DISCRETE TIME SCHEMES FOR OPTIMAL CONTROL PROBLEMS WITH MONOTONE CONTROLS*

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Abstract. In this article we consider the Hamilton-Jacobi-Bellman (HJB) equation associated to the optimization problem with monotone controls. The problem deals with the infinite horizon case and costs with update coefficients.
We study the numerical solution through the discretization in time by finite differences. Without the classical semiconcavity-like assumptions, we prove that the convergence in this problem is of order h^γ in contrast with the order $h^{\frac{\gamma}{2}}$ valid for general control problems. This difference arises from the simple and precise way the monotone controls can be approximated. We illustrate the result on a simple example.

Key words. monotone optimal control problems, HJB equations, numerical solutions

AMS subject classifications. 49J15, 49M25

1. Introduction. Several important problems in economics can be formulated as control problems with monotone controls. Among them we can distinguish production inventory, cash management and adjustment theory of investment problems (see [3]). A particular importance is given to economics systems wich involve the exploitation of non renewable resources such as oil and minerals. Given the impossibility to produce such resources it is easy to see how these problems fit the monotone optimal control problems framework. Also, taking into account the big dificculty of replacing such resources by naturally sustainable ones, being the oil the most famous example, the analysis of these problems acquires critical importance.

In this work, we study a discrete time scheme for the numerical resolution of the infinite horizon monotone optimal control problem, through the analysis of the finite horizon problem, introducing two numerical schemes.

The considered system is governed by the following differential equation

$$\begin{cases} \dot{y}(s) = g(y(s), \alpha(s)), \quad s > 0, \\ y(0) = x, \end{cases} \tag{1.1}$$

where $\alpha(\cdot)$ is the control function, which throughout this article is restrained to be a non-decreasing monotone function defined in the set $[0, +\infty)$, with values in $[0, 1]$. For each $a \in [0, 1]$ let $\mathcal{A}(a)$ be the set of these functions with initial values higher or equal to a, i.e. $\alpha(0) \geq a$. At a point $s > 0$, the vector $y(s) \in \mathbb{R}^\nu$ is the state corresponding to the control $\alpha(\cdot)$ employed; $x \in \Omega$ is the initial state, with $\Omega \subset \mathbb{R}^\nu$ an open set. We assume that the evolution of the system always stays in Ω, no matter which control is selected.
The performance of the employed control is measured through the functional J

$$J(x, \alpha(\cdot)) := \int_0^\infty f(y(s), \alpha(s)) e^{-\lambda s} \mathrm{d}s,$$

where $f : \Omega \times [0, 1] \to \mathbb{R}$ is the instantaneous cost and $\lambda > 0$ is the discount factor. The problem consists in finding for each pair $(x, a) \in \Omega \times [0, 1]$, a control $\bar{\alpha} \in \mathcal{A}(a)$

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that attains the minimum of the functional J. Therefore, the value function u is

$$u(x,a) := \inf_{\alpha\in\mathcal{A}(a)} J(x,\alpha(\cdot)), \tag{1.2}$$

which allows to construct optimal or suboptimal policies in feedback [1, 2].
We assume the following Lipschitzian and boundeness hypotheses on the functions g and f: there exist positive constants L_g, M_g, L_f and M_f such that $\forall\, x,\bar{x}\in\Omega,\ \forall\, a,\bar{a}\in[0,1]$,

$$\|g(x,a)-g(\bar{x},\bar{a})\| \le L_g\left(\|x-\bar{x}\|+|a-\bar{a}|\right), \qquad \|g(x,a)\|\le M_g, \tag{1.3}$$

$$|f(x,a)-f(\bar{x},\bar{a})| \le L_f\left(\|x-\bar{x}\|+|a-\bar{a}|\right), \qquad |f(x,a)|\le M_f. \tag{1.4}$$

Under these assumptions, using classic arguments it can be proven that the function u is bounded and Hölder continuous in both variables.

We consider the quasi-variational HJB inequality associated to the problem, given by the equation

$$\min\left(Lu(x,a), \frac{\partial u(x,a)}{\partial a}\right) = 0, \qquad \text{in } \Omega\times(0,1), \tag{1.5}$$

where

$$Lu(x,a) = \frac{\partial u(x,a)}{\partial x} g(x,a) + f(x,a) - \lambda u(x,a),$$

and the boundary condition

$$u(x,1) = \int_0^\infty f(\eta(s),1)e^{-\lambda s}\mathrm{d}s, \tag{1.6}$$

where $\eta(s)$ is the trajectory corresponding to the control $\alpha\equiv 1$, and x is the initial value.
In general, this equation does not admit a solution in $C^1(\Omega\times(0,1))$, so the notion of viscosity solutions comes into play (see [7, 8, 4]). Specifically, u is the unique viscosity solution of the HJB equation (1.5) with boundary conditions (1.6) (see [1, 9]).

We would like to obtain discretization schemes in order to numerically solve the equation (1.5). For this aim, we introduce an auxiliary optimal control problem where the policies have the additional restriction of being uniformly step functions with values in a discrete, equi-spaced set. Specifically, the control variable a takes values in the set

$$I_h := \left\{ ih | i = 0 \dots \frac{1}{h} \right\}.$$

We also define $I_h(a) = I_h \cap [a,1]$. In the space $C(\Omega\times I_h)$ we consider the operators

$$\begin{aligned} \left(A_b^h(w)\right)(x,a) &= (1-\lambda h)w(x+hg(x,a),b) + hf(x,a), \\ \left(A^h(w)\right)(x,a) &= \min_{b\in I_h(a)} \left(A_b^h(w)\right)(x,a), \end{aligned} \tag{1.7}$$

arising from the discretization of the HJB equation.

