GROUP-SYMMETRIC HOLOMORPHIC FUNCTIONS ON A BANACH SPACE

RICHARD ARON[1], PABLO GALINDO[2], DAMIÁN PINASCO [3], AND IGNACIO ZALDUENDO [3]

Abstract. We study the holomorphic functions on a complex Banach space E that are invariant under the action of a given group of operators on E. A great variety of situations occur depending, of course, on the group and the space. Nevertheless, in the examples we deal with they can be described in terms of a few natural ones and functions of a finite number of variables.

1. Introduction

Holomorphic functions that are invariant under the action of some group or semi-group of operators have been considered by several authors. In [14] functions invariant under the permutation of variables and their approximation by symmetric polynomials are studied on ℓ_p spaces. Some of those results are generalised in [9] to real separable rearrangement-invariant function spaces. In [1] the algebra of functions in the ball algebra $A(B_{\ell_p})$ which are invariant under permutation of variables is studied in detail, and its spectrum is described. Recently [5], [6], [7] the analogous situation for the space $\mathcal{H}_b(\ell_p)$ of holomorphic functions of bounded type has been studied, including a characterization of convolution operators on the algebra of symmetric functions through a symmetric convolution on the spectrum. A common feature of the symmetric homogeneous polynomials in all these examples is that they arise as compositions of finite variable polynomials with some natural examples of symmetric polynomials. Such a feature is also found in the new cases we treat in this paper: For instance, a polynomial P on $C([0,1])$ that is invariant under the action of homeomorphisms of $[0,1]$, i.e. $P(x) = P(x \circ \phi),\ x \in C([0,1])$ and ϕ a homeomorphism of $[0,1]$, turns out to be of the form $P(x) = q\big(x(0), x(1)\big)$, where q is a two-variable symmetric polynomial. Let us state that our polynomials are assumed to be continuous.

The requirement of "symmetry" reduces the number of existing functions: sometimes it is so restrictive that only the constant ones are "symmetric" while in others, the resulting space is infinite-dimensional or even it is algebraically generated by an infinite set.

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In section 2 we consider the following general situation. Let E be a Banach space, $U \subset E$ an open subset and $\mathcal{H}(U)$ an algebra of holomorphic functions defined on U (i.e., $\mathcal{H}(U)$ could be $\mathcal{H}_b(E)$, $\mathcal{H}(E)$, $A(B_E)$, $H^\infty(B_E)$, etc...). All the algebras we consider will be m-convex (see [11] or [17]). Now suppose G is a subgroup of $L(E)$ leaving U fixed. We consider the action of G on $\mathcal{H}(U)$ as follows

$$G \times \mathcal{H}(U) \longrightarrow \mathcal{H}(U)$$
$$(\gamma, h) \mapsto h \circ \gamma$$

and we consider the subalgebra of $\mathcal{H}(U)$ consisting of the G-symmetric functions, i.e.,

$$\mathcal{H}_G(U) = \{h \in \mathcal{H}(U) : h \circ \gamma = h \text{ for all } \gamma \in G\}.$$

We address several questions about $\mathcal{H}_G(U)$: the existence of a symmetrization operator $\mathcal{H}(U) \longrightarrow \mathcal{H}_G(U)$, the extendibility of multiplicative linear functionals from $\mathcal{H}_G(U)$ to $\mathcal{H}(U)$, and others regarding the spectra of $\mathcal{H}(U)$ and $\mathcal{H}_G(U)$.

In case G is a compact group (and a related situation, see Theorem 2.5) with the SOT topology, i.e., the topology of pointwise convergence on $L(E)$, we prove the existence of a "symmetrization" device acting on the space of analytic functions which shows that the subspace of G-symmetric holomorphic functions is complemented. Such a "symmetrization" is not multiplicative in general but, however, it is nice enough to guarantee that any character in $\mathcal{H}_G(U)$ can be extended to a character in $\mathcal{H}(U)$.

The remaining sections are devoted to the study of some examples. In section 3, we address the example of groups arising from homeomorphisms: given K a compact set and $\mathcal{S}$ a group of homeomorphisms of K, we consider the Banach space $E = C(K)$, and the group

$$G = \{\gamma : E \longrightarrow E : \gamma(x) = x \circ \phi \text{ for some } \phi \in \mathcal{S}\}.$$

In section 4 we deal with groups arising from operators on sequence spaces. Let $E = \ell^p$ or c_0, and G the group generated by the sequence of operators $\{\gamma_n\}_{n\in\mathbb{N}}$ of the form

$$(x_1, \ldots, x_n, \ldots) \xmapsto{\gamma_n} (x_1, \ldots, \omega_n x_n, \ldots),$$

where ω_n is an n-th root of unity.

Finally, in section 5 we study groups given by measure preserving maps. Take $K = [0, 1]$, $\mathcal{S} = \{\phi : K \longrightarrow K \text{ Lebesgue measure-preserving maps}\}$, and $E = C(K)''$ or $L_p(K)$, and $G = \{\gamma : E \longrightarrow E : \gamma(x) = x \circ \phi \text{ for some } \phi \in \mathcal{S}\}$. Here and in section 3 we use the *canonical extension* to the bidual of either a polynomial or a multilinear mapping as constructed in [4] (see also [8, 6.2] for a perhaps more easily available reference).

For these specific examples, we are able to describe the G-symmetric polynomials in terms of the most natural ones and polynomials of a finite number of complex variables.

2. Generalities

It is a general fact that the Taylor series of a G-symmetric holomorphic function at a G-fixed point a is built with G-symmetric polynomials:

Remark 2.1. If $f = \sum_k P_{k,a}$ is the Taylor series of $f \in \mathcal{H}_G(U)$ at a and $\gamma(a) = a$ for all $\gamma \in G$, then $P_{k,a} \in \mathcal{H}_G(U)$ for all k.

Proof. Since $\gamma(a) = a$, there is an open ball $B(a, \varepsilon)$ such that $\gamma(B(a, \varepsilon))$ lies in the ball of convergence of f at $a, B(a, r_f)$. Thus for all $x \in B(a, \varepsilon)$,

$$\sum_{k \geq 0} P_{k,a}(x - a) = f(x) = f(\gamma(x)) = \sum_{k \geq 0} P_{k,a}(\gamma(x) - a)$$
$$= \sum_{k \geq 0} P_{k,a}(\gamma(x) - \gamma(a)) = \sum_{k \geq 0} (P_{k,a} \circ \gamma)(x - a).$$

Thus by the uniqueness of the Taylor series expansion, $P_{k,a} = P_{k,a} \circ \gamma$ for all k. □

We note that the condition $\gamma(a) = a$ for all $\gamma \in G$ cannot be dispensed with. A simple example may be seen by putting $E = \mathbb{C}^2$, $G = \{id, S\}$ where $S(z, w) = (-z, w)$, $f(z, w) = z^2 + w^2$, and $a = (1, 0)$.

In what follows, we endow the group G with the strong operator topology (SOT). Recall [12] that a subset $B \subset U$ is called U*-bounded* if it is both bounded and bounded away from the complement of U.

Lemma 2.2. If G is compact and B is U-bounded, then $G(B)$ is U-bounded

Proof. Indeed, note that for any $x \in E$, $\|(\cdot)(x)\| : G \longrightarrow \mathbb{R}$ is continuous. Thus by compactness

$$\|\gamma(x)\| \leq C_x \text{ for all } \gamma \in G.$$

By the uniform boundedness principle, $\|\gamma\| \leq C$ for all $\gamma \in G$. Now $\|\gamma(x)\| \leq C\|x\| \leq C \sup\{\|x\| : x \in B\}$ for all $\gamma \in G$ and $x \in B$. If $G(B)$ were not U-bounded, given any $\varepsilon > 0$, we would find $a \in U^c$, $\gamma \in G$ and $x \in B$, with $\|\gamma(x) - a\| < \frac{\varepsilon}{C}$; applying γ^{-1}, we would have $\|x - \gamma^{-1}(a)\| < \varepsilon$. So we would be led to $d(B, U^c) = 0$, a contradiction. Note that we used that $G(U) \subset U$ (and hence $G(U^c) \subset U^c$). □

In the following two theorems we consider $\mathcal{H}(U)$ to be $H_b(U)$, and later we will comment on the cases $A(B_E)$ and $H^\infty(B_E)$, which are simpler.

Theorem 2.3. Let $G \subset L(E)$ be a compact group. The mapping $\sigma_G : \mathcal{H}(U) \longrightarrow \mathcal{H}_G(U)$ defined by

$$\sigma_G(h)(u) = \int_G (h \circ \gamma)(u) \, d\mu_G(\gamma), \; u \in U,$$

where μ_G is the Haar measure on G, is a continuous linear projection such that $\sigma_G(fh) = f\sigma_G(h)$ for $f \in \mathcal{H}_G(U)$. Moreover, if T is a closed subgroup of G, then $\sigma_G \circ \sigma_T = \sigma_G$, and, further, $\sigma_T \circ \sigma_G = \sigma_G$ whenever T is, in addition, a normal subgroup of G, i.e., $\gamma T = T\gamma \;\; \forall \gamma \in G$.

Proof. Since G carries the topology of pointwise convergence on $L(E)$, the integral is defined because the function $\gamma \in G \mapsto (h \circ \gamma)(u)$ is continuous on G. Clearly, $\sigma_G(h)$ is holomorphic as it is continuous and Gateaux-holomorphic.

Since G is SOT-compact, by Lemma 2.2 $G(B)$ is U-bounded if B is, and we have $\|\sigma_G(h)\|_B \leq \|h\|_{G(B)}$ from which we deduce the continuity of σ_G.

