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ERGODIC THEOREM FOR AMENABLE GROUPS AND WEAKLY INTEGRABLE FUNCTIONS

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Abstract

We analyze averages of amenable groups along Følner sequences in the way considered by Lindenstrauss but for weakly integrable functions. The objective is to generalize the quantitative results of Haynes, adapting the technique of trimmed sums for classical ergodic averages.

I. Introduction

In the setting of classical ergodic theory, a probability space $(X, \mathcal{B}, \mu)$ and a measure preserving transformation $f : X \to X$ are considered. If $\varphi : X \to \mathbf{R},$ then the N-statistical sum at $x \in X$ is

$$\mathcal{S}_N, \varphi(x) := \frac{1}{N} \sum_{n=0}^{N-1} \varphi(f^n(x)).$$

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The Birkhoff ergodic theorem states the pointwise convergence of the sequence $\{\mathcal{S}_N, \varphi(x)\}$ for any x, μ-a.e. and for $\varphi \in L^1(X, \mu)$. If the transformation is ergodic, then $\lim_{N \to \infty} \mathcal{S}_N, \varphi(x) = \int \varphi d\mu$.

The extension of the ergodic theorem to the action of groups as dynamics was the matter of meaningful investigations. The classical ergodic theorem considers measure preserving **Z**-actions, thus the extension consists in dealing with more general Γ-actions. One capital contribution was the work of Lindenstrauss [5] who generalized the classical Birkhoff ergodic theorem to the action of amenable groups. Previous meaningful work about pointwise convergence of ergodic averages for the action of groups can be seen, for instance, in [7] and [4]. In the first one, Nevo and Stein proved the pointwise ergodic convergence for finite measure preserving actions of the free group F_r, $r \geq 2$; in the second one, Fujiwara and Nevo established an ergodic convergence for exponentially mixing actions of word-hyperbolic groups.

One formulation of amenability is the following: a topological locally compact group Γ is *amenable* if for any compact $K \subset \Gamma$ and for any $\delta > 0$, there exists a compact $F \subset \Gamma$ such that

$$\frac{| F \Delta KF |}{| F |} < \delta, \tag{1}$$

where $| \cdot |$ denotes the left invariant Haar measure on Γ. This measure will be also denoted by $m_\Gamma(\cdot)$.

The averages analyzed by Lindenstrauss in [5] for actions of amenable groups Γ on a Lebesgue space (X, μ) and map $\varphi : X \to \mathbf{R}$, are of the form

$$\frac{1}{| E |} \int_E \varphi(\gamma x) dm_\Gamma(\gamma),$$

with $E \subset \Gamma$. In that article, the convergence of these averages provided the existence of adequate sequences (F_n) in Γ, called *Følner sequences* was proved. More particularly, the so-called tempered sequences are considered.

A sequence (F_n) of compact subsets of Γ is a *Følner sequence* if for any compact subset K of Γ and for any $\delta > 0$, then

$$\frac{|F_n \Delta K F_n|}{|F_n|} < \delta, \tag{2}$$

for large enough n. A sequence (F_n) is *tempered* if there is a $C > 0$ such that

$$\frac{\left|\bigcup_{k=1}^{n} F_k^{-1} F_n\right|}{|F_n|} < C. \tag{3}$$

Lindenstrauss calls (3) the *Schulman condition*. In [5], it was proved that for Følner sequences (F_n) satisfying Schulman condition and for $\varphi \in L^1(X, \mu)$, there is a Γ-invariant map $\overline{\varphi}$ such that

$$S_{F_N, \varphi}(x) := \frac{1}{m_\Gamma(F_N)} \int_{F_N} \varphi(\gamma x) dm_\Gamma(\gamma) \to_N \overline{\varphi}(x), \tag{4}$$

for μ-a.e. x. The sequence $N \to |F_{N-1}|$ grows super-exponentially [5].

