ON THE CONSTRUCTION OF A FINITE SIEGEL SPACE

JOSÉ PANTOJA[1,3], JORGE SOTO ANDRADE[1], JORGE A. VARGAS[1,2]

Abstract. In this note we construct a finite analogue of classical Siegel's Space. Our approach is to look at it as a non commutative Poincare's half plane. The finite Siegel Space is described as the space of Lagrangians of a $2n$ dimensional space over a quadratic extension E of a finite base field F. The orbits of the action of the symplectic group $Sp(n, F)$ on Lagrangians are described as homogeneous spaces. Also, Siegel's Space is described as the set of anti-involutions of the symplectic group.

1. introduction

Classical Siegel's half space is a clever generalization of Poincaré's half plane. In [4], the starting idea is to replace the real base field $\mathbb{R}$ by the full matrix ring $M(n, \mathbb{R})$. Then Siegel's half space consists of all symmetric complex $n \times n$ matrices whose imaginary part is positive definite.

We address here the case of a finite base field. Our approach to obtain the finite analogue of Siegel's half space is to extend the universal (double cover of) Poincaré's half plane construction given in [9] to the case where the base field F is replaced by a ring A with involution denoted *, that we read "star". A ring with involution is also called involutive ring, as in [8, 10]. Instead of the group $G_F = SL(2, F)$ we have now its star-analogue $G_A = SL_*(2, A)$ introduced in [10]. A natural G_A- space is the star-plane $\mathcal{P}_A$ consisting of all points $x = (x_1, x_2) \in P = A \times A$ whose coordinates x_1 and x_2 star-commute, i.e. $x_1 x_2^* = x_2 x_1^*$. Notice *en passant* the analogy with Manin's $q-$plane, whose points have coordinates that anti-commute.

We introduce the canonical star - anti-hermitian form ω on P given by

$$\omega(x, y) = x_1 y_2^* - x_2 y_1^*$$

for all $x, y \in P$. We have then

$$\omega(y, x) = -\omega(x, y)^*$$

Date: May 18, 2014.
2010 *Mathematics Subject Classification.* Primary 22E46; Secondary 17B10.
Key words and phrases. Finite Siegel half space, star-analogue .
[1]Partially supported by FONDECYT 1120578.
[2]Partially supported by CONICET, SECYTUNC (Argentina) .
[3]Partially supported by Pontificia Universidad Católica de Valparaíso.

for all $x, y \in P$, and we see that the star-plane $\mathcal{P}_A$ consists of all isotropic vectors for ω.

We also notice that if we write

$$x^* = \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix}$$

for $x = (x_1, x_2) \in \mathcal{P}_A$, then we have

$$\omega(x, y) = xwy^*$$

where

$$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

.

The star-plane $\mathcal{P}_A$ is stratified by the family of G_A- subplanes $\mathcal{P}_A(J)$ of $\mathcal{P}_A$ given by the condition $Ax + Ay = J$ where J is a left ideal in A. In what follows we will be mainly interested in the generic case $J = A$, and we will take $A = M_n(F)$ endowed with the transpose map.

As a motivation for the construction below, recall that finite Poincaré half plane, more precisely the double cover of finite Poincaré half plane, may be realized as the set of lines through the origin in the usual plane $E^2 = E \times E, E$ a quadratic extension of the base finite field F, whose slope does not lie in $F \cup \{\infty\}$. Lines through the origin are however just the Lagrangians for the symplectic bilinear form *determinant* on E^2. and the constraint that the slope of a Lagrangian L does not lie in $F \cup \{\infty\}$. amounts to say that the symplectic form h given by Galois twisting of the determinant, given by

$$h(x, y) = x_1\bar{y}_2 - x_2\bar{y}_1$$

for $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ in E, is *non degenerate* when restricted to L.

Indeed, if the constraint on L is fulfilled, we may take a representative vector of the form $(z, 1) \in L \ \ (z \in E)$, so that $L = \{(zx_2], x_2)|x_2 \in E\}$ and then h on L is given by

$$h(\begin{pmatrix} zx_2 \\ x_2 \end{pmatrix}, \begin{pmatrix} zy_2 \\ y_2 \end{pmatrix}) = (z - \bar{z})x_2\bar{y}_2,$$

so h non degenerate means just $z \neq \bar{z}$.

Under the action of $SL(2, F)$ in the set $\mathcal{L}$ of all Lagrangians we have then the generic orbit consisting of all Lagrangians on which h is non degenerate and the residual orbit consisting of all Lagrangians on which h is degenerate, i. e. null in this case, so that $z = \bar{z}$, i.e. $z \in F$. More generally we will see below that the rank of the restriction h_L of h to L characterizes the $SL(2, F)-$ orbits in $\mathcal{L}$.

2. Preliminaries

2.1. **General setup.** We assume now that the involutive ring $(A, *)$ is a quadratic Galois extension of a sub involutive ring A_0, i.e. that the Galois group $\Gamma = Gal(A, A_0)$ is of order 2 and that $A_0 = Fix_A(\Gamma)$. We denote $a \mapsto \bar{a}$ the nontrivial element τ of Γ. Notice that τ extends naturally to the plane $A^2 = A \times A$ and to the star-plane $\mathcal{P}_A$, Our data is then $(\mathcal{P}_A, \omega, \tau)$.

We introduce the canonical star-τ-antihermitian form h on P given by

$$h(x, y) = \omega(x, \bar{y})$$

for all $x, y \in P$. We have

$$h(y, x) = -\overline{h(x, y)}^*$$

for all $x, y \in P$.

2.2. **The full matrix ring case.** We specialize now to the case where the involutive ring $(A, *)$ is the full matrix ring $M_n(E)$ over a finite field E endowed with the transpose mapping. We assume moreover that E is a quadratic extension of a subfield F with Galois group $\{Id, \tau\}$.

We have the big special linear group $G_E = SL_*(2, A)$ and the small special linear group $G_F = SL_*(2, A_0)$ that appears as the fixed point set of τ in G_E. The set of all lines through the origin in the plane $\mathcal{P}_A$ is denoted by $\mathcal{L}_A$ It follows from classical Witt's theorem that G_E acts transitively on $\mathcal{L}_A$.

Indeed the non-commutative 1- dimensional subspaces $L \in \mathcal{P}_A$ may be readily identified with classical Lagrangians in the symplectic space $V = E^{2n}$, endowed with the canonical symplectic form ω', that in terms of the canonical basis $e_1, \cdots, e_{2n}$ for V is given by $\omega(e_j, e_{n+j}) = -\omega(e_{n+j}, e_j) = 1, j = 1, \ldots, n$ and $\omega(e_k, e_s) = 0$ for $|k - s| \neq n$.

Recall [8] that Lagrangian subspaces L in V may be described as $L = L_{(a,b)} = \langle aP + bQ \rangle \quad (a, b \in A, aA + bA = A, ab^* = ba^*)$ where the column vectors P and Q are given by $P = (e_1, \cdots, e_n)^*, Q = (e_{n+1}, \cdots, e_{2n})^*$ and $\langle u \rangle$ stands for the vector subspace of V spanned by the components $u_1, ..., u_n$ of any $u \in M = V^n$.

Moreover $L_{(a,b)} = L_{(a',b')}$ if and only if $a' = ca$ and $b' = cb$ for a suitable $c \in A$. So classical Lagrangians correspond to non commutative lines through the origin in $\mathcal{P}_A$.

On the other hand regarding the action of G_E we have $g(L_{(a,b)}) = L_{(a,b)g}g$ for $g \in G_E$.

The set of classical Lagrangian subspaces for ω' in V will be denoted by $\mathcal{L}_V$ or just $\mathcal{L}$.

In what follows we will switch to the classical setting for Lagrangians in V for the case of $A = M_n(E)$.

We denote the set of symmetric matrices with coefficients in E^n by $Sym(E^n)$. The isotropy subgroup for the subspace L_+ spanned by $e_1, \ldots, e_n$ is the semidirect product of the subgroups

$$K := \{ \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix}, A \in GL_n(E)\}$$

$$P^+ = \{ \begin{pmatrix} I & B \\ 0 & I \end{pmatrix}, B \in Sym(E^n) \}$$

On the other hand, the isotropy subgroup for the subspace L_- spanned by the vectors $e_{n+1}, \ldots, e_{2n}$ is the semidirect product of K times the subgroup

$$P^- = \{ \begin{pmatrix} I & 0 \\ B & I \end{pmatrix}, B \in Sym(E^n) \}$$

Let $\mathcal{L} : Sym(E^n) \to \mathcal{L}_{E,2n}$ be the Siegel map defined by the formula

$$\mathcal{L}(Z) = \{ \begin{pmatrix} Zx \\ x \end{pmatrix}, x \in E^n \}$$

We would like to point out that in [8], a complete description of the Lagrangian subspaces of E^{2n}, E a finite field, is given, in the study of the groups $SL_*(2, A)$ (applied to $A = M_n(E)$ and $*$ the transposition of matrices). The Siegel Lagrangian $\mathcal{L}(Z)$ is L_{Z,I_n} in the notation of [8].

Following Siegel, we write sometimes (A, B, C, D) for the $2n \times 2n$ matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad A, B, C, D \in M_n(E)$$

Remark 1. *Whenever $F = \mathbb{R}$, we have that $\mathcal{L}(Z)$ is equal to the action on the subspace L_- of the exponential of the Lie algebra element $(0, Z, 0, 0) \in \mathfrak{sp}(n, \mathbb{C})$.*

We define $\epsilon = \epsilon_F$ by

$$\epsilon_F = \begin{cases} 1 & \text{if } -1 \text{ is an square in } F \\ -1 & \text{otherwise} \end{cases}$$

This is just the Lagrange symbol $(\frac{-1}{p})$ in the case of a finite field F of characteristic p.

