Weighting by iteration: the case of Ryll-Nardzewski's iterations

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Abstract

Aczél's and Ryll-Nardzewski's dyadic iterations are iterative procedures which associate to a given mean M a family of means $\{M^d : d \in \mathrm{Dyad}\,([0,1])\}$ parameterized by $\mathrm{Dyad}\,([0,1])$, the dyadic fractions of the interval $[0,1]$. Aczél's iterations exhibit a nice characteristic: when M is a strict continuous mean and $x < y$, the set $\{M^d(x,y) : d \in \mathrm{Dyad}\,([0,1])\}$ is dense in $[x,y]$. This fact is in the basis of the construction of an algorithm of weighting for an ample class of means. In pursuit of a similar algorithm using Ryll-Nardzewski's instead of Aczél's iterations, a series of obstacles is found, which motivates the detailed study of these last conducted along this paper. Among other result of interest, several conditions on the mean M are identified which make viable a weighting algorithm based on these iterations.

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1 Introduction

Let be given a real interval I and a function $F : I \times I \to I$. Denoting by $\mathrm{Dyad}([0,1])$ the set of dyadic fractions of the interval $[0,1]$; i.e.

$$\mathrm{Dyad}([0,1]) = \left\{ \frac{k}{2^n} : 0 \le k \le 2^n,\, n \in \mathbb{N}_0 \right\}, \tag{1}$$

let us define $F^{(d)}$ and $F^{[d]}$ for $d \in \mathrm{Dyad}([0,1])$ as certain sequences of compositions of F with itself. In the first place, the *Aczél's dyadic iterations* $F^{(d)}$ of F are inductively defined for $x, y \in I$ as follows:

$$F^{(0)}(x,y) \equiv x,\ F^{(1)}(x,y) \equiv y, \tag{2}$$

and, for every $n \in \mathbb{N}_0$ and $0 \le k \le 2^{n+1}$,

$$F^{\left(\frac{k}{2^{n+1}}\right)}(x,y) = \begin{cases} F^{\left(\frac{h}{2^n}\right)}(x,y), & k = 2h,\ 0 \le h \le 2^n \\ F(F^{\left(\frac{h}{2^n}\right)}(x,y), F^{\left(\frac{h+1}{2^n}\right)}(x,y)), & k = 2h+1,\ 0 \le h \le 2^n - 1 \end{cases}. \tag{3}$$

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In the second place, the *Ryll-Nardzewski's dyadic iterations* $F^{[d]}$ of F are defined for $x, y \in I$ by

$$F^{[0]}(x,y) \equiv x, \;\; F^{[1]}(x,y) \equiv y, \tag{4}$$

and, for every $n \in \mathbb{N}_0$ and $0 \leq k \leq 2^{n+1}$,

$$F^{\left[\frac{k}{2^{n+1}}\right]}(x,y) = \begin{cases} F(x, F^{\left[\frac{k}{2^n}\right]}(x,y)), & 0 \leq k \leq 2^n \\ F(F^{\left[\frac{k-2^n}{2^n}\right]}(x,y), y), & 2^n \leq k \leq 2^{n+1} \end{cases}. \tag{5}$$

Remarkably, both dyadic iterations arose in mathematical constructions related to a characterization of the quasiarithmetic means by using functional equations. Indeed, Aczél's dyadic iterations were employed in [1] (see also [2], Chap. 6 and [3], Chap. 17) to characterize quasiarithmetic means as the solutions to

$$F(F(x,y), F(u,v)) = F(F(x,u), F(y,v)), \;\; x, y, u, v \in I, \tag{6}$$

a functional equation of composite type named *bisymmetry equation*, while Ryll-Nardzewski's ones arose in [22] with the same purpose but using, instead of (6), the system of equations

$$\begin{cases} F(F(x,y), z) = F(F(x,z), F(y,z)) \\ F(x, F(y,z)) = F(F(x,y), F(x,z)) \end{cases} \;\; x, y, z \in I, \tag{7}$$

known as *self-distributivity equations*.

In this paper the above defined iterations are studied mainly in the case in which the function F is a mean, so that a relative series of notations and terminology is now presented. Remind that a (two variables) mean M defined on a real interval I is a function $M : I \times I \to I$ which is *internal*; i.e., it satisfies the property

$$\min\{x, y\} \leq M(x,y) \leq \max\{x, y\}, \;\; x, y \in I. \tag{8}$$

The mean is said to be *strict* when the inequalities in (8) are strict provided that $x \neq y$ (*strict internality*). In view of the equality

$$M(x,x) = x, \;\; x \in I, \tag{9}$$

holds for every internal function M, means are *reflexive* functions. A mean M is said to be *symmetric* when

$$M(x,y) = M(y,x), \;\; x, y \in I. \tag{10}$$

The *coordinate means* $X(x,y) \equiv x$ and $Y(x,y) \equiv y$ are the unique means depending on a sole variable. The functions at the leftmost and rightmost members of the inequalities (8) are the *extremal means* min and max.

A mean M is [*strictly*] *isotone* when $x \mapsto M(x,y)$ is [strictly] increasing for every $y \in I$ and $y \mapsto M(x,y)$ is [strictly] increasing for every $x \in I$. An isotone mean preserve the product order in I^2; i.e., $M(x,y) \leq M(x',y')$ provided that $(x,y) \leq (x',y')$. A strictly isotone mean is a strict mean.

If M is a mean defined on I and $f : I \to \mathbb{R}$ is a strictly monotonic and continuous function (i.e., a homeomorphism from I onto $f(I)$), the *f-conjugated* M_f of M is the mean defined on $f(I)$ by

$$M_f(x,y) = f(M(f^{-1}(x), f^{-1}(y)),\ x, y \in f(I).$$

When M is a given mean and f varies on the set of homeomorphism from I in $\mathbb{R}$, then M_f runs along the entire *class of conjugation* of M. The means which are conjugated of the arithmetic mean $A(x,y) = (x+y)/2$ are named *quasiarithmetic* means; thus, a quasiarithmetic mean has the form

$$A_f(x,y) = f\left(\frac{f^{-1}(x) + f^{-1}(y)}{2}\right),\ x, y \in I, \tag{11}$$

where $f : I \to \mathbb{R}$ is a strictly monotonic and continuous function.

When the interval I is a cone, a mean M defined on I is said to be *homogeneous* provided that $M_f = M$ for every linear map $f : I \to I$, $f \neq 0$.

A mean M is said to be *continuous* or *separately continuous* when $(x,y) \mapsto M(x,y)$ is continuous on I^2 or $x \mapsto M(x,y),\ y \in I$, and $y \mapsto M(x,y),\ x \in I$, are continuous on I, respectively. Means fulfilling special types of separate uniform Lipschitz-continuity will play a capital role in this paper. If, for every $x, y \in I$, the inequalities

$$|M(x,u) - M(x,u')| \leq |u - u'| \text{ and } |M(u,y) - M(u',y)| \leq |u - u'| \tag{12}$$

hold for every $u,\ u' \in I$, then the mean M is said to be *separately nonexpansive*, while it is said *separately quasicontractive* when it is separately nonexpansive and, for every $x, y \in I$, one inequality (12) at least is strict provided that $u,\ u' \in I,\ u \neq u'$. In view of the inequalities

$$\begin{aligned} |M(x,y) - M(x',y')| &\leq |M(x,y) - M(x',y)| + |M(x',y) - M(x',y')| \\ &\leq |x - x'| + |y - y'|,\ x, y, x', y' \in I, \end{aligned}$$

every separately nonexpansive mean M turns out to be a (L^1) Lipschitz-continuous function (with Lipschitz constant equal to 1). Extremal and coordinate means are simple examples of separately nonexpansive and separately quasicontractive means, respectively.

Separate nonexpansiveness and quasicontractiveness are properties of a mean which are not generally preserved by conjugacy (see Example 28 in the Appendix). Unlike these properties, the following ones remain invariant under conjugacy: a mean M is said to be *separately (C)-nonexpansive* [*separately (C)-quasicontractive*] when the class of conjugation of M contains a separately nonexpansive [separately quasicontractive] mean; i.e., when there exists a homeomorphism $f : I \to \mathbb{R}$ such that M_f is separately nonexpansive [separately quasicontractive] (on $f(I)$). Further discussion on these concepts is contained in the final Appendix.

Returning to the main exposition, it should be pointed out that, under the denomination of *dyadic iterations*, a particularized study of Aczél's dyadic iterations was conducted in [8] and [4] (see also [11]). In the last of this references, the following result was proved:

Theorem 1 *For a strictly internal and reflexive function M, the function $d \mapsto M^{(d)}(x,y)$ defined on* $\mathrm{Dyad}([0,1])$ *is monotonically extended to $\delta \in [0,1]$ by*

$$M^{(\delta)}(x,y) = \lim_{d_n \uparrow \delta} M^{(d_n)}(x,y), \ x,y \in I, \tag{13}$$

where $(d_n : n \in \mathbb{N}) \subseteq \mathrm{Dyad}([0,1])$ is an increasing sequence with $\lim_{n\uparrow+\infty} d_n = \delta$. The extension $\delta \mapsto M^{(\delta)}(x,y)$, which is a strictly monotonic function when $x \neq y$ (strictly increasing or decreasing depending on $x < y$ or $x > y$), turns out to be unique and continuous provided that M is continuous. In this case, $(x,y) \mapsto M^{(\delta)}(x,y)$ is continuous in I^2 for every $\delta \in [0,1]$.

Now, when M is a strict continuous mean, the map $\delta \mapsto M^{(\delta)}(x,y)$ in the statement of Theor. 1 exhibits a close similitude with the map

$$\delta \mapsto (1-\delta)x + \delta y. \tag{14}$$

In fact, besides of being strictly monotonic and continuous, both maps assume the values x and y at the extreme points $\delta = 0$ and $\delta = 1$, respectively. Furthermore, $M^{(1/2)}(x,y) = M(x,y)$, while the map (14) takes the value $(x+y)/2 = A(x,y)$ at $\delta = 1/2$. As it is well known, (14) is the weighted version of the arithmetic mean $A(x,y)$, so that the map $\delta \mapsto M^{(\delta)}$ can be thought as a sort of weighting of the mean M. In conformity with this thought, it can be inductively proved that, for every $d \in \mathrm{Dyad}([0,1])$,

$$A^{(d)}(x,y) = (1-d)x + dy,$$

and hence, that the continuous extension $A^{(\delta)}$ of $A^{(d)}$, whose existence and uniqueness is ensured by Theor. 1, is given by (the weighting) $A^{(\delta)}(x,y) = (1-\delta)x + \delta y$ of A.

The problem of constructing weighted versions of a given mean is an important problem in the Theory of Means. General references on this theme are [14] and [13], while [18], [19], [20] and [21] is list of articles in which the problem of weighting is solved for particular means. Even though this is a very incomplete list, the state of the art is revealed by its items: the weighting of a mean M (or, sometimes, a mean belonging to certain family $\mathcal{M}$ of means) is obtained by suitably modifying an analytical representation of M (or of the mean in $\mathcal{M}$). A theory of weightings of two variables means which is independent of particular representations has arisen in [5], while the case of n variables means was treated in [7]. An extensive discussion of the problem of weighting of means is also contained in this last reference. A generalization of the Aczél's dyadic iterations to three variables means is studied in [12].

Following [5], given a (two variables) mean M defined on I, by a *weighting* M^δ of M is understood a map $\delta \mapsto M^\delta$ defined on the unit interval $[0,1]$ such that

(CW1) M^δ is a mean for every $\delta \in [0,1]$; and

(CW2) for every $x, y \in I$, $\delta \mapsto M^\delta(x,y)$ assumes the particular values $M^0(x,y) = x$, $M^{1/2}(x,y) = M(x,y)$ and $M^1(x,y) = y$ at the points $\delta = 0,\ 1/2,\ 1$, respectively.

The problem of finding a weighting of a general (continuous) mean M becomes non trivial when continuity or monotonicity of the map $\delta \mapsto M^\delta$ is required. In this regard, Theor. 1 is a satisfactory result inasmuch as it provides, for the means belonging to a class as ample as that constituted by the continuous strict means, a continuous and strictly monotonic weighting.

In this paper, the question of determining the conditions under which the Ryll-Nardzewski's iterations of a mean M can be profited to construct a weighting of M is investigated. In this regard, it should be noted that the first iterations of Ryll-Nardzewski's coincide with the Aczél's first ones: $M^{[k/2^n]} = M^{(k/2^n)}$ if $n \le 2$ and $0 \le k \le 2^n$. Moreover, both dyadic iterations coincide on the class of quasiarithmetic means; namely, if M has the form (11), then

$$M^{[d]}(x,y) = f\left(\frac{(1-d)\,f^{-1}(x) + df^{-1}(y)}{2}\right) = M^{(d)}(x,y),\ \ x, y \in I.$$

On the other hand, it is easy to see that an alternative definition of Aczél's dyadic iterations can be given by $M^{(0)}(x,y) = x$, $M^{(1)}(x,y) = y$ and, for $n \in \mathbb{N}_0$ and $0 \le k \le 2^{n+1}$,

$$F^{\left(\frac{k}{2^{n+1}}\right)}(x,y) = \begin{cases} F^{\left(\frac{k}{2^n}\right)}(x, F(x,y)), & \text{if } 0 \le k \le 2^n \\ F^{\left(\frac{k-2^n}{2^n}\right)}(F(x,y), y), & \text{if } 2^n \le k \le 2^{n+1} \end{cases},\ x, y \in I. \quad (15)$$

In spite of the resemblance among (15) and (5), the properties of $M^{[d]}$ and $M^{(d)}$ differ considerably each other. As it will be seen along the next sections, there is for Ryll-Nardzewski's dyadic iterations no result so simple as Theor. 1 and, to a certain extent, the independent study of $M^{[d]}$ conducted along this paper is justified by this fact. This study is structured as follows: in Section 2 are gather together the elementary properties of $M^{[d]}$, while the study of the continuity of the monotonic extensions of the map $d \mapsto M^{[d]}(x,y)$ is developed along the Sections 3, 4 and 5: the study of continuity at rational points (Section 3) is followed by the corresponding study at the irrationals of $[0,1]$ (Section 4), while a dynamic characterization of continuity is given at Section 5. The comparatively complex behavior of the Ryll-Nardzewski's dyadic iterations is there explained. Some auxiliary results not devoid of intrinsic interest are gather together in the final Appendix.