Note now that if $f \in \mathcal{H}_G(U)$,

$$\begin{aligned}
\sigma_G(fh) &= \int_G (fh)\circ\gamma \, d\mu_G(\gamma) \\
&= \int_G (f\circ\gamma)(h\circ\gamma)\, d\mu_G(\gamma) \\
&= \int_G f(h\circ\gamma)\, d\mu_G(\gamma) \\
&= f\int_G h\circ\gamma\, d\mu_G(\gamma) \\
&= f\sigma_G(h).
\end{aligned}$$

Next, observe that for the subgroup T,

$$\begin{aligned}
\sigma_G(\sigma_T(h)) &= \int_G\int_T h\circ t\, d\mu_T(t)\circ\gamma\, d\mu_G(\gamma) \\
&= \int_G\int_T h\circ t\circ\gamma\, d\mu_T(t)\, d\mu_G(\gamma) \\
&= \int_T\int_G h\circ(t\circ\gamma)\, d\mu_G(\gamma)\, d\mu_T(t) \qquad (*) \\
&= \int_T\int_G h\circ\gamma\, d\mu_G(\gamma)\, d\mu_T(t) \quad \text{by invariance of } \mu_G \\
&= \int_G h\circ\gamma\, d\mu_G(\gamma) = \sigma_G(h).
\end{aligned}$$

Finally, if T is normal, then

$$\begin{aligned}
\sigma_T(\sigma_G(h)) &= \int_T\int_G h\circ(\gamma\circ t)\, d\mu_G(\gamma)\, d\mu_T(t) \\
&= \int_T\int_G h\circ(t'\circ\gamma)\, d\mu_G(\gamma)\, d\mu_T(t),
\end{aligned}$$

and the proof follows as from $(*)$ above. $\square$

The continuity for the cases $\mathcal{H}(U) = A(B_E)$ and $\mathcal{H}(U) = H^\infty(B_E)$ is immediate from the definition of σ_G.

Remark 2.4. The symmetrization operator above is not necessarily multiplicative: Just recall the two variables case: $\sigma(f)(x,y) = \frac{f(x,y)+f(y,x)}{2}$. Then, $\sigma(\pi_1) = \sigma(\pi_2)$, however, $\sigma(\pi_1)\sigma(\pi_2) \neq \sigma(\pi_1\pi_2)$.

When T is normal in G, given $f \in \mathcal{H}_T(U)$ and $\overline{\gamma} \in G/T$ we may define $f\circ\overline{\gamma} = f\circ\gamma$. This is well-defined for if $\overline{\gamma_1} = \overline{\gamma_2}$, then $\gamma_1\gamma_2^{-1} = t \in T$ and $\gamma_1 = t\gamma_2$; thus $f\circ\gamma_1 = f\circ t\circ\gamma_2 = f\circ\gamma_2$.

Then σ_G may be written

$$\sigma_G(f) = \int_{G/T} f \circ \overline{\gamma}\, d\mu_{G/T}(\overline{\gamma}),$$

where $\mu_{G/T}$ is the Haar measure on the quotient group G/T.

Theorem 2.5. If for the —perhaps non compact— group G there is an ascending chain $\mathcal{C}$ of compact subgroups such that

i) $\bigcup_{S\in\mathcal{C}} S$ is dense in G, and
ii) $G(B)$ is U-bounded for each U-bounded subset B,

then there is a continuous symmetrization operator

$$\sigma_G : \mathcal{H}(U) \longrightarrow \mathcal{H}_G(U)$$

such that $\sigma_G(fh) = f\sigma_G(h)$ and $\sigma_G(f) = f$ for all $f \in \mathcal{H}_G(U)$ and $h \in \mathcal{H}(U)$.

Proof. We take a free ultrafilter $\mathcal{U}$ on the index set of compact subgroups of G. Then for any $h \in \mathcal{H}(U)$ and $u \in U$, define

$$\sigma_G(h)(u) = \lim_{\mathcal{U}} \sigma_S(h)(u) = \lim_{\mathcal{U},\, S\in\mathcal{C}} \sigma_S(h)(u).$$

Note that the limit exists because all $\sigma_S(h)(u)$ are contained in the bounded set $\{h(\gamma(u)) : \gamma \in G\}$. It is easily checked that $\sigma_G(h) \in \mathcal{H}(U)$ because the family $\{\sigma_S(h)\} \subset \mathcal{H}(U)$ as a τ_0-bounded family is equicontinuous and $\sigma_G(h)$ is Gateaux-holomorphic as a cluster point of a τ_0-bounded family. For any $\gamma \in \bigcup_{S\in\mathcal{C}} S$, the function $\sigma_G(h)$ is invariant under the action of γ, i.e., $\sigma_G(h)(\gamma(u)) = \sigma_G(h)(u)$ for every $u \in U$. Now the density assumption assures that also $\sigma_G(h)(\gamma(u)) = \sigma_G(h)(u)$ for every $\gamma \in G$ and $u \in U$.

For each U-bounded subset B of U, and $x \in B$,

$$|\sigma_G(h)(u)| = \lim_{\mathcal{U}} |\sigma_S(h)(u)| \le \lim_{\mathcal{U}} \|\sigma_S(h)\|_B \le \lim_{\mathcal{U}} \|h\|_{S(B)} \le \|h\|_{G(B)},$$

so $\|\sigma_G(h)\|_B \le \|h\|_{G(B)}$, and σ_G is continuous.

Finally, if $f \in \mathcal{H}_G(U)$, then $f \in \mathcal{H}_S(U)$ for all subgroups S of G. So for all $h \in \mathcal{H}(U)$ and $u \in U$,

$$\sigma_G(fh)(u) = \lim_{\mathcal{U}} \sigma_S(fh)(u) = \lim_{\mathcal{U}} f(u)\sigma_S(h)(u) = f(u) \lim_{\mathcal{U}} \sigma_S(h)(u) = f(u)\sigma_G(h)(u).$$

Therefore, $\sigma_G(fh) = f\sigma_G(h)$. □

The proof for the cases $\mathcal{H}(U) = A(B_E)$ and $\mathcal{H}(U) = H^\infty(B_E)$ is again immediate and in fact does not require condition ii) above.

We will call σ_G the *symmetrization* operator associated to G. The existence of such a projection implies that $\mathcal{H}_G(U)$ is closed in $\mathcal{H}(U)$ and that if $f \in \mathcal{H}_G(U)$ may be approximated by some $h \in \mathcal{H}(U)$, f can be approximated by the G-symmetric functions $\sigma_G(h)$ as happens in [14, Theorem 12]. Observe that whenever h is an m-homogeneous polynomial, $\sigma_G(h)$ is also an m-homogeneous polynomial since it is holomorphic and m-homogeneous.

Notice that the symmetrization operator given in [5, Example 2.20] is the case of the group G of permutations of $\mathbb{N}$ acting on $E = \ell_1$ and $\mathcal{C}$ the family of finite permutations.

In view of Lemma 2.2, one may wonder if condition i) of the Theorem implies condition ii). The following Remark will help clarify the role of condition ii).

Remark 2.6. Let E be ℓ_p, c_0, or ℓ_∞, and e_k the canonical unit vectors. We define continuous operators $s_n : E \longrightarrow E$ by

$$\begin{aligned} s_n(e_{2n-1}) &= n e_{2n}, \\ s_n(e_{2n}) &= \frac{1}{n} e_{2n-1}, \\ \text{and } s_n(e_k) &= e_k \text{ for all other } k. \end{aligned}$$

Note that $s_n^2 = id$ (the identity operator), and that all s_n's commute. Take S_m to be the group generated by $\{s_1, \ldots, s_m\}$, and define $G = \bigcup_m S_m$. Each S_m is finite (has 2^m elements) and is therefore compact. We have the following.

a) For any E, $A = \{e_k : k \in \mathbb{N}\}$ is bounded but $G(A)$ is unbounded. Thus i) does not imply ii).

b) For $E = c_0$, there is an element $x \in c_0$ such that $G(x)$ is unbounded: take

$$x = (1, 0, 2^{-\frac{1}{2}}, 0, 3^{-\frac{1}{2}}, 0, 4^{-\frac{1}{2}}, 0, \ldots) = \sum_{k=1}^{\infty} \frac{e_{2k-1}}{\sqrt{k}}.$$

Then $s_n(x) = \sum_{k \neq 2n} x_k e_k + \sqrt{n}\, e_{2n}$.

c) For $E = \ell_1$, G is not locally compact (and thus, not amenable): a neighborhood basis of $id \in G$ in the SOT topology is given by sets of the form

$$V = \{\gamma \in G : \|\gamma(a) - a\|_1 < \varepsilon, \text{ (finitely many } a\text{'s)}\}$$

and since the subspace spanned by non-increasing sequences $a = (a_1, a_2, \ldots)$, $(a_k \geq a_{k+1})$ is dense in ℓ_1, we may suppose a is non-increasing. Now

$$\begin{aligned} s_n(a) - a &= (0, 0, \ldots, \frac{a_{2n}}{n} - a_{2n-1}, n a_{2n-1} - a_{2n}, 0, 0, \ldots), \text{ so} \\ \|s_n(a) - a\|_1 &= |\frac{a_{2n}}{n} - a_{2n-1}| + |n a_{2n-1} - a_{2n}| \\ &\leq \frac{|a_{2n}|}{n} + |a_{2n-1}| + n|a_{2n-1}| + |a_{2n}| \\ &= \frac{|a_{2n}|}{n} + |a_{2n-1}| + \frac{1}{2}(2n-1)|a_{2n-1}| + \frac{|a_{2n-1}|}{2} + |a_{2n}| < \varepsilon, \end{aligned}$$

for large enough n. Note that the third term in the last line tends to zero by Pringsheim's theorem [10]. Thus for any given V, the operators s_n are in V for large n. Hence, for the bounded set $A = \{e_k : k \in \mathbb{N}\}$, VA is unbounded. If G were locally compact we would have $VA \subset KA$ for some compact K. This cannot happen —as in Lemma 2.2— by the uniform boundedness principle. Thus G is not locally compact.