Having considered a discretization in time, we introduce the consistent problem of finding in the functional space $C(\Omega \times I_h)$

$$\text{Problem } P^h : \text{ Find the fixed point } u^h \text{ of the operator } A^h \tag{1.8}$$

From (1.7) it turns out that A^h is a contractive operator if $0 < h < \frac{1}{\lambda}$. From this property it is straightforward to prove that (1.8) has an unique solution u^h, which is bounded and uniformly Hölder continuous in the first variable (see [5]). If (1.3) and (1.4) are satisfied, the same techniques give the Hölder continuity in both variables.

Furthermore, $\forall x \in \Omega, a \in I_h$ (see [5] and [6]) it can be proven that

$$u^h(x,a) = \min_{\alpha \in \mathcal{A}^h(a)} J^h(x,\alpha), \tag{1.9}$$

where $\mathcal{A}^h(a)$ is the subset of $\mathcal{A}(a)$ of the controls $\alpha(\cdot)$ that have constant values (such values belong to I_h) in the interval $(\xi h, (\xi+1)h]$, $\xi = 0, 1, \ldots$ and $\alpha(0) = a$; being also

$$J^h(x,\alpha) = h \sum_{\xi=0}^{\infty} f(y_h(\xi), \alpha(\xi h))(1-\lambda h)^{\xi},$$

and the sequence $y_h(\xi)$ is given by the following recursive formula

$$\begin{cases} y_h(\xi+1) &= y_h(\xi) + hf(y_h(\xi), \alpha(\xi h)), \quad \xi = 0, 1, \ldots \\ y_h(0) &= x. \end{cases}$$

The main result of this paper is the h^γ-order convergence of the infinite horizon problem discretization scheme, where γ is the Hölder constant of the value function u, and is state in Theorem 4.3.

The paper is organized as follows. In section 2 we introduce the associated finite horizon problem and its discretization in time. In section 3 we state and prove some useful properties of the discrete value functions and the convergence of the discretization scheme of the finite horizon problem. In section 4 we prove the main result of this paper, namely the convergence of the discretization scheme in the infinite horizon case. In section 5 we present a numerical example. Finally in section 6 we state the conclussions and discuss future work.

2. The associated finite horizon problem. With the purpose of obtaining some technical results on convergence, we consider a similar problem whose main difference is that it deals with finite horizon. The problem consists in finding the value function u_T, defined in the following way: $\forall t \in [0,T]$, $\forall x \in \Omega$, $\forall a \in [0,1]$,

$$u_T(t,x,a) := \inf_{a \in \mathcal{A}_T(a)} J_T(t,x,\alpha(\cdot)), \tag{2.1}$$

where $\mathcal{A}_T(a)$ is the set of non-decreasing functions defined in $[t,T)$ with values in $[a,1]$, and the functional J_T is defined by

$$J_T(t,x,\alpha(\cdot)) := \int_t^T f(y(s),\alpha(s)) e^{-\lambda(s-t)} \mathrm{d}s \tag{2.2}$$

with $y(\cdot)$ being the solution of (1.1) with initial condition $y(t) = x$.
If conditions (1.3) and (1.4) hold, then

$$|u_T(t,x,a)| \leq \frac{M_f}{\lambda}(1 - e^{\lambda t})$$

and

$$|u_T(t,x,a) - u_T(t,\bar{x},\bar{a})| \leq L_T\left(\|x - \bar{x}\| + |a - \bar{a}|\right)$$

where $L_T = \frac{L_f}{\lambda - L_g}$ if $\lambda > L_g$, $L_T = \frac{L_f}{L_g - \lambda}e^{(L_g - \lambda)T}$ if $\lambda < L_g$, and $L_T = TL_f$ if $\lambda = L_g$.
Also, there exists a positive constant M_T such that

$$|u_T(t,x,a) - u_T(\bar{t},x,a)| \leq M_T|t - \bar{t}|.$$

The proof of these properties makes use of classic techniques.
For the finite horizon case, the associated HJB equation takes the form

$$\min\left(Lu_T(t,x,a), \frac{\partial u_T(t,x,a)}{\partial a}\right) = 0 \qquad \text{in } (0,T)\times\Omega\times(0,1), \tag{2.3}$$

where

$$Lu_T = \frac{\partial u_T}{\partial t} + \frac{\partial u_T}{\partial x}g + f - \lambda u_T,$$

with final condition

$$u_T(T,x,a) = 0, \qquad \forall (x,a) \in \Omega \times [0,1],$$

and boundary condition

$$u_T(t,x,1) = \int_t^T f(\eta(s),1)e^{-\lambda(s-t)}\mathrm{d}s, \tag{2.4}$$

where $\eta(s)$ is the trajectory with initial value $\eta(t) = x$ corresponding to the control $\alpha \equiv 1$.

The viscosity solution of the equation (2.3) is defined similarly to the infinite horizon case (see [7] and [8]), proving that the value function u_T is the unique viscosity solution.
A different approach to the finite horizon problem with monotone controls has been made in [10] establishing a Pontriagyn Maximum Principle.

In a similar way to the infinite horizon problem, we consider for the finite horizon case a discretization in time. Let $h > 0$ and I_h, $I_h(a)$ as in the infinite horizon case.

