])$

$$\int_{t_k}^{t_{k+1}} \phi_3(|h(s)|) ds \leq R(V_k)\gamma \left(\int_{t_k}^{t_{k+1}} \frac{|h(s)|}{R(V_k)} ds \right)$$

We next employ (16) and the above inequality to reach

$$\int_{t_k}^{t_{k+1}} |g(s, x(s), \sigma(s))| ds \leq R(V_k)\gamma\left(\int_{t_k}^{t_{k+1}} \frac{\eta_{\sigma(s)}(s, x(s))}{R(V_k)} ds \right)$$
$$\leq R(V_k)\gamma \left(\frac{V_k - V_{k+1}}{R(V_k)} + \int_{t_k}^{t_{k+1}} \frac{\phi_4(V(s, x(s)))|u(s)|}{R(V_k)} ds \right),$$

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where we have used (30) and (18).

Using (31) with t_0 replaced by t_k, it follows that

$$V(t, x(t)) \leq \phi_5^{-1}\left(\phi_5(V_k) + \int_{t_k}^{t} |u(s)|ds\right) \qquad (37)$$

for $t \in [t_k, t_{k+1}]$. Since ϕ_4 is nondecreasing, we may employ (37) to derive the following bound

$$\begin{aligned}\int_{t_k}^{t_{k+1}} &\phi_4(V(s, x(s)))|u(s)|ds \\ &\leq \int_{\phi_5(V_k)}^{\phi_5(V_k)+\int_{t_k}^{t_{k+1}}|u(s)|ds} \phi_4 \circ \phi_5^{-1}(\tau)d\tau \\ &= \phi_5^{-1}\left(\phi_5(V_k) + \int_{t_k}^{t_{k+1}} |u(s)|ds\right) - V_k,\end{aligned}$$

where the last equality follows because, according to Assumption 5, then $\int_0^s \phi_4 \circ \phi_5^{-1}(\tau)d\tau = \phi_5^{-1}(s)$. Following similar lines, using (17), (37), and since ω is nondecreasing, then

$$\begin{aligned}\int_{t_k}^{t_{k+1}} \left|G(s,x(s),\sigma(s))u(s)\right| ds &\leq \int_{t_k}^{t_{k+1}} \omega(V(s,x(s)))\, |u(s)|\, ds \\ &\leq \int_{\phi_5(V_k)}^{\phi_5(V_k)+\int_{t_k}^{t_{k+1}}|u(s)|ds} \omega \circ \phi_5^{-1}(\tau)d\tau \\ &= \int_{V_k}^{\phi_5^{-1}(\phi_5(V_k)+\int_{t_k}^{t_{k+1}}|u(s)|ds)} \frac{\omega(r)}{\phi_4(r)}dr,\end{aligned}$$

where the last equality follows via the change of variables $r = \phi_5^{-1}(\tau)$. Combining the bounds obtained into the inequality for V_{k+1}, and considering that $\int_{t_k}^{t_{k+1}} |u(s)|ds \leq R(V_k)u_k$ for ISS and that $\int_{t_k}^{t_{k+1}} |u(s)|ds = u_k$ for pISS, it follows that $V_{k+1} \leq h_1(V_k, u_k, V_{k+1})$. Evaluating (37) at $t = t_{k+1}$, then $V_{k+1} \leq \ell(V_k, u_k)$. From the definition of h_2, it follows that $V_{k+1} \leq h_2(V_k, u_k)$. The function $h_2(a, \cdot)$ is nondecreasing because $h_1(a, \cdot, c)$ is. Thus, $h_2(V_k, u_k) \leq h_2(V_k, U)$. Then, $h_2(V_k, U) < V_k$ follows from $V_k \geq \alpha(U)$ and (33). ∎

D. *Proof of Lemma 4.1*

The proof requires the following preliminary result.

Lemma A.2: Let X_M be the set of Lebesgue measurable functions $h : [0, 1] \to [0, M]$, where $M > 0$. Let $\rho : [0, M] \to [0, 1]$ be continuous, strictly increasing, and such that $\rho(0) = 0$ and $\rho(M) = 1$. Then, there exists $\gamma \in \mathcal{K}$ such that $\int_0^1 \rho(h(s))\, ds \leq \gamma\left(\int_0^1 h(s)\, ds\right)$ for all $h \in X_M$.

