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Soliton Almost Kähler Structures on 6-Dimensional Nilmanifolds for the Symplectic Curvature Flow

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Abstract The aim of this paper is to study self-similar solutions to the symplectic curvature flow on 6-dimensional nilmanifolds. For this purpose, we focus our attention on the family of symplectic two- and three-step nilpotent Lie algebras admitting a *minimal compatible metric* and give a complete classification of these algebras together with their respective metric. Such a classification is given by using our generalization of Nikolayevsky's nice basis criterion, which, for the convenience of the reader, will be repeated here in the context of canonical compatible metrics for geometric structures on nilmanifolds. By computing the Chern–Ricci operator P in each case, we show that the above distinguished metrics define a soliton almost Kähler structure. Many illustrative examples are carefully developed.

Keywords Symplectic curvature flow · Self-similar solutions · Geometric structures on nilmanifolds · Nilpotent Lie algebras · Convexity of the moment map · Nice basis

Mathematics Subject Classification Primary 57N16 · Secondary 22E25 · 22E45

1 Introduction

Let (g, J, ω) be an almost Kähler structure on a manifold M^{2n}. Let us denote by p its Chern–Ricci form and by ric the usual Ricci tensor of the Riemannian manifold

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(M^{2n}, g). The *symplectic curvature flow* (SCF) on a compact almost Kähler manifold $(M^{2n}, g_0, J_0, \omega_0)$ is given by the system of evolution equations

$$\begin{cases} \frac{\partial}{\partial t}\omega = -2\mathrm{p}; & \text{with } \omega(0) = \omega_0, \\ \frac{\partial}{\partial t} g = -2(\mathrm{p}^c(\cdot, J\cdot) - \mathrm{ric}^{ac}); & \text{with } g(0) = g_0. \end{cases} \tag{1.1}$$

Here, p^c is the complexified component of p (also called the J-invariant part of p) and ric^{ac} denotes the anti-complexified part of ric (also known as the anti-J-invariant part of ric).

This geometric flow was recently introduced by Streets and Tian in [18], where the short-time existence and uniqueness for this flow are proved. The solution to Eq. (1.1) preserves the almost Kähler structure, and if the initial almost Kähler structure is in fact Kähler, then such is a solution to the Kähler Ricci flow.

Let G be a simply connected Lie group admitting a left invariant almost Kähler structure (g_0, J_0, ω_0). Any left invariant almost Kähler structure on G is determined by an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ and a non-degenerate skew-symmetric bilinear form ω on $\mathfrak{g}$, here $\mathfrak{g} = \mathrm{Lie}(\mathrm{G})$ (the Lie algebra of G). One can consider the symplectic curvature flow on G, where (1.1) becomes a system of ordinary differential equations. The almost Kähler structure (g_0, J_0, ω_0) is called a *soliton* [12, Sect. 7] if the solution to the SCF starting at (g_0, J_0, ω_0) is (algebraically) self-similar, this is to say, the solution has the form

$$\begin{cases} \omega_t = c(t)\omega_0(\phi_t \cdot, \phi_t \cdot) \\ g_t = c(t) g_0(\phi_t \cdot, \phi_t \cdot) \end{cases} \tag{1.2}$$

for some $c(t) \in \mathbb{R}_{>0}$ and $\phi_t \in \mathrm{Aut}(\mathfrak{g})$, both differentiable at t, with $c(0) = 1$, $\phi_0 = \mathrm{Id}$ and $\phi_0' = D \in \mathrm{Der}(\mathfrak{g})$.

By following results given in [12], our aim in this work is to study soliton almost Kähler structures on 6-dimensional nilmanifolds. In some cases, such structures are determined by *minimal compatible metrics* on symplectic nilpotent Lie algebras (which are related with the anti-complexified Ricci flow introduced in [13]). This is the case of symplectic two-step nilpotent Lie algebras which are Chern–Ricci flat (it follows from results of Vezzoni [19] or Pook [16, Proposition 2]). In this way, we give a complete classification of minimal compatible metrics on symplectic three-step and two-step nilpotent Lie algebras and prove the main results of this paper:

Theorem A *All symplectic two-step Lie algebras of dimension* 6 *admit a minimal compatible metric and, in consequence, admit a soliton almost Kähler structure.*

Theorem B *Every minimal compatible metric on a symplectic three-step nilpotent Lie algebra of dimension 6 defines a soliton almost Kähler structure.*

In general, it is a difficult problem to know when a symplectic nilpotent Lie algebra admits a minimal compatible metric. This problem is equivalent to determining whether an orbit of the natural action of $\mathrm{Sp}(n, \mathbb{R})$ on $\Lambda^2(\mathbb{R}^{2n})^* \otimes \mathbb{R}^{2n}$ is *distinguished*, i.e., we must determine when an orbit contains a critical point of the norm-square of the moment map $m_{\mathfrak{sp}}$ associated with the action. By using convexity properties of the moment map and recent results of Jablonski [6], the author presented a criterion to knowing whether a *nice element* of a real reductive representation has a distinguished orbit [2]. Such a result can be considered as a generalization of Nikolayevsky's nice basis criterion [15, Theorem 3]. As Theorems A and B are proved by using such a criterion, we prove, for the convenience of the reader, the corresponding criterion in the general context of canonical compatible metrics for geometric structures on nilmanifolds (Sect. 2.2); which is considered the second aim of this paper.

2 Preliminaries

2.1 Soliton Solutions for the SCF on Lie Groups

Let G be a Lie group admitting a left invariant almost Kähler structure, i.e., there exist a symplectic structure ω on $\mathfrak{g}$, an almost complex structure J on $\mathfrak{g}$ and an inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ satisfying the *compatibility condition*

$$\omega(X,Y)=\langle JX,Y\rangle,\ \forall X,Y\in\mathfrak{g}.$$

By symplectic structure on $\mathfrak{g}$ we mean that the non-degenerate skew-symmetric bilinear form ω is *closed*, that is,

$$\omega([X,Y],Z)+\omega([Y,Z],X)+\omega([Z,X],Y)=0 \tag{2.1}$$

for any X, Y and Z in $\mathfrak{g}$.

Given a bilinear form on $\mathfrak{g}$, say $B\colon\mathfrak{g}\times\mathfrak{g}\longrightarrow\mathbb{R}$, the complexified part of B and the anti-complexified part of B, denoted by B^c and B^{ac} respectively, are defined to be

$$B^c(\cdot,\cdot)=\frac{1}{2}(B(\cdot,\cdot)+B(J\cdot,J\cdot)) \text{ and } B^{ac}(\cdot,\cdot)=\frac{1}{2}(B(\cdot,\cdot)-B(J\cdot,J\cdot)).$$

In the same way we define the complexified part and anti-complexified part of a linear map $T\colon\mathfrak{g}\longrightarrow\mathfrak{g}$, denoted by T^c and T^{ac} respectively, to be

$$T^c=\frac{1}{2}(T-JTJ) \text{ and } T^{ac}=\frac{1}{2}(T+JTJ).$$

From now on, the transpose of a linear map $T\colon\mathfrak{g}\longrightarrow\mathfrak{g}$ with respect to $\langle\cdot,\cdot\rangle$ and ω are denoted by T^{T} and T^{T_ω}, respectively (note that $T^{\mathrm{T}_\omega}=-JA^{\mathrm{T}}J$).

Let p be the Chern–Ricci form of the left invariant almost Kähler structure $(\langle\cdot,\cdot\rangle,J,\omega)$. It is proved in [19, Proposition 4.1] and [16, Sect. 3] that p is given by

$$\mathrm{p}(X,Y)=\frac{1}{2}\left(\mathrm{tr}(\mathrm{ad}_{J[X,Y]})-\mathrm{tr}(J\,\mathrm{ad}_{[X,Y]})\right). \tag{2.2}$$

The above expression has interesting consequences, among them let us mention, for instance, if $\mathfrak{g}$ is a two-step nilpotent Lie algebra ($\mathrm{ad}_{[\cdot,\cdot]}=0$), then $\mathrm{p}=0$, i.e., $(\mathfrak{g},\langle\cdot,\cdot\rangle,J,\omega)$ is Chern–Ricci flat ([19, Proposition 4.1] or [16, Proposition 2]).

From Eq. (2.2), it is easy to see that there exists an $\widehat{H} \in \mathfrak{g}$ such that

$$\mathrm{p}(X, Y) = \omega(\widehat{H}, [X, Y]) \tag{2.3}$$

because ω is non-degenerate. Such an $\widehat{H}$ can be taken to be $\frac{1}{2}\sum \mathrm{ad}_{e_i}^{\mathrm{T}} e_i + \frac{1}{2}\sum J\,\mathrm{ad}_{e_i}^{\mathrm{T}}(Je_i)$, where $\{e_1, \ldots, e_n\}$ is an orthonormal basis for $\mathfrak{g}$. Therefore, if P is the Chern–Ricci operator, i.e., P is the linear transformation of $\mathfrak{g}$ such that $\mathrm{p}(X, Y) = \omega(\mathrm{P}X, Y)$ for all $X, Y \in \mathfrak{g}$, then $\mathrm{P} = \mathrm{ad}_{\widehat{H}} + \mathrm{ad}_{\widehat{H}}^{\mathrm{T}_\omega}$ (it is immediate from Eq. (2.1)).

According to the above formula for P, we can say more about the Chern–Ricci form p. To do this, let us make a short digression on left-symmetric algebras.

Definition 2.1 [17] A *left-symmetric algebra structure* (LSA-structure) on a Lie algebra $\mathfrak{g}$ is a bilinear product $\centerdot : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ satisfying the conditions

(1) $[X, Y] = X \centerdot Y - Y \centerdot X$,
(2) $X \centerdot (Y \centerdot Z) - Y \centerdot (X \centerdot Z) - [X, Y] \centerdot Z = 0$

for all X, Y and $Z \in \mathfrak{g}$.

Given $X \in \mathfrak{g}$, let λ_X denote (respectively ρ_X) the left multiplication by X (respectively right multiplication by X) in the left-symmetric algebra: $\lambda_X(Y) = X \centerdot Y$ and $\rho_X(Y) = Y \centerdot X$ for all $Y \in \mathfrak{g}$. The LSA-structure is called *complete* if for every $X \in \mathfrak{g}$, the linear transformation $\mathrm{Id} + \rho_X$ is bijective.

Note that the LSA-structure conditions are equivalent to having a left-invariant affine connection on $\mathfrak{g}$ which is: (1) torsion free and (2) flat. These concepts play an important role in the study of affine crystallographic groups and of fundamental groups of affine manifolds, which are well-developed theories and have a rich history that includes challenging problems due to Louis Auslander and John W. Milnor. We refer the reader to [1] for a comprehensive review of the literature on such topics.

On the completeness of an LSA-structure, we have the following result due to Dan Segal.

Theorem 2.2 [17] *Let* $\mathfrak{g}$ *be a Lie algebra over a field* k *of characteristic zero and* $\centerdot : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ *be an LSA-structure on* $\mathfrak{g}$. *The following conditions are equivalent:*

(1) The LSA-structure is complete.
(2) The LSA-structure is right nilpotent, i.e., ρ_X *is a nilpotent linear transformation, for all* $X \in \mathfrak{g}$.
(3) $\mathrm{tr}(\rho_X) = 0$ *for all* $X \in \mathfrak{g}$.