2 First properties of Ryll-Nardzewski's dyadic iterations

In [8] and [4], the elementary properties of the Aczél's dyadic iterations of a function F were collected. The corresponding study for Ryll-Nardzewski's dyadic iterations is abridged in this section. Let us begin with the following:

Proposition 2 *Let $F : I \times I \to I$ be a function. The following assertions concerning $F^{[d]}$ holds:*

i) *If F is continuous or reflexive, then $F^{[d]}$ is, respectively, continuous or reflexive for every $d \in \mathrm{Dyad}([0,1])$;*

ii) *If F is internal, then $F^{[d]}$ is internal for every $d \in \mathrm{Dyad}([0,1])$. Moreover, if F is strictly internal, then $F^{[d]}$ is strictly internal for every $d \in \mathrm{Dyad}([0,1]),\ d \neq 0,\ 1$;*

iii) *If F is isotone, then $F^{[d]}$ is isotone for every $d \in \mathrm{Dyad}([0,1])$. Moreover, if F is strictly isotone, then $F^{[d]}$ is strictly isotone for every $d \in \mathrm{Dyad}([0,1]),\ d \neq 0,\ 1$;*

iv) *If F is a symmetric function, then*

$$F^{[d]}(y,x) = F^{[1-d]}(x,y),\ \ x,y \in I, \tag{16}$$

for every $d \in \mathrm{Dyad}([0,1])$.

Proof. The simple inductive proofs of the assertions will be, with exception of **iv)**, omitted. For $d = 0$ or $d = 1$ the equality (16) is true by (4). Assume that it was true for all dyadic fractions with denominator 2^n, then, (5) with $d = k/2^{n+1}$, $0 \leq k \leq 2^{n+1}$, and the symmetry of F yield

$$\begin{aligned} F^{\left[\frac{k}{2^{n+1}}\right]}(y,x) &= \begin{cases} F\left(F^{\left[1-\frac{k}{2^n}\right]}(x,y), y\right), & \text{if } 0 \leq k \leq 2^n \\ F\left(x, F^{\left[1-\frac{h-2^n}{2^n}\right]}(x,y)\right), & \text{if } 2^n \leq k \leq 2^{n+1} \end{cases} \\ &= \begin{cases} F^{\left[\frac{1-\frac{k}{2^n}+1}{2}\right]}(x,y), & \text{if } 0 \leq k \leq 2^n \\ F^{\left[\frac{1-\frac{h-2^n}{2^n}}{2}\right]}(x,y), & \text{if } 2^n \leq k \leq 2^{n+1} \end{cases} = F^{\left[1-\frac{k}{2^{n+1}}\right]}(x,y). \end{aligned}$$

This completes the inductive proof. ■

The simple proof of the following important result will be omitted.

Proposition 3 *Let $f : I \to \mathbb{R}$ a bijective function; then $(F_f)^{[d]} = \left(F^{[d]}\right)_f$ for every $d \in \mathrm{Dyad}([0,1])$.*

As a consequence, iterations and conjugacy commute each other and therefore, Ryll-Nardzewski's dyadic iterations of a homogeneous mean turn out to be homogeneous.

A function F is said to be *self-distributive* or *bisymmetric* depending on equation (7) or (6) is satisfied by F. Clearly, a bisymmetric function is also a self-distributive function. Ryll-Nardzewski's dyadic iterations coincide with Aczél's ones on the family of self distributive functions.

Proposition 4 *If F is a self-distributive function, then $F^{[d]} = F^{(d)}$, for every $d \in \mathrm{Dyad}([0,1])$.*

Since quasiarithmetic means are bisymmetric functions, $(A_f)^{[d]} = (A_f)^{(d)}$ for every $d \in \mathrm{Dyad}([0,1])$, as asserted in the Introduction.
Proof. It is inductively proved that, if F is a self-distributive function, then (5) can be replaced by (15) and therefore, $F^{[d]} = F^{(d)}$, for every $d \in \mathrm{Dyad}([0,1])$. The details are omitted. ■

Remark 5 *(On the proofs of Aczél's and Ryll-Nardzewski's theorems) The last result shown that Aczél's dyadic iterations can be replaced by Ryll-Nardzewski's ones in the proof of the characterization theorem of [1] and, correspondingly, that Aczél's dyadic iterations can be taken instead of Ryll-Nardzewski's ones in the characterization theorem of [22].*

3 Continuity at rational points

In what follows, the study of the Ryll-Nardzewski's dyadic iterations will be circumscribed to class of means. For a general mean M, the behavior of these iterations is appreciably different from that of Aczél's ones. First of all, unlike the map $d \mapsto M^{(d)}(x,y)$, the map $d \mapsto M^{[d]}(x,y)$ is not necessarily strictly monotonic (for $x \neq y$!) on the class of strict means. This fact is illustrated by the example below.

Example 6 *(A mean with non monotonic iterations) The counter-harmonic mean CH, defined by*

$$CH(x,y) = \frac{x^2+y^2}{x+y}, \ x,y>0,$$

is a symmetric continuous strict mean on $\mathbb{R}^+$. As a simple computation shows,

$$CH^{\left[\frac{9}{16}\right]}(x,y) - CH^{\left[\frac{1}{2}\right]}(x,y) = CH(CH(x,CH(x,CH(x,y))),y) - CH(x,y)$$

and, in view of the first term of the Taylor expansion of the function

$$\delta \mapsto CH(CH(\delta,CH(\delta,CH(\delta,2+\delta))),2+\delta) - CH(\delta,2+\delta)$$

is given by

$$CH(CH(\delta,CH(\delta,CH(\delta,2+\delta))),2+\delta) - CH(\delta,2+\delta) = -\frac{1}{2}\delta + O\left(\delta^2\right),$$

it is concluded that $CH^{\left[\frac{9}{16}\right]}(\delta,2+\delta) - CH^{\left[\frac{1}{2}\right]}(\delta,2+\delta) < 0$ for every small enough $\delta > 0$.

However, the iterations become monotonic when applied to an isotone mean.

Lemma 7 *If M is a [strictly] isotone mean, then the map $\mathrm{Dyad}([0,1]) \ni d \mapsto M^{[d]}(x,y)$ turns out to be [strictly] increasing when $x<y$ and [strictly] decreasing when $x>y$.*

In the above statement, Dyad([0,1]) is given the order induced by the usual order on $\mathbb{R}$. Of course, $M^{[d]}(x,x) \equiv x$ in I for every $d \in$ Dyad([0,1]).

Proof. The simple inductive proof of this lemma will be omitted. ■

As a consequence of this lemma and from the fact that Dyad([0,1]) is a dense subset of $[0,1]$, the map $d \mapsto M^{[d]}(x,y)$ can be extended to a [strictly] monotonic map defined on $[0,1]$ when M is [strictly] isotone. In fact, for a given $\delta \in [0,1]$, consider the sets

$$D^{-}(\delta) = \{d \in \text{Dyad}([0,1]) : d \leq \delta\}$$

and

$$D^{+}(\delta) = \{d \in \text{Dyad}([0,1]) : d \geq \delta\};$$

then, the map defined for every $x, y \in I$ by

$$M_{-}^{[\delta]}(x,y) = \begin{cases} \sup_{d \in D^{-}(\delta)} M^{[d]}(x,y), & \text{if } x < y \\ x, & \text{if } x = y \\ \inf_{d \in D^{-}(\delta)} M^{[d]}(x,y), & \text{if } x > y \end{cases}, \tag{17}$$

is the *left continuous* monotonic extension of $d \mapsto M_{-}^{[d]}(x,y)$. The *right continuous* monotonic extension $M_{+}^{[d]}(x,y)$ of $d \mapsto M^{[d]}(x,y)$ is similarly defined by substituting in (17) $\sup_{d \in D^{-}(\delta)}$ by $\inf_{d \in D^{+}(\delta)}$ and $\inf_{d \in D^{-}(\delta)}$ by $\sup_{d \in D^{+}(\delta)}$. Note that the equality

$$M_{-}^{[\delta]}(x,y) = \lim_{n \uparrow +\infty} M^{[d_n]}(x,y)$$

holds when $\{d_n : n \in \mathbb{N}\} \subseteq$ Dyad([0,1]) is any increasing sequence such that $d_n \uparrow \delta \in [0,1]$ when $n \uparrow +\infty$, and that, correspondingly, by taking decreasing sequences in Dyad([0,1]) with $d_n \downarrow \delta \in [0,1]$ when $n \uparrow +\infty$, the equality

$$M_{+}^{[d]}(x,y) = \lim_{n \uparrow +\infty} M^{[d_n]}(x,y)$$

is obtained.

Now, for a [strictly] isotone mean M, the extended maps $\delta \mapsto M_{-}^{[\delta]}(x,y)$ and $\delta \mapsto M_{+}^{[\delta]}(x,y)$ are both [strictly] monotonic for $x \neq y$. For example, consider $\delta \mapsto M_{-}^{[\delta]}(x,y)$ for a strictly isotone mean M and take $x < y$ and $\delta_1, \delta_2 \in [0,1]$ such that $\delta_1 < \delta_2$. Then there exist d, $d' \in$ Dyad([0,1]) such that $\delta_1 < d < d' < \delta_2$; thus, Lemma 7 and (17) imply

$$M_{-}^{[\delta_1]}(x,y) \leq M_{-}^{[d]}(x,y) < M_{-}^{[d']}(x,y) \leq M_{-}^{[\delta_2]}(x,y).$$

The case in which $x > y$ is similar, while $M_{-}^{[\delta]}(x,x) \equiv x$.

Clearly, the respectively left and right-continuous maps $\delta \mapsto M_{-}^{[\delta]}(x,y)$ and $\delta \mapsto M_{+}^{[\delta]}(x,y)$ have the same points of discontinuity. In this way, to study the continuity of the monotonic extensions of $d \mapsto M^{[d]}(x,y)$ it suffices to consider, as made in the remaining of this paper, the left continuous monotonic extension $\delta \mapsto M_-^{[\delta]}(x,y)$. As a first issue, the continuity of $\delta \mapsto M_-^{[\delta]}(x,y)$ at dyadic points of $[0,1]$ is established under the additional hypothesis of separated continuity of M.

Theorem 8 *Let M be a strictly isotone and separately continuous mean defined on I; then, for every $x, y \in I$, the following assertions hold:*

i) 0 *and* 1 *are points of continuity of* $\delta \mapsto M_-^{[\delta]}(x,y)$*;*

ii) *if* $\xi \in [0,1]$ *is a point of continuity of* $\delta \mapsto M_-^{[\delta]}(x,y)$*, then* $\xi/2$ *and* $(\xi+1)/2$ *are points of continuity as well;*

iii) $\delta \mapsto M_-^{[\delta]}(x,y)$ *is continuous on* $\mathrm{Dyad}([0,1])$*.*

Proof. Clearly, $\delta \mapsto M_-^{[\delta]}(x,y)$ is continuous (by the left) at 1. To show the continuity (by the right) at 0, it suffices to prove that $M^{[1/2^n]}(x,y) \to x$ when $n \uparrow +\infty$. Indeed, the bounded sequence $\left\{M^{[1/2^n]}(x,y) : n \in \mathbb{N}\right\}$ is increasing when $x < y$ and decreasing when $x > y$ (while $M^{1/2^n}(x,x) \equiv x$). Now, (5) yields

$$M^{[1/2^{n+1}]}(x,y) = M(x, M^{[1/2^n]}(x,y))$$

so that, setting $\lim_{n\uparrow+\infty} M^{[1/2^n]}(x,y) = \mu \geq x$ and taking into account the continuity of $y \mapsto M(x,y)$, a passage to the limit in this equality gives

$$\mu = M(x,\mu);$$

whence, in view of the strict isotonicity of M, the equality $\mu = x$ follows.

The proof of **ii)** easily follows from (5). In fact, for $\delta \in [0,1]$ let us consider a sequence $(d_n : n \in \mathbb{N}) \subseteq \mathrm{Dyad}([0,1])$ such that $d_n \uparrow \delta$ when $n \uparrow +\infty$; then, by (5) it can be written

$$M^{\left[\frac{d_n}{2}\right]}(x,y) = M(x, M^{[d_n]}(x,y)) \text{ and } M^{\left[\frac{d_n+1}{2}\right]}(x,y) = M(M^{[d_n]}(x,y), y), \quad (18)$$

whence (taking into account the separated continuity of M) a passage to the limit $n \uparrow +\infty$ yields

$$M_-^{\left[\frac{\delta}{2}\right]}(x,y) = M(x, M_-^{[\delta]}(x,y)) \text{ and } M_-^{\left[\frac{\delta+1}{2}\right]}(x,y) = M(M_-^{[\delta]}(x,y), y). \quad (19)$$

Now, if $\xi \in [0,1]$ is a point of continuity of $\delta \mapsto M_-^{[\delta]}(x,y)$ and $(d'_n : n \in \mathbb{N}) \subseteq \mathrm{Dyad}([0,1])$ is a sequence such that $d'_n \downarrow \xi$ when $n \uparrow +\infty$, then (19) and the continuity of M give

$$\lim_{n\uparrow+\infty} M^{\left[\frac{d'_n}{2}\right]}(x,y) = \lim_{n\uparrow+\infty} M(x, M^{[d'_n]}(x,y)) = M(x, M_-^{[\xi]}(x,y)) = M_-^{\left[\frac{\xi}{2}\right]}(x,y)$$

and, similarly,

$$\lim_{n\uparrow+\infty} M^{\left[\frac{d'_n+1}{2}\right]}(x,y) = \lim_{n\uparrow+\infty} M(M^{[d'_n]}(x,y),y) = M(M_-^{[\xi]}(x,y),y) = M_-^{\left[\frac{\xi+1}{2}\right]}(x,y),$$

which prove the continuity of $\delta \mapsto M_-^{[\delta]}(x,y)$ at $\xi/2$ and $(\xi+1)/2$, respectively.

iii) follows from **i)** and **ii**) by induction: $\delta \mapsto M_-^{[\delta]}(x,y)$ is continuous at $\delta = 0,1$ and, if it was continuous at the points of the form $k/2^n$ with $k = 0,1,\ldots,2^n$, then it turns out to be continuous at the points of the form $k/2^{n+1}$ with $k = 0,1,\ldots,2^{n+1}$. This completes the proof. ■

The conclusions of the above theorem are not maintained when strict isotonicity is replaced by isotonicity: coordinate or extremal means illustrate this fact.

As shown by the Example 14 below, the continuity of $\delta \mapsto M_-^{[\delta]}(x,y)$ at a rational point $r \notin \mathrm{Dyad}([0,1])$ can not be generally ensured under the hypotheses of Theor. 8. In the following paragraphs, a series of concepts is introduced to facilitate the study of the continuity at such points.