The possibility of extending the ergodic theory to functions outside the class of integrable maps was contemplate, for the classical case, by several researchers. In [1], Aaronson and Nakada considered ergodic sums for a **R**-valuated ergodic stationary process $(X_1, X_2, ...)$ with $E(X_1) = \infty$. The aim of that article was to analyze the possibility of a weak law of large numbers for the statistical sums $S_n = \sum_{k=1}^{n} X_k$, since there is not a strong law of large numbers due to $E(X_1) = \infty$. Weak law means that there exists a sequence (b_n) such that $\frac{S_n}{b_n}$ stochastically converges. In [1], the authors used technique of "trimming" which, roughly speaking, consists into excluding (trimming) the maximal terms of $\{X_1, X_2, ..., X_n\}$ in each sum

S_n. In that article, a law of large numbers for dependent process in which one term is removed from the statistical sums is proved. The main theorem in [1] generalizes a previous result of Mori [6] for i.i.d. random variables and one by Diamond and Vaaler [2] for the particular case of continued fractions. The assumption on the process is the condition of continued fraction mixing. In [3], Haynes established a quantitative version of the classical Birkhoff ergodic theorem for non-integrable maps using trimmed sums. In the context of classical dynamical systems, the stationary process is $X_n = \varphi(T^n(x))$. The class of potentials φ is the weakly integrable maps and of the dynamics is imposed a kind of continued fraction mixing. The objective of this work is to do a similar analysis of [3] but in the context of amenable action groups on a probability space (X, μ).

Let $\mathcal{A} = \{A_i\}_{i \in \mathbf{N}}$ be a measurable partition of (X, μ) and let $a(\gamma) : X \to X$ be the Γ-action on X for any element γ. The following kind of weak-mixing condition is considered. Let $g : \mathbf{R}^+ \to [0, \infty)$ and a tempered Følner sequence (F_n) such that

$$\int_{\gamma F_N} \frac{\mu(A_i \cap a(\gamma) A_j)}{\mu(A_i)\mu(A_j)} dm_\Gamma(\gamma) \leq g(m_\Gamma(F_N)) + m_\Gamma(F_N). \tag{5}$$

Definition. Let $\phi : \mathbf{R}^+ \to \mathbf{R}^+$ be an increasing function, the class of ϕ-*weakly integrable functions* is constituted by maps φ such that

$$\sup\{\phi(t)\mu(\{x : |\varphi(x)| > t\})\} < \infty. \tag{6}$$

The class of ϕ-weakly integrable functions will be denoted by $L^{\phi, w}$. If $\phi = id$, then we have the usual class of weakly integrable functions. For $\phi : \mathbf{R}^+ \to \mathbf{R}^+$, increasing and continuous, let us consider the map

$$T_{\phi, \varepsilon}(t) = \phi^{-1}(t \log^{1/2+\varepsilon} t),$$

for $\varphi \in L^{\phi, w}$ and for Følner sequence (F_n), denote

$$H_1(N) = \int_{\{\varphi \le T_{\phi, \varepsilon}(|F_N|)\}} \varphi d\mu,$$

$$H_2(N) = \int_{\varphi \le T_{\phi, \varepsilon}(|F_N|)} \varphi^2 d\mu$$

and

$$H_3(N) = H_1(N)^2(N + G(N)) + H_2(N),$$

where

$$G(N) = \sum_{n=1}^{N} (g(m_\Gamma(F_n))).$$

The following theorem is the main result to be proved.

Theorem. *Let* Γ *be an amenable group action on a probability space* (X, μ), *let* φ *be a measurable and non-negative map in the class* $L^{\phi, \omega}$ *and assume the existence of a map g satisfying equation* (5). *If*

$$(N+1)H_1(N+1) - NH_1(N) \ll H_3(N+1) - H_3(N) \ll H_3(N)^{2/3}, \quad (7)$$

then for a given $\varepsilon > 0$ *and for a tempered sequence* (F_N), *there is a cutoff function* $\delta_{\varepsilon, F_N} : X \to \{0, 1\}$ *such that*

$$S_{F_N, \varphi}(x) = H_1(N) + \delta_{\varepsilon, F_N}(x) \max_{\gamma \in F_N} \varphi(\gamma x) + O_\varepsilon(I_3(N)^{2/3} \log H_3(N)^{1/3+\varepsilon}),$$

for almost every x.