Proof: Let $\gamma^* : [0, M] \to [0, 1]$ be defined by

$$\gamma^*(l) = \sup\left\{\int_0^1 \rho(h(s))\, ds : \ h \in X_M \ \wedge \int_0^1 h(s)\, ds \leq l\right\}.$$

Note that γ^* is nondecreasing. Consider the strictly decreasing sequence $\{s_k\}_{k=0}^{\infty}$, $s_k = \rho^{-1}(2^{-k})$. Note that $s_0 = M$. Let $\varepsilon > 0$ and $k \in \mathbb{N}_0$ be such that $2^{-k+1} < \varepsilon$. Suppose that $l > 0$ satisfies $l < \delta = 2^{-k}s_k$. Let $h \in X_M$ be such that $\int_0^1 h(s)\, ds = l$ and let $A_1 = \{s \in [0, 1] : h(s) \leq s_k\}$ and $A_2 = \{s \in [0, 1] : h(s) > s_k\}$. The fact $|A_2|s_k \leq \int_{A_2} h(s)\, ds \leq l$, implies that $|A_2| \leq l/s^k < 2^{-k}$, where $|A_2|$ denotes the Lebesgue measure of A_2. Then, taking into account that ρ is strictly increasing, $\int_0^1 \rho(h(s))\, ds = \int_{A_1} \rho(h(s))\, ds + \int_{A_2} \rho(h(s))\, ds \leq \int_{A_1} \rho(s_k)\, ds + |A_2| \leq 2^{-k} + 2^{-k} = 2^{-k+1} < \varepsilon$. Therefore $\gamma^*(l) < \varepsilon \quad \forall l < \delta$ and $\lim_{l\to 0^+} \gamma^*(l) = 0$. Since γ^* is nondecreasing, $\gamma^*(0) = 0$, and $\lim_{l\to 0^+} \gamma^*(l) = 0$, then there exists $\gamma \in \mathcal{K}$ such that $\gamma^*(s) \leq \gamma(s)$ for all $s \geq 0$. Thus, for all $h \in X_M$, we have $\int_0^1 \rho(h(s))\, ds \leq \gamma^*\left(\int_0^1 h(s)ds\right) \leq \gamma\left(\int_0^1 h(s)ds\right)$. ∎

Proof of Lemma 4.1: i) $\Rightarrow$ ii). For every $r \geq 1$, consider the function $h_r \in L^1([0, 1])$ defined by $h_r(s) = r$ if $s \in [0, 1/r]$ and $h_r(s) = 0$ otherwise. We have $\int_0^1 \phi(|h_r(s)|)ds = \frac{\phi(r)}{r} \leq \gamma(1)$, for all $r \geq 1$. As a consequence, $\limsup_{r\to\infty} \frac{\phi(r)}{r} \leq \gamma(1) < \infty$.

ii) $\Rightarrow$ i). We can suppose, without loss of generality, that ϕ is, in addition, strictly increasing. If it is not, just replace ϕ by any class-$\mathcal{K}$ function which majorizes ϕ and satisfies item ii) of the lemma. Let $\bar{r} > 0$ and $K > 0$ satisfy $\phi(r) \leq Kr$ for all $r \geq \bar{r}$. For every $h \in L^1([0, 1])$, define $f(h) = \{s \in [0, 1] : |h(s)| < \bar{r}\}$, $g(h) = \{s \in [0, 1] : |h(s)| \geq \bar{r}\}$ and $\underline{h}(s) = h(s)$ if $s \in f(h)$ and $\underline{h}(s) = 0$ otherwise. Note that $[0, 1] = f(h) \cup g(h)$ and $\int_0^1 \phi(|h(s)|)ds = \int_{f(h)} \phi(|h(s)|)ds + \int_{g(h)} \phi(|h(s)|)ds = \int_0^1 \phi(|\underline{h}(s)|)ds + \int_{g(h)} \phi(|h(s)|)ds$. Define $\rho : [0, \bar{r}] \to [0, 1]$ via $\rho(s) = \phi(s)/\phi(\bar{r})$. By Lemma A.2, there exists $\tilde{\gamma} \in \mathcal{K}$ such that $\int_0^1 \rho(|\underline{h}(s)|)ds \leq \tilde{\gamma}\left(\int_0^1 |\underline{h}(s)|ds\right)$ and hence $\int_0^1 \phi(|\underline{h}(s)|)ds \leq \phi(\bar{r})\tilde{\gamma}\left(\int_0^1 |h(s)|ds\right)$. In addition, $\int_{g(h)} \phi(|h(s)|)ds \leq \int_{g(h)} K|h(s)|ds \leq K \int_0^1 |h(s)|ds$. Then, $\int_0^1 \phi(|h(s)|)ds \leq \phi(\bar{r})\tilde{\gamma}\left(\int_0^1 |h(s)|ds\right) + K \int_0^1 |h(s)|ds \leq \gamma\left(\int_0^1 |h(s)|ds\right)$, with $\gamma(s) := \phi(\bar{r})\tilde{\gamma}(s) + Ks$. ∎

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