(We note that, in the real case, we always have $\epsilon_F = -1$)

Proposition 1. *We have the decomposition into G_F- invariant subsets*

$$\mathcal{L}_V = \bigcup_{0 \le r \le n} \mathcal{H}_r,$$

where $\mathcal{H}_r$ stands for the set of all $W \in \mathcal{L}_V$ such that the rank of h_E restricted to $W \times W$ is r.

Next, we consider the hermitian form

$$h_0 : E^{2n} \times E^{2n} \to E$$

defined so that the canonical basis is an orthogonal basis for h_0, $h_0(e_j, e_j) = -1$ for $1 \le j \le n$ and $h_0(e_j, e_j) = 1$ for $n + 1 \le j \le 2n$.

We consider the group

$$Sp_0(n, F) := U(E^{2n}, h_0) \cap Sp(n, E).$$

Later on, for a finite field F we construct a generalized Cayley transform, that is, we show there exists an element C in $Sp(n,E)$ which conjugates $Sp(n,F)$ into $Sp_0(n,F)$. That is, $C^{-1}Sp_0(n,F)C = Sp(n,F)$. Thus, we verify that $Sp_0(n,F)$ is isomorphic to $Sp(n,F)$, (a well known result for $F=\mathbb{R}$, see page 242 of [6]).

Among the objectives of this note are, for a finite field F, to determine the orbits of both groups $Sp(n,F), Sp_0(n,F)$ in $\mathcal{L}_{E,2n}$ and the intersection of each orbit with the image of the Siegel map. When $F=\mathbb{R}, E=\mathbb{C}$ this problem has been considered and solved by [7], [5] and references therein.

It is known that for a finite field E and a hermitian form (W,h) on a finite dimensional vector space W over E, there always exists an ordered basis $w_1,\dots$ of W and a nonnegative integer r so that $h(w_k,w_s)=\delta_{ks}$ for $k,s\le r$ and $h(w_k,w_s)=0$ for $k>r$ or $s>r$.

In this situation we define the type of the form (W,h) to be r.

Let $\mathcal{O}_r$ the set of Lagrangian subspaces $W\in\mathcal{L}_{E,2n}$ so that the form h_0 restricted to W is of type r. Obviously $Sp_0(n,F)$ leaves invariant the subset $\mathcal{O}_r$ and $\mathcal{L}_{E,2n}=\mathcal{O}_n\cup\mathcal{O}_{n-1}\cup\cdots\cup\mathcal{O}_0$.
One of the main results of this work is:

Theorem 1. *Assume F is a finite field, then*

- *The orbits of $Sp_0(n,F)$ in $\mathcal{L}_{E,2n}$ are exactly the sets $\mathcal{O}_j, j=0,\cdots,n$.*
- *The orbits of $Sp(n,F)$ in $\mathcal{L}_{E,2n}$ are exactly the sets $\mathcal{H}_j, j=0,\cdots,n$.*
- *Any orbit of either $Sp(n,F)$ or $Sp_0(n,F)$ intersects the image of the Siegel map.*
- *Except for $n=1$, no orbit of $Sp_0(n,F)$ is contained in the image of the Siegel map.*
- *$\mathcal{H}_n$ is the unique orbit of $Sp(n,F)$ contained in the image of the Siegel map.*
- *$C\mathcal{H}_j = C\mathcal{O}_j$.*

3. Proofs

In order to write down the proof of theorem 1 we need to set up some notation and recall some known facts.

${}^tA$ denotes the transpose of the matrix A. Vectors v in E^k are column vectors, so that we write ${}^tv$ for the row vector corresponding to v

In particular, we will use

$$E^{2n}\ni v=\begin{pmatrix} x\\ y\end{pmatrix}, x,y\in E^n, E^{2n}\ni w=\begin{pmatrix} r\\ s\end{pmatrix}, r,s\in E^n,$$

Let I_n denote the $n\times n$ identity matrix and 0 denotes the zero matrix. We set

$$J:=\begin{pmatrix} 0 & I_n\\ -I_n & 0\end{pmatrix}.$$

Hence, $\omega(v,w) = {}^t\!xs - {}^t\!yr = {}^t\!vJw$. Thus,

$$(A,B,C,D) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, A,B,C,D \in M_n(E)$$

belongs to $Sp(n,E)$ if and only if

$${}^tAC = {}^tCA,\ \ {}^tDB = {}^tBD,\ \ {}^tAD - {}^tCB = I_n.$$

Let $G_n(E^{2n})$ denote the Grassmanian of the $n-$dimensional subspaces of E^{2n}. Hence, any of the the groups $Sp(n,E), Sp(n,F), Sp_0(n,F)$ acts on $G_n(E^{2n})$ by $TW = T(W)$.

A $n-$dimensional linear subspace W of (V,ω) is a Lagrangian subspace if and only if for every $v,w \in W$, ${}^t\!vJw = 0$ if and only if ${}^t\!xs - {}^t\!yr = 0$ for every $v,w \in W$. We fix $R,S \in E^{n\times n}$ and consider the subspace $W = \{\left(\begin{smallmatrix} Rx \\ Sx \end{smallmatrix}\right), x \in E^n\}$. Then, W is Lagrangian if and only if ${}^tRS - {}^tSR = 0$ and the matrix $\left(\begin{smallmatrix} R \\ S \end{smallmatrix}\right)$ has rank n. Actually, any Lagrangian subspace may be written as in the previous example (see also [8]). Particular examples of Lagrangian subspaces are $L_+, L_-, \mathcal{L}(Z)$, $(Z \in Sym(E^n))$. Needles to say, the image of $\mathcal{L}$ is equal to the orbit L_- under the subgroup P^+, hence, Bruhat's decomposition yields that the image of $\mathcal{L}$ is "open and dense" in $\mathcal{L}_{E,2n}$. Let $p : E^{2n} \to E^n$ denotes projection onto the second component. That is, $p\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = y$. It easily follows that:
A subspace $W \in G_n(E^{2n})$ belongs to the image of $\mathcal{L}$ if and only if W is Lagrangian and $p(W)$ is equal to E^n. We are ready for,

Lemma 1. *Let G be either $Sp(n,F)$ or $Sp_0(n,F)$ and fix $Z \in Sym(E^n)$. Then the orbit $G\mathcal{L}(Z)$ is contained in the image of $\mathcal{L}$ if and only if for every $(A,B,C,D) \in G$ the matrix $(CZ+D)$ is invertible.*

Proof: The subspace $(A,B,C,D)\mathcal{L}(Z) = \{\left(\begin{smallmatrix} (AZ+B)x \\ (CZ+D)x \end{smallmatrix}\right), x \in E^n\}$ is $n-$dimensional, Lagrangian and its image under p is equal to the image of $CZ+D$. Hence, if $(CZ+D)$ is invertible, by a change of variable we have that $(A,B,C,D)\mathcal{L}(Z)$ is equal to $\mathcal{L}(Z_1)$ for $Z_1 = (AZ+B)(CZ+D)^{-1}$. Conversely, if the orbit $G\mathcal{L}(Z)$ is contained in the image of the Siegel map, for each $g = (A,B,C,D) \in G$ there exists $Z_g \in Sym(E^n)$ so that

$$\{\left(\begin{smallmatrix} (AZ+B)x \\ (CZ+D)x \end{smallmatrix}\right), x \in E^n\} = \{\left(\begin{smallmatrix} Z_g x \\ x \end{smallmatrix}\right), x \in E^n\}.$$

Thus, the image of $CZ+D$ is equal to E^n.

□

Corollary 1. *$(A,B,C,D)\mathcal{L}(Z)$ belongs to the image of $\mathcal{L}$ if and only if $(CZ+D)$ is an invertible matrix.*

Example 1. *Orbits of $Sp_0(1,F)$ in the space of Lagrangians $\mathcal{L}_{E,2}$. We assume F is a finite field. Let $N(e) = e\bar{e}$ be the norm of the extension E/F.*

The hypothesis on F implies N is a surjective map onto F. After a computation, we obtain that $Sp_0(1,F)$ is the set of matrices

$$\{\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} : \alpha, \beta \in E, \alpha\bar{\alpha} - \beta\bar{\beta} = 1\}.$$

In this case $\mathcal{L}_{E,2} = G_1(E^2)$, a typical element of $G_1(E^2)$ is denoted by $E\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$ with $a \neq 0$ or $b \neq 0$. Since $h_0(\left(\begin{smallmatrix} z \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} w \\ 1 \end{smallmatrix}\right)) = 1 - z\bar{w}$, it readily follows:

$$\mathcal{O}_1 = \{E\left(\begin{smallmatrix} z \\ 1 \end{smallmatrix}\right), z \in E, N(z) \neq 1\} \cup \{L_+\},$$
$$\mathcal{O}_0 = \{E\left(\begin{smallmatrix} z \\ 1 \end{smallmatrix}\right), z \in E, N(z) = 1\}$$

For z so that $N(z) \neq 1$ we have $(1 - z\bar{z})^{-1} = t\bar{t}, t \in E$. If we define the matrix

$$A := \begin{pmatrix} \bar{t} & zt \\ \bar{z}\bar{t} & t \end{pmatrix}$$

then $A\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} zt \\ t \end{smallmatrix}\right)$. Obviously $A \in Sp_0(1,F)$. We are left to transform $E\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ into $E\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$. For this, we fix $z \neq 0$ such that $N(z^{-1}) \neq 1$. Then, by means of A the line $E\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ is transformed into the line $E\left(\begin{smallmatrix} 1 \\ \bar{z} \end{smallmatrix}\right)$, which is equal to the line $E\left(\begin{smallmatrix} \bar{z}^{-1} \\ 1 \end{smallmatrix}\right)$. From the previous calculation the last line is transformed into the line $E\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. Thus, $Sp_0(1,F)$ acts transitively in $\mathcal{O}_1$.