Remark 2.1. *In what follows, we consider that h^{-1} is an integer and that the horizon $T = \mu h$, with μ an integer.* In order to find the solution of (2.3) we employ recursive approximation schemes. A natural discretization (by finite differences) of the HJB equation is given in the following recursive formulation.

Scheme 1.

$$\begin{cases} \hat{u}_T^h(\mu,x,a) &= 0, \qquad \text{for } x \in \Omega,\ a \in I_h, \\ \hat{u}_T^h(n-1,x,a) &= \min\left\{(1-\lambda h)\hat{u}_T^h(n, x+hg(x,a),a) + hf(x,a); \min\limits_{b\in I_h}\hat{u}_T^h(n-1,x,b)\right\}, \\ & \qquad\qquad\qquad \text{for } n = 1,\dots,\mu,\ x \in \Omega, a \in I_h \cap (0,1), \\ \hat{u}_T^h(n-1,x,1) &= (1-\lambda h)\hat{u}_T^h(n,x+hg(x,1),1) + hf(x,1),\ \text{for } n = 1,\dots,\mu,\ x\in\Omega. \end{cases}$$

It can be proven that

$$\hat{u}_T^h(n,x,a) = \min_{\alpha \in \mathcal{A}_T((a)} \hat{J}^h(n,x,a),$$

where

$$\hat{J}^h(n,x,a) = h \sum_{\xi=n}^{\mu-1} f(\hat{y}_h(\xi), \alpha((\xi+1)h))(1-\lambda h)^{\xi-n},$$

and the sequence $\hat{y}_h(\xi)$ is given by

$$\begin{cases} \hat{y}_h(\xi+1) &= \hat{y}_h(\xi) + hg(\hat{y}_h(\xi), \alpha((\xi+1)h)), \qquad \xi = n, \dots, (\mu-1), \\ \hat{y}_h(n) &= x. \end{cases}$$

Following [5, 6], we can define a simpler scheme as follows.

Scheme 2.

$$\begin{cases} u_T^h(\mu,x,a) &= 0. \quad \forall x \in \Omega, \, \forall a \in I_h, \\ u_T^h(n-1,x,a) &= \min\limits_{b \in I_h(a)} \left\{(1-\lambda h) u_T^h(n, x+hg(x,a), b) + hf(x,a)\right\}, \\ & \qquad\qquad\qquad \text{for } n = 1, \dots, \mu. \end{cases} \tag{2.5}$$

It is clear that u_T^h is the solution of the following optimization problem

$$u_T^h(n,x,a) = \min_{\alpha \in \mathcal{A}_T^h(a)} J_T^h(n,x,\alpha), \tag{2.6}$$

where $\mathcal{A}_T^h(a)$ is the set of restrictions of the controls $\alpha(\cdot) \in \mathcal{A}^h(a)$ to the interval $[0,T]$,

$$J_T^h(n,x,\alpha) = h \sum_{\xi=0}^{\mu-1} f(y_h(\xi), \alpha(\xi h))(1-\lambda h)^{\xi-n},$$

and the sequence $y_h(\xi)$ is given by the following recursive formula

$$\begin{cases} y_h(\xi+1) &= y_h(\xi) + hg(y_h(\xi), \alpha(\xi h)), \qquad \xi = n, \dots, (\mu-1), \\ y_h(n) &= x. \end{cases} \tag{2.7}$$

Next theorem shows that both schemes are equivalent, in the sense that their difference is of order h.

Theorem 2.2. *The difference between the solutions of the two schemes can be bounded in the following way*

$$\left| u_T^h(n,x,a) - \hat{u}_T^h(n,x,a) \right| \leq L_f e^{L_g(T-nh)} h$$

Proof. Let α and $\hat{\alpha}$ the optimal controls for u_T^h and $\hat{u}_T^h$ respectively, then (since $0 < h < \frac{1}{\lambda}$),

$$\begin{aligned}
u_T^h(n,x,a) - \hat{u}_T^h(n,x,a) &\leq J^h(n,x,\hat{\alpha}) - \hat{J}^h(n,x,\hat{\alpha}) \\
&= h\sum_{\xi=n}^{\mu-1} \left(f(y_h(\xi),\hat{\alpha}(\xi h)) - f(\hat{y}_h(\xi),\hat{\alpha}((\xi+1)h))\right)(1-\lambda h)^{\xi-n} \\
&\leq h\sum_{\xi=n}^{\mu-1} |f(y_h(\xi),\hat{\alpha}(\xi h)) - f(\hat{y}_h(\xi),\hat{\alpha}((\xi+1)h))| \\
&\leq h\sum_{\xi=n}^{\mu-1} L_f\left(\|y_h(\xi)-\hat{y}_h(\xi)\| + |\hat{\alpha}(\xi h) - \hat{\alpha}((\xi+1)h)|\right) \\
&\leq hL_f\left(1 + \sum_{\xi=n}^{\mu-1}\|y_h(\xi)-\hat{y}_h(\xi)\|\right). \qquad (2.8)
\end{aligned}$$

since, from the monotonicity assumption on the controls, it holds that

$$\sum_{\xi=n}^{\mu} |\hat{\alpha}(\xi h) - \hat{\alpha}((\xi+1)h)| \leq 1.$$

We see now that

$$\begin{aligned}
&\|y_h(\xi+1) - \hat{y}_h(\xi+1)\| \\
&\qquad \leq \|y_h(\xi)-\hat{y}_h(\xi)\| + h\|g(y_h(\xi),\hat{\alpha}(\xi h)) - g(\hat{y}_h(\xi),\hat{\alpha}((\xi+1)h))\| \\
&\qquad \leq \|y_h(\xi)-\hat{y}_h(\xi)\| + hL_g\left(\|y_h(\xi)-\hat{y}_h(\xi)\| + |\hat{\alpha}(\xi h) - \hat{\alpha}((\xi+1)h)|\right).
\end{aligned}$$