We now study the spectrum of $\mathcal{H}_G(U)$, (i.e., the set of continuous scalar-valued homomorphisms) which we will denote $\mathcal{M}(\mathcal{H}_G)$. We define

$$\rho : \mathcal{M}(\mathcal{H}(U)) \longrightarrow \mathcal{M}(\mathcal{H}_G(U)) \text{ where } \rho(\varphi) = \varphi|_{\mathcal{H}_G(U)},$$

and

$$(\gamma, \varphi) \in G \times \mathcal{M}(\mathcal{H}(U)) \longrightarrow \varphi_\gamma \in \mathcal{M}(\mathcal{H}(U)) \text{ where } \varphi_\gamma(h) = \varphi(h \circ \gamma).$$

Corollary 2.7. With $\mathcal{H}(U)$ and G as in Theorem 2.5, the restriction mapping

$$\rho : \mathcal{M}(\mathcal{H}(U)) \longrightarrow \mathcal{M}(\mathcal{H}_G(U))$$

is surjective.

Proof. We need to show that every character $\alpha : \mathcal{H}_G(U) \longrightarrow \mathbb{C}$ extends to a character $\varphi : \mathcal{H}(U) \longrightarrow \mathbb{C}$. Let $I_\alpha = Ker\alpha \subset \mathcal{H}_G(U)$, and let I be the ideal of $\mathcal{H}(U)$ generated by I_α. We show that I is a proper ideal: if not, we would have $f_1, \ldots, f_n \in I_\alpha$ and $h_1, \ldots, h_n \in \mathcal{H}(U)$ such that

$$1 = f_1 h_1 + \cdots + f_n h_n.$$

Apply $\alpha \circ \sigma_G$ and we have

$$\begin{aligned} 1 &= \alpha(\sigma_G(1)) \\ &= \alpha(\sigma_G(f_1 h_1 + \cdots + f_n h_n)) \\ &= \alpha(f_1 \sigma_G(h_1) + \cdots + f_n \sigma_G(h_n)) \\ &= \alpha(f_1)\alpha(\sigma_G(h_1)) + \cdots + \alpha(f_n)\alpha(\sigma_G(h_n)) \\ &= 0. \end{aligned}$$

The properties of σ_G show that $\sigma_G(I) \subset I_\alpha$, hence I is contained in the hyperplane $Ker(\alpha \circ \sigma_G)$ which is closed since both α and σ_G are continuous. Thus $\overline{I}$ is a closed proper ideal. For any closed proper ideal the quotient algebra is also an m-convex algebra, so $\overline{I}$ is contained in a closed maximal ideal. Now the Gelfand-Mazur theorem for m-convex algebras [2] implies that this closed maximal ideal is the kernel of a character $\varphi \in \mathcal{M}(\mathcal{H}(U))$. □

Some comments are in order:

Remark 2.8. Consider $\mathcal{H}(U) = \mathcal{H}_b(E)$. This is a barrelled m-convex algebra. Thus even though all its closed maximal ideals have codimension one, since some of its elements have unbounded spectra, by [18] $\mathcal{H}_b(E)$ contains non-closed maximal ideals of codimension larger than one.

Note also that in general (i.e., without the existence of σ_G) $\mathcal{H}_G(U)$ is an inverse-closed subalgebra of $\mathcal{H}(U)$: if $h \in \mathcal{H}_G(U)$ and h is invertible in $\mathcal{H}(U)$, then it is invertible in

$\mathcal{H}_G(U)$. Indeed, for all $\gamma \in G$,

$$\begin{aligned} 1 = 1 \circ \gamma &= (hh^{-1}) \circ \gamma \\ &= (h \circ \gamma)(h^{-1} \circ \gamma) \\ &= h(h^{-1} \circ \gamma), \end{aligned}$$

thus $h^{-1} \circ \gamma = h^{-1}$ by the uniqueness of inverses, and $h^{-1} \in \mathcal{H}_G(U)$.

However, this in itself does not imply the extendibility of multiplicative linear functionals [15].

We end this section with a few remarks regarding the structure of the spectrum of $\mathcal{H}_G(U)$.

Remark 2.9. There may, of course, be many characters extending α. If φ extends α, then the orbit $\mathcal{O}_\varphi = \{\varphi_\gamma : \gamma \in G\}$ is contained in the fiber $\rho^{-1}(\alpha)$ over α: for every $\gamma \in G$ and $f \in \mathcal{H}_G(U)$,

$$\varphi_\gamma(f) = \varphi(f \circ \gamma) = \varphi(f) = \alpha(f).$$

The orbit $\mathcal{O}_\varphi$ is in general, smaller than the fiber $\rho^{-1}(\rho(\varphi))$. An example where the orbit is small: take φ to be evaluation at zero. Then $\mathcal{O}_\varphi = \{\varphi\}$.

Another example where an orbit differs from the fiber is the following: Let T be a non-surjective hypercyclic operator on E, and G the group generated by T. Then the only G-invariant holomorphic functions are the constants, and $\mathcal{M}(\mathcal{H}_G(E))$ is a single point. Let φ be evaluation at x, and let $y \in E$ be a point not in the image of T. Then evaluation at y is not in $\mathcal{O}_\varphi$.

The fiber $\rho^{-1}(\rho(\psi))$ is a disjoint union of orbits. Indeed, if $\rho(\varphi) = \rho(\psi)$ we have already seen that $\mathcal{O}_\varphi \subset \rho^{-1}(\rho(\psi))$. But all orbits are disjoint: if $\psi_a = \varphi_b$, then $\psi = \varphi_{a^{-1}b}$.

When G is a compact group, all orbits in any given fiber have the same barycenter (which is not in general a character). Indeed,

$$\begin{aligned} \varphi(\sigma_G(h)) &= \varphi\left(\int_G h \circ \gamma \, d\mu_G(\gamma)\right) \\ &= \int_G \varphi(h \circ \gamma) \, d\mu_G(\gamma) \\ &= \int_G \varphi_\gamma(h) \, d\mu_G(\gamma) \\ &= \left(\int_G \varphi_\gamma \, d\mu_G(\gamma)\right)(h), \end{aligned}$$

thus $\varphi \circ \sigma_G$ is the barycenter of the orbit $\mathcal{O}_\varphi$.

3. Groups arising from homeomorphisms

In this section we study the case of $E = C(K)$, for a compact space K and the group $G \subset L(E)$ of composition operators on $C(K)$ arising from all homeomorphisms ϕ of K, i.e., $G = \{\gamma : E \longrightarrow E : \gamma(x) = x \circ \phi$ for some homeomorphism $\phi\}$.

Theorem 3.1. Suppose $K = [0,1]$. A holomorphic function $f : E \to \mathbb{C}$ is G-symmetric if and only if there is an analytic symmetric function $\mathcal{F} \in H(\mathbb{C}^2)$ such that $f(x) = \mathcal{F}(x(0), x(1))$.

Proof. Let $\phi_n(t) = t^n$ and let $x \in E$. The sequence $(x \circ \phi_n)_n$ is a bounded pointwise convergent one to the function $\vartheta_x := x(0)\chi_{[0,1[} + x(1)\chi_{\{1\}}$ that can be seen as an element in E''. Moreover, by the Lebesgue Dominated Convergence Theorem the sequence $(x\circ\phi_n)_n$ is a $w(E, E')$-Cauchy sequence that is also $w(E'', E')$-convergent to ϑ_x.

Recall that the bidual space of E is also a $C(K)$ type space, thus it also has the Dunford-Pettis property. Therefore for any multilinear form A on E its canonical extension [4] $\tilde{A}$ to E'' is weakly sequentially continuous, and therefore, maps weakly Cauchy sequences into convergent ones. So for any G-symmetric polynomial $P : E \to \mathbb{C}$, we have

$$\tilde{P}(x(0)\chi_{[0,1[} + x(1)\chi_{\{1\}}))) = \tilde{P}(\vartheta_x) = \lim_n P(x \circ \phi_n) = P(x).$$

Notice that for the homeomorphism $\kappa(t) = 1 - t$,

$$P(x) = P(x \circ \kappa) = \tilde{P}(x(1)\chi_{[0,1[} + x(0)\chi_{\{1\}}),$$

that shows that $P(x)$ is a symmetric function of the variables $\{x(0), x(1)\}$.