In the first place, recall that a dyadic rational $\varepsilon \in \mathrm{Dyad}([0,1])$, $\varepsilon \neq 1$, admits a finite binary representation of the form $(0.\varepsilon_1\cdots\varepsilon_n)_2$ with $\varepsilon_i \in \{0,1\}$, $i = 1,\ldots,n$, and $\varepsilon_n = 1$. In general, a rational number $\xi \in [0,1]$ has a *binary expansion* of the form

$$\xi = (0.\xi_1\cdots\xi_n\overline{\varepsilon_1\cdots\varepsilon_p})_2\,, \tag{20}$$

where $\xi_1\cdots\xi_n$ is the non-periodic part while $\overline{\varepsilon_1\cdots\varepsilon_p}$ denotes the periodic part of the expansion. The finite sequence of digits $\varepsilon_1\cdots\varepsilon_p$ is usually named period of the expansion and it is understood that it is minimal; i.e., there is no finite sequence of digits $\varepsilon_1\cdots\varepsilon_{p'}$ with $p' < p$ such that $\xi = (0.\xi_1\cdots\xi_n\overline{\varepsilon_1\cdots\varepsilon_{p'}})_2$. A rational $\xi \in [0,1]$ is said to be a *periodic dyadic fraction* when its binary expansion (20) does not contain a non-periodic part; i.e., when the binary expansion of ξ has the form $\xi = (0.\overline{\varepsilon_1\cdots\varepsilon_p})_2$.

The set of all finite sequences of digits will be denoted by $\mathcal{FS}\{0,1\}$; i.e. $\mathcal{FS}\{0,1\} = \bigcup_{n\in\mathbb{N}}\{0,1\}^n$. The *length* of the sequence $\varepsilon = (\varepsilon_1,\ldots,\varepsilon_n)$ is $\ell(\varepsilon) = n$. On $\mathcal{FS}\{0,1\}$ it is defined the operation '&' (*concatenation*) by

$$(\varepsilon_1,\ldots,\varepsilon_m)\&(\varepsilon'_1,\ldots,\varepsilon'_n) = (\varepsilon_1,\ldots,\varepsilon_m,\varepsilon'_1,\ldots,\varepsilon'_n).$$

If $(\varepsilon_1,\ldots,\varepsilon_p) \in \mathcal{FS}\{0,1\}$ is of form $(\varepsilon_1,\ldots,\varepsilon_p) = (\varepsilon_1,\ldots,\varepsilon_q)\&\cdots\&(\varepsilon_1,\ldots,\varepsilon_q)$ for a certain divisor q of p, then $(\varepsilon_1,\ldots,\varepsilon_p)$ is said to be a *repeating sequence*. Thus, the sequence of digits $(\varepsilon_1,\ldots,\varepsilon_q)$ corresponding to the periodic part of the binary expansion (20) is not a repeating sequence. An important role will correspond to the subset of $\mathcal{FS}\{0,1\}$ constituted by all non repeating sequences, so that it will be denoted by $\mathcal{NRFS}\{0,1\}$.

The sets $\{0,1\}^n$, $n \in \mathbb{N}$, will be equipped with the *lexicographic order* induced by the order $\preceq$ defined on $\{0,1\}$ by $\{(0,0),(0,1),(1,1)\}$. Recall that this order is defined by $(\varepsilon_1,\ldots,\varepsilon_n) \preceq (\varepsilon'_1,\ldots,\varepsilon'_n)$ if $\varepsilon_i \preceq \varepsilon'_i$ where $i = \min\{j : \varepsilon_j \neq \varepsilon'_j\}$. If $(0.\varepsilon_1\cdots\varepsilon_n)_2$ is a dyadic fraction, then $(0.\varepsilon_1\cdots\varepsilon_n)_2 = (0.\varepsilon_1\cdots\varepsilon_n 0\cdots 0)_2$ so that, by eventually adding zeros to any expansion, the length of two dyadic fractions $(0.\varepsilon_1 \cdots \varepsilon_m)_2$ and $(0.\varepsilon'_1 \cdots \varepsilon'_n)_2$ can be matched. Take, for instance, $m < n$; then, the inequality $(0.\varepsilon_1 \cdots \varepsilon_m)_2 \leq (0.\varepsilon'_1 \cdots \varepsilon'_n)_2$ holds if and only if $(\varepsilon_1, \ldots, \varepsilon_m, 0, \ldots, 0) \preceq (\varepsilon'_1, \ldots, \varepsilon'_n)$, where $m - n$ zeros have been added to the expansion $(0.\varepsilon_1 \cdots \varepsilon_m)_2$.

Now, let us associate to a finite sequence of digits $(\varepsilon_1, \ldots, \varepsilon_n) \in \mathcal{FS}\{0,1\}$ the map $h_{(\varepsilon)} : [0,1] \to [0,1]$ given by

$$h_{(\varepsilon)}(t) = (h_{\varepsilon_1} \circ \cdots \circ h_{\varepsilon_n})(t), \ t \in [0,1], \tag{21}$$

where

$$h_{\varepsilon_i}(t) = \begin{cases} \frac{t}{2}, & \text{if } \varepsilon_i = 0 \\ \frac{t+1}{2}, & \text{if } \varepsilon_i = 1 \end{cases}, \ t \in [0,1]. \tag{22}$$

Clearly, the equality

$$h_{(\varepsilon \& \varepsilon')} = h_{(\varepsilon)} \circ h_{(\varepsilon')}$$

holds for every $\varepsilon, \varepsilon' \in \mathcal{FS}\{0,1\}$. It should be noted that, if $(\varepsilon_1, \ldots, \varepsilon_n) \in \{0,1\}^n$ and $\varepsilon_n = 1$, then $(0.\varepsilon_1 \cdots \varepsilon_n)_2$ is the binary expansion of $\varepsilon = k/2^n \in \mathrm{Dyad}([0,1])$, where $k = \sum_{i=1}^n \varepsilon_i 2^{n-i}$. In view of this fact, for $\varepsilon \in \mathrm{Dyad}([0,1])$, $\varepsilon \neq 1$, let us define $h_{(\varepsilon)}$ also by (21)-(22) provided that $\varepsilon = (0.\varepsilon_1 \cdots \varepsilon_n)_2$.

Lemma 9 *If $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \mathcal{FS}\{0,1\}$, then*

$$h_{(\varepsilon)}(t) = \frac{t+k}{2^n}, \ t \in [0,1], \tag{23}$$

where $k = \sum_{i=1}^n \varepsilon_i 2^{n-i}$. In particular, if $\varepsilon = (0.\varepsilon_1, \ldots, \varepsilon_n)_2 \in \mathrm{Dyad}([0,1])$, $\varepsilon \neq 1$, then

$$h_{(\varepsilon)}(t) = \frac{t}{2^n} + \varepsilon, \ t \in [0,1]. \tag{24}$$

Proof. (23) is true for $\varepsilon = (0)$ and $\varepsilon = (1)$. Assuming that it was true for a given $n \geq 1$ and every $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{0,1\}^n$, then (21) yields

$$\begin{aligned} h_{(\varepsilon_1, \ldots, \varepsilon_n, \varepsilon_{n+1})}(t) &= h_{(\varepsilon_1, \ldots, \varepsilon_n)}\left(h_{\varepsilon_{n+1}}(t)\right) = \begin{cases} \frac{\frac{t}{2} + \sum_{i=1}^n \varepsilon_i 2^{n-i}}{2^n}, & \text{if } \varepsilon_{n+1} = 0 \\ \frac{\frac{t+1}{2} + \sum_{i=1}^n \varepsilon_i 2^{n-i}}{2^n}, & \text{if } \varepsilon_{n+1} = 1 \end{cases} \\ &= \frac{t+k}{2^{n+1}}, \end{aligned}$$

where $k = \sum_{i=1}^{n+1} \varepsilon_i 2^{n+1-i}$. This finishes the inductive proof of (23). Now, if $\varepsilon = (0.\varepsilon_1, \ldots, \varepsilon_n)_2 \in \mathrm{Dyad}([0,1])$, $\varepsilon \neq 1$, then $\varepsilon = k/2^n$ with $k = \sum_{i=1}^n \varepsilon_i 2^{n-i}$ and (24) follows from (23). ■

Suppose that M is a mean and fix a point $(x,y) \in I^2$. Denote by $I_{(x,y)}$ the interval $[\min\{x,y\}, \max\{x,y\}]$. As made before with the maps $h_{(\varepsilon)}$, for every $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \mathcal{FS}\{0,1\}$ let us define a map $m_{(\varepsilon)} : I_{(x,y)} \to I_{(x,y)}$ by

$$m_{(\varepsilon)}(u) = (m_{\varepsilon_1} \circ \cdots \circ m_{\varepsilon_n})(u), \ u \in I_{(x,y)}, \tag{25}$$

where

$$m_{\varepsilon_i}(u) = \begin{cases} M(x,u), & \text{if } \varepsilon_i = 0 \\ M(u,y), & \text{if } \varepsilon_i = 1 \end{cases}, \ u \in I_{(x,y)}. \tag{26}$$

Note that, when $x = y$, $m_{(\varepsilon)}$ is defined on the unique point $u = x$ and $m_{(\varepsilon)}(x) = x$. The equality

$$m_{(\varepsilon \& \varepsilon')} = m_{(\varepsilon)} \circ m_{(\varepsilon')}$$

holds for every pair of sequences $\varepsilon, \varepsilon' \in \mathcal{FS}\{0,1\}$.

If $\varepsilon = (0.\varepsilon_1 \cdots \varepsilon_n)_2 \in \mathrm{Dyad}([0,1])$, $m_{(\varepsilon)}$ is also defined by (25)-(26). Even if the maps $m_{(\varepsilon)}$ depend on the point $(x,y) \in I^2$, this dependence will turn out to be not relevant in the next developments.

In the next result, the main properties of the maps $m_{(\varepsilon)}$ are established.

Proposition 10 *Let M be a mean defined on I, then, the following assertions hold:*

i) *if M is separately continuous, then $m_{(\varepsilon)}$, $\varepsilon \in \mathcal{FS}\{0,1\}$, is continuous on $I_{(x,y)}$;*

ii) *if M is [strictly] isotone, then $m_{(\varepsilon)}$, $\varepsilon \in \mathcal{FS}\{0,1\}$, is [strictly] increasing on $I_{(x,y)}$ when $x \neq y$; moreover,*

iii) *if $\varepsilon, \varepsilon' \in \{0,1\}^n$, then*

$$m_{(\varepsilon)}(u) \leq m_{(\varepsilon')}(u), \ u \in I_{(x,y)}, \tag{27}$$

provided that $\varepsilon \preceq \varepsilon'$ and $x < y$; the inequality (27) is reversed when $x > y$. In particular, if $\varepsilon, \varepsilon' \in \mathrm{Dyad}([0,1])$, $\varepsilon \neq 1 \neq \varepsilon'$, then $m_{(\varepsilon)} \leq m_{(\varepsilon')}$ in $I_{(x,y)}$ provided that $\varepsilon \leq \varepsilon'$ and $x < y$ (while $m_{(\varepsilon')} \leq m_{(\varepsilon)}$ in $I_{(x,y)}$ when $x > y$).

Proof. **i)** follows from (25)-(26) and the separated continuity of M.

Since $m_0(u) = M(x,u)$ and $m_1(u) = M(u,y)$ are [strictly] increasing functions on $I_{(x,y)}$ by the [strict] isotonicity of M, the composition of a certain number of functions chosen among m_0 and m_1 turns out to be [strictly] increasing as well. This proves **ii)**.

Regarding **iii),** observe that

$$m_0(u) \leq M(x,y) \leq m_1(u) \tag{28}$$

for every $u \in I_{(x,y)}$ when $x < y$, while the inequalities (28) are reversed when $x > y$. Thus, if $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \preceq (\varepsilon'_1, \ldots, \varepsilon'_n) = \varepsilon'$ and $x < y$, then $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_{i-1}, 0, \varepsilon_{i+1} \ldots, \varepsilon_n)$ and $(\varepsilon_1, \ldots, \varepsilon_{i-1}, 1, \varepsilon'_{i+1}, \ldots, \varepsilon'_n) = \varepsilon'$, whence, in view of **ii)**,

$$\begin{aligned} m_{(\varepsilon)}(u) &= m_{(\varepsilon_1,\ldots,\varepsilon_{i-1})}\left(m_0(m_{(\varepsilon_{i+1}\ldots,\varepsilon_n)}(u))\right) \\ &\leq m_{(\varepsilon_1,\ldots,\varepsilon_{i-1})}\Big(m_1(m_{\left(\varepsilon'_{i+1},\ldots,\varepsilon'_n\right)}(u)\Big)) = m_{(\varepsilon')}(u) \end{aligned} \tag{29}$$

for every $u \in I_{(x,y)}$. Clearly, (29) holds in the opposite sense when $x > y$. The proof in the case in which $\varepsilon, \varepsilon' \in \mathrm{Dyad}([0,1])$ is derived from this and the discussion preceding Lemma 9. ∎

A relationship existing among the maps $h_{(\varepsilon)}$ and $m_{(\varepsilon)}$ and some of its consequences are established by the following result. The set of fixed points of a map f is to be denoted by $\mathrm{Fix}(f)$.

Proposition 11 *Let M be a mean defined on I; then, the following asseverations hold:*

i) *if $\varepsilon \in \mathcal{FS}\{0,1\}$ and $d \in \mathrm{Dyad}([0,1])$, $d \neq 1$, then*

$$M^{\left[h_{(\varepsilon)}(d)\right]}(x,y) = m_{(\varepsilon)}(M^{[d]}(x,y)),\ x,y \in I; \tag{30}$$

ii) *if $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \mathcal{NRFS}\{0,1\}$, then*

$$m_{(\varepsilon)}(x) = M^{\left[\frac{k}{2^n}\right]}(x,y),\ x,y \in I, \tag{31}$$

and

$$m_{(\varepsilon)}(y) = M^{\left[\frac{k+1}{2^n}\right]}(x,y),\ x,y \in I, \tag{32}$$

where $k = \sum_{i=1}^{n} \varepsilon_i 2^{n-i}$;

iii) $\mathrm{Fix}(m_{(\varepsilon)}) \neq \emptyset$ *for every $\varepsilon \in \mathcal{FS}\{0,1\}$ provided that M is separately continuous; furthermore, if M is strictly isotone, then*

$$\mathrm{Fix}(m_{(\varepsilon)}) \subseteq (\min\{x,y\}, \max\{x,y\})\,;$$

iv) *when M is strictly isotone and separately continuous, $\delta \mapsto M_-^{[\delta]}(x,y)$ is continuous on the whole interval $[0,1]$ if and only if it is continuous on a subinterval $\emptyset \neq (a,b) \subseteq [0,1]$.*

Note that the equalities

$$M^{\left[h^k_{(\varepsilon)}(d)\right]}(x,y) = m^k_{(\varepsilon)}(M^{[d]}(x,y)),\ k \in \mathbb{N}, \tag{33}$$

are derived by iterating the equality (30) of assertion **i)**. On its part, when M is strictly isotone and separately continuous, the set of discontinuities of the map $\delta \mapsto M_-^{[\delta]}(x,y)$ is dense in $[0,1]$ provided that it is non void. This is a restatement of the assertion **iv)**.