We now show $Sp_0(1,F)$ acts transitively in $\mathcal{O}_0$.

We fix $E\left(\begin{smallmatrix} a \\ 1 \end{smallmatrix}\right)$ so that $a\bar{a} = 1$. Let $E\left(\begin{smallmatrix} b \\ 1 \end{smallmatrix}\right)$ in $\mathcal{O}_0$. Then $N(a) = N(b)$, owing to theorem 90 of Hilbert we have $\frac{a}{b} = d\bar{d}^{-1}$. Since, the characteristic of F is different from two, the pair of vectors $\left(\begin{smallmatrix} a \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ -\bar{a} \end{smallmatrix}\right)$, as well as $\left(\begin{smallmatrix} b \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ -\bar{b} \end{smallmatrix}\right)$ determine two ordered basis for E^2. . Let T be the linear operator defined by $T(\left(\begin{smallmatrix} a \\ 1 \end{smallmatrix}\right)) = d\left(\begin{smallmatrix} b \\ 1 \end{smallmatrix}\right)$ and $T(\left(\begin{smallmatrix} 1 \\ -\bar{a} \end{smallmatrix}\right)) = d^{-1}\left(\begin{smallmatrix} 1 \\ -\bar{b} \end{smallmatrix}\right)$. One checks that $h_0(T\left(\begin{smallmatrix} a \\ 1 \end{smallmatrix}\right), T\left(\begin{smallmatrix} 1 \\ -\bar{a} \end{smallmatrix}\right)) = h_0(d\left(\begin{smallmatrix} b \\ 1 \end{smallmatrix}\right), d^{-1}\left(\begin{smallmatrix} 1 \\ -\bar{b} \end{smallmatrix}\right))$ and that $\omega(T\left(\begin{smallmatrix} a \\ 1 \end{smallmatrix}\right), T\left(\begin{smallmatrix} 1 \\ -\bar{a} \end{smallmatrix}\right)) = \omega(d\left(\begin{smallmatrix} b \\ 1 \end{smallmatrix}\right), d^{-1}\left(\begin{smallmatrix} 1 \\ -\bar{b} \end{smallmatrix}\right))$ to conclude that T lies in $U(E^2, h_0) \cap Sp(E^2, \omega) = Sp_0(1,F)$. Hence, $\mathcal{O}_0$ is an orbit of $Sp_0(1,F)$.

Remark 2. *The orbit $\mathcal{O}_0$ is contained in the image of the Siegel map, whereas the orbit $\mathcal{O}_1$ does contain a point in the complement to the image of the Siegel map. This observation shows that for a finite field F and $n = 1$ our conclusions are in concordance with the results obtained by other authors for the case of $F = \mathbb{R}$. More precisely in the real case, $\mathcal{O}_1$ splits in the union of two orbits, one orbit is the set of lines where h_0 is positive definite and the other is the set of lines where h_0 is negative definite. In this case the orbit corresponding to the set of lines where h_0 is positive definite is contained in the image of the Siegel map, whereas the orbit corresponding to the set of lines where h_0 is negative definite is not contained in the image of the Siegel map. The orbit corresponding to the set of lines where h_0 vanishes is contained in the image of the Siegel map.*

Remark 3. *The previous computations together with corollary 1 to lemma 1, let us conclude that* $\bar{\beta}z+\bar{\alpha}$ *is nonzero for every element of* $Sp_0(1,F)$ *such that* $-z\bar{z}+1=0$. *Whereas, for each* z *so that* $-z\bar{z}+1\neq 0$, *there exist an element of* $Sp_0(1,F)$ *so that* $\bar{\beta}z+\bar{\alpha}=0$, *it is the element that carries the line of direction* $(z,1)$ *onto the line of infinite slope!*

We have

Lemma 2. Z *be an element of* $Sym(E^n)$. *Then* $\mathcal{L}(Z)$ *belongs to* $\mathcal{H}_r$ *if and only if the anti-hermitian form on* E^n *defined by* $Z-\bar{Z}$ *has rank* r.

Proof. For the non-degenerate anti-hermitian form h_E on E^{2n}, given by $h_E(v,w):=w(v,\bar{w})={}^t x\bar{s}-{}^t y\bar{r}\ (v,w\in E^{2n})$, we have $Sp(n,F)=U(E^{2n},h_E)\cap Sp(n,E)$. Hence, $\mathcal{H}_j$ is invariant under the action of $Sp(n,F)$. It follows that

$$h_E(\left(\begin{smallmatrix} Zx \\ x \end{smallmatrix}\right),\left(\begin{smallmatrix} Zy \\ y \end{smallmatrix}\right))={}^t x(Z-\bar{Z})y \quad (x,y\in E^n),$$

from which the result. □

Example 2. *We now compute the orbits of* $Sp(1,F)$ *in* $\mathcal{L}_{E,2}$ *for a finite field F. For this we show that each* $\mathcal{H}_j$ *is an orbit of* $Sp(1,F)$. *In fact,*

$$\mathcal{H}_1=\{E\left(\begin{smallmatrix} z \\ 1 \end{smallmatrix}\right):z-\bar{z}\neq 0\} \ \text{ and } \mathcal{H}_0=\{E\left(\begin{smallmatrix} z \\ 1 \end{smallmatrix}\right):z\in F\}\cup\{E\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\}.$$

Since $J\in Sp(n,F)$ *we have that* $E\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ *is in the orbit of* $E\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. *Since the matrix* $(1,t,0,1)\in Sp(1,F)$ *and* $(1,t,0,1)(0,1)=(t,1)$ *we have that* $Sp(1,F)$ *acts transitively in* $\mathcal{H}_0$.

We now show that $Sp(1,F)$ *acts transitively in* $\mathcal{H}_1$. *Let* $E\left(\begin{smallmatrix} z \\ 1 \end{smallmatrix}\right),E\left(\begin{smallmatrix} w \\ 1 \end{smallmatrix}\right)$ *so that* $z-\bar{z}\neq 0, w-\bar{w}\neq 0$, *Since F is a finite field, there exists* $t_0\in E$ *so that* $z-\bar{z}=t_0\bar{t}_0(w-\bar{w})$. *We define*

$$A:=\frac{1}{z-\bar{z}}\begin{pmatrix} t_0w-\bar{t}_0\bar{w} & z\bar{t}_0\bar{w}-\bar{z}t_0w \\ t_0-\bar{t}_0 & z\bar{t}_0-\bar{z}t_0 \end{pmatrix}$$

The coefficients of A belong to F and

$$A\left(\begin{smallmatrix} z \\ 1 \end{smallmatrix}\right)=\frac{z}{z-\bar{z}}\begin{pmatrix} t_0w-\bar{t}_0\bar{w} \\ t_0-\bar{t}_0 \end{pmatrix}+\frac{1}{z-\bar{z}}\begin{pmatrix} z\bar{t}_0\bar{w}-\bar{z}t_0w \\ z\bar{t}_0-\bar{z}t_0 \end{pmatrix}=t_0\begin{pmatrix} w \\ 1 \end{pmatrix}.$$

$$detA=\frac{(z-\bar{z})(w-\bar{w})t_0\bar{t}_0}{(z-\bar{z})^2}=1.$$

We note that $\mathcal{H}_1$ *is contained in the image of the Siegel map, whereas* $\mathcal{H}_0$ *is not.*

We consider now

Let $g\in Sp_0(n,F)$, then

$$g^{-1}=diag(-I_n,I_n)\ {}^t\bar{g}\,diag(-I_n,I_n)$$

Therefore, the elements of $Sp_0(n,F)$ are the matrices

$$\begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix} A, B \in M_n(E),\ {}^t\bar{A}B = {}^tB\bar{A},\ {}^tA\bar{A} - {}^t\bar{B}B = I$$

Since $Sp_0(n,F)$ is invariant under the map $g \mapsto {}^tg$, we get the characterization of $Sp_0(n,F)$ obtained by [4], namely,
$(R,S,T,V) \in Sp_0(n,F)$ if and only if

$$T = \bar{S},\ V = \bar{R},\ R\ {}^tS = S\ {}^tR,\quad R\ {}^t\bar{R} - S\ {}^t\bar{S} = I_n. \qquad (S)$$

as is readily seen.

A simple computation shows:

$$Sp_0(n,F) \cap KP^+ = Sp_0(n,F) \cap KP^- = diag(A, \bar{A}), A \in U(n,E).$$

Now assuming that F is a finite field, we prove that any set $\mathcal{O}_r$ intersects nontrivially the image of the Siegel map, and for $r > 0$, that $\mathcal{O}_r$ contains a point of the complement of the image of the Siegel map.