This theorem implies the following additional property on the Chern–Ricci operator.

Proposition 2.3 *Let* $\mathfrak{g}$ *be a unimodular Lie algebra and* $(\langle\cdot,\cdot\rangle, J, \omega)$ *be an almost-Kähler structure on* $\mathfrak{g}$. *Then, its Chern–Ricci operator P is a nilpotent operator.*

Proof Consider the usual LSA-structure $\centerdot : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ on $\mathfrak{g}$ induced by the symplectic structure, which is defined implicitly by

$$\omega(H, [X, Y]) = -\omega(X \centerdot H, Y), \tag{2.4}$$

for any H, X and $Y \in \mathfrak{g}$.

Given $H \in \mathfrak{g}$, let P_H denote the linear transformation such that

$$\omega(H, [X, Y]) = \omega(P_H X, Y), \forall X, Y \in \mathfrak{g}. \tag{2.5}$$

From Eq. (2.1), it follows that

$$\mathrm{ad}_X^{\mathrm{T}\omega}(H) = \mathrm{ad}_H^{\mathrm{T}\omega}(X) + \mathrm{ad}_H(X)$$

and, in consequence, $P_H = \mathrm{ad}_H^{\mathrm{T}\omega} + \mathrm{ad}_H$. Since $\mathfrak{g}$ is unimodular ($\mathrm{tr}(\mathrm{ad}_Z) = 0, \ \forall Z \in \mathfrak{g}$), $\mathrm{tr}(P_H) = 2\mathrm{tr}(\mathrm{ad}_H) = 0$ for any $H \in \mathfrak{g}$.

By Eqs. (2.4) and (2.5), we have $\rho_H = -P_H$, and, on account of the above-mentioned, $\mathrm{tr}(\rho_H) = 0$ for any $H \in \mathfrak{g}$, which implies that the LSA-structure is complete and so, ρ_H is a nilpotent linear transformation for any $H \in \mathfrak{g}$. Since the Chern–Ricci operator P is equal to $P_{\widehat{H}} = -\rho_{\widehat{H}}$ for certain $\widehat{H} \in \mathfrak{g}$, this completes the proof. □

Remark 2.4 A direct proof of the above proposition can be given by using the equation:

$$\mathrm{tr}(\rho_X^k) = \mathrm{tr}(\rho_{X^k}), \ \forall X \in \mathfrak{g} \tag{2.6}$$

where $X^k = X^{k-1} \cdot X$ with $k \in \mathbb{N}$ (see [5, Proposition 15], [10, Theorem 2.2] or [17, Proposition 2])

Having disposed of this preliminary information on the Chern–Ricci form, we can now return to the main topic of this subsection.

Definition 2.5 [12, Definition 7.2] An almost Kähler structure $(\langle\cdot,\cdot\rangle, J, \omega)$ on a Lie algebra $\mathfrak{g}$ is called a *soliton* if for some $c \in \mathbb{R}$ and $D \in \mathrm{Der}(\mathfrak{g})$,

$$\begin{cases} \mathrm{P} = c\mathrm{Id} + \frac{1}{2}(D - JD^{\mathrm{T}}J) \\ \mathrm{P}^c + \mathrm{Ric}^{ac} = c\mathrm{Id} + \frac{1}{2}(D + D^{\mathrm{T}}). \end{cases} \tag{2.7}$$

It is proved in [12, Lemma 7.1] that the above definition is equivalent to saying that the solution to the SCF starting in $(\langle\cdot,\cdot\rangle, J, \omega)$ is self-similar in the sense of condition (1.2).

Note that, if the almost Kähler structure is in fact Kähler, then such a structure is a soliton if and only if $\langle\cdot,\cdot\rangle$ is a semi-algebraic Ricci soliton; $\mathrm{Ric} = c\mathrm{Id} + \frac{1}{2}(D + D^{\mathrm{T}})$ (because $\mathrm{P} = \mathrm{Ric}$). Here, we can highlight that it is now known from recent works of Michael Jablonski that semi-algebraic solitons are algebraic (see [8]).

Given that the SCF evolves the metric and the symplectic structure, preserving the compatibility, one expects that, in general, it is not enough to have a "distinguished" metric or a "distinguished" symplectic structure in order to have a soliton.

Example 2.6 Consider the Lie algebra $\mathfrak{g} := (\mathbb{R}^6, \mu)$ with

$$\mu = \begin{cases} [e_1, e_2] = e_1, [e_1, e_3] = e_1, [e_1, e_4] = -2\,e_6, [e_1, e_6] = -2\,e_5, \\ [e_2, e_5] = -2\,e_5, [e_2, e_6] = -e_6, [e_3, e_4] = 2\,e_4, [e_3, e_6] = e_6 \end{cases}$$

and the almost-Kähler structure on $\mathfrak{g}$ given by $(\langle\cdot,\cdot\rangle, J, \omega_{cn})$, where $\langle\cdot,\cdot\rangle$ is the canonical inner product of $\mathbb{R}^6$ and ω_{cn} is the canonical symplectic form of $\mathbb{R}^6$; $\omega_{cn} = e_1^* \wedge e_6^* + e_2^* \wedge e_5^* + e_3^* \wedge e_4^*$.

An easy computation shows that $(\langle\cdot,\cdot\rangle, J, \omega_{cn})$ is an Einstein strictly almost Kähler structure on $\mathfrak{g}$. In fact, $N_J(e_1, e_3) = 4e_1 \neq 0$, where N_J is the Nijenhuis tensor and

$$\begin{aligned}\mathrm{Ric} &= \mathrm{Diag}(-4,-3,-3,-2,2,0) - \frac{1}{2}\mathrm{Diag}(0,6,6,0,0,0) - \mathrm{Diag}(2,0,0,4,8,6)\\ &= -6\mathrm{Id}.\end{aligned}$$

In consequence, the Ricci tensor is J-invariant; $\mathrm{Ric}^{ac} = 0$.

The vector $\widehat{H}$ defined as $\widehat{H} := 3e_2 - e_3$ satisfies $\mathrm{p}(X, Y) = \omega_{cn}(\widehat{H}, [X, Y])$, so the Chern–Ricci operator P is given by

$$\mathrm{P} = \mathrm{Diag}(-6,-6,-2,-2,-6,-6).$$

In consequence, the Chern–Ricci operator is symmetric; $\mathrm{P}^c = \mathrm{P}$.

In this case, the soliton condition may then be reduced to $\mathrm{P} = c\mathrm{Id} + \frac{1}{2}(D + D^{\mathrm{T}})$, with $D \in \mathrm{Der}(\mathfrak{g})$. Since the algebra of derivations of $\mathfrak{g}$ is given by

$$\mathrm{Der}(\mathfrak{g}) = \mathrm{span}\left\{\begin{array}{l} E_{1,1} + 2E_{5,5} + E_{6,6}, E_{4,4} + E_{5,5} + E_{6,6}, E_{6,1} - E_{4,3},\\ E_{5,1} + \frac{1}{2}(E_{6,2} - E_{6,3}), (E_{1,2} + E_{1,3}) - 2(E_{6,4} + E_{5,6}, E_{5,2}) \end{array}\right\},$$

a trivial verification shows that $(\langle\cdot,\cdot\rangle, J, \omega)$ is not a soliton.

Example 2.7 Consider the family of strictly almost Kähler solvmanifolds given by the family of solvable Lie algebras $\mathfrak{g}(\lambda_1,\lambda_2,\lambda_3) := (\mathbb{R}^6, \mu_{\lambda_1,\lambda_2,\lambda_2})$, where

$$\mu_{\lambda_1,\lambda_2,\lambda_2} = \begin{cases} [e_1, e_3] = -\left({\lambda_1}^2 + {\lambda_2}^2\right)e_3, [e_1, e_4] = \lambda_3 e_3 + \left({\lambda_1}^2 + {\lambda_2}^2\right)e_4,\\ [e_2, e_6] = \left({\lambda_2}^2 - {\lambda_1}^2\right)e_2 + 2\,\lambda_1\lambda_2 e_5,\\ [e_5, e_6] = 2\,\lambda_1\lambda_2 e_2 + \left({\lambda_1}^2 - {\lambda_2}^2\right)e_5, \end{cases}$$

with strictly almost Kähler structure $(\langle\cdot,\cdot\rangle, J, \omega_{cn})$ given by the canonical inner product of $\langle\cdot,\cdot\rangle$ and the usual symplectic form ω_{cn} (note that $N_J(e_1, e_4) = \lambda_3 e_3 + 2(\lambda_1^2 + \lambda_2^2)e_4$).

The vector $\widehat{H}$ defined as $\widehat{H} := \frac{1}{2}\lambda_3 e_6$ satisfies $\mathrm{p}(X, Y) = \omega_{cn}(\widehat{H}, [X, Y])$. Since $e_6 \perp^{\omega} [\mathfrak{g}, \mathfrak{g}]$, the above family is Chern–Ricci flat ($\mathrm{p} = 0$).

An easy computation shows that

$$\mathrm{Ric}^{ac} = \frac{\lambda_3}{4}\mathrm{Diag}\left(-\lambda_3, 0, \begin{pmatrix} 2\lambda_3 & 4(\lambda_1^2+\lambda_2^2)\\ 4(\lambda_1^2+\lambda_2^2) & -2\lambda_3\end{pmatrix}, 0, \lambda_3\right)$$

and a straightforward computation of the algebra of derivations when $\lambda_3 \neq 0$ shows that the above structure is a soliton if and only if $\lambda_3 = 0$, and in that case, $\mathrm{Ric}^{ac} = 0$.

Some sufficient conditions to have a soliton have been given in [12]. These conditions are more easily verifiable than those in Definition 2.5.

Proposition 2.8 [12, Proposition 7.4] *If an almost-Kähler structure* $(\langle\cdot,\cdot\rangle, J, \omega)$ *on a Lie algebra* $\mathfrak{g}$ *satisfies any of the conditions*

$$\left\{ P + \mathrm{Ric}^{ac} = c\mathrm{Id} + D \right. \tag{2.8}$$

or

$$\begin{cases} P = c_1\mathrm{Id} + D_1 \\ \mathrm{Ric}^{ac} = c_2\mathrm{Id} + D_2 \end{cases} \tag{2.9}$$

with $c's \in \mathbb{R}$ *and* $D's \in \mathrm{Der}(\mathfrak{g})$*, then* $(\langle\cdot,\cdot\rangle, J, \omega)$ *defines a soliton with the same* c *and* D *in condition* (2.8)*, and* $c = c_1 + c_2$ *and* $D = D_1 + D_2$ *in condition* (2.9)*.*

Let $(\mathfrak{n}, \omega)$ be a symplectic nilpotent Lie algebra and $\langle\cdot,\cdot\rangle$ be a compatible metric with $(\mathfrak{n}, \omega)$. If the anti-complexified part of the Ricci operator of $\langle\cdot,\cdot\rangle$, Ric^{ac}, satisfies $\mathrm{Ric}^{ac} = c\mathrm{Id} + D$ for some $c \in \mathbb{R}$ and $D \in \mathrm{Der}(\mathfrak{n})$, then such a metric is *minimal* (see [11, Theorem 4.3]). Thus, in the nilpotent case, some soliton almost-Kähler structures are given by minimal compatible metrics with the Chern–Ricci operator being a derivation.