Proof. Let us prove **i)** by induction on the length $\ell(\varepsilon)$. If $\ell(\varepsilon) = 1$ and $d \in \mathrm{Dyad}([0,1])$, $d \neq 1$, then (22), (5) and (26) yield

$$\begin{aligned} M^{\left[h_{(\varepsilon_1)}(d)\right]}(x,y) &= \begin{cases} M^{\left[\frac{d}{2}\right]}(x,y), & \text{if } \varepsilon_1 = 0 \\ M^{\left[\frac{d+1}{2}\right]}(x,y), & \text{if } \varepsilon_1 = 1 \end{cases} \\ &= \begin{cases} M(x, M^{[d]}(x,y)), & \text{if } \varepsilon_1 = 0 \\ M(M^{[d]}(x,y), y), & \text{if } \varepsilon_1 = 1 \end{cases} \\ &= m_{(\varepsilon_1)}\left(M^{[d]}(x,y)\right),\ x,y \in I. \end{aligned}$$

Thus, assuming that (30) was true for $\ell(\varepsilon) = n \geq 1$ and every $d \in \mathrm{Dyad}([0,1])$, $d \neq 1$, for $\varepsilon \in \mathcal{FS}\{0,1\}$ with $\ell(\varepsilon) = n+1$, it can be written $\varepsilon = \varepsilon' \& (\varepsilon_{n+1})$ with

$\ell(\varepsilon') = n$ and thus,

$$\begin{aligned} M^{\left[h_{(\varepsilon)}(d)\right]}(x,y) &= M^{\left[h_{(\varepsilon'\&(\varepsilon_{n+1}))}(d)\right]}(x,y) \\ &= M^{\left[h_{(\varepsilon')}(h_{(\varepsilon_{n+1})}(d))\right]}(x,y) \\ &= m_{(\varepsilon')}\left(M^{\left[h_{(\varepsilon_{n+1})}(d)\right]}(x,y)\right) \\ &= m_{(\varepsilon')}\left(m_{(\varepsilon_{n+1})}\left(M^{[d)]}(x,y)\right)\right) \\ &= (m_{(\varepsilon')} \circ m_{(\varepsilon_{n+1})})\left(M^{[d)]}(x,y)\right) \\ &= m_{(\varepsilon'\&(\varepsilon_{n+1}))}\left(M^{[d)]}(x,y)\right),\ x,y \in I, \end{aligned}$$

which completes the inductive proof of **i)**.

Setting $d = 0$ in (30) yields

$$M^{\left[h_{(\varepsilon)}(0)\right]}(x,y) = m_{(\varepsilon)}(M^{[0]}(x,y)) = m_{(\varepsilon)}(x),\ x,y \in I,$$

whence, taking into account that $h_{(\varepsilon)}(0) = k/2^n$ by Lemma 9, equality (31) is obtained. By replacing $d = 1$ in (30), a similar reasoning shows that (32) holds. This finishes the proof of assertion **ii)**.

Regarding to **iii)**, observe that the map $m_{(\varepsilon)} : I_{(x,y)} \to I_{(x,y)}$ is continuous by Prop. 10-**i)**; thus, it has a fixed point at least. Now, if M is strictly isotone and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \mathcal{NRFS}\{0,1\}$, then $\varepsilon \neq (0, \ldots, 0)$ and $\varepsilon \neq (1, \ldots, 1)$; thus, $0 < k = \sum_{i=1}^{n} \varepsilon_i 2^{n-i} < 1 - 1/2^n$ and, for $x < y$, the equalities proved in **ii)** yield

$$\begin{aligned} x - m_{(\varepsilon)}(x) &= x - M^{\left[\frac{k}{2^n}\right]}(x,y) < 0, \\ y - m_{(\varepsilon)}(y) &= y - M^{\left[\frac{k+1}{2^n}\right]}(x,y) > 0. \end{aligned}$$

In this way, the extreme points of the interval $I_{(x,y)}$ are not fixed point of $m_{(\varepsilon)}$. These inequalities are reversed when $x > y$, so that the conclusion is maintained.

To prove **iv)**, assume that M is strictly isotone and separately continuous and fix $x, y \in I$. Replacing d in (30) by the terms of an increasing sequence $(d_n : n \in \mathbb{N})$ such that $d_n \uparrow \delta \in [0,1]$, it is obtained

$$M^{\left[h_{(\varepsilon)}(d_n)\right]}(x,y) = m_{(\varepsilon)}(M^{[d_n]}(x,y)),\ n \in \mathbb{N},$$

whence, passing to the limit $n \uparrow +\infty$, it is deduced

$$M_{-}^{\left[h_{(\varepsilon)}(\delta)\right]}(x,y) = m_{(\varepsilon)}(M_{-}^{[\delta]}(x,y)), \tag{34}$$

or, equivalently,

$$m_{(\varepsilon)}^{-1}\left(M_{-}^{\left[h_{(\varepsilon)}(\delta)\right]}(x,y)\right) = M_{-}^{[\delta]}(x,y), \tag{35}$$

since in view of Prop. 10-**i)** and **ii)** an inverse $m_{(\varepsilon)}^{-1}$ is admitted by $m_{(\varepsilon)}$. Observe that this last equality holds also for every $\varepsilon \in \mathrm{Dyad}([0,1]),\ \varepsilon \neq 1$. Now, assume that $\delta \mapsto M_-^{[\delta]}(x,y)$ is continuous on a subinterval $(a,b) \subseteq [0,1]$. There is no loss of generality in supposing that (a,b) is of the form $(k/2^n, (k+1)/2^n)$ for a certain $0 < k < 2^n - 1$, so that choosing $\varepsilon = k/2^n$, it turns out to be

$$h_{(\varepsilon)}(\delta) = \frac{k+\delta}{2^n} \in (k/2^n, (k+1)/2^n)$$

for every $\delta \in (0,1)$ and therefore, the map $\delta \mapsto m_{(\varepsilon)}^{-1}\left(M_-^{\left[h_{(\varepsilon)}(\delta)\right]}(x,y)\right)$ is continuous at every point $\delta \in (0,1)$. This fact and (35) prove that $\delta \mapsto M_-^{[\delta]}(x,y)$ is continuous on $(0,1)$. The continuity at $\delta = 0$ and $\delta = 1$ was proved in Theor. 8, so that $\delta \mapsto M_-^{[\delta]}(x,y)$ is continuous on the whole interval $[0,1]$. Since the converse is immediate, this concludes the proof of **iv)**. ■

The extreme points of $\mathrm{Fix}(m_{(\varepsilon)})$ when $\varepsilon \in \mathcal{NRFS}\{0,1\}$ are characterized by the following:

Proposition 12 *Let M be a strictly isotone and separately continuous mean and $\overline{\varepsilon} = (0.\overline{\varepsilon_1 \cdots \varepsilon_p})$ a periodic dyadic fraction. Then, $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_p) \in \mathcal{NRFS}\{0,1\}$ and $u_0 = M_-^{[\overline{\varepsilon}]}(x,y)$ turns out to be a fixed point of $m_{(\varepsilon)}$. Furthermore, $M_-^{[\overline{\varepsilon}]}(x,y)$ is the leftmost point of* $\mathrm{Fix}(m_{(\varepsilon)})$ *when $x < y$, while it is the rightmost one when $x > y$.*

Proof. Define, for every $k \in \mathbb{N}_0$, $\overline{\varepsilon}_k = \varepsilon(1 + 1/2^p + \cdots + 1/2^{kp})$; then, $\overline{\varepsilon}_k \uparrow \overline{\varepsilon}$ when $k \uparrow +\infty$, and therefore,

$$M_-^{[\overline{\varepsilon}]}(x,y) = \lim_{k\uparrow+\infty} M_-^{[\overline{\varepsilon}_k]}(x,y). \tag{36}$$

On the other hand, by Prop. 10, it can be written

$$M^{[\overline{\varepsilon}_k]}(x,y) = \overbrace{\left(m_{\varepsilon_1} \circ \cdots \circ m_{\varepsilon_p}\right) \circ \cdots \circ \left(m_{\varepsilon_1} \circ \cdots \circ m_{\varepsilon_p}\right)}^{k \text{ times}}(x) = m_{(\varepsilon)}^k(x), \tag{37}$$

and the limit $\lim_{k\uparrow+\infty} m_{(\varepsilon)}^k(x)$ (given by (36)) is a fixed point of the map m_ε. Furthermore, if $x < y$ and u_0 is the leftmost point of $\mathrm{Fix}(m_{(\varepsilon)})$, then $u < m_\varepsilon(u) < u_0$ for $x \le u < u_0$; thus $m_\varepsilon^k(x) \uparrow u_0$ when $k \uparrow +\infty$. Now, if $x > y$ and u_0 is the rightmost point of $\mathrm{Fix}(m_{(\varepsilon)})$, then $u_0 < m_\varepsilon(u) < u$ for $u_0 < u < x$ and therefore $m_\varepsilon^k(x) \downarrow u_0$ when $k \uparrow +\infty$. This finishes the proof. ■

For a rational point $\xi \in [0,1)$ with binary expansion given by (20), it can be written

$$M_-^{[\xi]}(x,y) = m_{(\xi_1,\ldots,\xi_n)}(M_-^{[\overline{\varepsilon}]}(x,y)), \tag{38}$$

where $\overline{\varepsilon} = 0.\overline{\varepsilon_1 \cdots \varepsilon_p}$. In fact, $\xi = (0.\xi_1 \cdots \xi_n\overline{\varepsilon_1 \cdots \varepsilon_p})_2 = (0.\xi_1 \cdots \xi_n)_2 + \frac{1}{2^n}\,(0.\overline{\varepsilon_1 \cdots \varepsilon_p})_2 = h_{(\xi_1,\ldots,\xi_n)}(0.\overline{\varepsilon_1 \cdots \varepsilon_p})$ and the equality (34) yields

$$\begin{aligned} M_-^{[\xi]}(x,y) &= M_-^{\left[h_{(\xi_1,\ldots,\xi_n)}(\overline{\varepsilon})\right]}(x,y) \\ &= m_{(\xi_1,\ldots,\xi_n)}(M_-^{[\overline{\varepsilon}]}(x,y)). \end{aligned}$$

As a consequence of (38), the continuity of the map $\delta \mapsto M_{-}^{[\delta]}(x,y)$ at the rational point ξ will depend on the dynamical properties of the map $m_{(\varepsilon)}$. This is the content of the following:

Theorem 13 *Let M be a strictly isotone and separately continuous mean defined on I. A necessary and sufficient condition in order that $\delta \mapsto M_{-}^{[\delta]}(x,y)$ be continuous at the rational point $\xi = (0.\xi_1 \cdots \xi_n \overline{\varepsilon_1 \cdots \varepsilon_p})_2$ is that the map $m_{(\varepsilon)}$ would have a unique fixed point in* $(\min\{x,y\}, \max\{x,y\})$.

In this way, $\delta \mapsto M_{-}^{[\delta]}(x,y)$ is continuous on $[0,1]\cap\mathbb{Q}$ if and only if, for every $\varepsilon \in \mathcal{NRFS}\{0,1\}$, the map $m_{(\varepsilon)}$ has a unique fixed point.

Proof. Let $\xi = (0.\xi_1 \cdots \xi_n \overline{\varepsilon_1 \cdots \varepsilon_p})_2$ be a rational number belonging to the interval $(0,1)$. To prove the sufficiency, fix a point $(x,y) \in I^2$ and assume that $m_{(\varepsilon)}$, $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_p) \in \mathcal{NRFS}\{0,1\}$, has in $(\min\{x,y\}, \max\{x,y\})$ a unique fixed point u_0 which, after Lemma 12, is given by $u_0 = M_{-}^{[\overline{\varepsilon}]}(x,y)$. Let us prove that $\delta \mapsto M_{-}^{[\delta]}(x,y)$ is continuous at $\overline{\varepsilon}$. First of all, observe that $0 < \overline{\varepsilon} < 1$ and therefore, for a small enough $\delta > 0$, there exists $d_0 \in \mathrm{Dyad}([0,1])$ such that $\overline{\varepsilon} < d_0 < \overline{\varepsilon} + \delta$. The decreasing sequence $\left\{h_{(\varepsilon)}^{m}(d_0) : m \in \mathbb{N}_0\right\}$ satisfies

$$h_{(\varepsilon)}^{m}(d_0) \downarrow \overline{\varepsilon} \tag{39}$$

when $m \uparrow +\infty$. In fact, by Lemma 9, $h_{(\varepsilon)}(t_0) = t_0$ if and only if

$$\frac{t_0 + k}{2^p} = t_0,$$

where $k = \sum_{i=1}^{p} \varepsilon_i 2^{p-i}$; hence, the unique fixed point t_0 of $h_{(\varepsilon)}$ is given by

$$t_0 = \frac{k}{2^p - 1} = \frac{\sum_{i=1}^{p} \varepsilon_i 2^{p-i}}{2^p - 1} = \overline{\varepsilon},$$

and therefore, $h_{(\varepsilon)}^{k}(d_0) \downarrow \overline{\varepsilon}$ when $k \uparrow +\infty$, as affirmed. On the other hand, $\lim_{k\uparrow+\infty} m_{(\varepsilon)}^{k}(M^{[d_0]}(x,y)) = u_0 = M_{-}^{[\overline{\varepsilon}]}(x,y)$, and then (33) implies

$$\lim_{k\uparrow+\infty} M^{\left[h_{(\varepsilon)}^{k}(d_0)\right]}(x,y) = \lim_{k\uparrow+\infty} m_{(\varepsilon)}^{k}(M^{[d_0]}(x,y)) = M_{-}^{[\overline{\varepsilon}]}(x,y),$$

or, in view of (39),

$$M_{+}^{[\overline{\varepsilon}]}(x,y) = M_{-}^{[\overline{\varepsilon}]}(x,y),$$

i.e., $\delta \mapsto M_{-}^{[\delta]}(x,y)$ is continuous at $\overline{\varepsilon}$.

Now, the continuity of $\delta \mapsto M_{-}^{[\delta]}(x,y)$ at $\delta = \xi$ follows from (38) and the continuity at $\delta = \overline{\varepsilon}$.