II. Lindenstrauss Theory

Let us consider a measure space (X, μ) and an amenable group Γ acting on it. The way to establish the theorem is the following: Firstly, it is proved that for functions $\varphi \in L^2(X, \mu)$, the error in $\left| S_{F_N, \varphi}(x) - \int \varphi d\mu \right|$ can be estimated as $O_\varepsilon(I_3(N)^{2/3}\log I_3(N)^{1/3+\varepsilon})$, with $I_3(N) = I_1^2(N + G(N)) + I_2$, where in this case, $I_1 = \int_X \varphi d\mu$ and $I_2 = \int_X \varphi^2 d\mu$. This is the route followed in [3], but here we will use a version of the Lindenstrauss maximal inequality. Then it is proved that for φ, a weakly integrable map and with the existence of the function g, there are infinities N such that the set $\{\gamma \in F_N : \varphi(\gamma x) > T_{\phi,\varepsilon}(|F_N|)\}$ has zero Haar measure.

We review the necessary background from the Lindenstrauss theory. Let (F_n) be a tempered sequence in an amenable group and let F be a compact subset of Γ, Lindenstrauss introduced collections of right translates of sets $F_1, F_2, ..., F_N$, which cover F. The collection $\overline{\mathcal{F}}$ is specified by subsets A_j of Γ with $F_j A_j \subset F$, $j = 1, 2, ..., N$, and

$$\overline{\mathcal{F}} = \{F_j a : a \in A_j,\ j = 1, 2, ..., N\}.$$

Then random subcollections of $\overline{\mathcal{F}}$ are introduced in the following way, let Ω be

$$\mathcal{F} : \Omega \to \mathcal{P}(\overline{\mathcal{F}}),$$

and a counting function is defined as

$$\Lambda^{\varpi} : F \to \mathbf{N},$$

$$\Lambda^{\varpi}(\gamma) = \sum_{B \in \mathcal{F}(\varpi)} I_B(\gamma),$$

with I_B the characteristic function of B. For any $B \in \mathcal{P}(\overline{\mathcal{F}})$, let $\| S \| = \sum_{B \in \mathcal{S}} |B|$.

The covering lemma of Lindenstrauss [5] says that for a given $\delta > 0$, the map $\mathcal{F}$ can be chosen such that

(i) $\mathbf{E}(\Lambda^{\varpi}(\gamma) | \Lambda^{\varpi}(\gamma) \geq 1 |) \leq 1 + \delta$,

(ii) $\mathbf{E}(\| \mathcal{F}(\varpi) \|) \leq h(\delta, C) \left| \bigcup_{j=1}^{N} A_j \right|$, where $h(\delta, C) = \frac{\delta}{1 + C\delta}$, C the constant for the tempered sequence (F_N) and $\mathbf{E}$ the expectation value.

It can be chosen as a compact set $\widetilde{F} \subset \Gamma$ such that $F = \bigcup_{j=1}^{N} F_j \widetilde{F}$ and for $\varepsilon > 0$, $|F| \leq (1 + \varepsilon) |\widetilde{F}|$.

For a fixed left invariant Haar measure *m*, the modular function of Γ is a map $\Delta : \Gamma \to \mathbf{R}$ with

$$\Delta(\gamma) \int_{E\gamma} \psi dm = \int_{E} R_{\gamma} \psi dm, \text{ for any map } \psi \in C^{0}(\Gamma), \tag{8}$$

where R_{γ} is the right translation. In particular,

$$\Delta(\gamma) = \frac{|E\gamma|}{|E|},$$

for any measurable set *E*.