Example 2.9 Consider the nilpotent Lie algebra $\mathfrak{n} := (\mathbb{R}^8, \mu)$ with

$$\mu = \begin{cases} [e_1, e_2] = \frac{\sqrt{14}}{14}e_4, [e_2, e_5] = \frac{\sqrt{14}}{14}e_8, [e_2, e_6] = \frac{\sqrt{14}}{14}e_3, [e_3, e_7] = \frac{\sqrt{14}}{14}e_4, \\ [e_5, e_7] = -\frac{\sqrt{14}}{14}e_6, [e_6, e_7] = -\frac{\sqrt{14}}{14}e_1, [e_7, e_8] = \frac{\sqrt{14}}{14}e_3 \end{cases}$$

and the almost Kähler structure given by $(\langle\cdot,\cdot\rangle, J, \omega_{cn})$, where $\langle\cdot,\cdot\rangle$ is the canonical inner product of $\mathbb{R}^8$ and ω_{cn} is the usual symplectic form of $\mathbb{R}^8$; $\omega_{cn} = e_1^* \wedge e_8^* + e_2^* \wedge e_7^* + e_3^* \wedge e_6^* + e_4^* \wedge e_5^*$. The vector $\widehat{H} := \frac{\sqrt{14}}{28}e_7$ satisfies $\mathrm{p}(X, Y) = \omega_{cn}(\widehat{H}, [X, Y])$. Since $e_7 \perp^{\omega} [\mathfrak{n}, \mathfrak{n}]$, $(\mathfrak{n}, \langle\cdot,\cdot\rangle, J, \omega)$ is Chern–Ricci flat.

The anti-complexified part of the Ricci operator of $(\mathfrak{n}, \langle\cdot,\cdot\rangle)$ is such that

$$\begin{aligned} \mathrm{Ric}^{ac} &= \frac{1}{56}\mathrm{Diag}(0, 1, 2, 4, -4, -2, -1, 0) \\ &= -\frac{3}{56}\mathrm{Id} + \frac{1}{56}\mathrm{Diag}(3, 4, 5, 7, -1, 1, 2, 3) \end{aligned}$$

with $\mathrm{Diag}(3, 4, 5, 7, -1, 1, 2, 3)$ being a derivation of $\mathfrak{n}$.

It follows from Proposition 2.8 that $(\langle\cdot,\cdot\rangle, J, \omega_{cn})$ is a soliton almost Kähler structure on $\mathfrak{n}$.

Remark 2.10 In the theory of nilsoliton metrics (minimal metrics on nilpotent Lie algebras), it is well known that the eigenvalues of the *Einstein derivation* are all positive integers (up to a positive multiple). More precisely, if $\langle\cdot,\cdot\rangle$ is a nilsoliton metric on a nilpotent Lie algebra $\mathfrak{n}$ with $\mathrm{Ric} = c\mathrm{Id} + D$, where $c \in \mathbb{R}$ and $D \in \mathrm{Der}(\mathfrak{n})$, then there exists a positive constant k such that all the eigenvalues of D lie in $k\mathbb{N}$ [3, Theorem 4.14]. The above example shows a subtle difference between nilsoliton metrics on nilpotent Lie algebras and minimal compatible metrics with symplectic Lie algebras.

2.2 Canonical Compatible Metrics for Geometric Structures on Nilmanifolds

In this section we give a brief exposition of *minimal compatible metrics* for geometric structures on nilmanifolds [11]. Such an approach is a way to study the problem of finding "the best metric" which is compatible with a fixed geometric structure γ on a simply connected nilpotent Lie group. By using strong results from real geometric invariant theory (real GIT), the properties that make a minimal metric "special" are given in [11]: a minimal metric is unique (up to isometry and scaling) when it exists, and it can be characterized as a *soliton solution* of the *invariant Ricci flow* [11, Theorem 4.4].

By using results given in [2], we introduce the notion of a *nice basis* (Definition 2.18) in the context of minimal metrics and give the corresponding criterion for knowing whether a geometric structure γ on a nilpotent Lie algebra admitting a γ-nice basis has a minimal compatible metric.

Let (N, γ) be a class-γ nilpotent Lie group: N is a simply connected nilpotent Lie group and γ is an *invariant geometric structure* on N (see [11, Definition 2.1]). We identify $\mathfrak{n}$ with $\mathbb{R}^n$ and so the structure of the Lie algebra on $\mathfrak{n}$ is given by an element $\mu \in \Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$; $\mathfrak{n} = (\mathbb{R}^n, \mu)$ and the geometric structure γ is given by the left translation of a tensor on $\mathbb{R}^n$ which we denote also by γ. In the same way, any left invariant compatible metric with (N, γ) is defined by an inner product $(\cdot, \cdot)$ on $\mathbb{R}^n$.

By definition, there is no loss of generality in assuming that the canonical inner product of $\mathbb{R}^n$, $\langle \cdot, \cdot \rangle$, also defines a compatible metric with (N, γ). Since the reductive group

$$\mathrm{G}_\gamma = \{g \in \mathrm{GL}_n(\mathbb{R}) : g \cdot \gamma = \gamma\}$$

is self adjoint with respect to any compatible metric (this also follows easily from the definition), then G_γ is compatible with the usual Cartan decomposition of $\mathrm{GL}_n(\mathbb{R})$, that is, $\mathrm{G}_\gamma = \mathrm{K}_\gamma \exp(\mathfrak{a}_\gamma)\mathrm{K}_\gamma$ with K_γ a subgroup of the Orthogonal group $\mathrm{O}(n)$ and $\mathfrak{a}_\gamma$ a subalgebra of the algebra of the diagonal matrices $\mathfrak{a}$.

Example 2.11 From [9, Theorem 5], we consider the free 2-step nilpotent Lie algebra of rank3:

$$\mathfrak{n}_{18} := \{[e_1, e_2] = e_4, [e_1, e_3] = e_5, [e_2, e_3] = e_6$$

and the symplectic structure $\omega_2(t)$ on $\mathfrak{n}_{18}$ given by

$$\omega_2(t) = e_1^* \wedge e_5^* + te_1^* \wedge e_6^* - te_2^* \wedge e_5^* + e_2^* \wedge e_6^* - 2te_3^* \wedge e_4^*.$$

In general, it is well known that, for any symplectic form ω, there exists a suitable change of basis such that ω is given by the "canonical symplectic form". In this case, we can try to make a change of the basis having the form

$$g = \mathrm{Diag}\left(m_{1,1}, m_{2,2}, m_{3,3}, m_{4,4}, \begin{pmatrix} m_{5,5} & m_{5,6} \\ m_{6,5} & m_{6,6} \end{pmatrix}\right)$$

and to solve $g \cdot \omega_2(t) = \omega_2(t)(g^{-1}\cdot, g^{-1}\cdot) = \omega_{cn}$ for $\{m_{1,1}, \ldots, m_{6,6}\}$. Here, $\omega_{cn} = e_1^* \wedge e_6^* + e_2^* \wedge e_5^* + e_3^* \wedge e_4^*$.

The solution to this equation is

$$\left\{ m_{4,4} = -\frac{1}{2}\frac{1}{t m_{3,3}}, m_{5,5} = -\frac{t}{(t^2+1)\, m_{2,2}}, \right.$$
$$\left. m_{5,6} = \frac{1}{(t^2+1)\, m_{1,1}}, m_{6,5} = \frac{1}{(t^2+1)\, m_{2,2}}, m_{6,6} = \frac{t}{(t^2+1)\, m_{1,1}} \right\}.$$

Hence, we can take the particular solution defined by $m_{1,1} = m_{2,2} = m_{3,3} = 1$, which defines a symplectomorphism from $(\mathfrak{n}_{18}, \omega_2(t))$ to $(\mathbb{R}^6, \mu_t, \omega_{cn})$ with

$$\mu_t = \{[e_1, e_2] = -2\, t\, e_4, [e_1, e_3] = -t\, e_5 + e_6, [e_2, e_3] = e_5 + t\, e_6\ .$$

With respect to $\mathrm{G}_{\omega_{cn}}$, we have the usual presentation of the *symplectic group*, $\mathrm{Sp}(3, \mathbb{R})$ by

$$\mathrm{Sp}(3, \mathbb{R}) = \left\{g \in \mathrm{GL}_6(\mathbb{R})/g^{\mathrm{T}} J g = J\right\},$$

where $Je_1 = e_6$, $Je_2 = e_5$, $Je_3 = e_4$ and $J^2 = -\mathrm{Id}$. "The" maximal compact subgroup of $\mathrm{Sp}(3, \mathbb{R})$ is the unitary group $\mathrm{U}(3)$ and a Cartan decomposition is given by $\mathrm{Sp}(3, \mathbb{R}) = \mathrm{U}(3) \exp(\mathfrak{a}_{\omega_{cn}})\mathrm{U}(3)$ with

$$\mathfrak{a}_{\omega_{cn}} = \{\mathrm{Diag}(-x_1, -x_2, -x_3, x_3, x_2, x_1) : x_i \in \mathbb{R}\}.$$

Definition 2.12 [11, Definition 2.2] Let $(\cdot, \cdot)$ be a *compatible metric* with the class-γ nilpotent Lie group (N, γ). Consider the orthogonal projection $\mathrm{Ric}^{\gamma}_{(\cdot,\cdot)}$ of the Ricci operator $\mathrm{Ric}_{(\cdot,\cdot)}$ on $\mathfrak{g}_\gamma = \mathrm{Lie}(\mathrm{G}_\gamma)$ with respect to the inner product $((\cdot, \cdot))$ of $\mathfrak{gl}_n(\mathbb{R})$ induced by $(\cdot, \cdot)$, i.e., for any A, B in $\mathfrak{gl}_n(\mathbb{R})$, $((A, B)) = \mathrm{tr}(AB^{\mathrm{T}})$, where B^{T} denotes the transpose of B with respect to $(\cdot, \cdot)$. $\mathrm{Ric}^{\gamma}_{(\cdot,\cdot)}$ is said to be an *invariant Ricci operator*, and the corresponding *invariant Ricci tensor* is defined by $\mathrm{ric}^\gamma = (\mathrm{Ric}^\gamma \cdot, \cdot)$.

Example 2.13 In the symplectic case, it is easy to see that the invariant Ricci operator coincides with the anti-complexified Ricci tensor, i.e., if $(\cdot, \cdot)$ is a compatible metric with $(\mathfrak{n}, \omega)$, then

$$\mathrm{Ric}^{\omega}_{(\cdot,\cdot)} = \mathrm{Ric}^{ac}_{(\cdot,\cdot)} = \frac{1}{2}\left(\mathrm{Ric}_{(\cdot,\cdot)} + J_{(\cdot,\cdot)}\mathrm{Ric}_{(\cdot,\cdot)}J_{(\cdot,\cdot)}\right),$$

where $J_{(\cdot,\cdot)}$ is the linear transformation such that $\omega(\cdot, \cdot) = (J_{(\cdot,\cdot)}\cdot, \cdot)$.