Next observe that if we denote by ι the linear mapping

$$(\alpha, \beta) \in \mathbb{C}^2 \to \alpha\chi_{[0,1[} + \beta\chi_{\{1\}} \in E'',$$

it turns out that $\|\iota\| \leq 2$ and that $\tilde{P} \circ \iota$ is a symmetric homogeneous polynomial for which

$$P(x) = \left(\tilde{P} \circ \iota\right)(x(0), x(1)).$$

Finally, any entire function f can be written as its Taylor series $f(x) = \sum P_m(x)\ \forall x \in E$. Thus $f(x) = \sum \left(\tilde{P}_m \circ \iota\right)(x(0), x(1)) = f(x(0)\kappa + x(1)(1 - \kappa))$. This yields that f factors through $\mathbb{C}^2$: Indeed, if $\mathcal{F}(u, v) := f(u(1-\kappa) + v\kappa)$, $\mathcal{F}$ is an entire function which is symmetric because for the homeomorphism κ, one has $(u(1-\kappa)+v\kappa)\circ\kappa = v(1-\kappa)+u\kappa$, hence $\mathcal{F}(v, u) = f(v(1 - \kappa) + u\kappa) = f(u(1 - \kappa) + v\kappa) = \mathcal{F}(u, v)$. Therefore,

$$f(x) = f\left(x(0)\kappa + x(1)(1 - \kappa)\right) = \mathcal{F}(x(0), x(1)) \qquad \square$$

Let's denote by $\mathcal{A}$ the algebra of G-symmetric analytic functions (necessarily of bounded type because of Theorem 3.1) on $E = C([0,1])$ endowed with the topology of uniform convergence on bounded subsets of E.

Proposition 3.2. The spectrum of $\mathcal{A}$ identifies with the quotient set $\mathbb{C}^2/\sim$, where $(a, b) \sim (c, d)$ if $\{a, b\} = \{c, d\}$.

Proof. Given $(a,b) \in \mathbb{C}^2/\sim$, we define the homomorphism $\varphi_{(a,b)}$ according to $\varphi_{(a,b)}(f) := \mathcal{F}(a,b)$ where $f(x) = \mathcal{F}(x(0), x(1))$. It is well defined, since if $f(x) = \mathcal{G}(x(0), x(1))$, for some symmetric $\mathcal{G} \in H(\mathbb{C}^2)$, then $\mathcal{F}(x(0), x(1)) = \mathcal{G}(x(0), x(1))$, for any $x \in E$. And in particular for some affine function x_0 with $x_0(0) = a$ and $x_0(1) = b$.

Clearly $\varphi_{(a,b)}$ is linear and multiplicative. It is indeed continuous because if $\lambda \geq \max\{|a|, |b|\}$, then $|\varphi_{(a,b)}(f)| = |\mathcal{F}(x_0(0), x_0(1))| = |f(x_0)| \leq \|f\|_{\lambda B_E}$.

Let φ belong to the spectrum of $\mathcal{A}$. Consider the multiplicative linear one-to-one and continuous map

$$\mathcal{F} \in H_s(\mathbb{C}^2) \overset{\Lambda}{\rightsquigarrow} (\mathcal{F}(x(0), x(1))) \in \mathcal{A}.$$

Then $\varphi \circ \Lambda$ is a continuous homomorphism of $H_s(\mathbb{C}^2)$, so there is a point $(a,b) \in \mathbb{C}^2$ such that $(\varphi \circ \Lambda)(\mathcal{F}) = \mathcal{F}(a,b)$. Then for any $f \in \mathcal{A}$, $f(x) = \mathcal{F}(x(0), x(1))$, we have $\varphi(f) = (\varphi \circ \Lambda)(\mathcal{F}) = \mathcal{F}(a,b)$. $\square$

Corollary 3.3. Every continuous endomorphism T of $\mathcal{A}$ is a composition operator, that is, there is a continuous $\Phi : E \to E$ such that $Tf(x) = (f \circ \Phi)(x)$.

Proof. To begin with, realize that every continuous endomorphism $\mathcal{T} : H_s(\mathbb{C}^2) \to H_s(\mathbb{C}^2)$ arises from an analytic mapping $\Upsilon = (\upsilon_1, \upsilon_2) : \mathbb{C}^2 \to \mathbb{C}^2$ such that $\mathcal{T}(\mathcal{G}) = \mathcal{G} \circ \Upsilon$ and $\Upsilon(z,w) \sim \Upsilon(w,z)$ are a permutation of each other. Then either

$$\upsilon_1(z,w) = \upsilon_1(w,z) \text{ and } \upsilon_2(z,w) = \upsilon_2(w,z)\ \forall (z,w) \text{ or}$$

$$\upsilon_1(z,w) = \upsilon_2(w,z) \text{ and } \upsilon_2(z,w) = \upsilon_1(w,z)\ \forall (z,w).$$

That is, either υ_1 and υ_2 are symmetric or $\upsilon_1(z,w) = \upsilon_2(w,z)\ \forall (z,w)$.

Now if $T : \mathcal{A} \to \mathcal{A}$ is a continuous homomorphism, then $\mathcal{T} := \Lambda^{-1} \circ T \circ \Lambda$ is a continuous homomorphism of $H_s(\mathbb{C}^2)$, so there is Υ such that $\Lambda^{-1} \circ T \circ \Lambda(\mathcal{G}) = \mathcal{G} \circ \Upsilon$. Thus, $(T \circ \Lambda)(\mathcal{G})(x) = \Lambda(\mathcal{G} \circ \Upsilon)(x)$. Hence

$$T(f)(x) = T\left(\Lambda(\mathcal{F})\right)(x) = \Lambda(\mathcal{F} \circ \Upsilon)(x) = \mathcal{F}\left(\upsilon_1(x(0), x(1)), \upsilon_2(x(0), x(1))\right).$$

In case both υ_1 and υ_2 are symmetric we consider the self-map of E, $x \mapsto \Phi(x) := \kappa\upsilon_1 \circ (x, x \circ \kappa) + (1-\kappa)\upsilon_2 \circ (x, x \circ \kappa)$. Then we have

$$f(\Phi(x)) = \mathcal{F}(\Phi(x)(0), \Phi(x)(1)) = \mathcal{F}\left(\upsilon_1(x(0), x(1)), \upsilon_2(x(1), x(0)\right) = Tf(x).$$

While in the other case, we consider the self-map of E, $x \mapsto \Phi(x) := \upsilon_1 \circ (x, x \circ \kappa)$, we have that

$$f(\Phi(x)) = \mathcal{F}(\Phi(x)(0), \Phi(x)(1)) = \mathcal{F}(\upsilon_1(x(0), x(1)), \upsilon_1(x(1), x(0)) = Tf(x).$$

In each case, Φ is continuous since υ_1 and υ_2 are uniformly continuous on bounded subsets of $\mathbb{C}^2$. $\square$

The group G is not compact for the SOT topology: Suppose that the sequence $T_n(x) = x \circ \phi_n$ given by composition with $\phi_n(t) = t^n$ has a cluster point $\Gamma \in G$. Then for the identity function ι and any $t \in [0,1]$, we have $|t^n - \Gamma(\iota)(t)| \leq \|T_n(\iota) - \Gamma(\iota)\| \to 0$ as $n \to \infty$. Thus $\Gamma(\iota)(t) = 0$ for $0 < t < 1$ and consequently for all $t \in [0,1]$, something that does not happen for $t = 1$. Nevertheless, we still have

Proposition 3.4. There is a continuous non-multiplicative projection from $H_b(E)$ onto $\mathcal{A}$.

Proof. The mapping $g \in H_b(E) \overset{\Theta}{\mapsto} \mathcal{G}(z,w) := \tilde{g}(z\chi_{[0,1[} + w\chi_{\{1\}}) \in H(\mathbb{C}^2)$ is well defined and in case g is G-symmetric, $\mathcal{G}$ is symmetric since such holds for the G-symmetric homogeneous polynomials. Let's denote $\mathfrak{S}$ the symmetrization operator $H(\mathbb{C}^2) \to H_s(\mathbb{C}^2)$ defined by $\mathfrak{S}(\mathcal{F})(z,w) = \frac{1}{2}(\mathcal{F}(z,w)+\mathcal{F}(w,z))$. Then $\Lambda\circ\mathfrak{S}\circ\Theta$ is the desired projection; its continuity is straightforward bearing in mind that the canonical extension is a continuous operator. To see that it is not multiplicative, consider in $H_b(E)$ the elements δ_0 and δ_1. As linear functionals, they coincide with their canonical extension which in turn is multiplicative [16], so $\widetilde{\delta_0 \cdot \delta_1} = \delta_0 \cdot \delta_1$. Thus, $\Theta(\delta_0)(z,w) = z$, $\Theta(\delta_1)(z,w) = w$ and $\Theta(\delta_0 \cdot \delta_1)(z,w) = zw$. And $\Lambda \circ \mathfrak{S} \circ \Theta(\delta_0)(x) = \Lambda\big(\frac{1}{2}(z+w)\big)(x) = \frac{1}{2}(x(0)+x(1))$, and also $\Lambda \circ \mathfrak{S} \circ \Theta(\delta_1)(x) = \Lambda\big(\frac{1}{2}(z+w)\big)(x) = \frac{1}{2}(x(0)+x(1))$, whereas $\Lambda \circ \mathfrak{S} \circ \Theta(\delta_0 \cdot \delta_1)(x) = \Lambda\big((zw)\big)(x) = (x(0),x(1))$. ☐

Remark 3.5. If we replace the group G by the group of linear isometries of $C(K), K = [0,1]$, then it turns out that any $f \in \mathcal{A}$ must be constant.