Therefore, if we write $\Delta_{\xi+1} = \|y_h(\xi+1) - \hat{y}(\xi+1)\|$, we have $\Delta_n = 0$ and recursively:

$$\Delta_{\xi+1} \leq \Delta_\xi(1+L_g h) + hL_g|\hat{\alpha}(\xi h) - \hat{\alpha}((\xi+1)h)|.$$

Hence,

$$\Delta_{\xi+1} \leq hL_g\sum_{j=n}^{\xi}(1+L_g h)^{\xi-j}|\hat{\alpha}(jh) - \hat{\alpha}((j+1)h)| \leq hL_g(1+L_g h)^{\xi+1-n}.$$

Replacing in (2.8), we obtain

$$\begin{aligned}
u_T^h(n,x,a) - \hat{u}_T^h(n,x,a) &\leq hL_f\left(1 + \sum_{\xi=n}^{\mu-1} hL_g(1+L_g h)^{\xi-n}\right) \\
&= hL_f(1+L_g h)^{\mu-n} \leq hL_f e^{L_g(T-nh)}.
\end{aligned}$$

In an analogous way, taking the control α, we obtain

$$\hat{u}_T^h(n,x,a) - u_T^h(n,x,a) \leq hL_f e^{L_g(T-nh)}$$

and the result follows.□

3. Properties of the value functions and convergence in the finite horizon case.

3.1. Properties of the function u_T^h. PROPOSITION 3.1. *u_T^h is bounded and Lipschitz continuous. Specifically, there exist $L_n^x > 0$, $L_n^a > 0$ such that, for $h \in (0, \frac{1}{\lambda})$, $x, \bar{x} \in \Omega$, and $a, \bar{a} \in [0,1] \cap I_h$, the function u_T^h satisfies*

$$\begin{aligned}\left|u_T^h(n,x,a) - u_T^h(n,\bar{x},a)\right| &\le L_n^x \|x - \bar{x}\|,\\ \left|u_T^h(n,x,a) - u_T^h(n,x,\bar{a})\right| &\le L_n^a |a - \bar{a}|\end{aligned}$$

and

$$\left|u_T^h(n,x,a)\right| \le M_f \left(\frac{1-(1-\lambda h)^{\mu-n}}{\lambda}\right), \tag{3.1}$$

where

$$L_n^x \le \begin{cases} L_f \frac{1}{L_g - \lambda} e^{(L_g-\lambda)(T-nh)} & \text{if} \quad L_g > \lambda,\\ L_f \frac{1}{\lambda - L_g} & \text{if} \quad L_g < \lambda,\\ L_f(T-nh) & \text{if} \quad L_g = \lambda \end{cases}$$

and

$$L_n^a \le \frac{L_{n+1}^x L_g + L_f}{\lambda}.$$

Proof. The proof of these properties uses classic comparison techniques. Inequality (3.1) is straightforward from (1.4), (2.6) and (2.7). It is evident that for $n = \mu$ we have $L_\mu = 0$, since by definition $u_T^h(\mu, x, a) \equiv 0$.
In order to complete the induction procedure, we consider the following inequality

$$\begin{aligned}u_T^h(n,x,a) - u_T^h(n,\bar{x},a) \le{}& (1-\lambda h) u_T^h(n+1, x + hg(x,a), b) + hf(x,b)\\ &- \big((1-\lambda h) u_T^h(n+1, \bar{x} + hg(\bar{x},b), b) + hf(\bar{x},b)\big),\end{aligned}$$

where b makes the minimum in (2.5) for $u_T^h(n,\bar{x},a)$. Then

$$u_T^h(n,x,a) - u_T^h(n,\bar{x},a) \le \big((1-\lambda h)(1+L_g h) L_{n+1}^x + L_f h\big) \|x - \bar{x}\|.$$

In an analogous way we obtain an inequality for $u_T^h(n,\bar{x},a) - u_T^h(n,x,a)$, consequently

$$L_n^x \le (1-\lambda h)(1+L_g h) L_{n+1}^x + L_f h. \tag{3.2}$$

Now, if $L_g = \lambda$, $(1-\lambda h)(1+L_g h) = 1 - (\lambda h)^2 \le 1$, then

$$L_n^x \le L_{n+1}^x + L_f h,$$

which implies

$$L_n^x \le L_f(T-nh).$$

In the case that $L_g < \lambda$, from the formula (3.2) we have

$$L_n^x \le (1-(\lambda - L_g)h) L_{n+1}^x + L_f h,$$

then

$$L_n^x \le L_f h \frac{1-(1-(\lambda-L_g)h)^{\mu-n}}{h(\lambda-L_g)} \le L_f \frac{1}{\lambda-L_g}.$$