To prove the converse, assume that $\mathrm{Fix}(m_{(\varepsilon)})$ was not a singleton for a certain point $(x,y) \in I^2$. Then, assuming $x < y$, the sequence $\left\{M^{[\overline{\varepsilon}_k]}(x,y)\right\}$ where $\overline{\varepsilon}_k$ is defined like in the proof of Lemma 12 satisfies

$$M_{-}^{[\overline{\varepsilon}]}(x,y) = \lim_{k\uparrow+\infty} M^{[\overline{\varepsilon}_k]}(x,y) = u_1$$

where u_1 is the leftmost fixed point of $m_{(\varepsilon)}$. On the other hand, the sequence $\left(M^{\left[h^k_{(\varepsilon)}(d_0)\right]}(x,y) : k \in \mathbb{N}\right)$ with d_0 defined as before satisfies

$$M_+^{[\overline{\varepsilon}]}(x,y) = \lim_{k\uparrow+\infty} M^{h^k_{(\varepsilon)}(d_0)}(x,y) = u_2$$

where u_2 is the rightmost fixed point of $m_{(\varepsilon)}$. Since $u_1 \neq u_2$, $\delta \mapsto M_-^{[\delta]}(x,y)$ is not continuous at $\delta = \overline{\varepsilon}$ and, in view of (38), neither at $\delta = \xi$. ■

As a corollary of the proof of Theor. 13, at a rational point $\xi = (0.\xi_1 \cdots \xi_n \overline{\varepsilon_1 \cdots \varepsilon_p})_2$ it can be written

$$\left[M_-^{[\xi]}(x,y), M_+^{[\xi]}(x,y)\right] = m_{(\xi_1,\ldots,\xi_n)}\left(\mathrm{conv}(\mathrm{Fix}(m_{(\varepsilon_1,\ldots,\varepsilon_p)}))\right),$$

where $\mathrm{conv}(A)$ denotes the convex hull of a set A. Thus, naming

$$E = \mathrm{conv}(\mathrm{Fix}(m_{(\varepsilon_1,\ldots,\varepsilon_p)})$$

and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_p)$, we have

$$\sum \left|m_{(\xi_1,\ldots,\xi_n)}(E)\right| \leq |y-x|,$$

where the sum of the left hand-side is taken over all $(\xi_1, \ldots, \xi_n) \in \mathcal{FS}\{0,1\} \setminus \{\varepsilon, \varepsilon\&\varepsilon, \varepsilon\&\varepsilon\&\varepsilon, \ldots\}$ and $|[a,b]| = b-a$ denotes the length of the interval $[a,b]$.

Before continuing the study of the continuity of the map $\delta \mapsto M_-^{[\delta]}(x,y)$, a mean with discontinuous iterations is exhibited in the following example.

Example 14 *(A mean defined on $I = [0,1]$ with $\delta \mapsto M_-^{[\delta]}(0,1)$ discontinuous at rational points) Let define two maps μ_0, $\mu_1 : [0,1] \to [0,1]$ by*

$$\mu_0(u) = \begin{cases} \frac{1}{5}u & \text{if } 0 \leq u \leq \frac{5}{8} \\ u - \frac{1}{2} & \text{if } \frac{5}{8} \leq u \leq 1 \end{cases}$$

and

$$\mu_1(u) = \begin{cases} u + \frac{1}{2} & \text{if } 0 \leq u \leq \frac{3}{8} \\ \frac{1}{5}u + \frac{4}{5} & \text{if } \frac{3}{8} \leq u \leq 1 \end{cases}.$$

Prop. 30 from the Appendix shows that there exist a strictly isotone, symmetric and continuous mean M on $[0,1]$ such that, for every $u \in [0,1]$,

$$m_0(u) = M(0,u) = \mu_0(u) \text{ and } m_1(u) = M(u,1) = \mu_1(u).$$

The composition $m_0 \circ m_1$ is promptly computed:

$$(m_0 \circ m_1)(u) = \begin{cases} \frac{1}{5}u + \frac{1}{10} & \text{if } 0 \leq u \leq \frac{1}{8} \\ u & \text{if } \frac{1}{8} \leq u \leq \frac{3}{8} \\ \frac{1}{5}u + \frac{3}{10} & \text{if } \frac{3}{8} \leq u \leq 1 \end{cases};$$

hence, $\mathrm{Fix}(m_0 \circ m_1) = [1/8, 3/8]$ and therefore, Theor. 13 shows that every rational point ξ with binary expansion of the form $(0.\xi_1 \cdots \xi_n \overline{01})_2$ is a discontinuity point of $\delta \mapsto M_-^{[\delta]}(0,1)$.

A separately quasicontractive mean M give rise to a family of maps $\{m_{(\varepsilon)} : \varepsilon = (\varepsilon_1, \ldots, \varepsilon_p) \in \mathcal{NRFS}\{0,1\}\}$ with a unique fixed point in $I_{(x,y)}$. In fact, if M is separately quasicontractive then, for every u, $u' \in I_{(x,y)}$, one at least of the inequalities

$$|m_0(u) - m_0(u')| = |M(x,u) - M(x,u')| < |u - u'|\,,$$

$$|m_1(u) - m_1(u')| = |M(u,y) - M(u',y)| < |u - u'|\,,$$

hold, and therefore, for $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_p) \in \mathcal{NRFS}\{0,1\}$,

$$\begin{aligned} & |m_{\varepsilon_1} \circ \cdots \circ m_{\varepsilon_n}(u) - m_{\varepsilon_1} \circ \cdots \circ m_{\varepsilon_n}(u')| \\ \leq \quad & |m_{\varepsilon_2} \circ \cdots \circ m_{\varepsilon_n}(u) - m_{\varepsilon_2} \circ \cdots \circ m_{\varepsilon_n}(u')| \\ \leq \quad & \cdots \leq |u - u'|\,, \end{aligned}$$

being strict at least one of these inequalities, so that the existence of two fixed points $u_1 \neq u_2$ has a contradictory consequence:

$$|u_2 - u_1| = \left| m_{(\varepsilon)}(u_2) - m_{(\varepsilon)}(u_1) \right| < |u_2 - u_1|\,.$$

Theorem 15 *$\delta \mapsto M_-^{[\delta]}(x,y)$ turns out to be continuous on $[0,1] \cap \mathbb{Q}$ provided that M is a strictly monotonic and separately (C)-quasicontractive mean.*

Proof. If M is separately (C)-quasicontractive, then M_f is separately quasicontractive for a certain homeomorphism $f : I \to \mathbb{R}$; thus, the previous discussion and Theor. 13 show that $\delta \mapsto (M_f)_-^{[\delta]}(x,y)$ is continuous on $[0,1] \cap \mathbb{Q}$ for every $x, y \in f(I)$. Now, $(M_f)^{[d]} = \left(M^{[d]}\right)_f$, $d \in \text{Dyad}([0,1])$, by Prop. 3 and therefore $(M_f)_-^{[\delta]} = \left(M_-^{[\delta]}\right)_f$, $\delta \in [0,1]$, by the continuity of f. Thus, $\delta \mapsto M_-^{[\delta]} = \left((M_f)_-^{[\delta]}\right)_{f^{-1}}$ is continuous on $[0,1] \cap \mathbb{Q}$ for every $x, y \in I$. ■

4 Continuity at irrational points

As a first fact concerning the continuity of $\delta \mapsto M_-^{[\delta]}(x,y)$ at irrational points it should be noted that the set of irrational discontinuities of this map is at most numerable. Now, let ξ be an irrational number belonging to the interval $[0,1]$; then, for every $n \in \mathbb{N}$ there exits $0 \leq k_n < n$ such that

$$\frac{k_n}{2^n} < \xi < \frac{k_n + 1}{2^n}.$$

Clearly, $k_n = \sum_{i=1}^n \xi_i 2^{n-i}$ provided that $(0.\xi_1\xi_2\cdots)_2$ is the binary expansion of ξ and

$$\frac{k_n}{2^n} \uparrow \xi \text{ and } \frac{k_n + 1}{2^n} \downarrow \xi$$

when $n \uparrow +\infty$. In this way, substituting ε by $(\xi_1, \xi_2, \cdots, \xi_n)$ successively in (31) and (32) yield

$$m_{(\xi_1,\xi_2,\cdots,\xi_n)}(x) = M^{\left[\frac{k_n}{2^n}\right]}(x,y),\ x,y \in I, \tag{40}$$

and

$$m_{(\xi_1,\xi_2,\cdots,\xi_n)}(y) = M^{\left[\frac{k_n+1}{2^n}\right]}(x,y),\ x,y \in I. \tag{41}$$

Now, if M is strictly isotone, then, when $x < y$,

$$\lim_{n\uparrow+\infty} M^{\left[\frac{k_n}{2^n}\right]}(x,y) = M_-^{[\xi]}(x,y) \text{ and } \lim_{n\uparrow+\infty} M^{\left[\frac{k_n+1}{2^n}\right]}(x,y) = M_+^{[\xi]}(x,y), \tag{42}$$

while, when $x > y$,

$$\lim_{n\uparrow+\infty} M^{\left[\frac{k_n}{2^n}\right]}(x,y) = M_+^{[\xi]}(x,y) \text{ and } \lim_{n\uparrow+\infty} M^{\left[\frac{k_n+1}{2^n}\right]}(x,y) = M_-^{[\xi]}(x,y), \tag{43}$$

and the following result can be established:

Theorem 16 *Let M be a strictly isotone mean defined on I and $\xi \in [0,1]$ be an irrational number with binary expansion $(0.\xi_1\xi_2\cdots)_2$. Then, for a given $(x,y) \in I^2$, the map $\delta \mapsto M_-^{[\delta]}(x,y)$ is continuous at $\delta = \xi$ if and only if the sequence $\left\{m_{(\xi_1,\xi_2,\cdots,\xi_n)} : n \in \mathbb{N}\right\}$ converges to a constant map when $n \uparrow +\infty$.*

After the observation made at the beginning of the section it turns out to be that, with the possible exception of a sequence of irrational points, the sequence $\left\{m_{(\xi_1,\xi_2,\cdots,\xi_n)} : n \in \mathbb{N}\right\}$ always converges to a constant map when $n \uparrow +\infty$. Lemma 9 shows that the hypothesis of the above theorem are fulfilled when $h_{(\xi_1,\xi_2,\cdots,\xi_n)}$ is defined by (21)-(22) (in whose case $M = A$).

Proof. By Prop. 10-**ii)**, the map $m_{(\xi_1,\xi_2,\cdots,\xi_n)}$ is strictly increasing on $I_{(x,y)}$ when $x \neq y$; thus, assuming $x < y$, it turns out to be

$$m_{(\xi_1,\xi_2,\cdots,\xi_n)}(x) < m_{(\xi_1,\xi_2,\cdots,\xi_n)}(u) < m_{(\xi_1,\xi_2,\cdots,\xi_n)}(y),$$

for every $u \in I_{(x,y)}$. If $\delta \mapsto M_-^{[\delta]}(x,y)$ is continuous at $\delta = \xi$, then $M_-^{[\xi]}(x,y) = M_+^{[\xi]}(x,y)$ and taking into account (40), (41) and (42), a passage to the limit $n \uparrow +\infty$ in the above inequalities gives

$$M_-^{[\xi]}(x,y) \leq \lim_{n\uparrow+\infty} m_{(\xi_1,\xi_2,\cdots,\xi_n)}(u) \leq M_-^{[\xi]}(x,y),$$

so that there exists $\lim_{n\uparrow+\infty} m_{(\xi_1,\xi_2,\cdots,\xi_n)}(u)$ for every $u \in I_{(x,y)}$ and the limit function does not depend on u. When $x > y$, all above inequalities hold in reverse sense, so that the conclusion remains unchanged.

Conversely, if the sequence $\left\{m_{(\xi_1,\xi_2,\cdots,\xi_n)} : n \in \mathbb{N}\right\}$ converge to a constant map when $n \uparrow +\infty$, then $\lim_{n\uparrow+\infty} m_{(\xi_1,\xi_2,\cdots,\xi_n)}(x) = \lim_{n\uparrow+\infty} m_{(\xi_1,\xi_2,\cdots,\xi_n)}(y)$ and therefore, (40), (41) and (42) show that $M_-^{[\xi]}(x,y) = M_+^{[\xi]}(x,y)$. ■

In practice, the convergence to a constant of the sequence $\left\{m_{(\xi_1,\xi_2,\cdots,\xi_n)} : n \in \mathbb{N}\right\}$ is barely verifiable. In order to prove an operative sufficient condition of continuity of $\delta \mapsto M_-^{[\delta]}(x,y)$ at irrational points let us introduce some suitable notation. Recall that the *(one-sided) shift transformation* $S : [0,1] \to [0,1]$ is defined by (cf., for instance, [16], pg. 48)

$$S(t) = 2t \pmod 1 = \begin{cases} 2t, & 0 \le t < 1/2 \\ 2t-1, & 1/2 \le t \le 1 \end{cases}, \tag{44}$$

and that, if $(0.\xi_1\xi_2\cdots)_2$ is the binary expansion of $\xi \in [0,1]$, then $S(\xi) = (0.\xi_2\xi_3\cdots)_2$. Now, for a strictly isotone and separately continuous mean M, define a map $\phi : I_{(x,y)} \to I_{(x,y)}$ by

$$\phi(u) = \begin{cases} m_0^{-1}(u), & u \in [x, M(x,y)) \\ m_1^{-1}(u), & u \in [M(x,y), y] \end{cases} \tag{45}$$

when $x < y$, and

$$\phi(u) = \begin{cases} m_1^{-1}(u), & u \in [y, M(x,y)] \\ m_0^{-1}(u), & u \in (M(x,y), x] \end{cases} \tag{46}$$

when $x > y$. Since the maps m_0 and m_1 are both strictly increasing and continuous, the definition of ϕ is justified by the inequality (28). When $x < y$, ϕ turns out to be strictly increasing and continuous on each subinterval $[x, M(x,y))$ and $[M(x,y), y]$ and, moreover, $\phi(u) > u,\ u \in (x, M(x,y))$, and $\phi(u) < u,\ u \in (M(x,y), y]$. Similar properties are enjoyed by ϕ when $x > y$.

Finally, for every $\xi \in [0,1]$, denote by $G[\xi]$ the gap of the map $\delta \mapsto M_-^{[\delta]}(x,y)$ at the point $\delta = \xi$; i.e.

$$G[\xi] = \left[M_-^{[\xi]}(x,y), M_+^{[\xi]}(x,y)\right]. \tag{47}$$

Lemma 17 *Let M be a strictly isotone mean; then, for every $\delta \in [0,1]$,*

$$\phi(G[\delta]) = G[S(\delta)]. \tag{48}$$

In words, the gap at the point $S(\delta)$ is the image under the map ϕ of the gap at the point δ.