Definition 2.14 (*Minimal compatible metric*) [11, Definition 2.3] A left invariant metric $\langle \cdot, \cdot \rangle$ compatible with a class-γ nilpotent Lie group (N_μ, γ) is called *minimal* if

$$\mathrm{tr}\left(\mathrm{Ric}^{\gamma}_{\langle\cdot,\cdot\rangle}\right)^2 = \min\left\{\mathrm{tr}\left(\mathrm{Ric}^{\gamma}_{(\cdot,\cdot)}\right)^2 : \begin{array}{l}(\cdot, \cdot) \text{ is a compatible metric with } (\mathrm{N}_\mu, \gamma)\\ \text{and } \mathrm{sc}((\cdot, \cdot)) = \mathrm{sc}(\langle \cdot, \cdot \rangle)\end{array}\right\}.$$

Now we study the natural action of $\mathrm{GL}_n(\mathbb{R})$ (and G_γ) on $V := \Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$ given by the change of basis:

$$g \cdot \mu(X, Y) = g\mu(g^{-1}X, g^{-1}Y),\ X, Y \in \mathbb{R}^n,\ g \in \mathrm{GL}_n(\mathbb{R}),\ \mu \in V.$$

The corresponding representation of $\mathfrak{gl}_n(\mathbb{R})$ on V is

$$A \cdot \mu(X, Y) = A\mu(X, Y) - \mu(AX, Y) - \mu(X, AY),\ A \in \mathfrak{gl}_n(\mathbb{R}),\ \mu \in V.$$

Consider the usual inner product $\langle\cdot,\cdot\rangle$ on V defined by the canonical inner product of $\mathbb{R}^n$ as

$$\langle \mu, \lambda \rangle = \sum_{ijk} \langle \mu(e_i, e_j), e_k\rangle \langle \lambda(e_i, e_j), e_k\rangle,\ \mu, \lambda \in V$$

and let $\langle\langle\cdot,\cdot\rangle\rangle$ be the canonical inner product of $\mathfrak{gl}_n(\mathbb{R})$ induced by the canonical inner product of $\mathbb{R}^n$ (as in Definition 2.12).

We are now in a position to define the moment map of the above-mentioned action. This map is implicitly defined by

$$\begin{aligned} m_{\mathfrak{gl}_n(\mathbb{R})} : V &\longrightarrow \mathfrak{gl}_n(\mathbb{R}) \\ \langle\langle m_{\mathfrak{gl}_n(\mathbb{R})}(\mu), A\rangle\rangle &= \langle A \cdot \mu, \mu\rangle, \end{aligned} \tag{2.10}$$

for all $A \in \mathfrak{gl}_n(\mathbb{R})$ and $\mu \in V$.

Let $\mathrm{Proj}_{\mathfrak{g}_\gamma}$ denote the orthogonal projection of $\mathfrak{gl}_n(\mathbb{R})$ to $\mathfrak{g}_\gamma$ with respect to the inner product $\langle\langle\cdot,\cdot\rangle\rangle$. It is easy to see that the moment map for the action of G_γ on V, $m_{\mathfrak{g}_\gamma}$, is $\mathrm{Proj}_{\mathfrak{g}_\gamma} \circ m_{\mathfrak{gl}_n(\mathbb{R})}$. The following result illustrates the relationship between minimal metrics and the moment map.

Proposition 2.15 [11, Proposition 4.2] *Let* (N_μ, γ) *be a class-*γ *nilpotent Lie group. Then*

$$4Ric_{g\cdot\langle\cdot,\cdot\rangle} = m_{\mathfrak{gl}_n(\mathbb{R})}(g^{-1} \cdot \mu),\ \forall g \in \mathrm{GL}_n(\mathbb{R}),\ \textit{and} \tag{2.11}$$

$$4Ric^{\gamma}_{h\cdot\langle\cdot,\cdot\rangle} = m_{\mathfrak{g}_\gamma}(h^{-1} \cdot \mu),\ \forall h \in \mathrm{G}_\gamma, \tag{2.12}$$

where $\mathrm{Ric}_{g\cdot\langle\cdot,\cdot\rangle}$ *is the Ricci operator of the Riemannian manifold* $(\mathrm{N}_\mu, g\cdot\langle\cdot,\cdot\rangle)$ *with respect to the orthonormal basis* $\{g \cdot e_1, \ldots, g \cdot e_n\}$ *and* $\mathrm{Ric}^{\gamma}_{h\cdot\langle\cdot,\cdot\rangle}$ *is the invariant Ricci operator of* $(\mathrm{N}_\mu, \gamma, h\cdot\langle\cdot,\cdot\rangle)$ *with respect to the orthonormal basis* $\{h \cdot e_1, \ldots, h \cdot e_n\}$.

Hence, the problem of finding a minimal compatible metric with (N_μ, γ) is equivalent to finding a minimum value of $||m_{\mathfrak{g}_\gamma}||^2$ along the G_γ-orbit of μ (recall that any compatible metric is of the form $h \cdot \langle\cdot,\cdot\rangle$ with $h \in \mathrm{G}_\gamma$). The above is exactly to know if the orbit $\mathrm{G}_\gamma \cdot \mu$ is *distinguished* for the action of G_γ on V ($\mathrm{G}_\gamma \cdot \mu$ contains a critical point of $||m_{\mathfrak{g}_\gamma}||^2$).

Theorem 2.16 [11, Theorems 4.3 and 4.4] *Let* $(\mathrm{N}_{\mu}, \gamma)$ *be a class-*γ *nilpotent Lie group.* $(\mathrm{N}_{\mu}, \gamma)$ *admits a minimal compatible metric if and only if the* G_{γ}*-orbit of* μ *is distinguished for the natural action of* G_{γ} *on* V*. Moreover, there is at most one minimal compatible metric on* (N, γ) *up to isometry (and scaling).*

Remark 2.17 The last part of the above theorem follows from strong results on critical points of the norm-square of a moment map. In the proof of [11, Proposition 4.3 and 4.4] a result of Marian [14, Theorem 1] is used to prove such a part. However, there is an error in the proof of Marian's result. A correct proof can be found in [7, Theorem 5.1] and [4, Corollary 6.12].

We are now in a position to introduce the notion of a nice basis.

Definition 2.18 (γ**-nice basis**). We say that the canonical basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$ is a γ-nice basis for $(\mathbb{R}^n, \mu, \gamma)$ if for any metric of the form $a \cdot \langle\cdot,\cdot\rangle$ with $a \in \exp(\mathfrak{a}_{\gamma})$, $\mathrm{Ric}^{\gamma}_{a\cdot\langle\cdot,\cdot\rangle} \in \mathfrak{a}_{\gamma}$ holds, where $\mathrm{Ric}^{\gamma}_{a\cdot\langle\cdot,\cdot\rangle}$ is represented with respect to the orthonormal basis $\{a \cdot e_1, \ldots, a \cdot e_n\}$ of $(\mathbb{R}^n, \mu, a \cdot \langle\cdot,\cdot\rangle)$.

Remark 2.19 By Proposition 2.15 and [2, Proposition 4.8], the above definition is equivalent to saying that μ is a nice-element for the natural action of G_{γ} on V [2, Definition 3.3].

Remark 2.20 In general, it is difficult to know whether a pair (N, γ) admits a γ-nice basis, even when $\gamma = 0$ (nilsoliton case). The author in [2, Sect. 4] investigated this problem in the general case of real reductive representations, and some of the obtained results will be very useful for the study of minimal metrics.

Notation 2.21 Consider $(\mathbb{R}^n, \mu, \gamma)$, where $\{e_1, \ldots, e_n\}$ is a γ-nice basis. Denote by $\mathfrak{R}_{\gamma}(\mu)$ an *ordered set of weights related with* μ to the action of G_{γ} on V (see [2, Notation 2.5]), i.e., if $\{C^k_{i,j}\}$ are the structural constants of $(\mathbb{R}^n, \mu)$ in the basis $\{e_1, \ldots, e_n\}$, then

$$\mathfrak{R}_{\gamma}(\mu) := \left\{\mathrm{Proj}_{\mathfrak{g}_{\gamma}}(E_{k,k} - E_{i,i} - E_{j,j}) : C^k_{i,j} \neq 0\right\},$$

where $\{E_{i,j}\}$ is the canonical basis of $\mathfrak{gl}_n(\mathbb{R})$.

We denote by β^{γ}_{μ} the *minimal convex combination* (mcc) of the convex hull of $\mathfrak{R}_{\gamma}(\mu)$, i.e., β^{γ}_{μ} is the unique vector closest to the origin in the mentioned hull.

The Gram matrix of $(\mathfrak{R}_{\gamma}(\mu), \langle\langle\cdot,\cdot\rangle\rangle)$ will be denoted by $\boldsymbol{U}^{\gamma}_{\mu}$, that is, if $\mathfrak{R}_{\gamma}(\mu)_p$ is the pth element of $\mathfrak{R}_{\gamma}(\mu)$, then

$$\boldsymbol{U}^{\gamma}_{\mu}(p, q) = \langle\langle \mathfrak{R}_{\gamma}(\mu)_p, \mathfrak{R}_{\gamma}(\mu)_q \rangle\rangle.$$

By using the above notation, it follows from [2, Theorem 3.14] a goal in this work.

Theorem 2.22 *Let* $(\mathbb{R}^n, \mu, \gamma)$ *be such that* $\{e_1, \ldots, e_n\}$ *is a* γ*-nice basis.* (N_μ, γ) *admits a compatible minimal metric if and only if the equation*

$$\boldsymbol{U}^\gamma_\mu [x_i] = \lambda[1] \tag{2.13}$$

has a positive solution $[x_i]$ *for some* $\lambda \in \mathbb{R}$. *Moreover, in that case, there exists an* $a \in \exp(\mathfrak{a}_\gamma)$ *such that* $a \cdot \langle\cdot,\cdot\rangle$ *defines a minimal compatible metric with* (N_μ, γ).

Remark 2.23 The proof of [2, Theorem 3.14] says even more, namely, if $(\mathbb{R}^n, \mu, \gamma)$ admits a minimal compatible metric, then one can find such a metric by solving the equation

$$m_{\mathfrak{g}_\gamma}(a \cdot \mu) = \beta^\gamma_\mu \tag{2.14}$$

for $a \in \exp(\mathfrak{a}_\gamma)$. Since

$$\beta^\gamma_\mu = \frac{1}{\sum x_p} \left(\sum x_p \mathfrak{R}_\gamma(\mu)_p \right),$$

where $[x_i]$ is any positive solution to Eq. (2.13), in practice it is sometimes easy to solve Eq. (2.14).

3 Soliton Almost Kähler Structures

In this section, we present those soliton almost Kähler structures on two- and three-step nilpotent Lie algebras of dimension 6 that are obtained by minimal compatible metrics with symplectic structures. It may be that all SCF-solitons on any nilmanifold are of this kind with the Chern–Ricci operator being a derivation of the respective nilpotent Lie algebra.