Proof. According to the Banach-Stone theorem, now f has to be also invariant under all multiplication operators $x \in C(K) \mapsto xh \in C(K)$ given by all $h \in C(K)$ such that $|h(t)| = 1\ \forall t \in K$. That is, $f(xh) = f(x)$, which according to Theorem 3.1 yields $\mathcal{F}\big(x(0)h(0), x(1)h(1)\big) = \mathcal{F}\big(x(0),x(1)\big)$. Therefore since any element in $\mathbb{C}^2$ can be described as $(x(0),x(1))$ for some $x \in C(K)$, and for every couple (α,β) in the torus, there is $h \in C(K)$ with $|h(t)| = 1$ for all $t \in K$ such that $h(0) = \alpha$, $h(1) = \beta$, we have that $\mathcal{F}(\alpha z_1, \beta z_2) = \mathcal{F}(z_1,z_2)$, $\forall z_1,z_2 \in \mathbb{C}$. Using the identity principle, the former identity holds for every $(\alpha,\beta) \in \mathbb{C}^2$, and thus $\mathcal{F}(\alpha,\beta) = \mathcal{F}(1,1)$. ☐

Example 3.6. For $K := \mathbb{N} \cup \{\infty\}$, the one point compactification of $\mathbb{N}$, and the space $E = C(\mathbb{N} \cup \{\infty\}) = c$, the space of convergent sequences, the G-symmetric analytic functions $f : E \to \mathbb{C}$ are of the form $\mathcal{F}(\lim x_n)$ where $\mathcal{F} \in H(\mathbb{C})$ and $(x_n) \in c$.

Proof. In this case the homeomorphisms ϕ of K are the permutations of $\mathbb{N}$, since any homeomorphism maps isolated points into isolated points, that is $\phi(\mathbb{N}) = \mathbb{N}$ and $\phi(\infty) = \infty$, and, indeed, every permutation s of $\mathbb{N}$ leads to a homeomorphism ϕ since s is continuously extended to K by mapping ∞ into itself.

Next, we observe that the linear form $(x_n) \in c \to \lim_n x_n \in \mathbb{C}$ is symmetric. And to complete the proof, consider the sequence of homeomorphisms (ϕ_k) defined by $\phi(k+i) = k+i$ and reversing the order in the interval $[1,\ k]$. Then for every $n \in \mathbb{N}$, we have that $\phi_{n+i}(n) = i+1$. Thus $\phi_k(n) = k-n+1$, and so for every $n \in \mathbb{N}$, the sequence $(x \circ \phi_k(n))_k$ converges to $x(\infty) = \lim_n x_n$. Thus $(x \circ \phi_k)_k$ is weakly convergent to the constant sequence $\lim_n x_n$. Finally use the same arguments as in the proof of Theorem 3.1. ☐

Example 3.7. If $K = S^1$, the unit sphere in $\mathbb{C}$, then any G-symmetric analytic function $f : E \to \mathbb{C}$ is a constant one.

Proof. Recall that the mappings $T_a(z) = \frac{z-a}{1-\bar{a}z}$ appear among the homeomorphisms of S^1. And also that if $|\lambda| = 1$ $\lim_{t\to -1} T_{t\lambda}(z) = \lambda$ for all $z \in S^1$. Therefore for any $x \in E$, $\lim_{t\to -1} x \circ T_{t\lambda}(z) = x(\lambda)$ for all $z \in S^1$. Thus the sequence $(x\circ T_{t\lambda})_{t\to -1}$ converges weakly to the constant function $x(\lambda)\mathbf{1}$. So for any G-symmetric polynomial $P : E \to \mathbb{C}$, that is necessarily weakly sequentially continuous, we have $P(x(\lambda)\mathbf{1}) = \lim_{t\to -1} P(x \circ T_{t\lambda}) = P(x)$. Hence $P(x) = P(x(1)\mathbf{1}) = P(x(\lambda)\mathbf{1})$.

Now, consider the linear mapping $\iota : \mathbb{C} \to E$ given by $\iota(\lambda) := \lambda\mathbf{1}$. If $\mathcal{F} := P \circ \iota$, it is an homogeneous polynomial and $P(x) = \mathcal{F}(x(\lambda))$ for any $\lambda \in S^1$. We check that $\mathcal{F}(\lambda) = 0$. Indeed, pick $x \in E$ such that $x(1) = \lambda$ and $x(-1) = 0$. Then

$$\mathcal{F}(\lambda) = P(\lambda\mathbf{1}) = P(x(1)\mathbf{1}) = P(x) = P(x(-1)\mathbf{1}) = P(0) = 0.$$

Finally, apply Remark 2.1 to see that f must be constant. □

Remark 3.8. Assume that $K = [0, 1]^2 \subset \mathbb{C}$. Any analytic G-symmetric function $f : E \to \mathbb{C}$ is constant.

Proof. Recall that K is homeomorphic to the closed unit disc of $\mathbb{C}$ because of the Riemann mapping theorem (see [13], p. 179). So we can replace K by $\overline{\Delta}$. Now, the arguments of Remark 3.7 lead to the result. □

Example 3.9. Assume that K is the T-shaped space, that is $K = [-1, 1] \cup [0, i] \subset \mathbb{C}$. An analytic function $f : E \to \mathbb{C}$ is G-symmetric if and only if there is an analytic function $\mathcal{F} \in H(\mathbb{C}^4)$ symmetric with respect to the last three variables such that $f(x) = \mathcal{F}(x(0), x(1), x(-1), x(i))$.

Proof. Let ϕ_n, n odd, be the homeomorphism of K given by $\phi_n(t) = t^n$ if $t \in [-1, 1]$ and $\phi_n(it) = it^n$ if $it \in [0, i]$. Argue as in Proposition 3.1. The sequence $(x \circ \phi_n)_n$ is a bounded pointwise convergent one to the function $\vartheta_x := x(0)\chi_{]-1,1[\cup]0,i[} + x(1)\chi_{\{1\}} + x(-1)\chi_{\{-1\}} + x(i)\chi_{\{i\}}$ that can be seen as an element in E''. Thus for homogeneous polynomials P,

$$P(x) = \tilde{P}(x(0)\chi_{]-1,1[\cup]0,i[} + x(1)\chi_{\{1\}} + x(-1)\chi_{\{-1\}} + x(i)\chi_{\{i\}}).$$

Since suitable rotations would permute the points $\{-1, 1, i\}$, P is symmetric with respect to $x(-1), x(1), x(i)$.

This leads to the "if" part of the statement. While for the converse, it is enough to realize that any homeomorphism of K must have the set $\{-1, 1, i\}$ invariant and 0 as a fixed point by a connectedness argument. □

4. Groups arising from operators acting on sequence spaces

Most of the existing research on symmetric analytic functions has been done on the classical sequence spaces ℓ_p, $1 \le p < \infty$. See [1], [5], [6], [7], [9], and [14]. It was shown in [9] that there are no non-null polynomials on c_0 that are symmetric for the group G of permutations of $\mathbb{N}$. Above in Example 3.6 we pointed out the case of c.

In this section we deal with symmetric holomorphic functions given by the invariance under the action of the group of operators on $E = c_0$ generated by $\{\gamma_m\}_{m\in\mathbb{N}}$ where

$\gamma_m : c_0 \to c_0$ is the linear operator defined by $\gamma_m(e_j) = e^{i2\pi\delta_{mj}/m} e_j$ $(j \geq 1)$. We will denote by $\Pi_k : c_0 \to \mathbb{C}^k$ the projection defined by $\Pi_k\left((z_j)_{j=1}^{\infty}\right) = (z_1, \ldots, z_k)$, and by $\iota_k : \mathbb{C}^k \to c_0$ the inclusion $\iota_k(z_1, \ldots, z_k) = \sum_{j=1}^k z_j e_j$. We begin by proving the following assertion.

Lemma 4.1. Let $P_k : c_0 \to \mathbb{C}$ be a k-homogeneous polynomial such that $P_k = P_k \circ \gamma_m$ for all $m \geq 1$, then $P_k = P_k \circ \iota_k \circ \Pi_k$.

Proof. Given $(z_j)_{j \geq 1} \in c_0$ and $k < m$, let us consider the one variable polynomial

$$\mathbb{C} \xrightarrow{J_m} c_0 \xrightarrow{P_k} \mathbb{C}$$

$$\theta \mapsto (z_j)_{j\geq 1} + (\theta - 1) z_m e_m \mapsto P_k((z_j)_{j\geq 1} + (\theta - 1) z_m e_m).$$

This polynomial has degree at most k. Since $J_m(e^{i2\pi n/m}\theta) = (\gamma_m \circ \overset{n}{\cdots} \circ \gamma_m)\big(J_m(\theta)\big)$ the symmetry of P_k yields that $P_k \circ J_m(\theta) = P_k \circ J_m(e^{i2\pi n/m}\theta)$ for $n = 1, 2, \ldots, m$. Since $k < m$, then $P_k \circ J_m$ must be constant. In particular, if we define $w = (w_j)_{j\geq 1}$ such that $w_j = z_j$ if $j \neq m$ and $w_m = 0$, then we have that $P_k(w) = P_k(z)$. In other words, for $n \geq k$,

$$P_k \circ \iota_n \circ \Pi_n(z) = P_k \circ \iota_k \circ \Pi_k(z).$$

Now, since $\iota_n \circ \Pi_n(z) \underset{n\to\infty}{\longrightarrow} z$, and P_k is continuous, we conclude that

$$P_k(z) = P_k \circ \iota_k \circ \Pi_k(z)$$

□

By abuse of notation, for $1 \leq m \leq k$, we continue to write γ_m for the operators defined from $\mathbb{C}^k$ to $\mathbb{C}^k$, defined by $\gamma_m(z_1, \ldots, z_m) = (z_1, \ldots, e^{i2\pi/m} z_m, \ldots, z_m)$.