We observe that the form h_0 restricted to $\mathcal{L}(diag(d_1,\ldots,d_n))$ is the diagonal form

$$(1 - d_1\bar{d}_1)x_1\bar{y}_1 + \cdots + (1 - d_n\bar{d}_n)x_n\bar{y}_n.$$

Thus, F being a finite field, allow us to find d so that $d\bar{d} = 1$, from which we obtain that $\mathcal{L}(diag(0,\ldots,0,d,\ldots d))$ (r zeros) belongs to $\mathcal{O}_r$.

We fix now $0 < r \le n$ and $d \in E$ such that $d\bar{d} = 1$. Let W_r denote the subspace spanned by the vectors $e_1,\ldots,e_r, de_{r+1} + e_{n+r+1},\ldots, de_n + e_{2n}$. Then W_r is $n-$dimensional and isotropic for ω. The matrix of the form h_0 restricted to W_r, on the above basis, is $diag(-1,\ldots,-1,0,\ldots,0)$, (here -1 occurs r-times). Hence, W_r belongs to $\mathcal{O}_r$. Moreover, the dimension of $p(W_r)$ (p as defined before lemma 1) is $n - r < n$. Therefore W_r does not belong to the image of the Siegel map.

Remark 4. *For any permutation matrix T we have that the matrix $\begin{pmatrix} {}^tT^{-1} & 0 \\ 0 & T \end{pmatrix}$ belongs to $Sp(n,F)_0 \cap Sp(n,F)$.*

For the time being we assume -1 is not a square in F.

We now show that for odd $n > 1$, $\mathcal{O}_0$ contains points in the image of the Siegel map, and contain points in the complement of the image of the Siegel map.

To begin with, we consider $n = 3$.

We fix $d, c \in E$ so that $0 = 1 + c\bar{c} + d\bar{d}$ and $c\bar{d} \in F$.

We set

$$A := \begin{pmatrix} 1 & 0 & -c \\ 0 & 1 & -d \\ \bar{c} & \bar{d} & 1 \end{pmatrix} \quad B := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c & d & 0 \end{pmatrix}$$

Then,

$${}^{t}AB = \begin{pmatrix} 1+c\bar{c} & \bar{c}d & 0 \\ c\bar{d} & 1+d\bar{d} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad {}^{t}BA = \begin{pmatrix} 1+c\bar{c} & c\bar{d} & 0 \\ \bar{c}d & 1+d\bar{d} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Given that $\bar{c}d \in F$, both matrices are equal. Thus, the subspace $L := \{(Ax, Bx), x \in E^n\}$ is Lagrangian.

Since

$${}^{t}A\bar{A} = \begin{pmatrix} 1+c\bar{c} & \bar{c}d & 0 \\ c\bar{d} & 1+d\bar{d} & 0 \\ 0 & 0 & c\bar{c}+d\bar{d}+1 \end{pmatrix}, \quad {}^{t}B\bar{B} = \begin{pmatrix} 1+c\bar{c} & c\bar{d} & -c \\ \bar{c}d & 1+d\bar{d} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

both matrices are equal, and therefore h_0 restricted to L is the zero form. On the other hand, $detA = 1 + d\bar{d} + c\bar{c} = detB = 0$. This shows that L is an element of $\mathcal{O}_0$ which is not in the image of the Siegel map.

Now, in order to produce an element of $\mathcal{O}_0$ in the complement of the image of the Siegel map for odd n with $n > 3$, we write $n = 3+n-3$. Then the subspace $L \oplus E(e_4 + e_{n+4}) \oplus \cdots \oplus E(e_n + e_{2n})$ satisfies our requirement.

Finally, the subspace $\mathcal{L}(I_n)$, is an element of $\mathcal{O}_0$ which is in the image of the Siegel map.

For n even, $\mathcal{O}_0$ contains points in the complement of image of the Siegel map.

Let us take $c, b \in E$ such that $b\bar{b} = -1$. We set

$$A := \begin{pmatrix} -bc & -b \\ c & 1 \end{pmatrix} \qquad B := \begin{pmatrix} 1 & 0 \\ b & 0 \end{pmatrix}$$

Then ${}^{t}AB = {}^{t}BA = (0, 0, 0, 0)$

Thus, $W := \{(Ax, Bx), x \in E^2\}$ is a Lagrangian subspace.

Given that ${}^{t}A\bar{A} = {}^{t}B\bar{B} = (0, 0, 0, 0)$, we see that, h_0 restricted to W is the null form, that is, $W \in \mathcal{O}_0$.

Further, neither A nor B is invertible, and so W is not in the image of the Siegel map.

For $n = 2k$, it readily follows that the subspace $W \oplus \cdots \oplus W_2$ ($k-$times) belongs to $\mathcal{O}_0$ and it does not belong to the image of the Siegel map.

We compute now an example of points in $\mathcal{O}_n$ which are outside the image of the Siegel map, and also compute an element in $Sp_0(n, F)$ which carries these points into the image of the Siegel map.

For this, we notice that $(\alpha I_n, \beta I_n, \bar{\beta} I_n, \bar{\alpha} I_n)$ belong to $Sp_0(n, F)$ if and only if $\alpha\bar{\alpha} - \beta\bar{\beta} = 1$.

We fix an integer k so that $1 < k < n$. The subspace Z_k spanned by $e_1, \ldots, e_k, e_{n+k+1}, \ldots, e_n$ is Lagrangian, and h_0 is non degenerate on it. Obviously, Z_k does not belong to the image of the Siegel map. We may choose nonzero α, β so that $\alpha\bar{\alpha} - \beta\bar{\beta} = 1$. Then $(\alpha I_n, \beta I_n, \bar{\beta}, \bar{\alpha})$ takes Z_k into a subspace which belongs to the image of the Siegel map.

We will use bellow the following involution: for a matrix A, $A^\star =^t \bar{A}$

Lemma 3. *$Sp_0(n, F)$ acts transitively on $\mathcal{O}_n$.*

Proof. We have that $L_- = \mathcal{L}(0)$ is an element of $\mathcal{O}_n$. First, we will prove that given $\mathcal{L}(Z) \in \mathcal{O}_n$, there is an element of $Sp_0(n, F)$ which carries $\mathcal{L}(Z)$ onto L_-.

The matrix of the form h_0 restricted to $\mathcal{L}(Z)$ is $I_n - Z\bar{Z}$. Choosing an adequate basis, there exists an invertible matrix A so that $A(I_n - Z\bar{Z})\,{}^t\bar{A} = I_n$. Let set $B := -AZ$. Then, since

$$A\ {}^t(-AZ) = -AZ\,{}^tA, \text{ and } A\ {}^t\bar{A} - (-AZ)(-\,{}^t(\bar{A}\bar{Z}) = A(I_n - Z\bar{Z})\,{}^t\bar{A} = I_n$$

the matrix $(A, B, \bar{B}, \bar{A})$ belongs to $Sp_0(n, F)$ (it satisfies (S)).

On the other hand,

$$(A, B, C, D)\mathcal{L}(Z) = \{\left(\begin{smallmatrix} (AZ+(-AZ))x \\ (\bar{B}Z+\bar{A})x \end{smallmatrix}\right), x \in E^n\} = \{\left(\begin{smallmatrix} 0 \\ \bar{A}(I_n-\bar{Z}Z)x \end{smallmatrix}\right), x \in E^n\},$$

By above, the matrix $\bar{A}(I_n - \bar{Z}Z)$ is invertible, so that $\mathcal{L}(Z)$ belongs to the orbit of L_-.

Next, we will show that if $W = \{\left(\begin{smallmatrix} Rx \\ Sx \end{smallmatrix}\right) : x \in E^n\} \in \mathcal{O}_n$, then there exists an element g in $Sp_0(n, F)$ so that $gW \in Image(\mathcal{L})$.

In fact, we will show there exists $g \in Sp_0(n, F)$ so that $gW = \{(Cx, Dx) : x \in E^n\}$ with C invertible, and then by means of a matrix $(0, dI_n, \bar{d}I_n, 0)$ we will transform gW into an element of the image of the Siegel map.

Since W is in $\mathcal{O}_n$, there exists an invertible matrix A such that

$$A(-R^\star R + S^\star S)A^\star = I_n.$$

Let us consider $g = (-AR^\star, AS^\star, \bar{A}\bar{S}^\star, -\bar{A}\bar{R}^\star)$. Then

$$gW = \{((A^\star)^{-1}x, (\bar{A}\bar{S}^\star R - \bar{A}\bar{R}^\star S)x)x \in E^n\}.$$

Since

$$-AR^\star(-AR^\star)^\star - AS^\star(AS^\star)^\star = A(-R^\star R + S^\star S)A^\star = I_n$$

Also ${}^tRS = {}^tSR$, (because W is a Lagrangian subspace), hence we have $-AR^\star\ {}^t(AS^\star) = -AR^\star\bar{S}\,{}^tA = -A\,{}^t\bar{S}\bar{R}\,{}^tA = AS^\star\ {}^t(AR^\star)$, and so the matrix g belongs to $Sp_0(n, F)$This concludes the proof that $\mathcal{O}_n$ is the orbit of L_- under the group $Sp_0(n, F)$. □

Proposition 2. *There exists element C in $Sp(n,E)$ so that C^{-1} conjugates $Sp_0(n,F)$ onto $Sp(n,F)$.*

Proof. When -1 is not an square in F the proof follows quite close to the real case. We fix $i \in E$ a square root for -1. We consider the $2n \times 2n$ matrix

$$C_n := \frac{1}{\sqrt{-2}} \begin{pmatrix} iI_n & I_n \\ I_n & iI_n \end{pmatrix}$$