Finally, if $L_g > \lambda$, the inequality (3.2) gives

$$L_n^x \le L_f h \frac{(1+(L_g-\lambda)h)^{\mu-n}}{h(L_g-\lambda)},$$

therefore

$$L_n^x \le L_f \frac{1}{L_g-\lambda} e^{(L_g-\lambda)(T-nh)}.$$

We consider now $a, \bar{a} \in [0,1] \cap I_h, a > \bar{a}$. Note that $I_h(a) \subset I_h(\bar{a})$ and $b \in I_h(\bar{a})$ implies that $\max(a,b) \in I_h(a)$. Hence,

$$\begin{aligned} u_T^h(n,x,\bar{a}) - u_T^h(n,x,a) \le & (1-\lambda h) u_T^h(n+1, x+hg(x,\bar{a}), b) + hf(x,\bar{a}) \\ & - \left((1-\lambda h) u_T^h(n+1, x+hg(x,a), b) + hf(x,a)\right), \end{aligned}$$

where b makes the minimum for $u_T^h(n,x,a)$, and

$$\begin{aligned} u_T^h(n,x,a) - u_T^h(n,x,\bar{a}) \le & (1-\lambda h) u_T^h(n+1, x+hg(x,a), \max(a,\bar{b})) + hf(x,a) \\ & - \left((1-\lambda h) u_T^h(n+1, x+hg(x,\bar{a}), \bar{b}) + hf(x,\bar{a})\right), \end{aligned}$$

where $\bar{b}$ makes the minimum for $u_T^h(n,x,\bar{a})$. Then

$$|u_T^h(n,x,a) - u_T^h(n,x,\bar{a})| \le \left((1-\lambda h)\left(L_{n+1}^x L_g h + L_{n+1}^\alpha\right) + L_f h\right) |a-\bar{a}|,$$

so

$$L_n^a \le (1-\lambda h)\left(L_{n+1}^x L_g h + L_{n+1}^\alpha\right) + L_f h.$$

From the last statement we conclude that

$$L_n^a \le \frac{L_{n+1}^x L_g + L_f}{\lambda}.$$

□

3.2. The convergence in the case of finite horizon. Definition 3.2. *Given a control $\alpha(s)$ and a partition of the interval $[0,T]$ in μ subintervals of length $h = \frac{T}{\mu}$, we define its discretization and we will call it $\alpha^h \in \mathcal{A}_T^h(a)$*

$$\alpha^h(s) := \left[\frac{\alpha([\frac{s}{h}]h)}{h}\right] h,$$

where $[\cdot]$ represents the integer part.

Lemma 3.3. *Let $y(\cdot)$ the trajectory associated to the control $\alpha(\cdot)$. If α^h is the discretization of α and y^h its response, then*

$$\|y(t) - y^h(t)\| \le 2e^{L_g t} h.$$

Proof.

$$\begin{aligned}
\|y(t) - y^h(t)\| &\leq \int_0^t \|g(y(s), \alpha(s)) - g(y^h(s), \alpha^h(s))\| \mathrm{d}s \\
&\leq L_g \int_0^t \left[\|y(s) - y^h(s)\| + |\alpha(s) - \alpha^h(s)|\right] \mathrm{d}s \\
&\leq L_g ht + \int_0^t L_g \|y(s) - y^h(s)\| \mathrm{d}s
\end{aligned}$$

Then, by the Gronwäll inequality,

$$\|y(t) - y^h(t)\| \leq L_g ht + \int_0^t L_g^2 hse^{L_g(t-s)} \mathrm{d}s \leq \left(L_g t + e^{L_g t}\right) h \leq 2e^{L_g t} h.$$

□

REMARK 3.4. *In a similiar way, an application of the Gronwäll inequality yields, for some $K > 0$,*

$$\|y(t) - y_h([t/h])\| \leq Ke^{L_g t} h, \tag{3.3}$$

where y is the response of a control $\alpha \in \mathcal{A}_T^h(a)$ and y_h is given by (2.7), with the same control. The following theorem gives us a bound for the discretization in time of the problem with finite horizon.

THEOREM 3.5.

$$\left|u_T(nh, x, a) - u_T^h(n, x, a)\right| \leq C\phi(n)h, \tag{3.4}$$

where $\phi(n)$ is defined by

$$\phi(n) = \begin{cases} e^{(L_g - \lambda)T + \lambda nh} & \text{if } L_g > \lambda, \\ Te^{L_g nh} & \text{if } L_g = \lambda, \\ e^{L_g nh} & \text{if } L_g < \lambda \end{cases}$$

and $C > 0$ is a constant independent of h.

Proof. For J_T defined in (2.2), let

$$u^e(n, x, a) = \min_{\alpha^h \in \mathcal{A}_T^h(a)} J_T(nh, x, \alpha^h)$$

In consequence, u^e is the optimal cost for the original continuous problem when the controls are restrained to be step functions (that is they belong to $\mathcal{A}_T^h(a)$). By the triangle inequality,

$$\left|u_T(nh, x, a) - u_T^h(n, x, a)\right| \leq |u_T(nh, x, a) - u^e(n, x, a)| + \left|u^e(n, x, a) - u_T^h(n, x, a)\right|. \tag{3.5}$$

Observe that

$$\begin{aligned}\left|u^e(n,x,a)-u_T^h(n,x,a)\right| &= \left|\min_{\alpha\in\mathcal{A}_T^h(a)} J_T(nh,x,\alpha)-\min_{\alpha\in\mathcal{A}_T^h(a)} J_T^h(n,x,\alpha)\right| \\ &\le \sup_{\alpha\in\mathcal{A}_T^h(a)} \left|J_T(nh,x,\alpha)-J_T^h(n,x,\alpha)\right|. \end{aligned} \tag{3.6}$$