Proof. In the course of the proof of Theor. 8-**ii)** was proved that, for every $\delta \in (0,1]$,

$$M_-^{\left[\frac{\delta}{2}\right]}(x,y) = M(x, M_-^{[\delta]}(x,y)) \text{ and } M_-^{\left[\frac{\delta+1}{2}\right]}(x,y) = M(M_-^{[\delta]}(x,y), y).$$

Analogously, if $\delta \in [0,1)$ and $(d'_n : n \in \mathbb{N}) \subseteq \mathrm{Dyad}([0,1])$ such that $d'_n \downarrow \delta$ when $n \uparrow +\infty$, then, (18) holds for d'_n instead of d_n, so that a passage to the limit $n \uparrow +\infty$ yields

$$M_+^{\left[\frac{\delta}{2}\right]}(x,y) = M(x, M_+^{[\delta]}(x,y)) \text{ and } M_+^{\left[\frac{\delta+1}{2}\right]}(x,y) = M(M_+^{[\delta]}(x,y), y).$$

The above equalities can be equivalently written as follows:

$$M_-^{[\delta]}(x,y) = \begin{cases} m_0\left(M_-^{[2\delta]}(x,y)\right), & \text{if } \delta \in [0,1/2) \\ m_1(M_-^{[2\delta-1]}(x,y)), & \text{if } \delta \in [1/2,1] \end{cases} \tag{49}$$

and

$$M_+^{[\delta]}(x,y) = \begin{cases} m_0(M_+^{[2\delta]}(x,y)), & \text{if } \delta \in [0,1/2) \\ m_1(M_+^{[2\delta-1]}(x,y)), & \text{if } \delta \in [1/2,1] \end{cases}. \tag{50}$$

In this way,

$$G(2\delta) = m_0^{-1}\left(G\left[\delta\right]\right) \tag{51}$$

when $\delta \in [0,1/2)$, while

$$G(2\delta-1) = m_1^{-1}\left(G\left[\delta\right]\right) \tag{52}$$

when $\delta \in [1/2,1]$.

Now, observing that

$$G\left[\delta\right] \subseteq \begin{cases} [x, M(x,y)), & \text{if } \delta \in [0,1/2) \\ [M(x,y), y], & \text{if } \delta \in [1/2,1] \end{cases}$$

when $x < y$, and that

$$G\left[\delta\right] \subseteq \begin{cases} (M(x,y), x], & \text{if } \delta \in [0,1/2) \\ [y, M(x,y)], & \text{if } \delta \in [1/2,1] \end{cases}$$

when $x > y$, it turns out to be that (51) and (52) can be compactly written in the form (48). ■

Theorem 18 *Let M be a strictly isotone and separately (C)-nonexpansive mean defined on I; then, the maps $\delta \mapsto M_-^{[\delta]}(x,y)$, $x,y \in I$, are continuous at every irrational point of* $[0,1]$.

Proof. First suppose that M is strictly isotone and separately nonexpansive and fix a point $(x,y) \in I^2$. Then, for every $u,u' \in I$,

$$|m_0(u) - m_0(u')| \leq |u-u'| \text{ and } |m_1(u) - m_1(u')| \leq |u-u'|,$$

a pair of inequalities which (taking into account that m_0 and m_1 are both strictly increasing functions and, moreover, that

$$m_0([\min\{x,y\}, \max\{x,y\}]) = [\min\{x, M(x,y)\}, \max\{x, M(x,y)\}]$$

and

$$m_1([\min\{x,y\}, \max\{x,y\}]) = [\min\{M(x,y), y\}, \max\{M(x,y), y\}]$$

for every $x,y \in I$) can be equivalently written as

$$\left|m_0^{-1}(u) - m_0^{-1}(u')\right| \geq |u-u'|, \; u,u' \in [\min\{x, M(x,y)\}, \max\{x, M(x,y)\}],$$

and

$$\left|m_1^{-1}(u) - m_1^{-1}(u')\right| \geq |u - u'|\,,\ u, u' \in [\min\{M(x, y), y\}, \max\{M(x, y), y\}].$$

Using these inequalities it can be easily shown that, for every $\delta \in [0, 1]$,

$$|\phi(G[\delta])| \geq |G[\delta]|\,,$$

whence, using Lemma 17, the following inequality is derived:

$$|G[S(\delta)]| \geq |G[\delta]|\,,\ \delta \in [0, 1]. \tag{53}$$

Now, suppose that $\xi \in [0, 1]$ is an irrational number with binary expansion given by $(0.\xi_1\xi_2\cdots)_2$. If the equality $S^m(\xi) = S^n(\xi)$ was true for $m, n \in \mathbb{N},\ m < n$, then, setting $p = n - m$, the equality $S^{kp}(S^n(\xi)) = S^n(\xi)$ would be true for every $k \in \mathbb{N}$ and therefore, $S^n(\xi) = (0.\xi_{n+1}\xi_{n+2}\cdots)_2$ would have the form $(0.\overline{\varepsilon_1\cdots\varepsilon_p})_2$. In other terms, $(0.\xi_1\cdots\xi_n\overline{\varepsilon_1\cdots\varepsilon_p})_2$ would be the form of the binary expansion of ξ, contradicting its supposed irrationality. In consequence, $S^m(\xi) \neq S^n(\xi)$ when $n \neq m$ and therefore, the positive orbit $\{S^n(\xi) : n \in \mathbb{N}_0\}$ of an irrational number ξ contains an infinity of different points. In this way, if an irrational $\xi \in [0, 1]$ was a discontinuity of the map $\delta \mapsto M_-^{[\delta]}(x, y)$, then $|G[\xi]| > 0$ and, iterating inequality (53), it is deduced

$$|G[S^n(\delta)]| \geq |G[\delta]| > 0,\ n \in \mathbb{N},$$

which, taking into account that $\sum_{n=1}^{+\infty} |G[S^n(\delta)]| \leq |y - x| < +\infty$, turns out to be a contradiction. Thus, $\delta \mapsto M_-^{[\delta]}(x, y)$ can not be discontinuous at any irrational point, as asserted.

When M is separately (C)-nonexpansive then, for a certain homeomorphism f, the map $\delta \mapsto (M_f)_-^{[\delta]}(x, y) = \left(M_-^{[\delta]}\right)_f (x, y)$ turns out to be continuous at every irrational point, so that the same is true for the map $\delta \mapsto M_-^{[\delta]}(x, y)$. The proof follows from the arbitrariness of the point $(x, y) \in I^2$. ■

Remark 19 *(Sufficient condition of continuity for $\delta \mapsto M_-^{[\delta]}(x, y)$ at irrational points) A version of the last theorem can be established which provides, when $(x, y) \in I^2$ is fixed, a sufficient condition of continuity of the map $\delta \mapsto M_-^{[\delta]}(x, y)$ at irrational points. Indeed, if M is strictly isotone and the inequalities*

$$|M(x, u) - M(x, u')| \leq |u - u'| \ \text{ and } \ |M(u, y) - M(u', y)| \leq |u - u'| \tag{54}$$

are satisfied for every $u, u' \in I_{(x,y)}$, then $\delta \mapsto M_-^{[\delta]}(x, y)$ is continuous at every irrational point of $[0, 1]$. For instance, the functions μ_0 and μ_1 in Example 1 are Lipschitz-continuous with Lipschitz constant equal to 1, so that inequalities (54) hold for the corresponding mean M at the point $(0, 1)$ and therefore, $\delta \mapsto M_-^{[\delta]}(0, 1)$ is, in this case, continuous at every irrational point. An analogous remark can be made when the inequalities (54) are satisfied by a certain f-conjugated M_f of M.

5 Another approach to continuity

Let M be a strictly isotone and separately continuous mean defined on I. Taking into account the definition (45)-(46) of the map ϕ, the equality (49) can be rewritten in the form

$$\phi \circ M_{-}^{[\delta]}(x,y) = M_{-}^{[S(\delta)]}(x,y). \tag{55}$$

This equality enable us to consider the question of continuity of the map $\delta \mapsto M_{-}^{[\delta]}(x,y)$ from a dynamical point of view.

Firstly, let us write the equality (55) in the form

$$\phi \circ h_{(x,y)} = h_{(x,y)} \circ S, \tag{56}$$

where $h_{(x,y)}(\delta) = M_{-}^{[\delta]}(x,y)$, $\delta \in [0,1]$. It is clear from (56) that if $\delta \mapsto M_{-}^{[\delta]}(x,y) = h_{(x,y)}(\delta)$ was continuous, then $h_{(x,y)}$ would be an homeomorphism from $[0,1]$ onto $I_{(x,y)}$ and therefore, ϕ would turn out to be *topologically conjugated* ([16], pg. 68) to the shift S. Furthermore, $h_{(x,y)}$ is an increasing homeomorphism when $x < y$ while it is decreasing when $x > y$. Since the converse is also true, the following result can be established:

Theorem 20 *Let M be a strictly isotone and separately continuous mean defined on I. Then, the map $\delta \mapsto M_{-}^{[\delta]}(x,y)$ is continuous on $[0,1]$ if and only if, for every $x, y \in Y$, $x \neq y$, the map ϕ given by (45)-(46) is topologically conjugated to the shift transformation S through an homeomorphism $H_{(x,y)}$ such that $H_{(x,y)}$ is increasing when $x < y$ and decreasing when $x > y$.*

Proof. The necessity was proved in the discussion preceding the statement of the theorem. Conversely, suppose that for every $x, y \in I$, $x \neq y$, there exists a homeomorphism $h_{(x,y)}$ from $[0,1]$ onto $I_{(x,y)}$ with $h_{(x,y)}$ increasing when $x < y$ and decreasing when $x > y$, such that equality (56) is satisfied by $h_{(x,y)}$; i.e.,

$$\phi\left(h_{(x,y)}(\delta)\right) = \begin{cases} h_{(x,y)}(2\delta), & 0 \le \delta < 1/2 \\ h_{(x,y)}(2\delta - 1), & 1/2 \le \delta \le 1 \end{cases}. \tag{57}$$

In this way, assuming that $x < y$, it turns out to be $\phi\left(h_{(x,y)}(\delta)\right) > h_{(x,y)}(\delta)$ when $0 < \delta < 1/2$, while $\phi\left(h_{(x,y)}(\delta)\right) < h_{(x,y)}(\delta)$ when $1/2 \le \delta < 1$ and thus, from (45) it is deduced that

$$h_{(x,y)}(\delta) \in \begin{cases} [x, M(x,y)), & 0 \le \delta < 1/2 \\ (M(x,y), y], & 1/2 \le \delta \le 1 \end{cases},$$

whence

$$\phi(h_{(x,y)}(\delta)) = \begin{cases} m_0^{-1}(h_{(x,y)}(\delta)), & 0 \le \delta < 1/2 \\ m_1^{-1}(h_{(x,y)}(\delta)), & 1/2 \le \delta < 1 \end{cases}. \tag{58}$$

From (57), (58) and (26) it is deduced

$$h_{(x,y)}(\delta) = \begin{cases} M(x, h_{(x,y)}(2\delta)), & 0 \le \delta < 1/2 \\ M(h_{(x,y)}(2\delta - 1), y), & 1/2 \le \delta \le 1 \end{cases}, \tag{59}$$

which, taking into account that $h_{(x,y)}(0) = x$ and $h_{(x,y)}(1) = y$, shows that $h_{(x,y)}(d) = M^{[d]}(x,y)$ for every $d \in \mathrm{Dyad}([0,1])$. Hence $M_-^{[\delta]}(x,y) = h_{(x,y)}(\delta) = M_-^{[\delta]}(x,y)$ for every $\delta \in [0,1]$ and $\delta \mapsto M_-^{[\delta]}(x,y)$ turns out to be a continuous map. A similar reasoning produces the same equality (59) when $x > y$, so that the proof is completed. ■

Before finishing this section, a series of remarks and examples is given.

Remark 21 *(Theorem 20 for maps on the circle S^1). An equivalent statement of Theor. 20 can be given when the maps entering in equality (56) are considered as being maps on the real numbers* mod 1*; i.e., on $\mathbb{R} \setminus \mathbb{Z} \approx S^1$. With this purpose, for every $(x,y) \in I^2$ define the map $\sigma_{(x,y)} : [0,1] \to [\min\{x,y\}, \max\{x,y\}]$ by*

$$\sigma_{(x,y)}(\delta) = (1-\delta)\min\{x,y\} + \delta\max\{x,y\}, \ \delta \in [0,1], \tag{60}$$

and set $\Phi_{(x,y)} = \sigma_{(x,y)}^{-1} \circ \phi \circ \sigma_{(x,y)}$ and $H_{(x,y)} = \sigma_{(x,y)}^{-1} \circ h_{(x,y)}$. Using these notations, equality (55) takes the form

$$\Phi_{(x,y)} \circ H_{(x,y)} = H_{(x,y)} \circ S, \tag{61}$$

in which $\Phi_{(x,y)}$, $H_{(x,y)}$ and S are maps from $[0,1]$ into itself, so that by identifying the extreme points of the interval $[0,1]$, all maps entering in (55) become maps defined on the circle S^1. As a map from S^1 into S^1, the shift S turns out to be continuous and the same is true for $\Phi_{(x,y)}$, $x, y \in I$ (since M was supposed strictly isotone and separately continuous). Furthermore, both maps have degree *([16], pg. 72) equal to 2; i.e. $\deg(\Phi_{(x,y)}) = 2 = \deg(S)$, while*

$$\deg(H_{(x,y)}) = \begin{cases} 1, & x < y \\ -1, & x > y \end{cases}. \tag{62}$$

Theor. 20 can be restated in the following form: for a strictly isotone and separately continuous mean M defined on I, the map $\delta \mapsto M_-^{[\delta]}(x,y)$ is continuous on $[0,1]$ if and only if, for every $x, y \in Y$, $x \neq y$, the above defined map $\Phi_{(x,y)}$ is topologically conjugated to the shift S through an homeomorphism $H_{(x,y)}$ whose degree satisfies (62). Moreover, the statement is simplified when M is a symmetric mean: $\delta \mapsto M_-^{[\delta]}(x,y)$ is continuous on $[0,1]$ if and only if, for every $x, y \in Y$, $x < y$, the above defined map $\Phi_{(x,y)}$ is topologically conjugated to the shift S through an homeomorphism $H_{(x,y)}$ with $\deg(H_{(x,y)}) = 1$. In fact, if $x > y$, then interchanging the roles of x and y in (61) yields

$$\left(\Phi_{(y,x)} \circ H_{(y,x)}\right)(\delta) = \left(H_{(y,x)} \circ S\right)(\delta), \ \delta \in S^1,$$

and thus

$$\left(\Phi_{(y,x)} \circ H_{(y,x)}\right)(1-\delta) = \left(H_{(y,x)} \circ S\right)(1-\delta), \ \delta \in S^1.$$

*Taking into account that $S(1-\delta) = 1 - S(\delta)$, $\delta \in S^1$ and that, by (a passage to the limit in) Prop. 2-**iv)**, $M_-^{[\delta]}(y,x) = M_+^{[1-\delta]}(x,y)$, it is deduced that*

$H^*_{(x,y)}(\delta) = H_{(y,x)}(1-\delta) = \sigma^{-1}_{(y,x)}\left(M_-^{[1-\delta]}(y,x)\right) = \sigma^{-1}_{(y,x)}\left(M_+^{[\delta]}(x,y)\right)$ *is a continuous map with degree* $\deg(H^*_{(x,y)}) = -1$*. Now, in view of* $\sigma_{(y,x)} = \sigma_{(x,y)}$ *and (45)-(46),* $\Phi_{(y,x)} = \sigma^{-1}_{(y,x)} \circ \phi \circ \sigma_{(y,x)} = \sigma^{-1}_{(x,y)} \circ \phi \circ \sigma_{(x,y)} = \Phi_{(x,y)}$ *and therefore,*

$$\left(\Phi_{(x,y)} \circ H^*_{(x,y)}\right)(\delta) = \left(H^*_{(x,y)} \circ S\right)(\delta)\,,\ \ \delta \in S^1.$$

Remark 22 *(Difference among Aczél's and Ryll-Nardzewski's dyadic iterations) By using the representation given by (15), an equality can be written which is to the Aczél's dyadic iterations as equality (55) is to the Ryll-Nardzewski's ones. Indeed, defining a transformation* $T_{(\delta)} : I^2 \to I^2$ *by*

$$T_{(\delta)}(x,y) = \begin{cases} T_0(x,y), & \delta \in [0,1/2) \\ T_1(x,y), & \delta \in [1/2,1] \end{cases}\ ,\ x,y \in I,$$

where $T_0, T_1 : I^2 \to I^2$ *are the* mean-type *maps respectively given by*

$$T_0(x,y) = (x, M(x,y)) \ \text{ and } T_1(x,y) = (M(x,y), y),\ \ x,y \in I,$$

it turns out to be

$$M_-^{(\delta)} = M_-^{(S(\delta))} \circ T_{(\delta)}. \tag{63}$$

The qualitative differences evidenced by (15) and (63) explains the different behaviors of these iterations.