It readily follows that the matrix $C_n \in Sp(n,E)$. Let τ_F denote -1 if -2 is not a square in F and 1 if -2 is a square in F. We now verify the equality

$$\tau_F i h_E(v,w) = h_0(C_n v, C_n w) \text{ for every } v, w \in E^{2n}.$$

For this, we note that

$$ {}^t C_n diag(-I_n, I_n)\bar{C}_n = \tau_F iJ, \qquad \bar{C}_n J^{-1}\, {}^t C_n = i\tau_F diag(-I_n, I_n)$$

Indeed,

$$ {}^t C_n diag(-I_n, I_n)\bar{C}_n = \frac{1}{-2\tau_F} \left(\begin{smallmatrix} -i & 1 \\ -1 & i \end{smallmatrix}\right) \left(\begin{smallmatrix} -i & 1 \\ 1 & -i \end{smallmatrix}\right) = \frac{-2i}{-2\tau_F} J.$$

$$\bar{C}_n J^{-1}\, {}^t C_n = \frac{1}{-2\tau_F} \left(\begin{smallmatrix} -i & 1 \\ 1 & -i \end{smallmatrix}\right) \left(\begin{smallmatrix} -1 & -i \\ i & 1 \end{smallmatrix}\right) = i\tau_F diag(-I_n, I_n).$$

Hence, for $g \in Sp(n,F)$ we have

$$ {}^t(C_n g C_n^{-1}) diag(-I_n, I_n)\overline{CgC^{-1}} = \tau i\, {}^t C^{-1}\, {}^t g J g \bar{C}^{-1}$$
$$= \tau i (\tau i\ diag(-I_n, I_n))^{-1} = diag(-I_n, I_n).$$

Thus, $CgC^{-1} \in Sp_0(n,F)$. Owing to the equalities of above we deduce, $h_0(v,w) = h_0(CgC^{-1}v, Cgc^{-1}w) = \tau_F i h_E(v,w)$. Tracing back the computation, we arrive to $C^{-1}gC \in Sp(n,F)$ for $g \in Sp_0(n,F)$. Hence, we have proved the proposition when $-1 \notin F$. In case $-1 \in F$ we follow the proof in [6]. We choose $v \in E$ so that $N(v) = -1, b \in E^\times : b + \bar{b} = 0$. We define

$$C_n := \frac{1}{\sqrt{b(v^2-1)}} \begin{pmatrix} vI_n & bI_n \\ I_n & vbI_n \end{pmatrix}$$

Then, $C_n \in Sp(n,E)$ and $(b(v^2-1))\, {}^t\bar{C}_n C_n = (v+\bar{v})bJ$. A similar computation gives $h_0(Cv, Cw) = -(v+\bar{v})b\, h_E(v,w)$. □

Corollary 2. *The group $Sp(n,F)$ acts transitively on $\mathcal{H}_n$.*

Proof. Since the groups $Sp(n,F)$ and $Sp_0(n,F)$ are conjugated by the Cayley transform and the Cayley transform is a conformal map for the pair of bilinear forms h_0, h_E the corollary follows □

Remark 5. *If -1 is not a square in F.*

$$C_n^{-1} = -\tau_F \bar{C}_n.$$

For a subset W of E^{2n}, we define $\overline{W} = \{\bar{w}, w \in W\}$. For the linear subspace W, we denote by r_W the rank of the form h_E restricted to W.

Lemma 4. *For a Lagrangian subspace W of E^{2n} we have:*

$$dim(W+\overline{W}) = n + r_W$$

$$dim(W\cap\overline{W}) = n - r_W$$

Furthermore, $W\cap\overline{W} = (W+\overline{W})^{\perp_\omega} = W^{\perp_{h_E}}$.

Proof. We use the identities

$$Z^{\perp_\omega}\cap U^{\perp_\omega} = (Z+U)^{\perp_\omega}, (Z\cap U)^{\perp_\omega} = Z^{\perp_\omega} + U^{\perp_\omega}.$$

Since $W, \overline{W}$ are Lagrangian subspaces we have

$$W\cap\overline{W} = W^{\perp_\omega}\cap\overline{W}^{\perp_\omega} = (W+\overline{W})^{\perp_\omega}.$$

Fix $y=\bar{z}\in W\cap\overline{W}, z\in W,$ and $x\in W$, hence $h_E(x,y)=\omega(x,\bar{y})=\omega(x,z)=0$. Hence, $y\in W^{\perp_{h_E}}$. Next, for $y\in W^{\perp_{h_E}}$, we have $\omega(\bar{x},y)=0$ for every $x\in W$. The hypothesis W is Lagrangian forces $\bar{y}\in W$, hence $y=\bar{\bar{y}}\in W\cap\overline{W}$. □

Proposition 3. *For a finite field F and $k=0,\dots,n$, the group $Sp(n,F)$ acts transitively on $\mathcal{H}_k$.*

Proof. We make the following induction hypothesis: for every $m<n$ and for every $k\le m$ the group $Sp(m,F)$ acts transitively on the $\mathcal{H}_k$ determinate by the corresponding form h_E on (E^{2m},ω).

Since, we have already shown that $Sp(1,F)$ acts transitively on $\mathcal{H}_k, k=0,1$, the first step of the induction process follows.

We recall also that for n and $k=n$ we have shown that $Sp(n,F)$ acts transitively on $\mathcal{H}_n$. We are left to consider $r<n$.

We fix $W,Y\in\mathcal{H}_r$ with $r=r_W<n$, we must find $g\in Sp(n,F)$ so that $gW=Y$.

Since, each of the subspaces $W\cap\overline{W}, W+\cap\overline{W}$ are invariant under the Galois automorphism, it follows that the subspaces are the complexification of, respectively, $F^{2n}\cap W\cap\overline{W}, F^{2n}\cap(W+\cap\overline{W})$. We notice that the quotient space $(W+\cap\overline{W})/(W\cap\overline{W})$ is of dimension $n+r-(n-r)=2r<2n$. Now, by above we have that the push forward to $(W+\overline{W})/(W\cap\overline{W})$ of the form ω is a non degenerate form, and the same holds for h_E.

Thus, the inductive hypothesis gives a linear transform

$$T: F^{2n}\cap(W+\overline{W})/(F^{2n}\cap W\cap\overline{W}) \to F^{2n}\cap(Y+\overline{Y})/(F^{2n}\cap Y\cap\overline{Y})$$

such that $T^\star\omega=\omega$, and the complex extension transforms $W/(W\cap\overline{W})$ onto $Y/(Y\cap\overline{Y})$. We lift T to a linear transform

$$T: F^{2n}\cap(W+\overline{W}) \to F^{2n}\cap(Y+\overline{Y})$$

so that $T^\star\omega=\omega$ and the complex extension transforms W onto Y. Now we apply the theorem of Witt to T to get an element g of $Sp(n,F)$ which carries W into Y. This completes the induction process and we have the result □

Corollary 3. *$Sp_0(n,F)$ acts transitively in $\mathcal{O}_k, k=1,\dots,n$*

Lemma 5. *$\mathcal{H}_n$ is contained in the image of the Siegel map.*

Proof. Let $W \in \mathcal{H}_n$. We may choose representatives R and S for W and write then $W = \{\left(\begin{smallmatrix} Rx \\ Sx \end{smallmatrix}\right), x \in E^n\}$ with ${}^tRS - {}^tSR = 0$, $\left(\begin{smallmatrix} R \\ S \end{smallmatrix}\right)$ of rank n . Since $W \in \mathcal{H}_n$ the matrix ${}^tR\bar{S} - {}^tSR$ is invertible.

The matrix S has rank r, with $0 \leq r \leq n$. We will show that $r = n$.

We choose two $n \times n$ permutation matrices matrices P, Q in $GL_n(F)$ such that

$$PSQ = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$$

where B_1 is an invertible $r \times r$ matrix. We may write $W = \{\left(\begin{smallmatrix} RQx \\ SQx \end{smallmatrix}\right), x \in E^n\}$ and

$$\begin{pmatrix} ({}^tP)^{-1} & 0 \\ 0 & P \end{pmatrix} W = \{\left(\begin{smallmatrix} ({}^tP)^{-1}RQx \\ (B_1,B_2,B_3,B_4)x \end{smallmatrix}\right), x \in E^n\}$$

Since (B_1, B_2, B_3, B_4) has rank r performing column operations, we may assume B_2 and B_4 are the zero matrices. This amounts to a new change of representatives for W. Thus, $W = \{\left(\begin{smallmatrix} Ax \\ (B_1,0,B_3,0)x \end{smallmatrix}\right), x \in E^n\}$

Write $A := (A_1, A_2, A_3, A_4)$ with $, A_1 \in M_r(F), A_4 \in M_{n-r}(F)$. The hypothesis W is a Lagrangian, implies ${}^tA(B_1, 0, B_3, 0) = {}^t(B_1, 0, B_3, 0)A$ from which ${}^tA_1 + {}^tA_3B_3 = A_1 + {}^tB_3A_3, A_2 + {}^tB_3A_4 = 0$. The hypothesis the rank of $(A, (B_1, 0, B_3, 0))$ is n implies A_4 is invertible. Hence, replacing x by $(diag(I_r), 0, 0, A_r^{-1})x$ gives $W = \{\left(\begin{smallmatrix} Cx \\ Dx \end{smallmatrix}\right), x \in E^n\}$, with $C = (A_1, 0, A_3, I_{n-r})$ and $D = (diag(I_r), 0, B_3, 0)$. The matrix of h_E in this new coordinates is

$${}^tC\bar{D} - {}^tD\bar{C} = \begin{pmatrix} \bullet & 0 \\ \bullet & 0 \end{pmatrix}.$$

The hypothesis h_E restricted W has rank n implies then $n - r = 0$. $\square$

Corollary 4. *For any element $Z \in Sym(E^n)$ such that $Z - \bar{Z}$ is invertible and for any $(A, B, C, D) \in Sp(n, F)$, the matrix $CZ + D$ is invertible.*

We have completed the proof of theorem 1.