By following the classification given in [9] for 6-dimensional symplectic nilpotent Lie algebras, a simple inspection of such a classification list reveals that many pairs $(\mathfrak{n}, \omega)$ are written in an ω_{cn}-nice basis or by using a suitable change of basis, these pairs can be written in a nice basis. Here ω_{cn} is the canonical symplectic form of $\mathbb{R}^6$: $\omega_{cn} = e_1^* \wedge e_6^* + e_2^* \wedge e_5^* + e_3^* \wedge e_4^*$.

We denote by $m_{\mathfrak{sp}}$ the moment map corresponding to the action of the symplectic group $\mathrm{G}_{\omega_{cn}} = \mathrm{Sp}(3, \mathbb{R})$ on V and we have

$$\mathfrak{a}_{\omega_{cn}} = \{\mathrm{Diag}(-x_1, -x_2, -x_3, x_3, x_2, x_1) : x_i \in \mathbb{R}\}.$$

Theorem 2.22 has been applied to each mentioned algebra individually. We present in detail only those examples that we consider representative. The remaining cases are established in an entirely analogous way.

3.1 Symplectic Three-Step Nilpotent Lie Algebras

In this part, first we give a complete classification of minimal compatible metrics on symplectic three-step nilpotent Lie algebras. Afterwards, we compute the respective

Chern–Ricci operator, which happens to be a derivation in this case. From Proposition 2.8, Theorem B follows.

Example 3.1 We consider the nilpotent Lie algebra $\mathfrak{n}_{11}$ given by $[e_1, e_2] = e_4, [e_1, e_4] = e_5, [e_2, e_3] = e_6, [e_2, e_4] = e_6$, which carries two curves of non-equivalent symplectic structures, namely, $\omega_1(\lambda) = e_1^* \wedge e_6^* + e_2^* \wedge e_5^* + \lambda e_2^* \wedge e_6^* - e_3^* \wedge e_4^*$, with $\lambda \in \mathbb{R}$ and $\omega_2(\lambda) = -\omega_1(\lambda)$ (by [9, Theorem 5]). Let us look at the case for $\omega_1(\lambda)$; similar considerations apply to the other case.

In our approach, we need that the canonical inner product defines a compatible metric, which is similar to finding a basis for $\mathfrak{n}_{11}$, where the symplectic structure is defined by ω_{cn}. To do this, we perform a change of basis having the form

$$g^{-1} = \mathrm{Diag}\left(\begin{pmatrix} m_{1,1} & m_{1,2} \\ 0 & m_{2,2} \end{pmatrix}, \begin{pmatrix} m_{3,3} & m_{3,4} \\ m_{4,3} & m_{4,4} \end{pmatrix}, \begin{pmatrix} m_{5,5} & m_{5,6} \\ 0 & m_{6,6} \end{pmatrix}\right).$$

Since we also need an ω_{cn}-nice basis, we can also try to get that $m_{\mathfrak{sp}}(\exp(X) \cdot g \cdot \mathfrak{n}_{11}) \in \mathfrak{a}_{\omega_{cn}}$ for any $X \in \mathfrak{a}_{\omega_{cn}}$. If $\lambda \neq 0$ then by solving such a system of equations, we have, for instance, a solution given by

$$\Big\{m_{1,1} = -\tfrac{1}{2}\tfrac{\lambda}{m_{5,6}}, m_{1,2} = -\tfrac{1}{2} m_{2,2}\lambda, m_{2,2} = m_{2,2}, m_{3,3} = -{m_{4,4}}^{-1}, m_{3,4} = 0,$$
$$m_{4,3} = \tfrac{1}{2} {m_{4,4}}^{-1}, m_{4,4} = m_{4,4}, m_{5,5} = {m_{2,2}}^{-1}, m_{5,6} = m_{5,6}, m_{6,6} = -2\tfrac{m_{5,6}}{\lambda}\Big\}.$$

If we let $m_{2,2} = 1$, $m_{4,4} = 1$, and $m_{5,6} = 1$ then

$$g = \left(\begin{pmatrix} -\frac{2}{\lambda} & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix}, \begin{pmatrix} 1 & \frac{\lambda}{2} \\ 0 & -\frac{\lambda}{2} \end{pmatrix},\right)$$

defines a symplectomorphism from $(\mathfrak{n}_{11}, \omega_1(\lambda))$ to $(\mathbb{R}^6, \mu_\lambda, \omega_{cn})$, where

$$\mu_\lambda := \begin{cases} [e_1, e_2] = -\frac{1}{2}\lambda e_4, [e_1, e_3] = -\frac{1}{4}\lambda e_5, [e_1, e_4] = -\frac{1}{2}\lambda e_5, \\ [e_2, e_3] = -\frac{1}{2}\lambda e_5 + \frac{1}{4}\lambda e_6, [e_2, e_4] = -\frac{1}{2}\lambda e_6 \end{cases}$$

and $(\mathbb{R}^6, \mu_\lambda, \omega_{cn})$ is written in an ω_{cn}-nice basis. For all $\lambda \in \mathbb{R} \setminus \{0\}$, the Gram matrix is

$$\boldsymbol{U}_{\mu_\lambda}^{\omega_{cn}} = \frac{1}{2}\begin{bmatrix} 3 & 1 & 1 \\ 1 & 5 & 3 \\ 1 & 3 & 3 \end{bmatrix}$$

and since the general solution to $\boldsymbol{U}_{\mu_\lambda}^{\omega_{cn}} X = [1]$ is $X = \frac{1}{2}[1, 0, 1]^{\mathrm{T}}$, for any $\lambda \neq 0$, $(\mathfrak{n}_{11}, \omega_1(\lambda))$ does not admit a minimal metric.

When $\lambda = 0$, on the contrary, $(\mathfrak{n}_{11}, \omega_1(\lambda = 0))$ admits a minimal metric. In fact, like above, we consider $g = \mathrm{Diag}\Big(1, 1, \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & -1 \end{pmatrix}, 1, 1\Big)$. g defines a symplectomorphism from $(\mathfrak{n}_{11}, \omega_1(0))$ to $(\mathbb{R}^6, \mu, \omega_{cn})$ with

$$\mu := \left\{ [e_1, e_2] = \tfrac{1}{2}\, e_3 - e_4, [e_1, e_4] = -e_5, [e_2, e_3] = e_6, [e_2, e_4] = -\tfrac{1}{2}\, e_6, \right.$$

where $(\mathbb{R}^6, \mu, \omega_{cn})$ is written in an ω_{cn}-nice basis. The Gram matrix is

$$\boldsymbol{U}_{\mu}^{\omega_{cn}} = \frac{1}{2}\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

and the general solution to $\boldsymbol{U}_{\mu_\lambda}^{\omega_{cn}} X = [1]$ is $X = \frac{1}{2}[1, 1]^{\mathrm{T}}$. Since $X' = X$ is a positive solution, $(\mathfrak{n}_{11}, \omega_1(0))$ admits a minimal metric. To find such a metric, we solve the equation

$$4\mathrm{Ric}^{ac}(\exp(Y) \cdot \mu) = m_{\mathfrak{sp}}(\exp(Y) \cdot \mu) = \mathrm{mcc}(\mathfrak{R}_{\omega_{cn}}(\mu))$$

with $Y \in \mathfrak{a}_{\omega_{cn}}$. Let $Y = \mathrm{Diag}(\ln(2) + \frac{1}{4}\ln(3), 0, -\frac{1}{4}\ln(3) + \frac{1}{2}\ln(2), \frac{1}{4}\ln(3) - \frac{1}{2}\ln(2), 0, -\ln(2) - \frac{1}{4}\ln(3))$. The change of basis given by $\exp(Y)$ defines

$$\widetilde{\mu} := \left\{ [e_1, e_2] = \tfrac{\sqrt{6}}{12} e_3 - \tfrac{\sqrt{2}}{4} e_4, [e_1, e_4] = -\tfrac{\sqrt{6}}{6} e_5, [e_2, e_3] = \tfrac{\sqrt{2}}{4} e_6, [e_2, e_4] = -\tfrac{\sqrt{6}}{12} e_6. \right.$$

Since

$$m_{\mathfrak{gl}}(\widetilde{\mu}) = \frac{1}{6}\mathrm{Diag}(-4, -4, -1, -1, 2, 2),$$

it follows that

$$\begin{aligned} m_{\mathfrak{sp}}(\widetilde{\mu}) &= \frac{1}{2}(m_{\mathfrak{gl}}(\widetilde{\mu}) + J.m_{\mathfrak{gl}}(\widetilde{\mu}).J) \\ &= \frac{1}{2}\mathrm{Diag}(-1, -1, 0, 0, 1, 1) \\ &= -\mathrm{Id} + \underbrace{\frac{1}{2}\mathrm{Diag}(1, 1, 2, 2, 3, 3)}_{\text{Derivation}} \end{aligned}$$

and thus, the canonical inner product of $\mathbb{R}^6$ defines a minimal metric on $(\mathbb{R}^6, \widetilde{\mu}, \omega_{cn})$.

A straightforward computation shows that $(\mathbb{R}^6, \mu_\lambda, \langle\cdot,\cdot\rangle, J, \omega_{cn})$ and $(\mathbb{R}^6, \widetilde{\mu}, \langle\cdot,\cdot\rangle, J, \omega_{cn})$ are Chern–Ricci flat, where $\langle\cdot,\cdot\rangle$ is the canonical inner product of $\mathbb{R}^6$.

Example 3.2 We consider the nilpotent Lie algebra $\mathfrak{n}_{13}$ given by $[e_1, e_2] = e_4, [e_1, e_3] = e_5, [e_1, e_4] = e_6, [e_2, e_3] = e_6$ and the curve of non-equivalent symplectic structures $\omega_2(\lambda)$ with $\lambda \neq 0$:

$$\omega_2(\lambda) = e_1^* \wedge e_6^* + \lambda e_2^* \wedge e_4^* + e_2^* \wedge e_5^* + e_3^* \wedge e_5^*.$$

The change of basis given by $g = \left(1, \begin{pmatrix} \lambda - \frac{1}{2} & -\frac{1}{2} \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & 1 \\ 1 & 0 \end{pmatrix}, 1\right)$ defines a symplectomorphism from $(\mathfrak{n}_{13}, \omega_2(\lambda))$ to $(\mathbb{R}^6, \mu_\lambda, \omega_{cn})$ with

$$\mu_\lambda := \begin{cases} [e_1, e_2] = -\frac{1}{2\lambda} e_4 + \frac{1}{\lambda} e_5, [e_1, e_3] = \frac{(4\lambda - 1)}{4\lambda} e_4 + \frac{1}{2\lambda} e_5, [e_1, e_5] = e_6, \\ [e_2, e_3] = \frac{1}{\lambda} e_6. \end{cases}$$

It is a simple matter to see that $(\mathbb{R}^6, \mu_\lambda, \omega_{cn})$ is written in an ω_{cn}-nice basis and that if $\lambda \neq \frac{1}{4}$ then the Gram matrix is

$$\boldsymbol{U}_{\mu_\lambda}^{\omega_{cn}} = \frac{1}{2} \begin{bmatrix} 3 & 3 & 3 & 1 \\ 3 & 5 & 1 & 0 \\ 3 & 1 & 5 & 2 \\ 1 & 0 & 2 & 5 \end{bmatrix}.$$

The general solution to $\boldsymbol{U}_{\mu_\lambda}^{\omega_{cn}} X = [1]$ is $X = \frac{1}{25}[10 - 50\,t, 4 + 25\,t, 25\,t, 8]^{\mathrm{T}}$. Since $X' = \frac{2}{25}[3, 3, 1, 4]^{\mathrm{T}}$ is a positive solution, $(\mathfrak{n}_{13}, \omega_2(\lambda))$ admits a minimal metric.