Lemma 4.2. Let $\widetilde{Q}_k : \mathbb{C}^k \to \mathbb{C}$ be a k-homogeneous polynomial such that

$$\widetilde{Q}_k(z_1, z_2, \ldots, z_k) = \widetilde{Q}_k \circ \gamma_m(z_1, z_2, \ldots, z_k)$$

for all $1 \leq m \leq k$. Then there exists $Q_k : \mathbb{C}^k \to \mathbb{C}$ such that

$$\widetilde{Q}_k(z_1, z_2, \ldots, z_k) = Q_k(z_1, z_2^2, \ldots, z_k^k).$$

Proof. Let us denote by $\Lambda_k = \{(\alpha_m)_{m=1}^k \subset \mathbb{N}_0 : \alpha_1 + \cdots \alpha_k = k\}$. Since $\{z^\alpha\}_{|\alpha|=k} = \{z_1^{\alpha_1} \cdots z_k^{\alpha_k} : \alpha_1 + \cdots \alpha_k = k\}$ is a basis for the space of k-homogeneous polynomials over $\mathbb{C}^k$, we can write

$$\widetilde{Q}_k(z_1, \ldots, z_k) = \sum_{|\alpha|=k} a_\alpha z^\alpha.$$

From the symmetry of $\widetilde{Q}_k$, given $1 \leq m \leq k$, $\widetilde{Q}_k(z) = \widetilde{Q}_k \circ \gamma_m(z)$. Then we have

$$\sum_{|\alpha|=k} a_\alpha z^\alpha = \widetilde{Q}_k(z) = \widetilde{Q}_k \circ \gamma_m(z) = \sum_{|\alpha|=k} a_\alpha e^{i2\pi\alpha_m/m} z^\alpha.$$

We deduce that $a_\alpha = 0$ whenever m is not a divisor of α_m. Let us consider D_k, a subset of Λ_k, defined by $D_k = \{(\alpha_m)_{m=1}^k \subset \mathbb{N}_0 : \sum_{k=1}^k \alpha_j = k, \text{ and } m \mid \alpha_m \text{ for all } 1 \leq m \leq k\}$. Given $\alpha \in D_k$, if we write $\alpha_m = m\widetilde{\alpha}_m$, then we obtain

$$\widetilde{Q}_k(z_1, \ldots, z_k) = \sum_{\alpha \in D_k} a_\alpha z_1^{\widetilde{\alpha}_1} z_2^{2\widetilde{\alpha}_2} \cdots z_k^{k\widetilde{\alpha}_k}.$$

It is clear that $Q_k(z_1, z_2, \ldots, z_k) = \sum_{\alpha \in D_k} a_\alpha z_1^{\widetilde{\alpha}_1} z_2^{\widetilde{\alpha}_2} \cdots z_k^{\widetilde{\alpha}_k}$ is the desired polynomial. $\square$

Corollary 4.3. Let $P_k : c_0 \to \mathbb{C}$ be a k-homogeneous polynomial such that $P_k = P_k \circ \gamma_m$ for all $m \geq 1$. Then there exists $Q_k : \mathbb{C}^k \to \mathbb{C}$ such that $P_k(z) = Q_k(z_1, z_2^2, \ldots, z_k^k)$.

Remark 4.4. The previous corollary fails if we replace c_0 by ℓ_∞ or c. In the first case, let $B : \ell_\infty \to \ell_\infty$ be the backward shift. Given a Banach limit L, since $L((z_j)_{j=1}^\infty) = L(B((z_j)_{j=1}^\infty)$, it is clear that $L = L \circ \gamma_m$ for all $m \in \mathbb{N}$. Then, if Corollary 4.3 holds on ℓ_∞, we obtain that $L((z_j)_{j=1}^\infty) = Q(z_1)$ for some linear functional $Q : \mathbb{C} \to \mathbb{C}$, such that $Q(z_1) = \alpha_1 z_1$. Note that $L_{|c_0} \equiv 0$, then $\alpha_1 = 0$ and $L \equiv 0$, which is a contradiction. Working on c, the linear functional $L((z_j)_{j=1}^\infty) = \lim_{j\to\infty} z_j$ satisfies $L \circ \gamma_m = L$ and it is clear that it is not possible to write $L((z_j)_{j=1}^\infty) = \alpha_1 z_1$.

5. Groups arising from measure-preserving maps

We begin this section by considering $E = L_p[0,1]$, and studying those k-homogeneous polynomials $P_k : E \to \mathbb{C}$ satisfying $P_k(x) = P_k(x \circ \phi)$ for all $x \in E$ and any (Lebesgue) measure preserving map $\phi : [0,1] \to [0,1]$,

Note that $\int_0^1 x^k = \int_0^1 (x \circ \phi)^k$ for any measure preserving map ϕ, any $k \in \mathbb{N}_0$ and any measurable function $x : [0,1] \to \mathbb{C}$. So, given any polynomial $P_k : \mathbb{C}^k \to \mathbb{C}$ the mapping

$$x \mapsto P\left(\int_0^1 x, \int_0^1 x^2, \ldots, \int_0^1 x^k\right),$$

defined on $L_p[0,1]$ for $p \geq k$, has the desired property.

Now, given $N \in \mathbb{N}_0$, let us consider the $2^N + 1$ nodes $\left\{j/2^N\right\}_{j=0}^{2^N} \subset [0,1]$ and take a regular partition of $[0,1]$ using them. Write $I_j^{(N)} = \left((j-1)2^{-N}; j2^{-N}\right)$, for $j = 1, \ldots, 2^N$ and note that the measure of any of these sub-intervals is 2^{-N}. Let us denote by S_N the space of $N-$*level* step functions, defined by

$$S_N = \left\{ x : [0,1] \to \mathbb{C} : x(t) = \sum_{j=1}^{2^N} a_j \chi_j^{(N)}(t) \text{ for some finite sequence } \{a_j\}_{j=1}^{2^N} \subset \mathbb{C} \right\},$$

where $\chi_j^{(N)}(t) = \chi_{I_j^{(N)}}(t)$.

Let us begin by studying k-homogeneous polynomials defined on $E = L_p[0,1]$ for $k \leq p$. Any permutation σ of a 2^N elements set induces a measure preserving map

$\phi_\sigma : [0,1] \to [0,1]$ defined by

$$\begin{aligned}
\phi_\sigma(j2^{-N}) &= j2^{-N} && \text{for } 0 \leq j \leq 2^N,\\
\phi_{\sigma|_{I_j^{(N)}}} : I_j^{(N)} &\longrightarrow I_{\sigma(j)}^{(N)} && \text{for } 1 \leq j \leq 2^N,\\
\phi_{\sigma|_{I_j^{(N)}}}\left((1-\theta)(j-1)2^{-N} + \theta j 2^{-N}\right) &= (1-\theta)(\sigma(j)-1)2^{-N} + \theta\sigma(j)2^{-N} && \text{for } 0 < \theta < 1.
\end{aligned}$$

We have $P_k(x) = P_k(x \circ \phi_\sigma)$ for any $x \in S_N$. We can consider the following commutative diagram:

$$\begin{tikzcd}
\mathbb{C}^{2^N} \arrow[rr, "\widetilde{\sigma}"] \arrow[dr, "\iota_N"] \arrow[ddr, "P_{k|S_N}\circ\iota_N"'] & & \mathbb{C}^{2^N} \arrow[dl, "\iota_N"'] \arrow[ddl, "P_{k|S_N}\circ\iota_N"] \\
 & S_N \arrow[d, "P_{k|S_N}"] & \\
 & \mathbb{C} &
\end{tikzcd} \tag{1}$$

where, given $z = (z_j)_{1\leq j\leq 2^N}$, $\widetilde{\sigma}(z) = (z_{\sigma(j)})_{1\leq j\leq 2^N}$ and $\iota_N : \mathbb{C}^{2^N} \to S_N$, defined by

$$z \mapsto x(t) = \sum_{j=1}^{2^N} z_j \chi_j^{(N)}(t).$$

Hence,

$$P_k \circ \iota_N(z) = P_k\left(\sum_{j=1}^{2^N} z_j \chi_j^{(N)}(t)\right) = P_k\left(\sum_{j=1}^{2^N} z_{\sigma(j)} \chi_j^{(N)}(t)\right) = P_k \circ \iota_N(\widetilde{\sigma}(z)).$$

Then, $P_k \circ \iota_N$ is a symmetric polynomial on $\mathbb{C}^{2^N}$, and it is possible to represent it using any set of generators of the algebra of symmetric polynomials over $\mathbb{C}^{2^N}$. For instance, we can use the set

$$\left\{\sum_{j=1}^{2^N} z_j; \sum_{j=1}^{2^N} z_j^2; \ldots; \sum_{j=1}^{2^N} z_j^k; \ldots; \sum_{j=1}^{2^N} z_j^{2^N}\right\}.$$

Since $P_k \circ \iota_N$ is a k-homogeneous polynomial, it will be enough to take the set

$$\left\{\sum_{j=1}^{2^N} z_j; \sum_{j=1}^{2^N} z_j^2; \ldots; \sum_{j=1}^{2^N} z_j^k\right\}.$$

It follows that there exists $\widetilde{Q_N} : \mathbb{C}^k \to \mathbb{C}$ such that

$$P_k \circ \iota_N(z) = \widetilde{Q_N}\left(\sum_{j=1}^{2^N} z_j; \sum_{j=1}^{2^N} z_j^2; \ldots; \sum_{j=1}^{2^N} z_j^k\right).$$

For $x(t) = \iota_N(z)$, it means

$$P_k(x) = \widetilde{Q_N}\left(2^N \int_0^1 x; 2^N \int_0^1 x^2; \ldots; 2^N \int_0^1 x^k\right) = Q_N\left(\int_0^1 x; \int_0^1 x^2; \ldots; \int_0^1 x^k\right),$$

where Q_N is defined by $Q_N(\omega_1; \ldots; \omega_k) = \widetilde{Q_N}(2^N\omega_1; \ldots; 2^N\omega_k)$ for all $(\omega_1; \ldots; \omega_k) \in \mathbb{C}^k$.