Furthermore, we have the following facts:

Remark 6. *For a symmetric matrix Z such that $Z - \bar{Z}$ is not invertible, there exists $(A, B, C, D), (M, N, R, S) \in Sp(n, F)$ such that $CZ + D$ is invertible and $RZ + S$ is not invertible.*

This follows from Corollary 1 to lemma 1 and theorem 1.

Remark 7. *For $n > 1$ and any symmetric matrix Z there exists (A, B, C, D), $(M, N, R, S) \in Sp(n, F)_0$ so that $CZ + D$ is invertible and $RZ + S$ is not invertible.*

This follows from lemma 1 and theorem 1

4. Isotropy subgroups

The purpose of this section is to explicitly compute the structure of $\mathcal{O}_k, \mathcal{H}_k, k = 0, \dots, n$ as homogeneous spaces. For the real case, this has been accomplished by [11] [7] and references therein.

An element of $\mathcal{O}_{n-k}$ is constructed as follows: we define V_k to be the subspace spanned by the vectors $e_1+e_{n+1}, \dots, e_k+e_{n+k}, e_{k+1}, \dots, e_n$. Then, $V_0 = L_+$. A simple computation shows that the form h_E restricted to $V_k \times V_k$ is the null form, whereas the type of the form h_0 restricted to $V_k \times V_k$ is $n-k$,. Obviously V_k is a lagrangian subspace. Henceforth, for $x \in Sp(n, E)$, $Ad(x)$ denotes the inner automorphism defined by x. Let t_k be the partial Cayley transform

$$t_k := \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix}$$

where, $D_1 = D_4 = diag(\frac{\sqrt{2}}{2} I_k, I_{n-k}), D_2 = diag(-\frac{\sqrt{2}}{2} I_k, 0), D_3 = -D_2$. Then, t_k is an element of $Sp(n, E)$ and $t_k L_+ = t_k V_0 = V_k$. A computation gives

$$t_k^{-1} = \begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \end{pmatrix}$$

where, $L_1 = L_4 = diag(\frac{\sqrt{2}}{2} I_k, I_{n-k}), L_2 = diag(\frac{\sqrt{2}}{2} I_k, 0), L_3 = -L_2$.

Let $\mathcal{E}_{Sp_0(n,F)}(V_k)$ denote the set stabilizer of V_k in $Sp_0(n, F)$. The equality $\mathcal{E}_{Sp(n,E)}(V_0) = KP^+$ implies

$$\mathcal{E}_{Sp_0(n,F)}(V_k) = Ad(t_k)\mathcal{E}_{Sp(n,E)}(V_0) \cap Sp_0(n, F) = Ad(t_k)(KP^+) \cap Sp_0(n, F).$$

The stabilizer of V_0 in $Sp(n, F)$ is $KP_+ \cap Sp_0(n, F) = K \cap Sp_0(n, F) = \{diag(T, \bar{T}) : T \in U(n, E)\}$. Thus, the stabilizer of V_0 in $Sp_0(n, F)$ is isomorphic to $U(n, E)$.

The main result of this section is

Theorem 2. *The stabilizer group $\mathcal{E}_{Sp_0(n,F)}(V_k)$ is isomorphic to the semidirect product of the group $O(k, F) \times U(n-k, E)$ times the unipotent subgroup $Ad(t_k)(P^+) \cap Sp_0(n, F)$.*

The proof of the result requires some computations, which we carry out.

First, we verify that the subgroup of $Sp_0(n, F)$, $diag(S, T, S, \bar{T}), S$ in $O(k, F), T$ in $U(n-k, E)$ is contained in $\mathcal{E}_{Sp_0(n,F)}(V_k)$. For this, we write for $v \in V_k, v = \begin{pmatrix} x \\ y \\ x \\ 0 \end{pmatrix}$ with $x \in E^k, y \in E^{n-k}$. Hence,

$$diag(S, T, S, \bar{T})v = \begin{pmatrix} Sx \\ Ty \\ Sx \\ 0 \end{pmatrix} \in V_k.$$

Is clear that the unipotent subgroup is contained in $\mathcal{E}_{Sp_0(n,F)}(V_k)$.

For a matrix $T \in E^{n\times n}$ we write

$T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}, T_1 \in E^{k\times k}, T_2 \in E^{k\times n-k}, T_3 \in E^{n-k\times k}, T_4 \in E^{n-k\times n-k}$

And for $(A, B, 0, D) \in KP^+$ we have

$$Ad(t_k)(A, B, 0, D)$$
$$= \begin{pmatrix} \frac{1}{2}(A_1 - B_1 + D_1) & \frac{\sqrt{2}}{2}A_2 & \frac{1}{2}(A_1 + B_1 - D_1) & \frac{\sqrt{2}}{2}(B_2 - D_2) \\ \frac{\sqrt{2}}{2}(A_3 - B_3) & A_4 & \frac{\sqrt{2}}{2}(A_3 + B_3) & B_4 \\ \frac{1}{2}(A_1 - B_1 - D_1) & \frac{\sqrt{2}}{2}A_2 & \frac{1}{2}(A_1 + B_1 + D_1) & \frac{\sqrt{2}}{2}(B_2 + D_2) \\ -\frac{\sqrt{2}}{2}D_3 & 0 & \frac{\sqrt{2}}{2}D_3 & D_4 \end{pmatrix}.$$

Next, we show that $Ad(t_k)K \cap Sp_0(n, F)$ is equal to the subgroup $\{diag(S, T, S, {}^tT^{-1}) : S \in O(k, F), T \in U(n, E)\}$. In fact, the computation for $Ad(t_k)X$ gives for $S \in O(k, F), T \in U(n, E)$ that $Ad(t_k)(diag(S, T, S, {}^tT^{-1})) = diag(S, T, S, {}^tT^{-1})$.

Now for $diag(A, D) = diag(A, {}^tA^{-1}) \in K$, such that $Ad(t_k)(diag(A, D)) \in Sp_0(n, F)$, (1.2) and the formula for $Ad(t_k)X$ imply the equalities

$$\overline{(A_1 + D_1)} = A_1 + D_1, \quad \bar{A}_2 = D_2 \quad \bar{A}_3 = D_3, \quad \bar{A}_4 = D_4$$

and

$$\overline{A_1 - D_1} = A_1 - D_1, \quad A_2 = -\bar{D}_2, \quad A_3 = -\bar{D}_3$$

So

$$D_2 = A_2 = 0, \quad D_3 = A_3 = 0, \quad , \bar{A}_1 = A_1, \bar{D}_1 = D_1.$$

Hence, $A_1 \in O(n, F)$. Finally, the equality $D = {}^tA^{-1}$ yields, $A_1 = D_1$, which shows the claim.

Now $Ad(t_k)P^+ \cap Sp_0(n, F) = \{Ad(t_k)(I_n, B, 0, I_n) : {}^tB = B \text{ and } \bar{B}_1 = -B_1, B_3 = {}^tB_2 = 0, B_4 = 0\}$. In fact, the formula for $Ad(t_k)X$ leads us to

$$Ad(t_k)(I, B, 0, I) = \begin{pmatrix} \frac{1}{2}(2I - B_1) & 0 & \frac{1}{2}B_1 & \frac{\sqrt{2}}{2}B_2 \\ -\frac{\sqrt{2}}{2}B_3 & I & \frac{\sqrt{2}}{2}B_3 & B_4 \\ -\frac{1}{2}B_1 & 0 & \frac{1}{2}(2I + B_1) & \frac{\sqrt{2}}{2}B_2 \\ 0 & 0 & 0 & I \end{pmatrix}$$

From (1.2) we get $\bar{B}_1 = -B_1, \quad B_2 = 0, \quad B_4 = 0$, and the equality follows.

(E) We will show at this point the equality $\mathcal{E}_{Sp_0(n,F)}(V_k) = (Ad(t_k)K \cap Sp_0(n, F))(Ad(t_k)P^+ \cap Sp_0(n, F))$.