Although it is difficult to give an explicit formula for such a curve of metrics in this case, we can say that if such metrics have scalar curvature equal to $-\frac{1}{4}$ then

$$4\mathrm{Ric}^{ac} = m_{\mathfrak{sp}} = -\frac{25}{22}\mathrm{Id} + \frac{5}{11}\mathrm{Diag}(1, 2, 2, 3, 3, 4).$$

Furthermore, they are given in the family of symplectic nilpotent Lie algebras $(\mathbb{R}^6, \mu_t, \omega_{cn})$ with

$$\mu_t := \begin{cases} [e_1, e_2] = -t e_4 \pm \frac{1}{22}\sqrt{99 - 1452\,t^2} e_5, [e_1, e_3] = \pm\frac{1}{22}\sqrt{55 - 1452\,t^2} e_4 + t e_5, \\ [e_1, e_5] = \frac{\sqrt{22}}{11} e_6, [e_2, e_3] = 2\,t e_6. \end{cases}$$

The Chern–Ricci operator of $(\mathbb{R}^6, \mu_t, \langle\cdot,\cdot\rangle, J, \omega_{cn})$ is

$$\mathrm{P} = \frac{t\sqrt{22}}{22}(E_{4,1} - E_{6,3}) \pm \frac{\sqrt{22}}{484}\sqrt{99 - 1452\,t^2}(E_{5,1} - E_{6,2}),$$

which is easily seen to be a derivation of $(\mathbb{R}^6, \mu_t)$. From Proposition 2.8, $(\langle\cdot,\cdot\rangle, J, \omega_{cn})$ defines a soliton almost Kähler structure on $(\mathbb{R}^6, \mu_t)$.

If $\lambda = \frac{1}{4}$, one can proceed as above and show that $(\mathfrak{n}_{13}, \omega_2(\lambda = \frac{1}{4}))$ admits a soliton almost Kähler structure.

Example 3.3 For a final example, consider the nilpotent Lie algebra $\mathfrak{n}_{12}$ given by $[e_1, e_2] = e_4, [e_1, e_4] = e_5, [e_1, e_3] = e_6, [e_2, e_3] = -e_5, [e_2, e_4] = e_6$ and the curve of non-equivalent symplectic structures $\omega_1(\lambda) = \lambda e_1^* \wedge e_5^* + e_2^* \wedge e_6^* + (\lambda + 1) e_3^* \wedge e_4^*$ (with $\lambda \in \mathbb{R} \setminus \{-1, 0\}$). Consider the change of basis given by

$$g = \mathrm{Diag}\left(1, 1, 1, \lambda + 1, \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}\right),$$

which defines a symplectomorphism from $(\mathfrak{n}_{12}, \omega_1(\lambda))$ to $(\mathbb{R}^6, \mu_\lambda, \omega_{cn})$, where

$$\mu_\lambda := \begin{cases} [e_1, e_2] = (\lambda + 1)\, e_4, [e_1, e_3] = e_5, [e_1, e_4] = \frac{\lambda}{\lambda+1} e_6, \\ [e_2, e_3] = -\lambda\, e_6, [e_2, e_4] = \frac{1}{\lambda+1} e_5. \end{cases}$$

As above, $(\mathbb{R}^6, \mu_\lambda, \omega_{cn})$ is written in an ω_{cn}-nice basis and, by Theorem 2.22, one can show that $(\mathfrak{n}_{12}, \omega_1(\lambda))$ admits a minimal compatible metric. By solving $m_{\mathfrak{sp}}(\exp(Y) \cdot \mu_\lambda) = \mathrm{mcc}(\mathfrak{R}_{\omega_{cn}}(\mu_\lambda))$, we have a symplectomorphism defined by $\exp(Y)$ from $(\mathbb{R}^6, \mu_\lambda, \omega_{cn})$ to $(\mathbb{R}^6, \widetilde{\mu}_\lambda, \omega_{cn})$ with

$$\widetilde{\mu_\lambda} := \begin{cases} [e_1, e_2] = \frac{\sqrt{2}}{4} \frac{(\lambda+1)\sqrt{\lambda^2+\lambda+1}}{\lambda^2+\lambda+1} e_4, [e_1, e_3] = \frac{\sqrt{2}}{4} \frac{\sqrt{\lambda^2+\lambda+1}}{\lambda^2+\lambda+1} e_5, \\ [e_1, e_4] = \frac{\sqrt{2}}{4} \operatorname{sign}\left(\frac{\lambda}{\lambda+1}\right) e_6, [e_2, e_3] = -\frac{\sqrt{2}}{4} \frac{\lambda \sqrt{\lambda^2+\lambda+1}}{\lambda^2+\lambda+1} e_6, \\ [e_2, e_4] = \frac{\sqrt{2}}{4} \operatorname{sign}(\lambda + 1) e_5, \end{cases}$$

where the canonical inner product defines a minimal compatible metric with $(\mathbb{R}^6, \widetilde{\mu}_\lambda, \omega_{cn})$.

This example is interesting in the following sense. Let P_λ be the Chern–Ricci operator of $(\mathbb{R}^6, \widetilde{\mu}_\lambda, \langle \cdot, \cdot \rangle, J, \omega_{cn})$. We have

$$\mathrm{P}_\lambda = \frac{1}{16} \frac{(1 + \operatorname{sign}(\lambda))\, |\lambda + 1|\, \sqrt{\lambda^2 + \lambda + 1}}{\lambda^2 + \lambda + 1} (E_{5,1} - E_{6,2}).$$

It is easy to see that P_λ is a Derivation of $(\mathbb{R}^6, \widetilde{\mu}_\lambda)$ and, moreover, $(\mathbb{R}^6, \widetilde{\mu}_\lambda, \omega_{cn})$ is Chern–Ricci flat if and only if λ is a negative number (with $\lambda \neq -1$).

Theorem 3.4 *The classification of minimal metrics on 6-dimensional symplectic three-step nilpotent Lie algebras is given in the Table 1. In each case, such a metric defines a soliton almost Kähler structure, where each Chern–Ricci operator is a derivation of the respective nilpotent Lie algebra (see Table 2).*

In the Table 1, each Lie algebra defines a symplectic three-step Lie algebra given by $(\mathbb{R}^6, \widetilde{\mu}, \omega_{cn})$ and the canonical inner product on $\mathbb{R}^6$ defines a minimal metric of scalar curvature equal to $-\frac{1}{4}$. In the column $||\beta||^2$ we give the squared norm of the *stratum* associated with the minimal metric and, in the Derivation column, we give the derivation of $(\mathbb{R}^6, \widetilde{\mu})$ in such a way that

$$m_{\mathfrak{sp}_6(\mathbb{R})}(\widetilde{\mu}) = -||\beta||^2 \mathrm{Id} + \text{Derivation}.$$

In the last column, we give the dimension of the automorphism group of the symplectic three-step Lie algebra $(\mathbb{R}^6, \widetilde{\mu}, \omega_{cn})$. A line means that such symplectic nilpotent Lie algebra does not admit a minimal metric.

Table 1 Classification of minimal compatible metrics on symplectic three-step Lie algebras of dimension 6

Not.	Critical point	Derivation	$\|\|\beta\|\|^2$	dim Aut
10.1	$[e_1, e_2] = -\frac{\sqrt{2}}{4}e_4, [e_1, e_3] = -\frac{\sqrt{2}}{4}e_5, [e_1, e_4] = \frac{\sqrt{2}}{4}e_6,$ $[e_2, e_4] = -\frac{\sqrt{2}}{4}e_5$	$\frac{1}{2}\mathrm{Diag}(1, 1, 2, 2, 3, 3)$	1	5
10.2	$[e_1, e_2] = -\frac{\sqrt{2}}{4}e_4, [e_1, e_4] = \frac{\sqrt{2}}{4}e_6, [e_2, e_3] = \frac{\sqrt{2}}{4}e_6,$ $[e_2, e_4] = -\frac{\sqrt{2}}{4}e_5$	$\frac{1}{2}\mathrm{Diag}(1, 1, 2, 2, 3, 3)$	1	5
11.1 $\lambda \neq 0$	–	–	–	5
11.1 $\lambda = 0$	$[e_1, e_2] = -\frac{\sqrt{6}}{12}e_3 + \frac{\sqrt{2}}{4}e_4, [e_1, e_3] = \frac{\sqrt{2}}{4}e_5,$ $[e_1, e_4] = -\frac{\sqrt{6}}{12}e_5, [e_2, e_4] = -\frac{\sqrt{6}}{6}e_6$	$\frac{1}{2}\mathrm{Diag}(1, 1, 2, 2, 3, 3)$	1	6
11.2 $\lambda \neq 0$	–	–	–	5
11.2 $\lambda = 0$	$[e_1, e_2] = \frac{\sqrt{6}}{12}e_3 + \frac{\sqrt{2}}{4}e_4, [e_1, e_3] = \frac{\sqrt{2}}{4}e_5,$ $[e_1, e_4] = \frac{\sqrt{6}}{12}e_5, [e_2, e_4] = \frac{\sqrt{6}}{6}e_6$	$\frac{1}{2}\mathrm{Diag}(1, 1, 2, 2, 3, 3)$	1	6
12.1	$[e_1, e_2] = f_1(\lambda)\,(\lambda + 1)\,e_4, [e_1, e_3] = f_1(\lambda)e_5,$ $[e_2, e_3] = f_1(\lambda)(-\lambda)e_6,$ $[e_1, e_4] = \frac{\sqrt{2}}{4}\mathrm{sign}\left(\frac{\lambda}{\lambda+1}\right)e_6, [e_2, e_4] = \frac{\sqrt{2}}{4}\mathrm{sign}\,(\lambda + 1)e_5$	$\frac{1}{2}\mathrm{Diag}(1, 1, 2, 2, 3, 3)$	1	5
13.1	$[e_1, e_2] = f_2(\lambda)\,(1 - \lambda)\,e_4, [e_1, e_3] = f_2(\lambda)e_5,$ $[e_2, e_3] = f_2(\lambda)(\lambda)e_6, [e_2, e_4] = \frac{\sqrt{6}}{6}\mathrm{sign}\,(\lambda - 1)\,e_5$	$\frac{1}{6}\mathrm{Diag}(5, 3, 6, 8, 11, 9)$	$\frac{7}{6}$	7
13.2 $\lambda \neq \frac{1}{4}$	see Example 3.2	$\frac{5}{11}\mathrm{Diag}(1, 2, 2, 3, 3, 4)$	$\frac{25}{22}$	6
13.2 $\lambda = \frac{1}{4}$	$[e_1, e_2] = -\frac{\sqrt{165}}{66}e_4 + \frac{\sqrt{11}}{11}e_5, [e_1, e_3] = \frac{\sqrt{165}}{66}e_5,$ $[e_1, e_5] = \frac{\sqrt{22}}{11}e_6, [e_2, e_3] = \frac{\sqrt{165}}{33}e_6$	$\frac{5}{11}\mathrm{Diag}(1, 2, 2, 3, 3, 4)$	$\frac{25}{22}$	6
13.3	–	–	–	7