For any $\alpha \in \mathcal{A}_T^h(a)$, definining $\zeta = -(\lambda h)^{-1}\log(1-\lambda h)$, we obtain

$$\begin{aligned}|J_T(nh,x,\alpha)-J_T^h(n,x,\alpha)| &\le \int_{nh}^{T} \left|f(y(s),\alpha(s))e^{-\lambda(s-nh)} - f(y_h([\frac{s}{h}]),\alpha([\frac{s}{h}]h))e^{-\zeta\lambda h([\frac{s}{h}]-n)}\right| \mathrm{d}s \\ &\le L_f \int_{nh}^{T} \left|y(s)-y_h([\frac{s}{h}])\right| e^{-\lambda(s-nh)}\mathrm{d}s + M_f \int_{nh}^{T} \left|e^{-\lambda(s-nh)} - e^{-\zeta\lambda h([\frac{s}{h}]-n)}\right| \mathrm{d}s. \end{aligned} \tag{3.7}$$

From (3.3),

$$L_f \int_{nh}^{T} \left|y(s)-y_h([\frac{s}{h}])\right| e^{-\lambda(s-nh)}\mathrm{d}s \le L_f K h \int_{nh}^{T} e^{(Lg-\lambda)s+\lambda nh}\mathrm{d}s. \tag{3.8}$$

Following [6], is not hard to prove that, for some $\bar{K}>0$,

$$M_f \int_{nh}^{T} \left|e^{-\lambda(s-nh)} - e^{-\zeta\lambda h([\frac{s}{h}]-n)}\right| \mathrm{d}s \le \bar{K}h. \tag{3.9}$$

On the other hand, in order to estimate the firs term in (3.5), let α be the optimal control for u_T and α^h its discretization. Since α^h is a step control, we have

$$\begin{aligned} 0 &\le u^e(n,x,a) - u_T(nh,x,a) \le J_T(n,x,\alpha^h) - J_T(nh,x,\alpha) \\ &= \int_{nh}^{T} f(y^h(s),\alpha^h(s))e^{-\lambda(s-nh)}\mathrm{d}s - \int_{nh}^{T} f(y(s),\alpha(s))e^{-\lambda(s-nh)}\mathrm{d}s \\ &\le \int_{nh}^{T} \left|f(y^h(s),\alpha^h(s)) - f(y(s),\alpha(s))\right| e^{-\lambda(s-nh)}\mathrm{d}s \\ &\le L_f \int_{nh}^{T} \left(|y^h(s)-y(s)| + |\alpha(s)-\alpha^h(s)|\right) e^{-\lambda(s-nh)}\mathrm{d}s \\ &\le L_f \left(\int_{nh}^{T} 2he^{(L_g-\lambda)s+\lambda nh}\mathrm{d}s + \frac{h}{\lambda}\right). \end{aligned} \tag{3.10}$$

Therefore, from (3.5), (3.7), (3.8), (3.9) and (3.10), defining $K_1 = L_f(K+2)$ and $K_2 = \bar{K} + L_f/\lambda$, we obtain

$$\left|u_T(nh,x,a) - u_T^h(n,x,a)\right| \le \left(K_1 \int_{nh}^{T} e^{(L_g-\lambda)s+\lambda nh}\mathrm{d}s + K_2\right) h.$$

Now we analize the different cases, where a large enough scalar C is considered. For $\lambda > L_g$,

$$\left(K_1\int_{nh}^{T} 2e^{(L_g-\lambda)s+\lambda nh}\mathrm{d}s + K_2\right) h = \left(K_1\frac{e^{L_g nh} - e^{(L_g-\lambda)T+\lambda nh}}{\lambda - L_g} + K_2\right) h$$
$$\leq \left(K_1\frac{e^{L_g nh}}{\lambda - L_g} + K_2\right) \leq Ce^{L_g nh}h.$$

For $\lambda < L_g$,

$$\left(K_1\int_{nh}^{T} 2e^{(L_g-\lambda)s+\lambda nh}\mathrm{d}s + K_2\right) h \leq \left(K_1\frac{e^{(L_g-\lambda)T+\lambda nh} - e^{L_g nh}}{L_g - \lambda} + K_2\right) h$$
$$\leq \left(K_1\frac{e^{(L_g-\lambda)T+\lambda nh}}{L_g - \lambda} + K_2\right) h \leq Ce^{(L_g-\lambda)T+\lambda nh}h.$$

For $\lambda = L_g$,

$$\left(K_1\int_{nh}^{T} 2e^{(L_g-\lambda)s+\lambda nh}\mathrm{d}s + K_2\right) h = \left(K_1 e^{\lambda nh}(T-nh) + K_2\right) h \leq Ce^{\lambda nh}Th.$$

□

4. Convergence in the case of infinite horizon. The procedure to get a bound in the convergence in this case is based mainly in the result obtained for the case of finite horizon and in the following lemmas.

Lemma 4.1. *Under the hypotheses* (1.3) *and* (1.4), *we have*

$$|u(x,a) - u_T(t,x,a)| \leq \frac{M_f}{\lambda}e^{-\lambda(T-t)}. \tag{4.1}$$

Lemma 4.2. *Under the hypotheses* (1.3) *y* (1.4), *we have*

$$|u^h(x,a) - u_T^h(n,x,a)| \leq \frac{M_f}{\lambda}e^{-\lambda(T-nh)}. \tag{4.2}$$

The proofs are straightforward from definitions (1.2), (2.1), (1.9), (2.5). They are essentially identical and employ simple analysis reasonings.