Let $X \in KP^+$ so that $Ad(t_k)X \in Sp_0(n, F)$. Condition (1.2) gives us the following equalities,

$$\bar{A}_1 - \bar{B}_1 + \bar{D}_1 = A_1 + B_1 + D_1, \quad B_2 + D_2 = \bar{A}_2, \quad \bar{A}_3 - \bar{B}_3 = D_3, \quad \bar{A}_4 = D_4$$
$$\bar{A}_1 + \bar{B}_1 - \bar{D}_1 = A_1 - B_1 - D_1, \quad \bar{B}_2 - \bar{D}_2 = A_2, \quad \bar{A}_3 + \bar{B}_3 = -D_3, \quad B_4 = 0.$$

From the second equality on each line, we deduce $D_2 = 0$. Thus, $B_2 = \bar{A}_2$. From the third equality in both lines we obtain $\bar{A}_3 = 0$. Hence $A_3 =$

0 and $B_3 = -\bar{D}_3$. Next ${}^tAD - {}^tB0 = I$ give us $D = {}^tA^{-1}$. Explicitly $D = ({}^tA_1^{-1}, 0, -{}^t(A_1^{-1}A_2A_4^{-1}), {}^tA_4^{-1})$. Since $(A, B, 0, D) \in Sp(n, E)$ and so ${}^tBD = {}^tDB$. The computation of the last equality lead us to

$$\begin{pmatrix} A_1^{-1}B_1 - {}^tY\bar{Y} & A_1^{-1}\bar{A}_2 \\ A_4^{-1}\bar{Y} & 0 \end{pmatrix} = \begin{pmatrix} {}^tB_1\,{}^tA_1^{-1} - {}^t\bar{Y}Y & {}^t\bar{Y}\,{}^tA_4^{-1} \\ -\,{}^t\bar{A}_2^{-1}\,{}^tA_1^{-1} & 0 \end{pmatrix}$$

where $Y := {}^t(A_1^{-1}A_2A_4^{-1})$

Now, the equality of the (2,1)-coefficients gives $A_4^{-1}\,{}^t\bar{A}_4^{-1}\,{}^t\bar{A}_2\bar{A}_1^{-1} = -\,{}^t\bar{A}_2\,{}^tA_1^{-1}$, which, after we transpose both members of the last equality, we obtain

$$\bar{A}_1^{-1}\bar{A}_2\bar{A}_4^{-1}\,{}^tA_4^{-1} = -A_1^{-1}\bar{A}_2.$$

From, equality of the (1,2)-coefficients implies

$$\bar{A}_1^{-1}\bar{A}_2\bar{A}_4^{-1}\,{}^tA_4^{-1} = A_1^{-1}\bar{A}_2.$$

Thus, $A_2 = 0$ and we have that

$$(A, B, 0, D) = (diag(A_1, A_4), diag(B_1, 0), 0, diag({}^tA_1^{-1}, {}^tA_4^{-1})).$$

The hypothesis $Ad(t_k)(A, B, 0, D) \in Sp_0(n, F)$ let us conclude that $A_1 \in O(k, F)$,
$A_4 \in U(n-k, E)$. From here, (E) is shown, and the theorem follows.

5. Anti-involutions in $Sp(n, F)$.

In this section we analyze the structure on the set of anti-involutions in the group $Sp(n, F)$. We will show that this set is a homogeneous space for $Sp(n, F)$.
The denote by $\mathcal{C}(n, F)$ the set of anti-involutions ,i.e.,

$$\mathcal{C}(n, F) = \{T \in Sp(n, F) : T^2 = -1\}.$$

Proposition 4. *$\mathcal{C}(n, F)$ is equivariant isomorphic to $\mathcal{H}_n$ when -1 is not a square in F, whereas is isomorphic to*
$Sp(n, F)/(Sp(n, F) \cap K)$ when -1 is a square in F.

It is clear that $\mathcal{C}(n, F)$ is invariant under conjugation. Since $J = (0, I_n, -I_n, 0)$ is an element of $Sp(n, F)$ we have that JT is an element of $Sp(n, F)$. The poof of the proposition will follow from the next three lemmas

Lemma 6. *i) Let T be an involution, then JT is a symmetric matrix. That is, ${}^t(JT) = JT$*

ii) For $T \in Sp(n, F)$, such that JT is symmetric, we have that T is an involution.

Proof: Recall ${}^tJ = -J$, ${}^tTJT = J, T^2 = -1$ Hence, ${}^t(JT) = -\,{}^tTJ = -JT^{-1} = JT$. For the second statement, we have ${}^t(JT) = JT$ hence $J = -\,{}^tT^{-1}JT = {}^tTJT$ thus $T^2 = -I$.

□

According to lemma 6, to each involution T in $Sp(n,F)$ we naturally associate a symmetric non-degenerate bilinear form b_T on F^{2n}. The matrix of the form b_T in the canonical basis is JT.

Now, from the classification of symmetric non-degenerate bilinear forms on F^{2n} we have that b_T is either equivalent to the Euclidean form $x_1^2+\cdots+x_{2n}^2$ or to the non-Euclidean form $x_1^2+\cdots+x_{2n-1}^2+cx_{2n}^2$ where $c\in F$ is not a square.

Since $det(JT)=1$. we obtain

Remark 8. *The form b_T is always equivalent to the Euclidean form.*

The group $Sp(n,F)$ acts on $Sp(n,F)\cap Sym(F^{2n})$ by the formula

$$(g,S)\to {}^t\!(g^{-1})Sg^{-1}.$$

It readily follows that the map $\mathcal{C}(n,F)\ni T\to JT\in Sp(n,F)\cap Sym(F^{2n})$ intertwines the respective actions of $Sp(n,F)$.

Hence, for $g\in Sp(n,F)$ the forms b_T and $b_{gTg^{-1}}$ are equivalent.

To continue with, we split up the analysis of $\mathcal{C}(n,F)$ into the two possible cases, namely, -1 is either a square in F or -1 is not a square in F.

We assume first that -1 is not an square in F. Let us fix a square root $i\in E$ of -1.

For an anti involution $T\in Sp(n,F)$ we have that T is a semisimple linear map with possible eigenvalues $i,-i$ because the minimal polynomial of T divides x^2+1.

Let $V_i(T)$ (resp $V_{-i}(T)$) the corresponding possible eigenspace in E^{2n}. Hence, $E^{2n}=V_i(T)\oplus V_{-i}(T)$, and we have

Proposition 5. *i) Both subspaces $V_i(T),V_{-i}(T)$ are nonzero.*

ii) $\overline{V_i(T)}=V_{-i}(T)$.

iii) $F^{2n}\cap V_i(T)=F^{2n}\cap V_{-i}(T)=\{0\}$.

iv) The map $F^{2n}\ni v\to v-iTv\in V_i(T)$ is linear bijection over F.

v) $V_i(T)$ (resp $V_{-i}(T)$) is a lagrangian subspace.

vi) $h_E(v-iTv,w-iTw)=2\omega(v,w)+2ib_T(v,w),\,for\,v,w\in F^{2n}$.

vii) The decomposition $E^{2n}=V_i(T)\oplus V_{-i}(T)$ is orthogonal with respect to h_E.

viii) h_E restricted to $V_i(T)$ is non degenerate.

Proof: The result from the facts $T\in U(h_{\mathbb{R}},E^{2n})\cap Sp(n,E)$ and $i\notin F.Inparticular$, viii) follows from vii) and that h_E is non degenerate. For $x,y\in V_i(T),\omega(x,y)=\omega(Tx,Ty)=ii\omega(x,y)=-\omega(x,y)$.

□

Let $v_j-iTv_j, j=1,\ldots,n$ denote an orthonormal basis of $V_i(T)$ for the restriction of $\frac{1}{2i}h_E$. Then, $v_1,\ldots,v_n$ span a lagrangian subspace of F^{2n} and $v_1,\ldots,v_n,Tv_1,\ldots,Tv_n$ is a basis for F^{2n}.

In fact, from vi) we obtain $w(v_k, v_s) = 0, b_T(v_k, v_s) = \delta_{k,s}$. The last statement follows from $T^2 = -1$ applied to $\sum_{1\leq j\leq n} c_j v_j + d_j T v_j = 0$ for $c_j, d_j \in F$ and a short computation.

Lemma 7. *Assume -1 is not a square in F. Then, the action of $Sp(n, F)$ in $\mathcal{C}(n, F)$ is transitive.*

Proof. Proposition 6 gives rise to a map from $\mathcal{C}(n, F)$ to $\mathcal{L}_{E,2n}$ by the rule

$$\mathcal{C}(n, F) \ni T \longrightarrow V_i(T)$$

From viii) we have the image of the map is contained in $\mathcal{H}_n$. For $g \in Sp(n, F)$ we have the equality $gV_i(T) = V_i(gTg^{-1})$, which shows that the map is equivariant. The maps is obviously injective. Since $\mathcal{H}_n$ is an orbit of $Sp(n, F)$ (Theorem 1) we have that the map is a bijection and hence the result □

Next, we assume $-1 = i^2$ with $i \in F$. Then, due to the semisimplicity of T we have the decomposition $F^{2n} = (F^{2n} \cap V_i(T)) \oplus (F^{2n} \cap V_{-i}(T))$.

From the equalities $\omega(x, y) = -\omega(x, y)$ for $x, y \in V_i(T)$, we have that the subspaces $F^{2n} \cap V_i(T), F^{2n} \cap V_{-i}(T)$ are isotropic, Corollary 3 pag 81 in [1] gives us that both subspaces are lagrangian. Therefore, the anti hermitian form h_E restricted to $F^{2n} \cap V_i(T)$ is the null form, which forces to $V_i(T)$ to be an element of $\mathcal{H}_0$.

Lemma 8. *Assume -1 is a square in F. Then, $\mathcal{C}(n, F)$ is a homogeneous space equivalent to $Sp(n, F)/(Sp(n, F) \cap K)$.*

Remark 9. *The map $\mathcal{C}(n, F) \ni T \longrightarrow V_i(T) \in \mathcal{H}_0$ is equivariant for $Sp(n, F)$ and in this case is no longer injective (c.f. example 3-a) , due to theorem 1 $\mathcal{H}_0$ is a homogeneous space for $Sp(n, F)$, hence, the map is surjective.*

We now show lemma 7. We set

$$H := \begin{pmatrix} iI_n & 0 \\ 0 & -iI_n \end{pmatrix}.$$

Then, $H \in \mathcal{C}(n, F)$. Let T be an anti involution in $Sp(n, F)$ we will show that T is conjugated in $Sp(n, F)$ to the matrix H. For this, we define $D := J^{-1}TJ$, which is another anti involution in $Sp(n, F)$.