The following proposition is a version of the Lindenstrauss maximal inequality:

Proposition 1. *Let*

$$D_{N, \alpha} = \left\{ x : \max_{j=1, \ldots, N} \left| S_{F_N, \varphi}(x) - \int \varphi d\mu \right| > \alpha \right\}. \tag{9}$$

Then

$$\mu(D_{N, \alpha}) \leq \frac{C}{\alpha} \| \Phi \|_1.$$

Proof. For a fixed $x \in X$, set

$$A_j = \left\{ \gamma \in \widetilde{F} : \left| S_{F_N, \varphi}(x) - \int \varphi d\mu \right| > \alpha \right\}. \tag{10}$$

Thus, we have, for any $a \in A_j$,

$$\alpha | F_j a | \leq | F_j a | \left| \int_{F_j} \frac{1}{| F_j |} \varphi(\gamma x) dm(\gamma) - \int \varphi d\mu \right|,$$

and so

$$\frac{| F_j a |}{| F_j |} \left| \int_{F_j} \varphi(\gamma x) dm(\gamma) - | F_j | \int \varphi d\mu \right|$$

$$= \left| \Delta(a) \left| \int_{F_j} \varphi(\gamma a x) dm(\gamma) - \Delta(a) | F_j | \int \varphi d\mu \right| \right|$$

$$= \left| \int_{F_j a} \varphi(\gamma x) dm(\gamma) - | F_j a | \int \varphi d\mu \right|.$$

Then

$$\alpha \| \mathcal{F}(\varpi) \| \leq \sum_{B \in \mathcal{F}(\varpi)} \left| \int_B \varphi(\gamma x) dm(\gamma) - | B | \int \varphi d\mu \right|.$$

By the covering lemma, with $\delta = 1$,

$$\alpha h(1, C) \left| \bigcup_{j=1}^{N} A_j \right| \leq \mathbf{E}(\| \mathcal{F}(\varpi) \|),$$

but also

$$\int_{\widetilde{F}} I_{D_{N, \alpha}} dm(\gamma) = \left| \bigcup_{j=1}^{N} A_j \right|,$$

so that

$$\alpha h(1, C) \left| \bigcup_{j=1}^{N} A_j \right| = \alpha h(1, C) \int_{\widetilde{F}} I_{D_{N, \alpha}} dm(\gamma) \leq \mathbf{E}(\| \mathcal{F}(\varpi) \|). \qquad (11)$$

We have

$$\begin{aligned}
&\mathbf{E}(\| \mathcal{F}(\varpi) \|) \\
&\leq \mathbf{E}\left(\sum_{B \in \mathcal{F}(\varpi)} \left| \int_B \varphi(\gamma x) dm(\gamma) - | B | \int \varphi d\mu \right| \right) \\
&= \mathbf{E}\left(\sum_{B \in \mathcal{F}(\varpi)} \left| \int_B \left(\varphi(\gamma x) - \int \varphi d\mu \right) dm(\gamma) \right| \right) \leq \mathbf{E}\left(\sum_{B \in \mathcal{F}(\varpi)} \int_B \left| \varphi(\gamma x) - \int \varphi d\mu \right| dm(\gamma) \right) \\
&= \mathbf{E}\left(\int_F \Lambda^{\varpi}(\gamma) \left| \varphi(\gamma x) - \int \varphi d\mu \right| dm(\gamma) \right) \leq 2 \int_F | \Phi(\gamma x) | dm(\gamma),
\end{aligned}$$

where, in the last inequality, the covering lemma is used with $\delta = 1$ and $\Phi(x) = \varphi(x) - \int \varphi d\mu.$

Therefore,

$$\int_{\widetilde{F}} I_{D_{N, \alpha}} dm(\gamma) \leq \frac{K}{\alpha} \int_F | \Phi(\gamma x) | dm(\gamma),$$

with $K = \dfrac{2}{h(1, C)}.$

Then we have

$$\begin{aligned}
\mu(D_{N, \alpha}) &= \frac{1}{\widetilde{F}} \int_{\widetilde{F}} \int_X I_{D_{N, \alpha}} dm(\gamma) d\mu(x) \\
&\leq \frac{K}{\alpha} \frac{1}{| \widetilde{F} |} \int_X \int_F | \Phi(\gamma x) | dm(\gamma) d\mu(x) \\
&= \frac{K}{\alpha} \frac{1}{| \widetilde{F} |} | F | \int_X | \Phi(\gamma x) | d\mu(x) \leq \frac{K}{\alpha} (1 + \varepsilon) \| \Phi \|_1,
\end{aligned}$$

recalling that $| F | \leq (1 + \varepsilon) | \widetilde{F} |.$ □

III. Proof of the Theorem

The next proposition extends to amenable groups as a result in [3].