Theorem 4.3. *The following estimate of the discretization error in time holds*

$$|u(x,a) - u^h(x,a)| \leq \mathcal{C}h^{\gamma}, \tag{4.3}$$

where

$$\gamma = \begin{cases} 1 & \textit{if } \lambda > L_g \\ \dfrac{\lambda}{L_g} & \textit{if } \lambda < L_g \\ \in (0,1) & \textit{if } \lambda = L_g. \end{cases}$$

Proof. We make the following decomposition

$$|u(x,a)-u^h(x,a)| \le |u(x,a)-u_T(0,x,a)|+|u_T(0,x,a)-u_T^h(0,x,a)|+|u_T^h(0,x,a)-u^h(x,a)|.$$

By Lemma 4.1, we have

$$|u(x,a) - u_T(0,x,a)| \le \frac{M_f}{\lambda} e^{-\lambda T}$$

and by Theorem 3.5,

$$|u_T(0,x,a) - u_T^h(0,x,a)| \le \bar{M}\phi(0)h.$$

Finally, since by Lemma 4.2 we have

$$|u_T^h(0,x,a) - u^h(x,a)| \le \frac{M_f}{\lambda} e^{-\lambda T},$$

we then obtain

$$|u(x,a) - u^h(x,a)| \le \bar{M}\phi(0)h + \frac{2M_f}{\lambda} e^{-\lambda T}. \tag{4.4}$$

In order to prove (4.3), we study the different cases. If $L_g > \lambda$, by (4.4),

$$|u(x,a) - u^h(x,a)| \le M_1 \left(e^{-\lambda T} + h e^{(L_g - \lambda)T} \right). \tag{4.5}$$

The expression (4.5) has a minimum given by

$$T = \frac{1}{L_g} \log \left(\frac{\lambda}{L_g - \lambda} h^{-1} \right),$$

then, replacing in (4.5) and calling $\gamma = \frac{\lambda}{L_g}$, we obtain

$$|u^h(x,a) - u(x,a)| \le M_1 K_1 h^\gamma + M_1 K_1 h h^{\gamma - 1} = 2M_1 K_1 h^\gamma, \tag{4.6}$$

where

$$K_1 = \max \left\{ (\frac{\gamma}{1-\gamma})^{1-\gamma}, (\frac{1-\gamma}{\gamma})^{\gamma} \right\}.$$

From (4.6) we have

$$|u(x,a) - u^h(x,a)| \le \mathcal{C} h^\gamma.$$

In the case that $L_g < \lambda$, we have from (4.4)

$$|u(x,a) - u^h(x,a)| \le \frac{2M_f}{\lambda} e^{-\lambda T} + \bar{M}h$$

and passing to the limit it turns out that

$$|u(x,a) - u^h(x,a)| \le \bar{M}h.$$

Finally, for $L_g = \lambda$, inequality (4.4) gives

$$|u(x,a) - u^h(x,a)| \leq \bar{M}Th + \frac{2M_f}{\lambda} e^{-\lambda T} \leq M_2(e^{-\lambda T} + hT). \tag{4.7}$$

The minimum of the expression (4.7) is given by $T = -\frac{1}{\lambda}\log(\frac{h}{\lambda})$, if $h < \lambda$, and then (4.7) becomes

$$|u(x,a) - u^h(x,a)| \leq -M_2 \frac{h}{\lambda} \log(\frac{h}{\lambda}) + M_2 \frac{h}{\lambda},$$

that can be transformed into the form

$$|u(x,a) - u^h(x,a)| \leq \mathcal{C} h^\gamma$$

with $\gamma \in (0,1)$ and where $\mathcal{C}$ depends on γ ($\mathcal{C} \to +\infty$ when $\gamma \to 1$).
□

Remark 4.4. *The estimate* (4.3) *has been reached in this problem with monotone controls without having used semiconcavity-like hypotheses (see [6, Condition 4.9]).*

5. Example. We consider the infinite horizon monotone optimal control problem. The controlled dynamic is:

$$\begin{cases} \dot{y}(s) = g(y(s), \alpha(s)), & s > 0, \\ y(0) = x \end{cases} \tag{5.1}$$

where $\mathcal{A}(a)$ is defined as in the general case and $\Omega = (1, +\infty)$. The cost to minimize is given by

$$u(x,a) = \inf_{\alpha \in \mathcal{A}(a)} \int_0^\infty f(y(s), \alpha(s)) e^{-3s} \mathrm{d}s.$$

Here

$$g(x,a) = f(x,a) := \begin{cases} xa & x < e \\ ea & x \geq e \end{cases}$$

which clearly verify the general hypotheses. Here, the discount factor is $\lambda = 3 > e$ and the Lipschitz continuity constant of g is e.