Example 23 *(A mean M defined on* $I = [0,1]$ *with* $\delta \mapsto M_-^{[\delta]}(0,1)$ *discontinuous at an irrational point* $\xi \in [0,1]$*) If* $\xi \in [0,1]$ *is an irrational number, then the orbit* $O(\xi)$ *of* ξ *under the shift transformation; i.e.,* $O(\xi) = \{S^n(\{\xi\}) : n \in \mathbb{Z}\}$*, has the following form:* $O(\xi) = A \cup \bigcup_{n\in\mathbb{N}} B_{-n}$*, where* $A = O_+(\xi) = \{\alpha_n : \alpha_n = S^n(\{\xi\}), n \in \mathbb{N}_0\}$ *is the positive orbit of* ξ *and the negative orbit* $O_-(\xi) = \bigcup_{n\in\mathbb{N}} B_{-n}$ *splits in the (mutually disjoint) sets* $B_{-n} = \{x \in [0,1] : S^n(x) = \xi\}$*. After (21), (22) and (44), the set of solutions to the equation* $S^n(x) = \xi$ *can be written in the form* $\{\beta_{n,k} : 0 \le k < 2^n\}$ *where* $\beta_{n,k}$ *is given by*

$$\beta_{n,k} = h_{(\varepsilon)}(\xi),$$

provided that $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \mathcal{FS}\{0,1\}$ *satisfies (*$\ell(\varepsilon) = n$ *and)* $\sum_{i=1}^n \varepsilon_i 2^{n-i} = k$*. Now, for a doubly infinite sequence* $(\gamma_n : n \in \mathbb{Z}) \subseteq \mathbb{R}^+$ *such that* $\sum_{n=-\infty}^{+\infty} \gamma_n = 1$ *let us consider the function* $H : [0,1] \to [0,1]$ *defined by* $H(0) = 0$ *and*

$$H(u) = \frac{1}{2}\left(u + \sum_{\alpha_n \le u} \gamma_n + \sum_{\beta_{n,k} \le u} \frac{\gamma_n}{2^n}\right),\ \ u \in (0,1].$$

Clearly, H *is a strictly increasing and left continuous function whose unique discontinuities are the points of the orbit* $O(\xi)$*. The set* $J = [0,1] \setminus \overline{H([0,1])}$ *is an open subset of* $\mathbb{R}$ *with an infinite number of components; more precisely,* $J = \bigcup_{n\in\mathbb{N}_0} J_n \cup \bigcup_{n\in\mathbb{N},\, 0\le k<2^n} J_{-n}^{(k)}$ *where* $J_n = (H(\alpha_n^-), H(\alpha_n^+)),\ n \in \mathbb{N}_0,$ *and*

$J_{-n}^{(k)} = (H(\beta_{n,k}^-), H(\beta_{n,k}^+))$, $n \in \mathbb{N}$, $0 \le k < 2^n$. *Now, define a map* $f : [0,1] \to [0,1]$ *by*

$$f(u) = \begin{cases} H(S(v)), & \text{if } u = H(v) \text{ with } v \in [0,1] \setminus O(\xi) \\ H((S(v))^-) + \frac{(u - H(v^-))(H((S(v))^+) - H((S(v))^-))}{H(v^+) - H(v^-)}, & \text{if } u \in [H(v^-), H(v^+)] \text{ with } v \in O(\xi) \end{cases},$$

and realize that f *really has the form*

$$f(u) = \begin{cases} f_0(u), & u \in [0, H(1/2)) \\ f_1(u), & u \in [H(1/2), 1] \end{cases},$$

where the maps f_0 *and* f_1 *are strictly increasing and continuous on* $[0, H(1/2))$ *and* $[H(1/2), 1]$*, respectively. Additionally,* $f_0(u) > u$, $0 < u < H(1/2)$*, and* $f_1(u) < u$, $H(1/2) < u < 1$*, while* $f_0((H(1/2))^-) = 1$ *and* $f_1(H(1/2)) = 0$ *and therefore, the hypotheses of Prop. 30 are satisfied by the pair of maps* $\mu_0 = f_0^{-1}$, $\mu_1 = f_1^{-1}$*. In conclusion, there exists a strictly isotone, symmetric and continuous mean* M *on* $[0,1]$ *such that, for every* $u \in [0,1]$,

$$M(0,u) = \mu_0(u) \text{ and } M(u,1) = \mu_1(u).$$

Since the equality $\phi \circ H = H \circ S$ *holds for this mean and* H *is strictly increasing and left continuous, it turns out to be* $H(\delta) = M_-^{[\delta]}(0,1)$, $\delta \in [0,1]$*, so that* $\delta \mapsto M_-^{[\delta]}(0,1)$ *is discontinuous at the point* ξ.

6 Weightings by using Ryll-Nardzewski's dyadic iterations

The results proved in the preceding sections enable us to establish the following theorems on weightings based on Ryll-Nardzewski's dyadic iterations.

Theorem 24 *Let* M *be a strictly isotone and separately continuos mean defined on* I*. In order that the map* $\delta \mapsto M_-^{[\delta]}(x,y)$ *be continuous for a given* $(x,y) \in I$ *it is necessary and sufficient that the following conditions be satisfied:*

(i) *for every* $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \mathcal{NRFS}\{0,1\}$*, the map* $m_{(\varepsilon)} = m_{\varepsilon_1} \circ \cdots \circ m_{\varepsilon_n}$ *has a unique fixed point in* $I_{(x,y)}$*; and*

(ii) *for every irrational number* $\xi = (0.\xi_1\xi_2\cdots)_2 \in [0,1]$*, the sequence of maps* $\{m_{(\xi_1,\xi_2,\cdots,\xi_n)} : n \in \mathbb{N}\}$ *converges to a constant in* $I_{(x,y)}$ *when* $n \uparrow +\infty$.

*When conditions **(i)** and **(ii)** hold for every* $(x,y) \in I^2$*, the maps* $\delta \mapsto M_-^{[\delta]}(x,y)$, $x,y \in I$*, are all strictly monotonic and continuous on* $[0,1]$*. Moreover,* $M_-^{[0]}(x,y) = 0$, $M_-^{[1/2]}(x,y) = M(x,y)$ *and* $M_-^{[1]}(x,y) = y$*, so that* $\delta \mapsto M_-^{[\delta]}$ *is a continuous and strictly monotonic weighting of the mean* M*. The mean* $M_-^{[\delta]}$ *turns out to be continuous for every* $\delta \in [0,1]$ *provided that* M *is continuous.*

Proof. After Theors. 13 and 16, only the continuity of $M_{-}^{[\delta]}$ on I^2 requires some discussion. When M is a continuous mean, $M^{[d]}$ is continuous for every $d \in \mathrm{Dyad}\,([0,1])$ and taking into account (17), $M_{-}^{[\delta]}$ turns out to be lower semi-continuous when $x < y$ and upper semicontinuous when $x > y$. Analogously, $M_{+}^{[\delta]}$ turns out to be upper semicontinuous when $x < y$ and lower semicontinuous when $x > y$. Now, $M_{-}^{[\delta]} = M_{+}^{[\delta]}$ by the continuity of $\delta \mapsto M_{-}^{[\delta]}(x, y)$ and therefore, $M_{-}^{[\delta]}$ is continuous. ■

Theorem 25 *Let M be a strictly isotone and separately continuos mean defined on I. In order that the map $\delta \mapsto M_{-}^{[\delta]}(x, y)$ be continuous for a given $(x, y) \in I$ it is necessary and sufficient that the map ϕ given by (45)-(46) be topologically conjugated to the shift transformation S through an homeomorphism $H_{(x,y)}$ such that $H_{(x,y)}$ is increasing when $x < y$ and decreasing when $x > y$. When this condition is fulfilled by ϕ, the map $\delta \mapsto M_{-}^{[\delta]}$ turns out to be a continuous and strictly monotonic weighting of the mean M. The mean $M_{-}^{[\delta]}$ turns out to be continuous for every $\delta \in [0, 1]$ provided that M is continuous.*

Proof. The proof follows immediately from Theor. 20. As for the continuity of $M_{-}^{[\delta]}$, it follows from the same argument employed in the proof of Theor. 24. ■

Theorem 26 *Let M be a strictly isotone and separately (C)-quasicontractive mean defined on I; then the maps $\delta \mapsto M_{-}^{[\delta]}(x, y)$, $x, y \in I$, are strictly monotonic and continuous on $[0, 1]$. Moreover, $\delta \mapsto M_{-}^{[\delta]}$ is a continuous and strictly monotonic weighting of the mean M. The mean $M_{-}^{[\delta]}$ turns out to be continuous for every $\delta \in [0, 1]$.*

Proof. The proof of the assertions of the theorem is an immediate consequence of Theors. 15, 18. Since a separately (C)-quasicontractive mean is a Lipschitz-continuous function, the continuity of $M_{-}^{[\delta]}$ is proved as in the proof of the previous theorem. ■

It should be added that, when M is a symmetric mean satisfying the hypotheses of any one of the preceding theorems, then the weighting $\delta \mapsto M_{-}^{[\delta]}$ of M possesses the following symmetry property:

$$M_{-}^{[1-\delta]}(x, y) = M_{-}^{[1-\delta]}(y, x),\ \ x, y \in I,\ \delta \in [0, 1].$$

7 Appendix

A useful criterion of separated nonexpansiveness or quasicontractiveness can be formulated in terms of partial derivatives. To this end, consider a mean M defined on I such that there exist its partial derivatives M_x and M_y on I^2. If M is, in addition, separately nonexpansive, then passing to the limit $u' \to u$ in the inequalities (12) yields

$$|M_x| \leq 1 \text{ and } |M_y| \leq 1. \tag{64}$$

Conversely, if a mean M has partial derivatives M_x and M_y satisfying inequalities (64), then the Mean Value Theorem yields

$$|M(x,u) - M(x,u')| = |M_y(x,\theta)|\,|u-u'| \le |u-u'|\,,\ u,u' \in I,$$

where θ is an intermediate value, and in a similar way,

$$|M(u,y) - M(u',y)| \le |u-u'|\,,\ u,u' \in I,$$

so that M turns out to be separately nonexpansive. Clearly, M turns out to be separately quasicontractive provided that at least one of the inequalities (64) is strict.

Proposition 27 *Let M be a mean defined on I with partial derivatives existing at every point $(x,y) \in I^2$; then M is separately nonexpansive if and only if the inequalities (64) are satisfied by M_x and M_y. When one at least of these inequalities is strict, M turns out to be separately quasicontractive.*

Proof. See the previous discussion. ■

Using the above result, an example is exhibited which shows that separate (C)-nonexpansiveness is a property weaker than separate nonexpansiveness.

Example 28 *(A separately (C)-quasicontractive but not separately nonexpansive mean). The Heronian mean $\mathfrak{H}_{\mathfrak{E}}$ is the mean defined (cf. [13], pg. 399) by*

$$\mathfrak{H}_{\mathfrak{E}}(x,y) = \frac{x+y+\sqrt{xy}}{3},\ x,y>0.$$

Since $\left(\mathfrak{H}_{\mathfrak{E}}\right)_x(x,y) = \left(1+\sqrt{y/x}/2\right)/3 > 0$ and $\left(\mathfrak{H}_{\mathfrak{E}}\right)_y(x,y) = \left(1+\sqrt{x/y}/2\right)/3 > 0$, $\mathfrak{H}_{\mathfrak{E}}$ is strictly isotone. Moreover, $\{(x,y) \in \mathbb{R}^+ \times \mathbb{R}^+ : \left(\mathfrak{H}_{\mathfrak{E}}\right)_x(x,y) \le 1$ and $\left(\mathfrak{H}_{\mathfrak{E}}\right)_y(x,y) \le 1\} = \{(x,y) \in \mathbb{R}^+ \times \mathbb{R}^+ : 1/16 \le y/x \le 16\} \ne \mathbb{R}^+ \times \mathbb{R}^+$, and therefore $\mathfrak{H}_{\mathfrak{E}}$ is not a separately nonexpansive mean by Prop. 27. The conjugated $\left(\mathfrak{H}_{\mathfrak{E}}\right)_f$ of $\mathfrak{H}_{\mathfrak{E}}$ by the homeomorphism $f(x) = \sqrt{x}$, $x>0$, is clearly given by

$$\left(\mathfrak{H}_{\mathfrak{E}}\right)_f(x,y) = \sqrt{\frac{x^2+xy+y^2}{3}},\ x,y>0.$$

The mean $\left(\mathfrak{H}_{\mathfrak{E}}\right)_f = \mathcal{L}^{[2]}$ is known as the generalized logarithmic mean of order 2 *(cf. [13], pg. 385) and, as expected, $\mathcal{L}^{[2]}_x(x,y)$, $\mathcal{L}^{[2]}_y(x,y) > 0,\ x,y>0$, so that $\mathcal{L}^{[2]}$ is also strictly isotone. Now, by adding the partial derivatives of $\mathcal{L}^{[2]}$, for $x,y>0$ it is obtained*

$$\begin{aligned}\mathcal{L}^{[2]}_x(x,y),\, \mathcal{L}^{[2]}_y(x,y) &< \mathcal{L}^{[2]}_x(x,y) + \mathcal{L}^{[2]}_y(x,y)\\ &= \left(\sqrt{\frac{x^2+xy+y^2}{3}}\right)^{-1}\left(\frac{x+y}{2}\right) = \frac{A(x,y)}{\mathcal{L}^{[2]}(x,y)} \le 1.\end{aligned}$$

The last inequality is derived from the fact that $\mathcal{L}^{[2]}$ is a superarithmetic mean:

$$\mathcal{L}^{[2]}(x,y) \ge A(x,y),\ x,y>0.$$

In this way, Prop. 27 shows that $\mathcal{L}^{[2]}$ is separately quasicontractive.