Proposition 2. *Let* $\varphi : X \to \mathbf{R}^+$ *and belong to* $L^2(X, \mu)$. *Assume that*

$$NI_1 \ll I_3(N+1) - I_3(N) \ll I_3(N)^{2/3}.$$

Then

$$S_{F_N, \varphi}(x) - \int \varphi d\mu = O_\varepsilon(I_3(N)^{2/3} \log I_3(N)^{1/(3+\varepsilon)}).$$

Proof. Let $j \in \mathbf{N}$, $\varepsilon > \mathbf{0}$, and let us consider a sequence $\{N_j\}$ such that

$$(N_{j+1} - N_j)I_1 \ll I_3(N_{j+1}) - I_3(N_j) \ll j^2 \log^{1+\varepsilon}(j+1). \tag{12}$$

It can be possible if N_j is the smallest integer such that $I_3(N_j) > j^2 \log^{1+\varepsilon}(j+1)$. From above proposition, we get

$$\mu\left(\left\{x : \left| S_{F_{N_j}, \varphi}(x) - \int \varphi d\mu \right| > j^2 \log^{1+\varepsilon}(j+1)\right\}\right) \leq \frac{C}{j^2 \log^{1+\varepsilon}(j+1)} \| \Phi \|_1, \tag{13}$$

and so, by the Borel-Cantelli lemma, we obtain

$$\Psi_{N_j}(x) := S_{F_{N_j}, \varphi}(x) - \int \varphi d\mu = O_\varepsilon(j^2 \log^{1+\varepsilon}(j+1)), \text{ for large } j.$$

We have $N_j \leq N \leq N_{j+1}$ such that $\Psi_{N_j} \leq \Psi_N \leq \Psi_{N_{j+1}}$ and from the hypothesis, *N* can be interpolated *N* such that

$$\Psi_N(x) := S_{F_N, \varphi}(x) - \int \varphi d\mu = O_\varepsilon(j^2 \log^{1+\varepsilon}(j+1)), \text{ for } N_j \leq N \leq N_{j+1}.$$

For $N \geq N_{j+1}$, we have

$$j^2 \log^{1+\varepsilon}(j+1) \ll I_3(N)^{2/3} \log I_3(N)^{1/(3+\varepsilon)}. \qquad \square$$

To complete the proof of the theorem, it must be seen that for weakly integrable maps, $\{\gamma \in F_N : \varphi(\gamma x) > T_{\phi, \varepsilon}(| F_N |)\}$ has zero Haar measure for infinites *N*.

Lemma 1. *Let* $\varphi : X \to \mathbf{R}^+$ *be a measurable map in the class* $L^{\phi, w}$, *for a increasing and continuous function* $\phi : \mathbf{R}^+ \to \mathbf{R}^+$. *Let* $T_{\phi, \varepsilon}(t) = \phi^{-1}(t \log^{1/2+\varepsilon} t)$, *and* $g : \mathbf{R}^+ \to [0, \infty)$ *be a function satisfying*

$$\int_{\gamma F_N} \frac{\mu(A_i \cap a(\gamma) A_j)}{\mu(A_i)\mu(A_j)} dm_\Gamma(\gamma) \leq g(| F_N |) + | F_N |,$$

for a measurable partition $\{A_i\}_{i \in \mathbf{N}}$ *of* (X, μ). *Then* $m_\Gamma(\{\gamma \in F_N : \varphi(\gamma x) > T_{\phi, \varepsilon}(| F_N |)\}) > 0$ *for a.e.* $x \in X$ *and for finites N.*

Proof. Let

$$B_N = \int_{\gamma F_N} \int_{\overline{\gamma} F_N} \mu(\{x : \varphi(\gamma x) > T_{\phi, \varepsilon}(| F_N |), \varphi(\overline{\gamma} x) > T_{\phi, \varepsilon}(| F_N |)\}) dm_\Gamma(\gamma) dm_\Gamma(\overline{\gamma}).$$