Now we are ready to prove our first lemma.

Lemma 5.1. Given $k \in \mathbb{N}$, there exists $N_k \in \mathbb{N}_0$ such that $\Phi_k : S_{N_k} \to \mathbb{C}^k$, defined by

$$x \mapsto \left(\int_0^1 x; \ldots; \int_0^1 x^k\right)$$

is surjective.

Proof. The proof is by induction on k. If $k = 1$, it is easy to see that the range of Φ_1 is $\mathbb{C}$ considering the constant functions (or $0-$*level* step functions). So, $N_1 = 0$. Let us show that if the assertion holds for k, then also holds for $k + 1$.

First, using the surjectivity of Φ_k, let us fix $x_1, \ldots, x_k \in S_{N_k},\ x_l(t) = \sum_{j=1}^{2^{N_k}} a_{l,j}\chi_j^{(N_k)}(t)$ for $1 \le l \le k$, such that $\Phi_k(x_j) = e_j$, where $\{e_1, \ldots, e_k\}$ is the canonical basis of $\mathbb{C}^k$. Let ξ_{k+1} be a $(k + 1)$th primitive root of unity and let us choose any natural number N_{k+1} such that $k2^{N_k} + (k+1) \le 2^{N_{k+1}}$. Now, we can take a regular partition of $[0, 1]$ using $2^{N_{k+1}}$ nodes and take a subset of $N_{k+1}-$*level* step functions ψ defined in the following way:

$$\psi(t) = \sum_{l=1}^{k} \alpha_l \sum_{j=1}^{2^{N_k}} a_{l,j}\chi^{(N_{k+1})}_{(l-1)2^{N_k}+j}(t) + \alpha_{k+1} \sum_{j=1}^{k+1} \xi_{k+1}^j \chi^{(N_{k+1})}_{k2^{N_k}+j}(t) \quad \text{for } t \in [0, 1],$$

where $\alpha_1, \ldots, \alpha_{k+1} \in \mathbb{C}$. Since $k2^{N_k} + (k + 1) \le 2^{N_{k+1}}$, these functions are null for all

$$t \in \bigcup_{j > k2^{N_k}+(k+1)} I_j^{(N_{k+1})}.$$

For $1 \le r \le k$ and $1 \le l \le k$, a trivial verification and the inductive hypothesis show that

$$\int_0^1 \left(\sum_{j=1}^{2^{N_k}} a_{l,j}\chi_j^{(N_{k+1})}(t)\right)^r dt = \delta_{l,r}\, \frac{2^{N_k}}{2^{N_{k+1}}}. \text{ Further}$$

$$\int_0^1 \left(\sum_{j=1}^{k+1} \xi_{k+1}^j \chi^{(N_{k+1})}_{k2^{N_k}+j}(t)\right)^r dt = \int_0^1 \left(\sum_{j=1}^{k+1} \xi_{k+1}^{jr} \chi^{(N_{k+1})}_{k2^{N_k}+j}(t)\right) dt = 0 \text{ by the primitiveness of } \xi_{k+1}$$

and

$$\int_0^1 \left(\sum_{j=1}^{k+1} \xi_{k+1}^j \chi_{k2^{N_k}+j}^{(N_{k+1})}(t) \right)^{k+1} \; dt = \frac{k+1}{2^{N_{k+1}}}.$$

Also for $1 \le r \le k+1$,

$$\psi^r(t) = \sum_{l=1}^{k} \alpha_l^r \big(\sum_{j=1}^{2^{N_k}} a_{l,j} \chi_{(l-1)2^{N_k}+j}^{(N_{k+1})}(t) \big)^r + \alpha_{k+1}^r \big(\sum_{j=1}^{k+1} \xi_{k+1}^j \chi_{k2^{N_k}+j}^{(N_{k+1})}(t) \big)^r \quad \text{for } t \in [0,1],$$

from where one easily deduces that Φ_{k+1} is surjective. $\square$

Remark 5.2. Let $k \in \mathbb{N}_0, N_k$ as in Proposition 5, and $N > N_k$. Then, since $S_{N_k} \hookrightarrow S_N$, we have that

$$P_k(x) = Q_{N_k} \left(\int_0^1 x; \ldots ; \int_0^1 x^k \right) \text{ for all } x \in S_{N_k}$$

and also

$$P_k(x) = Q_N \left(\int_0^1 x; \ldots ; \int_0^1 x^k \right) \text{ for all } x \in S_{N_k},$$

using the surjectivity of Φ_k, we deduce that $Q_N \equiv Q_{N_k}$.

Suppose that we endow $\bigcup_{N\ge 1} S_N$ with some norm $\|\cdot\|$ such that $x \mapsto \int_0^1 x^j$ is a continuous polynomial for $1 \le j \le k$. If we complete $(\bigcup_{N\ge 1} S_N, \|\cdot\|)$ we are led to a Banach space E such that $\bigcup_{N\ge 1} S_N \subset E$ is dense and $x \mapsto \int_0^1 x^j$ is a continuous polynomial on E for any $1 \le j \le k$. We are ready to state the first theorem of this section.

Theorem 5.3. Let E be a Banach space as above. If $P_k : E \to \mathbb{C}$ is a k-homogeneous polynomial such that for any measure preserving map $\phi : [0,1] \to [0,1]$, we have $P_k(x) = P_k(x \circ \phi)$ for all $x \in E$, then there exists a polynomial $Q : \mathbb{C}^k \to \mathbb{C}$ such that

$$P_k(x) = Q \left(\int_0^1 x; \ldots ; \int_0^1 x^k \right).$$

Proof. Given P_k, let us fix $Q = Q_{N_k}$. The assertion follows from the following diagram,

$$\begin{tikzcd} \bigcup_{N\ge N_k} S_N \arrow[r, "\Phi_k"] \arrow[d, hook] & \mathbb{C}^k \arrow[d, "Q"] \\ E \arrow[ur, "\overline{\Phi}_k"] \arrow[r, "P_k"'] & \mathbb{C} \end{tikzcd}$$

where $\overline{\Phi}_k$ is defined using the density of $\cup_{N\ge N_k} S_N$ in E and the continuity of Φ_k. $\square$

The following Corollary can be obtained also from Theorem 9′ in [14].

Corollary 5.4. Let $p \geq k$ and $P_k : L_p[0,1] \to \mathbb{C}$ be a k-homogeneous polynomial such that for any measure preserving map $\phi : [0,1] \to [0,1]$, we have $P_k(x) = P_k(x \circ \phi)$ for all $x \in L_p[0,1]$. Then, there exists a polynomial $Q : \mathbb{C}^k \to \mathbb{C}$ such that

$$P_k(x) = Q\left(\int_0^1 x; \ldots ; \int_0^1 x^k\right).$$

Corollary 5.5. We can consider $S_N \subset L_\infty[0,1]$, and E the closure of $\bigcup_{N \in \mathbb{N}} S_N$ in $L_\infty[0,1]$. Following the same ideas we find that

$$P_{k|E}(x) = Q\left(\int_0^1 x; \ldots ; \int_0^1 x^k\right).$$

In particular this characterization is valid for $C[0,1]$ and for the space of Riemann integrable functions over $[0,1]$ with the sup-norm.

Corollary 5.6. Consider in $E = C''[0,1]$ the group $G \subset L(E)$ of the self-maps $x \in E \mapsto x \circ \phi \in E$, given by all measure preserving maps $\phi : [0,1] \to [0,1]$, where $x \circ \phi(\mu)$ is defined by $x(\mu \circ \phi^{-1})$. If P_k is a G-symmetric k-homogeneous polynomial on E, then, there exists $Q : \mathbb{C}^k \to \mathbb{C}$ such that

$$P_{k|C[0,1]}(x) = Q\left(\int_0^1 x; \ldots ; \int_0^1 x^k\right).$$

5.1. **Symmetry through the canonical extension.** We might think that the assumptions on P_k can be modified in order to obtain results restricted to $C[0,1]$. We can restrict ourselves to studying polynomials on the space $C[0,1]$ using their canonical extension [4] to define the symmetry. We say that P_k is *ext-symmetric* if for its canonical extension $\widetilde{P_k}$ we have $P_k(x) = \widetilde{P_k}(x \circ \phi)$ for all continuous functions x and any measure preserving map $\phi : [0,1] \to [0,1]$. Let us denote by λ the Lebesgue measure on $[0,1]$. If we endow $C''([0,1])$ with the algebra structure induced by the Arens product [3], we obtain the following result.

Theorem 5.7. Let $P_k : C[0,1] \to \mathbb{C}$ be an *ext-symmetric* k-homogeneous polynomial. Then, there exists a polynomial $Q : \mathbb{C}^k \to \mathbb{C}$ such that

$$\widetilde{P_k}(T) = Q\left(T(\lambda); T^2(\lambda); \ldots ; T^k(\lambda)\right) \qquad \forall \quad T \in C''([0,1]).$$

Proof. First, given a subinterval $I \subset [0,1]$, let us consider a bounded sequence of continuous functions $\{x_n\}_{n \in \mathbb{N}}$ pointwise convergent to $\chi_I(t)$. Moreover, by the Lebesgue Dominated Convergence Theorem the sequence $\{x_n\}_n$ is a weakly Cauchy sequence that is also weak* convergent to $\chi_I(t)$ in $C''([0,1])$.