Even if separate (C)-nonexpansiveness is a property considerably weaker than separate nonexpansiveness, it is far from being generally enjoyed by continuous means. The next example illustrates this fact.

Example 29 *(A family of continuous but not separately (C)-nonexpansive means) Suppose that a continuous mean M defined on I satisfies the following property: there exists an isolated point $(x_0, y_0) \in \operatorname{int}(I^2)$, $x_0 \neq y_0$, such that $M(x_0, y_0) = \min\{x_0, y_0\}$ or $M(x_0, y_0) = \max\{x_0, y_0\}$; in other words, M ceases of being a strict mean only in an isolated point $(x_0, y_0) \in \operatorname{int}(I^2)$. Let us show that M can not be separately (C)-nonexpansive. In fact, if M was separately (C)-nonexpansive, then there would exist a homeomorphism $f : I \to \mathbb{R}$ such that the inequalities*

$$|f(M(x,u)) - f(M(x,u'))| \leq |f(u) - f(u')| \tag{65}$$

and

$$|f(M(u,y)) - f(M(u',y))| \leq |f(u) - f(u')| \tag{66}$$

hold for every $x, y, u, u' \in I$. Suppose that $x_0 < y_0$, $M(x_0, y_0) = x_0$ and that, for a small enough $\delta > 0$, (x_0, y_0) is the unique (not diagonal!) point in the square $[x_0 - \delta, y_0 + \delta]^2$ with this property. In this way, the two points $u_1 = x_0$ and $u_2 = y_0$ are the unique fixed points of the function $m_0 : [x_0, y_0 + \delta] \to [x_0, y_0 + \delta]$ defined by $m_0(u) = M(x_0, u)$. Furthermore, $m_0(u) < u$ for every $u \in (x_0, y_0) \cup (y_0, y_0 + \delta)$ and therefore, the sequences $\{m_0^n(y_0 - \delta_0) : n \in \mathbb{N}\}$ and $\{m_0^n(y_0 + \delta_0) : n \in \mathbb{N}\}$ with $0 < \delta_0 < \min\{\delta, y_0 - x_0\}$ turns out to be monotonic and

$$m_0^n(y_0 - \delta_0) \downarrow x_0, \qquad m_0^n(y_0 + \delta_0) \downarrow y_0$$

when $n \uparrow +\infty$. Now, setting $x = x_0$ in the inequality (65) produces

$$|f(m_0(u)) - f(m_0(u')))| \leq |f(u) - f(u')|, \ u, u' \in [x_0, y_0 + \delta],$$

whence the inequality

$$|f(m_0^n(u)) - f(m_0^n(u'))| \leq |f(u) - f(u')|, \ u, u' \in [x_0, y_0 + \delta]$$

follows by iteration. Replacing in this last equality u by $y_0 - \delta_0/2$, u' by $y_0 + \delta_0/2$ and then passing to the limit $n \uparrow +\infty$, it is deduced

$$\begin{aligned} |f(x_0) - f(y_0)| &= \lim_{n \uparrow +\infty} |f(m_0^n(y_0 - \delta_0)) - f(m_0^n(y_0 + \delta_0))| \\ &\leq |f(y_0 - \delta_0) - f(y_0 + \delta_0)|, \ 0 < \delta_0 < \min\{\delta, y_0 - x_0\}. \end{aligned}$$

Since f is injective, $|f(x_0) - f(y_0)| > 0$ and then, the continuity of f at y_0 is contradicted by the above inequality. In the case in which $x_0 < y_0$, $M(x_0, y_0) = y_0$, a similar contradiction is obtained by considering the map $m_1(u) = M(u, y_0)$ in the interval $[x_0 - \delta, y_0]$ and inequality (66). The remaining cases are analogously treated. This proves that M can not be separately (C)-nonexpansive.

Since the mean M constructed in Example 23 is discontinuous at the points of the orbit of an irrational number, it can not be separately (C)-nonexpansive by Theor. 18; thus, an example of strict mean which is not separately (C)-nonexpansive is provided by M.

Now, let us prove the result used in constructing the means of Examples 14 and 23.

Proposition 30 *Let $\mu_0, \mu_1 : [0,1] \to [0,1]$ be two continuous and strictly increasing functions such that $\mu_0(u) < u < \mu_1(u)$ for all $u \in (0,1)$, and $\mu_0(0) = 0,\ \mu_1(1) = 1,\ \mu_0(1) = \alpha = \mu_1(0)$ with $0 < \alpha < 1$; then, there exists a strictly isotone, symmetric and continuous mean M on $[0,1]$ such that, for every $u \in [0,1]$,*

$$M(0,u) = \mu_0(u) \text{ and } M(u,1) = \mu_1(u). \tag{67}$$

Of course, there are infinite continuous symmetric means M defined on $[0,1]$ satisfying the boundary value conditions (67). For instance, if μ is the harmonic function in the triangle $0 < u < v < 1$ with boundary values given by $\mu(0,t) = \mu_0(t),\ \mu(t,1) = \mu_1(t),\ \mu(t,t) = t,\ (0 \le t \le 1)$, then, by the maximum principle (cf. [15]), $u \le \mu(u,v) \le v$ for every $u < v$, whence the symmetric extension to $[0,1]^2$ of μ, which is given by

$$M(u,v) = \begin{cases} \mu(u,v), & u \le v \\ \mu(v,u), & v \le u \end{cases},$$

turns out to be a symmetric continuous mean defined on $[0,1]$. The interest of the above proposition lies in the existence of a strictly isotone and continuous mean satisfying the boundary conditions of the statement.

Proof. In order to prove the proposition it is sufficient to define a strictly isotone and continuous mean M on the triangular domain $T = \{(u,v) \in [0,1]^2 : u \le v\}$. Indeed, extending M to the whole square $[0,1]^2$ by $M(u,v) = M(v,u)$ gives a strictly isotone, symmetric and continuous mean M defined on $[0,1]$.

Let us consider a splitting of the triangle T in four triangular subdomains $T_i,\ i = 1,2,3,4$, respectively defined by

$$\begin{aligned} T_1 &= \{(u,v) \in T : v \le \alpha\}, \quad T_2 = \{(u,v) \in T : v \ge \alpha,\ (1-\alpha)u + \alpha v \le \alpha\}, \\ T_3 &= \{(u,v) \in T : u \le \alpha,\ (1-\alpha)u + \alpha v \ge \alpha\}, \quad T_4 = \{(u,v) \in T : u \ge \alpha\}, \end{aligned}$$

and define M on T as follows:

$$M(u,v) = \begin{cases} (1-\frac{u}{v})\mu_0(v) + u, & (u,v) \in T_1 \\ (1-\frac{(1-\alpha)u}{\alpha(1-v)})\mu_0(v) + \frac{(1-\alpha)u}{1-v}, & (u,v) \in T_2 \\ (1-\frac{\alpha(1-v)}{(1-\alpha)u})\mu_1(u) + \frac{\alpha^2(1-v)}{(1-\alpha)u}, & (u,v) \in T_3 \\ (\frac{v-u}{1-u})\mu_1(u) + \frac{1-v}{1-u}u, & (u,v) \in T_4 \end{cases}. \tag{68}$$

For every $i = 1,2,3,4$, the function $M_i = M|_{T_i},\ i = 1,2,3,4$, is continuous and strictly increasing in both variables on its respective domain of definition.

Take, for example, $i = 2$; then, the continuity of M_2 follows from that of μ_0 and, regarding the monotonicity, we have the equality

$$\begin{aligned} M_2(u,v) &= (1 - \frac{(1-\alpha)u}{\alpha(1-v)})\mu_0(v) + \frac{(1-\alpha)u}{1-v} \\ &= \frac{1-\alpha}{1-v}\left(1 - \frac{\mu_0(v)}{\alpha}\right)u + \mu_0(v), \end{aligned}$$

in which, taking into account that $\mu_0(v) < \alpha < 1$ provided that $v < 1$, the inequality

$$\frac{1-\alpha}{1-v}\left(1 - \frac{\mu_0(v)}{\alpha}\right) > 0$$

holds for the coefficient of the variable u and thus, $u \mapsto M_2(u,v)$ turns out to be strictly increasing in the interval $[0, \frac{\alpha(1-v)}{1-\alpha}]$. Note that the interval $[0, \frac{\alpha(1-v)}{1-\alpha}]$ reduces to the point $u = 0$ when $v = 1$. Now, if $0 \leq u < \alpha$ and $v_1, v_2 \in [\alpha, 1 - \frac{(1-\alpha)u}{\alpha}]$, $v_1 < v_2$, in view of $v \mapsto (1-\alpha)u/(1-v)$ is strictly increasing on $[0,1)$ as well as μ_0, we have

$$\begin{aligned} M_2(u,v_1) &= (1 - \frac{(1-\alpha)u}{\alpha(1-v_1)})\mu_0(v_1) + \frac{(1-\alpha)u}{1-v_1} \\ &< (1 - \frac{(1-\alpha)u}{\alpha(1-v_2)})\mu_0(v_1) + \frac{(1-\alpha)u}{1-v_2} \\ &< (1 - \frac{(1-\alpha)u}{\alpha(1-v_2)})\mu_0(v_2) + \frac{(1-\alpha)u}{1-v_2} = M_2(u,v_2), \end{aligned}$$

which, taking into account that the interval $[\alpha, 1 - \frac{(1-\alpha)u}{\alpha}]$ reduces to the point α when $u = \alpha$, proves that $u \mapsto M_2(u,v)$ is strictly increasing in the interval $[\alpha, 1 - \frac{(1-\alpha)u}{\alpha}]$. The proof of the continuity and the strict monotonicity of the remaining M_i is similar.

Now, for every $i = 1,2,3,4$, the function M_i is internal in its respective domain of definition. For example, taking into account that $0 \leq \mu_0(v) \leq \alpha < 1$, it can be written

$$\begin{aligned} M_2(u,v) &\leq (1 - \frac{(1-\alpha)u}{\alpha(1-v)})\alpha + \frac{(1-\alpha)u}{1-v} \\ &= \alpha \leq v, \end{aligned}$$

and

$$M_2(u,v) \geq \frac{(1-\alpha)u}{1-v} \geq u$$

for every $(u,v) \in T_2$. The internality of the remaining M_i can be analogously proved. Note in passing that it is right here where the hypotheses $\mu_0(u) < u < \mu_1(u)$, $0 < u < 1$, are employed: the inequality $\mu_0(v) < v$ shows that $M_1(u,v) < v$, while the inequality $u < M_4(u,v)$ is a consequence of $u < \mu_1(u)$.

Now, observing that

$$\begin{aligned} T_1 \cap T_2 &= \{(u,v) \in T : v = \alpha,\, u \leq \alpha\}, \\ T_2 \cap T_3 &= \{(u,v) \in T : (1-\alpha)u + \alpha v = \alpha\}, \\ T_3 \cap T_4 &= \{(u,v) \in T : u = \alpha\}, \end{aligned}$$

while

$$T_1 \cap T_3 = T_1 \cap T_4 = T_2 \cap T_4 = \{(\alpha,\alpha)\},$$

it is easy to see that $M_i|_{T_i \cap T_j} = M_j|_{T_i \cap T_j}$ for every $i,j = 1,2,3,4,\ i \neq j$, and hence, that M defined by (68) turns out to be a strictly isotone and continuous mean M on the triangular domain T. This finishes the proof. ■

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References

[1] J. Aczél, On mean values, Bull. Amer. Math. Soc. **54**, (1948), 392-400.

[2] J. Aczél, *Lectures on Functional Equations and their Applications*, Academic Press, New York and London, (1966).

[3] J. Aczél, J. Dhombres, *Functional Equations in Several Variables*, Cambridge Univ. Press, Cambridge, 1989.

[4] L. R. Berrone, *A dynamical characterization of quasilinear means*, Aequationes Math. **84**, Issue **1** (2012), 51-70.

[5] L. R. Berrone, *The Aumann functional equation for general weighting procedures*, Aequationes Math., DOI 10.1007/s00010-015-0344-4, (2015).

[6] L. R. Berrone, *Generalized Cauchy means*, Aequationes Math., DOI 10.1007/s00010-015-0341-7, (2015).

[7] L. R. Berrone, *Weighting general means*, (to appear).

[8] L. R. Berrone, A. L. Lombardi, *A note on equivalence of means*, Publ. Math. Debrecen **58**, Fasc. **1-2**, (2001), 49-56.

[9] L. R. Berrone, J. Moro, *Lagrangian means*, Aequationes Math. **55**, (1998), 217-226.

[10] L. R. Berrone, J. Moro, *On means generated through the Cauchy mean value theorem*, Aequationes Math. **60**, (2000), 1-14.

[11] L. R. Berrone, G. E. Sbérgamo, *La familia de bases de una media continua y la representación de las medias cuasiaritméticas*, Bol. de la Asoc. Venezolana de Matemática, Vol. **XIX**, No. **1**, (2012), 3-18.

[12] L. R. Berrone, G. E. Sbérgamo, *Weightings general means by iteration*, (to appear).

[13] P. S. Bullen, *Handbook of Means and Their Inequalities*, Kluwer Academic Publishers, Dordrecht, 2010.

[14] P. S. Bullen, D. S. Mitrinović, P. M. Vasić, *Means and Their Inequalities*, D. Reidel Publishing Company, Dordrecht, 1988.

[15] D. Gilbarg, N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin, (1983).

[16] A. Katok, B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Encyclopedia of Math. and its Appl. v. 54, Cambridge Univ. Press, Cambridge, 1995.

[17] A. Katok, B. Hasselblatt, *A First Course in Dynamics (with a Panorama of Recent Developments)*, Cambridge Univ. Press, Cambridge, 2003.

[18] J. Matkowski, *On weighted extensions of Bajraktarević means*, Sarajevo J. Math. **6**, No. **19**, (2010), 169-188.

[19] M. Raïssouli, *Parameterized logarithmic mean*, Int. Journal of Math. Analysis, Vol. **6** no. 18, (2012), 863-869.

[20] M. Raïssouli, J. Sándor, *On a method of construction of new means with applications*, Journal of Inequalities and Applications, DOI 10.1186/1029-242X-2013-89, (2013).

[21] K. C. Richards, H. C. Tiedeman, *A note on weighted identric and logarithmic means*, J. Ineq. Pure and Appl. Math. **7(5)** Art. 157, (2006).

[22] C. Ryll-Nardzewski, *Sur les moyennes*, Studia Math. **11**, (1949), 31-37.