Table 1 continued

Not.	Critical point	Derivation	$\|\|\beta\|\|^2$	dim Aut
14.1	$[e_1, e_2] = \frac{\sqrt{55}}{22} e_5, [e_1, e_3] = \frac{3\sqrt{11}}{22} e_4, [e_1, e_4] = \frac{\sqrt{22}}{11} e_6$	$\frac{5}{11}\mathrm{Diag}(1, 2, 2, 3, 3, 4)$	$\frac{25}{22}$	6
14.2	$[e_1, e_2] = \frac{\sqrt{55}}{22} e_5, [e_1, e_3] = -\frac{3\sqrt{11}}{22} e_4, [e_1, e_4] = \frac{\sqrt{22}}{11} e_6$	$\frac{5}{11}\mathrm{Diag}(1, 2, 2, 3, 3, 4)$	$\frac{25}{22}$	6
14.3	$[e_1, e_2] = \frac{\sqrt{6}}{6} e_4, [e_1, e_3] = \frac{\sqrt{6}}{6} e_5, [e_1, e_4] = \frac{\sqrt{6}}{6} e_6$	$\frac{1}{6}\mathrm{Diag}(3, 5, 6, 8, 9, 11)$	$\frac{7}{6}$	7
15.1	–	–	–	5
15.2	–	–	–	5
15.3	$[e_1, e_2] = \frac{\sqrt{21}}{14} e_5, [e_1, e_5] = \frac{\sqrt{42}}{14} e_6, [e_2, e_3] = \frac{\sqrt{35}}{14} e_4$	$\frac{5}{28}\mathrm{Diag}(2, 4, 3, 7, 6, 8)$	$\frac{25}{28}$	4
21.1	$[e_1, e_2] = -\frac{\sqrt{66}}{44} e_4 - \frac{\sqrt{22}}{44} e_5, [e_1, e_3] = \frac{3\sqrt{22}}{44} e_4 + \frac{\sqrt{66}}{44} e_5,$ $[e_1, e_4] = -\frac{\sqrt{22}}{11} e_6, [e_2, e_3] = \frac{\sqrt{66}}{22} e_6$	$\frac{5}{11}\mathrm{Diag}(1, 2, 2, 3, 3, 4)$	$\frac{25}{22}$	6
21.2	$[e_1, e_2] = \frac{\sqrt{6}}{6} e_3, [e_1, e_3] = \frac{\sqrt{6}}{6} e_6, [e_2, e_4] = \frac{\sqrt{6}}{6} e_6$	$\frac{1}{6}\mathrm{Diag}(3, 5, 8, 6, 9, 11)$	$\frac{7}{6}$	7
21.3	$[e_1, e_2] = \frac{\sqrt{6}}{6} e_3, [e_1, e_3] = -\frac{\sqrt{6}}{6} e_6, [e_2, e_4] = \frac{\sqrt{6}}{6} e_6$	$\frac{1}{6}\mathrm{Diag}(3, 5, 8, 6, 9, 11)$	$\frac{7}{6}$	7
22.1	$[e_1, e_2] = \frac{1}{2} e_5, [e_1, e_5] = \frac{1}{2} e_6$	$\frac{1}{4}\mathrm{Diag}(2, 4, 5, 5, 6, 8)$	$\frac{5}{4}$	8

Here, $f_1(\lambda) = \frac{\sqrt{2}}{4} \frac{\sqrt{\lambda^2+\lambda+1}}{\lambda^2+\lambda+1}$ and $f_2(\lambda) = \frac{\sqrt{6}}{6} \frac{\sqrt{\lambda^2-\lambda+1}}{\lambda^2-\lambda+1}$

3.2 Symplectic Two-Step Nilpotent Lie Algebras

Here, we give the classification of minimal compatible metrics on two-step nilpotent Lie algebras, which determines immediately a soliton almost Kähler structure because the Chern–Ricci operator is always zero.

Example 3.5 Consider the free 2-step nilpotent Lie algebra of rank3:

$$\mathfrak{n}_{18} := \{[e_1, e_2] = e_4, [e_1, e_3] = e_5, [e_2, e_3] = e_6 .$$

By [9, Theorem 5], $\mathfrak{n}_{18}$ carries two curves of non-equivalent symplectic structures, namely, $\omega_1(s)$ and $\omega_2(t)$ (with $s \in \mathbb{R}\backslash\{0, 1\}$ and $t \in \mathbb{R}\backslash\{0\}$), and an isolated symplectic structure ω_3. In this example we want to prove that every pair $(\mathfrak{n}_{18}, \omega_i)$ admits a minimal compatible metric.

Table 2 Chern–Ricci operator of minimal compatible metrics on symplectic three-step Lie algebras of dimension 6

Not.	Chern–Ricci operator
10.1	Chern–Ricci flat
10.2	Chern–Ricci flat
11.1 ($\lambda = 0$)	Chern–Ricci flat
11.2 ($\lambda = 0$)	Chern–Ricci flat
12.1	$\mathrm{P}_\lambda = \frac{\sqrt{2}}{8}(1 + \mathrm{sign}(\lambda))\,\lvert\lambda + 1\rvert\, f_1(\lambda)(E_{5,1} - E_{6,2})$
13.1	$\mathrm{P}_\lambda = \frac{\sqrt{6}}{12}\lvert\lambda - 1\rvert f_2(\lambda)(E_{5,1} - E_{6,2})$
13.2	see Example 3.2
13.2 ($\lambda = \frac{1}{4}$)	$\mathrm{P} = \frac{\sqrt{30}}{132}(E_{4,1} - E_{6,3}) + \frac{\sqrt{2}}{22}(E_{5,1} - E_{6,2})$
14.1	$\mathrm{P} = \frac{3\sqrt{2}}{44}(E_{4,1} - E_{6,3})$
14.2	$\mathrm{P} = -\frac{3\sqrt{2}}{44}(E_{4,1} - E_{6,3})$
14.3	$\mathrm{P} = \frac{1}{12}(E_{5,1} - E_{6,2})$
15.3	$\mathrm{P} = \frac{3\sqrt{2}}{56}(E_{5,1} - E_{6,2})$
21.1	$\mathrm{P} = -\frac{3}{44}(E_{4,1} - E_{6,3}) + \frac{\sqrt{3}}{44}(E_{5,1} - E_{6,2})$
21.2	$\mathrm{P} = \frac{1}{12}(E_{5,1} - E_{6,2})$
21.3	$\mathrm{P} = -\frac{1}{12}(E_{5,1} - E_{6,2})$
22.1	$\mathrm{P} = \frac{1}{8}(E_{5,1} - E_{6,2})$

Here, $f_1(\lambda) = \frac{\sqrt{2}}{4}\frac{\sqrt{\lambda^2+\lambda+1}}{\lambda^2+\lambda+1}$ and $f_2(\lambda) = \frac{\sqrt{6}}{6}\frac{\sqrt{\lambda^2-\lambda+1}}{\lambda^2-\lambda+1}$

Consider the case of the first curve

$$\omega_1(s) = e_1^* \wedge e_6^* + s e_2^* \wedge e_5^* + (s-1) e_3^* \wedge e_4^*, \text{ with } s \in \mathbb{R} \setminus \{0, 1\}$$

and let $g = \mathrm{Diag}(1, 1, 1, s-1, s, 1)$. The change of basis given by g defines a symplectomorphism from $(\mathfrak{n}_{18}, \omega_1(s))$ to $(\mathbb{R}^6, \mu_s, \omega_{cn})$ with

$$\mu_s := \{[e_1, e_2] = (s-1)e_4, [e_1, e_3] = s e_5, [e_2, e_3] = e_6\ .$$

It is obvious that $(\mathbb{R}^6, \mu_s, \omega_{cn})$ is written in an ω_{cn}-nice basis because

$$\mathfrak{R}_{\omega_{cn}}(\mu_s) = \left\{\frac{1}{2}\mathrm{Diag}(-1, -1, -1, 1, 1, 1)\right\}$$

and, from this, $(\mathfrak{n}_{18}, \omega_1(s))$ admits a minimal compatible metric which can be found by solving the equation

$$m_{\mathfrak{sp}}(a \cdot \mu_s) = \frac{1}{2}\mathrm{Diag}(-1, -1, -1, 1, 1, 1)$$

for $a \in \exp(\mathfrak{a}_{\omega_{cn}})$.

Let $a = \exp(X)$ with

$$X = \frac{1}{2}\mathrm{Diag}(\ln(4\,s^2 - 4\,s + 4), 0, 0, 0, 0, -\ln(4\,s^2 - 4\,s + 4)).$$

The change of basis given by a defines the curve

$$\widetilde{\mu_s} := \begin{cases} [e_1, e_2] = \frac{1}{2}\ (s-1)\sqrt{\left(s^2 - s + 1\right)^{-1}} e_4, [e_1, e_3] = \frac{1}{2}\, s\sqrt{\left(s^2 - s + 1\right)^{-1}} e_5, \\ [e_2, e_3] = \frac{1}{2}\sqrt{\left(s^2 - s + 1\right)^{-1}} e_6 \end{cases}$$

and, since

$$m_{\mathfrak{gl}}(\widetilde{\mu_s}) = \frac{1}{2(s^2 - s + 1)}\mathrm{Diag}(-2\,s^2 + 2\,s - 1, -s^2 + 2\,s - 2, -(s^2+1), (s-1)^2, s^2, 1),$$

we get

$$\begin{aligned} m_{\mathfrak{sp}}(\widetilde{\mu_s}) &= \frac{1}{2}(m_{\mathfrak{gl}}(\widetilde{\mu_s}) + J.m_{\mathfrak{gl}}(\widetilde{\mu_s}).J) \\ &= \frac{1}{2}\mathrm{Diag}(-1, -1, -1, 1, 1, 1) \\ &= -\frac{3}{2}\mathrm{Id} + \underbrace{\mathrm{Diag}(1, 1, 1, 2, 2, 2)}_{\text{Derivation}}. \end{aligned}$$

The canonical inner product of $\mathbb{R}^6$ defines a minimal compatible metric on $(\mathbb{R}^6, \widetilde{\mu}_s, \omega_{cn})$.