Let us consider a sequence $\{i_j\}$ such that

$$\mu(\{x : \varphi(\gamma x) > T_{\phi, \varepsilon}(| F_N |)\}) = \bigcup_{j=1}^{\infty} A_{i_j}.$$

Thus,

$$B_N = \int_{F_N} \int_{\gamma F_N} \mu(\{x : \varphi(\gamma x) > T_{\phi, \varepsilon}(| F_N |), \varphi(\overline{\gamma^{-1}\gamma} x) > T_{\phi, \varepsilon}(| F_N |)\}) dm_\Gamma(\gamma) dm_\Gamma(\overline{\gamma})$$

$$\leq \sum_{j, k=1}^{\infty} \int_{F_N} \int_{\gamma F_N} \mu(A_{i_j} \cap a(\overline{\gamma}) A_{j_k}) dm_\Gamma(\gamma) dm_\Gamma(\overline{\gamma})$$

$$\leq \sum_{j, k=1}^{\infty} \mu(A_{i_j} \cap a(\overline{\gamma}) A_{j_k})(g(| F_N |) + | F_N |)$$

$$\leq \mu(\{x : \varphi(\gamma x) > T_{\phi, \varepsilon}(| F_N |)\})^2 | F_N |^2.$$

Since $\varphi \in L^{\phi, w}$, we have

$$\mu(\{x : \varphi(\gamma x) > T_{\phi, \varepsilon}(| F_N |)\})^2 | F_N | \leq \frac{\sup\{\phi(t)\mu(\{x : | \varphi(x) | > t\})\}}{\log^{1/(2+\varepsilon)}(| F_N |)}.$$

Therefore, $\sum_{m=1}^{\infty} B_{2^m} < \infty$ and, by the Borel-Cantelli lemma, there are just finites m such that $\varphi(\gamma x) > T_{\phi, \varepsilon}(|F_{2^m}|)$, $\varphi(\bar{\gamma} x) > T_{\phi, \varepsilon}(|F_{2^m}|)$, for $\gamma \neq \bar{\gamma}$. Considering $2^m \leq N < 2^{m+1}$, it can be concluded that there are only finites N such that $m_\Gamma(\{\gamma \in F_N : \varphi(\gamma x) > T_{\phi, \varepsilon}(|F_N|)\}) > 0$ for a.e. $x \in X$. □

Each N-ergodic average can be separated in sets of points of X in which $\varphi \leq T_{\phi, \varepsilon}(|F_N|)$ and sets in which $\varphi > T_{\phi, \varepsilon}(|F_N|)$. For $\varphi \leq T_{\phi, \varepsilon}(|F_N|)$, Proposition 2 applies, whereas for $\varphi > T_{\phi, \varepsilon}(|F_N|)$, Lemma 1 applies, so that the theorem is proved.

References

[1] J. Aaronson and H. Nakada, Trimmed sums for non-negative mixing stationary processes, Stochastic Process. Appl. 104(2) (2003), 173-192.

[2] H. Diamond and J. D. Vaaler, Estimates for partial sums of continued fraction partial quotients, Pacific J. Math. 122(1) (1986), 73-82.

[3] A. Haynes, Quantitative ergodic theorems for weakly integrable functions, Ergodic Theory Dynam. Systems 34 (2014), 534-542.

[4] K. Fujiwara and A. Nevo, Maximal and pointwise ergodic theorems for word-hyperbolic groups, Ergodic Theory Dynam. Systems 18 (1998), 843-858.

[5] E. Lindenstrauss, Pointwise theorems for amenable groups, Invent. Math. 146 (2001), 259-295.

[6] T. Mori, Stability for sums of i.i.d. random variables when extreme terms are excluded from sums, Z. Wahrsch. Verw. Gebiete 40(2) (1977), 159-167.

[7] A. Nevo and E. M. Stein, A generalization of Birkhoff's pointwise ergodic theorem, Acta Math. 173 (1994), 135-154.

[8] A. Nevo, Pointwise ergodic theorems for actions of groups, Handbook of Dynamical Systems, Elsevier B.V., Amsterdam, Vol. 1B, 2006, pp. 871-982.