The minimal polynomial of $J^{-1}TJ$ divides the polynomial $x^2 + 1 = (x - i)(x + i)$. Hence, $D := J^{-1}TJ$ is diagonalizable over F.

Let $W_{\pm i}$ the associated eigenspaces. Thus, $F^{2n} = W_i \oplus W_{-i}$.

Since for every $v, w \in F^{2n}, \omega(Dv, Dw) = \omega(v, w)$, we have that $W_{\pm i}$ are isotropic subspaces for ω. The hypothesis that ω is non degenerate forces, $W_{\pm i}$ to be lagrangian subspaces. Thus, there exists $P \in Sp(n, F)$ so that

$Pe_1, \dots, Pe_n$ is a basis for W_i, $\qquad Pe_{n+1}, \dots, Pe_{2n}$ is a basis for W_{-i}

We have

$$DPe_j = iPe_j = P(ie_j) = PH(e_j), j = 1, \dots, n,$$
$$DPe_j = -iPe_j = P(-ie_j) = PH(e_j), j = n + 1, \dots, 2n.$$

Hence, $DP = PH$. That is,

$$PH = DP = J^{-1}TJP.$$

Therefore,

$$H = P^{-1}J^{-1}TJP = (JP)^{-1}T(JP).$$

The matrices in $Gl(2n, F)$ which commute with H are the matrices $diag(A, B)$, $A, B, \in Gl_n(F)$. Thus, the isotropy at H is $Sp(n, F) \cap K$.

□

Remark 10. *A particular element of $Sp(n, F)$ which conjugates H onto J is the Cayley transform*

$$C(e_j) = \frac{1}{-2i}(e_j + ie_{n+j}), j = 1, \dots, n, \quad C(e_{n+j}) = e_j - ie_{n+j}, j = 1, \dots, n.$$

Example 3. *We assume $-1 = i^2, i \in F$.*

A simple calculation yields $\mathcal{C}(1, F)$ is

$$\{\begin{pmatrix} \pm i & x \\ 0 & - \pm i \end{pmatrix}, \begin{pmatrix} \pm i & 0 \\ y & - \pm i \end{pmatrix}, x \in F, y \in F^\times\}$$

union the set

$$\{\begin{pmatrix} a & -\frac{1+a^2}{c} \\ c & -a \end{pmatrix}, c \in F^\times, a \in F \backslash \{\pm i\}\}$$

Hence, the cardinal of the set of involutions is $2(q+q-1)+(q-2)(q-1) = q(q+1)$. The isotropy at $diag(i, -i)$ is the subgroup $diag(a, -a), a \in F^\times$. Hence $card(Sl(2, F_q))/card(F^\times) = q(q-1)(q+1)/(q-1) = card(\mathcal{C}(1, F))$. Also,

$$V_i(\begin{pmatrix} -i & 0 \\ x & i \end{pmatrix}) = F\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad V_i(\begin{pmatrix} i & x \\ 0 & -i \end{pmatrix}) = F\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

$$V_i(\begin{pmatrix} a & -\frac{1+a^2}{c} \\ c & -a \end{pmatrix}) = F\begin{pmatrix} \frac{1+a^2}{c} \\ a-i \end{pmatrix},$$

$$V_i(\begin{pmatrix} -i & x \\ 0 & i \end{pmatrix}) = F\begin{pmatrix} x \\ 2i \end{pmatrix}, V_i(\begin{pmatrix} i & 0 \\ x & -i \end{pmatrix}) = F\begin{pmatrix} 2i \\ x \end{pmatrix}.$$

5.1. **The case $T^2 = a, a$ square.** Let F be a field of odd characteristic, and let ω be a non degenerate alternating form in $V = F^{2n}$. We fix $a \in F$ and define

$$S_a := \{T \in Sp(w) : T^2 = aId\}$$

for $a = 1$ the identity matrix belongs to S_a

for $a = -1$ the matrix J belongs to S_a

Proposition 6. *For $a \notin \{1, -1\}$ and $a = b^2, b \in F$ the set S_a is empty.*

Proof. Let $T \in S_a$, then the eigenvalues of T belongs to the set $\pm b$. Let W_b, W_{-b} be the eigenspaces of V.

The equality $\frac{1}{2}(bI - T) + \frac{1}{2}(bI + T) = bI$ implies that $V = W_b \oplus W_{-b}$.

For $x, y \in W_b$, we have $\omega(x, y) = 0$ ($\omega(x, y) = \omega(Tx, Ty) = b^2\omega(x, y)$,) . Similarly, for $x, y \in W_{-b}$ we have $\omega(x, y) = 0$. Therefore, both subspaces are isotropic.

We now verify for $x, \in W_b, y \in W_{-b}$ that $\omega(x, y) = 0$. In fact, $\omega(x, y) = \omega(Tx, Ty) = b(-b)\omega(x, y) = -a\omega(x, y)$. Since $a \neq -1$, we get $\omega(x, y) = 0$.

Then, assuming S_a is not empty, unless $a \in \{1, -1\}$ we have ω equal to the null form, and the result follows.

Another proof follows along the following lines :

For a symplectic matrix, if λ is an eigenvalue, then $1/\lambda$ is also an eigenvalue.

So if $b, -b$ are the unique eigenvalues, and $b \notin \{\pm 1, \pm i\}$ we must have $-b = 1/b$ from which $b^2 = -1$ so $a = -1$. □

5.1.1. *The case* $a = 1$. Let W be any subspace of V such that ω restricted to W is non degenerate, so $V = W \oplus W^{\perp}$.

Define T_W to be the linear operator equal to the identity in W and equal to $-I$ in $W^{\perp}$.

It readily follows that $T_W \in Sp(n, F)$ and T_W is an involution.

Proposition 7. *Any involution T in $Sp(n, F)$ is equal to a T_W for a convenient W.*

Proof. In fact, the eigenvalues of T belongs to the set ± 1 Let W_1, W_{-1} be the eigenspaces of V the equality $\frac{1}{2}(I - T) + \frac{1}{2}(I + T) = I$ implies that $V = W_1 \oplus W_{-1}$.

For $x, \in W_1, y \in W_{-1}$ we have $\omega(x, y) = 0$. In fact, $\omega(x, y) = \omega(Tx, Ty) = 1(-1)\omega(x, y) = -1\omega(x, y)$.

It follows: ω restricted to any of the subspaces in non degenerate. Hence, $T = T_{W_1}$. □

Corollary 5. *The orbits of $Sp(n, F)$ in $\mathcal{C}_1(n, F)$ are parameterized by $k = 1, 2, \ldots, 2n$. Indeed, for each k the set of involutions T such that its $1-$eigenspace is of dimension k, is an orbit for $Sp(n; F)$.*

6. Acknowledgements

Part of the work was done during the meeting "Representation theory days in Patagonia" organized by University of Talca. The authors are grateful to the organizers of the workshop, Stephen Griffeth, Steen Ryom-Hansen, Jean F. van Diejen, Luc Lapointe, for their kind invitation to attend to such a nice meeting.

The authors want also to thank Pierre Cartier for illuminating discussions and suggestions related to this work.

References

[1] Bourbaki, N., Éléments de Mathématique, Livre II, Algèbre, Chapter IX, Herman et Cie, 1959.

[2] Dieudonne, J., Le Géométrie des groupes classiques, Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer Verlag , 1955.

[3] Roger Richardson, Gerhard Röhrle, and Robert Steinberg Parabolic,. Parbolic subgroups with Abelian unipotent radical. Invent Math. 110, 649-67l (1992)

[4] Siegel, C., Symplectic geometry, American Journal of Mathematics, Vol. 65, No. 1 , pp. 1-86,(Jan., 1943).

[5] Kaneyuki, S., Pseudo-hermitian symmetric spaces and Siegel domains over non degenerate cones, Hokkaido Math. Jour. Vol 20, 213-239, (1991).

[6] Soto Andrade, J., Bull. Soc. Math. France

[7] Wolf, J., Fine Structure of Hermitian Symmetric spaces, Symmetric Spaces, short courses presented at Washington University, (Boothby, Weiss, edts), pages 271-357, Marcel Dekker, Inc, New York, 1972.

[8] Pantoja, J. and Soto-Andrade, J., Représentations de $SL_*(2, A)$ et $SL(n, q)$, C.R. Acad. Sci. Paris, t. 323 Série I, p. 1109-1112, 1996.

[9] Soto-Andrade, J. and Vargas, J., Twisted spherical functions on the Poicaré Upper Half Plane, J. Algebra 248, 724-246, 2002.

[10] Pantoja, J. and Soto-Andrade, J., A Bruhat decomposition of the group $SL_*(2, A)$, J. Algebra 262, 401-412, 2003.

[11] Takeuchi, M., On Orbits in a Compact Hermitian Symmetric Space, American Journal of Mathematics, Vol. 90, No. 3 (Jul. 1968), 657-680.

Instituto de Matemáticas, PUCV;

Blanco Viel 596, Co. Barón, Valparaíso, Chile
E-mail address: `jpantoja@ucv.cl`

Departamento de Matemáticas, Fac. de Ciencias, Universidad de Chile;

Las Palmeras 3425 , Ñuñoa, Santiago, Chile
E-mail address: `sotoandrade@u.uchile.cl`

FAMAF-CIEM;

Ciudad Universitaria, 5000 Córdoba, Argentine
E-mail address: `vargas@famaf.unc.edu.ar`