Now, consider the case of

$$\omega_2(t) = e_1^* \wedge e_5^* + te_1^* \wedge e_6^* - te_2^* \wedge e_5^* + e_2^* \wedge e_6^* - 2te_3^* \wedge e_4^*.$$

From Example 2.11, we have that $(\mathfrak{n}_{18}, \omega_2)$ is equivalent to $(\mathbb{R}^6, \mu_t, \omega_{cn})$ with

$$\mu_t := \{[e_1, e_2] = -2\,t\,e_4, [e_1, e_3] = -t\,e_5 + e_6, [e_2, e_3] = e_5 + t\,e_6\ .$$

It is easy to see that $(\mathbb{R}^6, \mu_t, \omega_{cn})$ is written in an ω_{cn}-nice basis and that

$$\begin{aligned} \mathfrak{R}_{\omega_{cn}}(\mu) = \{&\mathrm{Diag}(-1, 0, -\tfrac{1}{2}, \tfrac{1}{2}, 0, 1), \mathrm{Diag}(0, -1, -\tfrac{1}{2}, \tfrac{1}{2}, 1, 0), \\ &\tfrac{1}{2}\mathrm{Diag}(-1, -1, -1, 1, 1, 1)\}. \end{aligned}$$

Like above, Theorem 2.22 implies that $(\mathfrak{n}_{18}, \omega_2(t))$ admits a minimal compatible metric and, proceeding in a similar way to the above, we get that $\widetilde{\mu}_t := a \cdot \mu$ with

$$a = \mathrm{Diag}\left(1, 1, 2\sqrt{3t^2 + 1}, \frac{1}{2\sqrt{3t^2 + 1}}, 1, 1\right)$$

is such that the canonical inner product of $\mathbb{R}^6$ defines a minimal compatible metric on $(\mathbb{R}^6, \widetilde{\mu}_t, \omega_{cn})$ for any $t \in \mathbb{R} \setminus \{0\}$.

In the latter case,

$$\omega_3 := e_3^* \wedge e_5^* - e_1^* \wedge e_6^* + e_2^* \wedge e_5^* + 2e_3^* \wedge e_4^*,$$

and we can now proceed analogously like above. We leave to the reader to verify that the following change of basis defines a minimal compatible metric with $(\mathfrak{n}_{18}, \omega_3)$:

$$g := \text{Diag}\left(-1, \begin{pmatrix} 1 & \frac{5}{6} \\ 1 & -\frac{1}{6} \end{pmatrix}, \begin{pmatrix} -2 & -\frac{1}{6} \\ 2 & \frac{7}{6} \end{pmatrix}, 1\right),$$
$$a := \text{Diag}\left(2\sqrt{3}, 1, 1, 1, 1, \frac{\sqrt{3}}{6}\right).$$

Proceeding in an entirely analogous way, we can study the remaining symplectic two-step Lie algebras in [9, Theorem 5.24] and obtain

Theorem 3.6 *All symplectic two-step Lie algebras of dimension* 6 *admit a minimal compatible metric and, in consequence, these admit a soliton almost Kähler structure.*

Remark 3.7 We must say that there are several mistakes in the classification given in [9]. For example, 16.(b) does not define a symplectic structure and 9. is not a curve of non-equivalent symplectic structures. Some errors have already been corrected by personal communication with authors, like the symplectic structure given in 23.(c).

In Table 3, each Lie algebra defines a symplectic two-step Lie algebra $(\mathbb{R}^6, \widetilde{\mu}, \omega_{cn})$ such that the canonical inner product on $\mathbb{R}^6$ defines a minimal metric of scalar curvature equal to $-\frac{1}{4}$. In the column $||\beta||^2$ we give the squared norm of the stratum associated with the minimal metric and, in the Derivation column, we give the derivation of $(\mathbb{R}^6, \widetilde{\mu})$ in such a way that

$$m_{\mathfrak{sp}_6(\mathbb{R})}(\widetilde{\mu}) = -||\beta||^2\text{Id} + \text{Derivation}.$$

In the last column, we give the dimension of the automorphism group of the symplectic two-step Lie algebra $(\mathbb{R}^6, \widetilde{\mu}, \omega_{cn})$.

4 Conclusions

The SCF was introduced by Jeffrey Streets and Gang Tian as a geometric analysis approach to understanding the topology and geometry of symplectic manifolds, to introduce the "Ricci flow philosophy" to symplectic geometry. Such flow evolves almost Kähler structures towards certain "canonical geometric structures" on symplectic manifolds.

Lie groups have always been a source for finding explicit examples for many concepts and notions in geometry. Our results are given with the idea of providing and understanding the "canonical geometries" to the SCF.

Table 3 Classification of minimal compatible metrics on symplectic two-step Lie algebras of dimension 6

Not.	Critical point	Derivation	$\|\|\beta\|\|^2$	dim Aut
16.1	$[e_1, e_2] = \frac{\sqrt{2}}{4} e_3, [e_1, e_5] = \frac{\sqrt{2}}{4} e_6,$ $[e_2, e_4] = \frac{\sqrt{2}}{4} e_6, [e_4, e_5] = \frac{\sqrt{2}}{4} e_3$	$\frac{1}{2}$Diag(1, 2, 3, 1, 2, 3)	1	6
17	$[e_1, e_3] = \frac{\sqrt{6}}{6} e_5, [e_1, e_4] = \frac{\sqrt{6}}{6} e_6,$ $[e_2, e_3] = \frac{\sqrt{6}}{6} e_6$	$\frac{1}{6}$Diag(3, 5, 6, 8, 9, 11)	$\frac{7}{6}$	7
18.1	$[e_1, e_2] = f_1(s)((s-1)\, e_4),$ $[e_1, e_3] = f_1(s)(s\, e_5),$ $[e_2, e_3] = f_1(s) e_6$	Diag(1, 1, 1, 2, 2, 2)	$\frac{3}{2}$	8
18.2	$[e_1, e_2] = f_2(t)(-2t\, e_4),$ $[e_1, e_3] = f_2(t)(-t\, e_5 + e_6),$ $[e_2, e_3] = f_2(t)(e_5 + t\, e_6)$	Diag(1, 1, 1, 2, 2, 2)	$\frac{3}{2}$	8
18.3	$[e_1, e_2] = \frac{\sqrt{3}}{12} e_4 - \frac{\sqrt{3}}{4} e_5,$ $[e_1, e_3] = \frac{\sqrt{3}}{4} e_4 - \frac{\sqrt{3}}{12} e_5$ $[e_2, e_3] = -\frac{\sqrt{3}}{6} e_6$	Diag(1, 1, 1, 2, 2, 2)	$\frac{3}{2}$	10
23.1	$[e_1, e_2] = \frac{1}{2} e_5, [e_1, e_3] = \frac{1}{2} e_6$	$\frac{1}{4}$Diag(4, 5, 6, 8, 9, 10)	$\frac{7}{4}$	9
23.2	$[e_1, e_2] = -\frac{1}{2} e_4, [e_2, e_3] = \frac{1}{2} e_6$	Diag(1, 1, 1, 2, 2, 2)	$\frac{3}{2}$	8
23.3	$[e_1, e_2] = \frac{1}{2} e_5, [e_1, e_3] = -\frac{1}{2} e_4$	Diag(1, 1, 1, 2, 2, 2)	$\frac{3}{2}$	8
24.1	$[e_1, e_4] = \frac{1}{2} e_6, [e_2, e_3] = \frac{1}{2} e_5$	$\frac{1}{2}$Diag(1, 1, 2, 2, 3, 3)	1	6
24.2	$[e_1, e_4] = \frac{1}{2} e_6, [e2, e3] = -\frac{1}{2} e_5$	$\frac{1}{2}$Diag(1, 1, 2, 2, 3, 3)	1	6
25	$[e_1, e_2] = \frac{\sqrt{2}}{2} e_6$	$\frac{1}{2}$Diag(3, 4, 5, 5, 6, 7)	$\frac{5}{2}$	12

Here, $f_1(s) = \frac{1}{2}\sqrt{(s^2 - s + 1)^{-1}}$ and $f_2(t) = \frac{1}{2}\sqrt{(3\,t^2 + 1)^{-1}}$

Although we have only considered a specific family of symplectic nilpotent Lie algebras in dimension 6, we must say that we have also studied symplectic solvable Lie algebras in dimension 6 and 8 (in a full-computational manner) and we have found evidence that soliton almost Kähler structures on nilpotent Lie algebras are determined by minimal compatible metrics with the respective Chern–Ricci operator being a derivation. We think that results in this direction can be important, because the problem of existence and uniqueness of such structures on nilmanifolds could be understood by using powerful results from real GIT. In fact, we think that many structural results on solvsolitons are true in this context.

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References

1. Burde, D.: Left-symmetric algebras, or pre-Lie algebras in geometry and physics. Cent. Eur. J. Math. **4**(3), 323–357 (2006)

2. Fernández-Culma, E.A.: On distinguished orbits of reductive representations. J. Algebra **396**, 61–81 (2013)
3. Heber, J.: Noncompact homogeneous Einstein spaces. Invent. Math. **133**, 279–352 (1998)
4. Heinzner, P., Schwarz, G.W., Stötzel, H.: Stratifications with respect to actions of real reductive groups. Compositio Math. **144**, 163–185 (2008)
5. Helmstetter, J.: Radical d'une algèbre symétrique à gauche. Ann. Inst. Fourier **29**(4), 17–35 (1979)
6. Jablonski, M.: Detecting orbits along subvarieties via the moment map. Münster J. Math. **3**, 67–88 (2010)
7. Jablonski, M.: Distinguished orbits of reductive groups. Rocky Mt. J. Math. **42**(5), 1521–1549 (2012)
8. Jablonski, M.: Homogeneous Ricci solitons are algebraic. Geom. Topol. (to appear) (2013)
9. Khakimdjanov, Yu., Goze, M., Medina, A.: Symplectic or contact structures on Lie groups. Diff. Geom. Appl. **21**, 41–54 (2004)
10. Kim, H.: Complete left-invariant affine structures on nilpotent Lie groups. J. Diff. Geom. **24**(3), 373–394 (1986)
11. Lauret, J.: A canonical compatible metric for geometric structures on nilmanifolds. Ann. Glo. Anal. Geom. **30**, 107–138 (2006)
12. Lauret, J.: Curvature flows of almost-Hermitian Lie groups. Trans. Am. Math. Soc. (to appear) (2013)
13. Le, H.V., Wang, G.: Anti-complexified Ricci flow on compact symplectic manifolds. J. Reine Angew. Math. **530**, 17–31 (2001)
14. Marian, A.: On the real moment map. Math. Res. Lett. **8**, 779–788 (2001)
15. Nikolayevsky, Y.: Einstein solvmanifolds and the pre-Einstein derivation. Trans. Am. Math. Soc. **363**, 3935–3958 (2011)
16. Pook, J.: Homogeneous and locally homogeneous solutions to symplectic curvature flow. arXiv.org (2012)
17. Segal, D.: The structure of complete left-symmetric algebras. Math. Ann. **293**(1), 569–578 (1992)
18. Streets, J., Tian, G.: Symplectic curvature flow. J. Reine Angew. Math. (to appear) (2010)
19. Vezzoni, L.: A note on canonical Ricci forms on 2-step nilmanifolds. Proc. Am. Math. Soc. **141**(1), 325–333 (2013)