Recall that $C''([0,1])$ has the Dunford-Pettis property. Therefore as in the proof of Theorem 3.1,

$$\widetilde{P_k}(\chi_I) = \lim_{n \to \infty} \widetilde{P_k}(x_n),$$

and then we can deduce that the equality $\widetilde{P_k}(\chi_I \circ \phi) = \widetilde{P_k}(\chi_I)$ holds for any measure preserving map ϕ, and it is also valid for the space of $N - level$ step functions. As in Corollary 5.5, we obtain that there exists a polynomial $Q : \mathbb{C}^k \to \mathbb{C}$ such that

$$\widetilde{P_{k|C[0,1]}}(x) = P_k(x) = Q\left(\int_0^1 x; \dots; \int_0^1 x^k\right) \qquad \forall\ x \in C[0,1].$$

Now, it remains to prove that

$$\widetilde{P_k}(T) = Q\left(T(\lambda); T^2(\lambda); \dots; T^k(\lambda)\right) \qquad \forall \quad T \in C''([0,1]).$$

Next, we check that the proposed polynomial satisfies the criterion given in [16]. Note that using the “algebraic properties” of the canonical extension, it is enough to show that the monomials $w_m(x) = \int_0^1 x^m$ are extended by $W_m(T) = T^m(\lambda)$ for $1 \le m \le k$. So, we have to prove that

(i) For each $x \in C[0,1]$, $DW_m(x)$ is weak* continuous, and

(ii) For each $T \in C''[0,1]$ and $(x_\alpha) \subset C[0,1]$ weak* convergent to T, $DW_m(T)(x_\alpha) \to DW_m(T)(T)$.

Recall that in spite of the non commutativity of the Arens product, the equality $xT = Tx$ holds whenever $x \in C[0,1]$ and $T \in C''[0,1]$. Both conditions are fulfilled because the Arens product is weak* continuous in the first variable and $DW_m(x)(T) = mTx^{k-1}(\lambda)$, $DW_m(T)(x_\alpha) = mx_\alpha T^{k-1}(\lambda)$. □

Corollary 5.8. The k-homogeneous polynomial $P_k : C[0,1] \to \mathbb{C}$ is *ext-symmetric* if and only if $\widetilde{P_k}$ is G-symmetric for the group G considered in Corollary 5.6 .

Proof. The 'if' part is obvious, while for the 'only if', notice that $\lambda \circ \phi^{-1} = \lambda$, so $(T \circ \phi)(\lambda) = T(\lambda \circ \phi^{-1}) = T(\lambda)$ for $T \in C''([0,1])$. □

Now, we construct an *ext-symmetrization* operator for k-homogeneous polynomials defined on $E = C[0,1]$. We need to define an operator $S : \mathcal{P}(^kE) \to \mathcal{P}_s(^kE)$. Here $\mathcal{P}_s(^kE)$ denotes the space of ext-symmetric k-homogeneous polynomials.

We begin by recalling, from Theorem 5.7, that we have to find a polynomial $R_k : \mathbb{C}^k \to \mathbb{C}$, such that

$$S(P_k)(x) = R_k\left(\int_0^1 x; \dots; \int_0^1 x^k\right) \qquad \text{for all } x \in C[0,1]. \tag{2}$$

We can identify $\mathbb{C}^{2^N}$ and the $N - level$ step functions as in (1). Now, we can consider the symmetrization of $Q_{k,N} = P_{k|S_N} \circ \iota_N$, namely

$$\left(Q_{k,N}\right)_s(z_1; \dots; z_{2^N}) = \frac{1}{(2^N)!} \sum_{\sigma \in \mathfrak{G}_{2^N}} P_{k|S_N} \circ \iota_N(z_{\sigma(1)}; \dots z_{\sigma(2^N)}),$$

where $\mathfrak{G}_{2^N}$ is the permutation group of 2^N elements. So, there exists $\overline{Q_{k,N}} : C^k \to C$, necessarily of degree not greater than k, such that

$$(Q_{k,N})_s (z_1; \ldots ; z_{2^N}) = \overline{Q_{k,N}} \left(\sum_{i=1}^{2^N} z_i; \sum_{i=1}^{2^N} z_i^2; \ldots ; \sum_{i=1}^{2^N} z_i^k \right).$$

For convenience, given N, let us define $R_{k,N} : C^k \to C$ according to $R_{k,N}(\omega_1; \ldots ; \omega_k) = \overline{Q_{k,N}}(2^N\omega_1; \ldots ; 2^N\omega_k)$, so that

$$\overline{Q_{k,N}} \left(\sum_{i=1}^{2^N} z_i; \sum_{i=1}^{2^N} z_i^2; \ldots ; \sum_{i=1}^{2^N} z_i^k \right) = R_{k,N} \left(\frac{1}{2^N} \sum_{i=1}^{2^N} z_i; \frac{1}{2^N} \sum_{i=1}^{2^N} z_i^2; \ldots ; \frac{1}{2^N} \sum_{i=1}^{2^N} z_i^k; \right).$$

Despite the fact that characterization (2) is valid for continuous functions, note that it is also valid for $N - level$ step functions.

Given an ultrafilter $\mathcal{U}$ on the set of natural numbers, we need to guarantee the existence of $\lim_{\mathcal{U}} R_{k,N} \subset \mathcal{P}({}^{\leq k}\mathbb{C}^k)$. For this, let us check that given a point $\omega = (\omega_1; \ldots ; \omega_k) \in C^k$, the values of $R_{k,N}(\omega)$ remain bounded. From Proposition 5.1 there exists $N_k \in N$ such that

$$S_{N_k} \longrightarrow C^k$$

$$x \mapsto \left(\int_0^1 x; \ldots ; \int_0^1 x^k \right)$$

is surjective. So, we can choose $x = \sum_{j=1}^{2^{N_k}} z_j \chi_j^{(N_k)} \in S_{N_k}$ satisfying $\omega = \left(\int_0^1 x; \ldots ; \int_0^1 x^k \right)$. If we denote by $x \circ \sigma = \sum_{j=1}^{2^{N_k}} z_{\sigma(j)} \chi_j^{(N_k)}$, then

$$\begin{aligned} R_{k,N_k}(\omega) =& R_{k,N_k} \left(\int_0^1 x; \ldots ; \int_0^1 x^k \right) = \overline{Q_{k,N_k}} \left(2^{N_k} \int_0^1 x; \ldots ; 2^{N_k} \int_0^1 x^k \right) \\ =& (Q_{k,N_k})_s (z_1; \ldots ; z_{2^{N_k}}) = \frac{1}{(2^{N_k})!} \sum_{\sigma \in \mathfrak{G}_{2^{N_k}}} P_{k|S_{N_k}} \circ \iota_{N_k} \left(z_{\sigma(1)}; \ldots z_{\sigma(2^{N_k})} \right) \\ =& \frac{1}{(2^{N_k})!} \sum_{\sigma \in \mathfrak{G}_{2^{N_k}}} P_k(x \circ \sigma). \end{aligned}$$

So,

$$|R_{k,N_k}(\omega)| \leq \left| \frac{1}{(2^{N_k})!} \sum_{\sigma \in \mathfrak{G}_{2^{N_k}}} P_k(x \circ \sigma) \right| \leq \|P_k\| \max_{\sigma \in \mathfrak{G}_{2^{N_k}}} \|x \circ \sigma\|^k.$$

Since $S_{N_k} \subset S_N$ for any $N \geq N_k$, in these cases we can also consider $x \in S_N$. If we need to estimate $|R_{k,N}(\omega)|$ for $N \geq N_k$, we will find that

$$|R_{k,N}(\omega)| \leq \left| \frac{1}{(2^N)!} \sum_{\sigma \in \mathfrak{G}_{2^N}} P_k(x \circ \sigma) \right| \leq \|P_k\| \max_{\sigma \in \mathfrak{G}_{2^{N_k}}} \|x \circ \sigma\|^k,$$

but for any step function x, the norm

$$\|x \circ \sigma\| = \sup_{\|\mu\|=1} \left| \int_0^1 x \circ \sigma \, d\mu \right| \leq \sup_{\|\mu\|=1} \|\mu\| \, \|x \circ \sigma\|_\infty \leq \|x\|_\infty.$$

From this we conclude that $|R_{k,N}(\omega)| \leq \|P_k\| \, \|x\|_\infty^k$, for all $N \geq N_k$, hence the sequence is bounded and there exists $R_k \in \mathcal{P}(^{\leq k}\mathbb{C}^k)$ defined by $R_k(\omega) := \lim_{\mathcal{U}} R_{k,N}(\omega) : \mathbb{C}^k \to \mathbb{C}$. Thus we can define the symmetrization operator for k-homogeneous polynomials by

$$S(P_k)(x) = R_k \left(\int_0^1 x; \ldots ; \int_0^1 x^k \right) \qquad \text{for all } x \in C[0,1].$$

If we consider a symmetric polynomial P_k, then from Remark 5.2 we find that $R_{k,N} = R_{k,N_k}$ for all $N \geq N_k$, and then $S(P_k) = P_k$.

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Richard Aron. Department of Mathematics and Computer Science, Kent State University, Kent, OH 44240.
E-mail address: aron@math.kent.edu

Pablo Galindo. Departamento de Análisis Matemático, Universidad de Valencia, Valencia, Spain.
E-mail address: pablo.galindo@uv.es

Damián Pinasco. Universidad Torcuato Di Tella. Av. Figueroa Alcorta 7350 (C1428BCW), Buenos Aires, Argentina.
E-mail address: dpinasco@utdt.edu

Ignacio Zalduendo. Universidad Torcuato Di Tella. Av. Figueroa Alcorta 7350 (C1428BCW), Buenos Aires, Argentina.
E-mail address: izalduendo@utdt.edu