Remark 5.1. *Note that function g does not verify the sufficient condition for [6, Condition 4.9] to be satisfied (see [6, Lemma 4.3]).* Let us calculate the value function. Let $(x,a) \in \Omega \times [0,1]$ and $\alpha \in \mathcal{A}(a)$. We name y_a the trajectory associated to the constant control a. Since f and g are a non-negative increasing functions w.r. to both variables we have

$$\int_0^\infty f(y(s), \alpha(s)) e^{-3s} \mathrm{d}s \geq \int_0^\infty f(y_a(s), a) e^{-3s} \mathrm{d}s,$$

since by direct comparison of its derivatives we conclude that $y(s) \geq y_a(s)$ and by monotonicity of the controls, $\alpha(s) \geq a$ for all $s > 0$. Being the constant control a admissible we conclude that

$$u(x,a) = \int_0^\infty f(y_a(s), a) e^{-3s} \mathrm{d}s.$$

If $a = 0$, then $u(x,a) = 0$. If not, we will distinguish two cases.

1. If $x < e$:
$$u(x,a) = \int_0^\infty f(y_a(s),a)e^{-3s}\mathrm{d}s = a\left[\int_0^\beta y_a(s)e^{-3s}\mathrm{d}s + \int_\beta^\infty ee^{-3s}\mathrm{d}s\right] \tag{5.2}$$
where β is such that $y_a(\beta) = e$. We compute $y_a(s)$ and β. The solution to the dynamic
$$\begin{cases} \dot{y}(s) = ay(s), & s \in [0,\beta], \\ y(0) = x, \end{cases}$$
is $y_a(s) = xe^{as}$ and so
$$y_a(\beta) = xe^{a\beta} = e \Rightarrow \beta = \frac{1-\log x}{a}.$$
The first addend of (5.2) is then
$$\int_0^\beta xe^{as}e^{-3s}\mathrm{d}s = \frac{x}{a-3}e^{(a-3)s}\Big|_0^\beta = \frac{x}{a-3}(e^{(a-3)\beta}-1) = \frac{x}{3-a} - \frac{e}{3-a}\left(\frac{e}{x}\right)^{-\frac{3}{a}},$$
and thesecond addend becomes
$$\int_\beta^\infty ee^{-2s}\mathrm{d}s = -\frac{e}{3}e^{-3s}\Big|_\beta^\infty = \frac{e}{3}e^{-3\beta} = \frac{e}{3}\left(\frac{e}{x}\right)^{-\frac{3}{a}}.$$
Therefore
$$u(x,a) = \frac{ax}{3-a} + a\left(\frac{e}{x}\right)^{-\frac{3}{a}}\left[\frac{3}{e} - \frac{3}{3-a}\right] = \frac{ax}{3-a} - \frac{a^2e}{3(3-a)}\left(\frac{e}{x}\right)^{-\frac{3}{a}}$$
2. If $x \geq e$ we have that
$$u(x,a) = \int_0^\infty aee^{-3s}\mathrm{d}s = \frac{ae}{3}.$$

We conclude that
$$u(x,a) = \begin{cases} 0 & \text{if } a = 0, \\ \dfrac{ax}{3-a} - \dfrac{a^2e}{3(3-a)}\left(\dfrac{e}{x}\right)^{-\frac{3}{a}} & \text{if } a > 0, x < e, \\ \dfrac{ea}{3} & \text{if } a > 0, x \geq e. \end{cases}$$
The continuity of $u(x,a)$ at $a = 0$ for any x and at $x = e$ for any $a > 0$ is straightforfard.

We consider now the time discretization. For $h > 0$, the discrete value function is
$$u^h(x,a) = \inf_{\alpha\in\mathcal{A}^h(a)} h\sum_{\xi=0}^\infty f(y_h(\xi),\alpha(\xi h))(1-3h)^\xi,$$
where the discrete trajectory is given by
$$\begin{cases} y_h(\xi+1) &=& y_h(\xi) + hg(y_h(\xi),\alpha(\xi h)), \\ y_h(0) &=& x. \end{cases}$$

Let $a \in I_h$ and α be a discrete control. We name y_h^a the trajectory associated to the constant discrete control a. We have

$$h\sum_{\xi=0}^{\infty} f(y_h(\xi),\alpha(\xi h))(1-3h)^{\xi} \geq h\sum_{\xi=0}^{\infty} f(y_h^a(\xi),a)(1-3h)^{\xi},$$

so

$$u^h(x,a) = h\sum_{\xi=0}^{\infty} f(y_h^a(\xi),a)(1-3h)^{\xi}. \tag{5.3}$$

We distinguish again two cases:

1. If $x < e$ then

$$h\sum_{\xi=0}^{\infty} f(y_h^a(\xi),a)(1-3h)^{\xi} = h\left\{\sum_{\xi=0}^{\hat{\xi}-1} y_h^a(\xi)a(1-3h)^{\xi} + \sum_{\xi=\hat{\xi}}^{\infty} ea(1-3h)^{\xi}\right\}, \tag{5.4}$$

where $\hat{\xi}$ is such that $y_h^a(\hat{\xi}-1) < e$ and $y_h^a(\hat{\xi}) \geq e$. By induction, it is easy to prove that

$$y_h^a(\xi) = x(1+ah)^{\xi}$$

for every $0 \leq \xi \leq \hat{\xi}$. Since the function $x(1+ah)^{\rho}$ is continuous and strictly increasing in ρ, there exists a unique $\rho \in (\hat{\xi}-1,\hat{\xi}]$ such that $x(1+ah)^{\rho} = e$, so

$$\rho = \frac{1-\log(x)}{\log(1+ah)} \Rightarrow \hat{\xi} = \left\lceil \frac{1-\log(x)}{\log(1+ah)} \right\rceil,$$

where $\lceil\cdot\rceil$ stands for the upper integer part. Hence,

$$(1+ah)^{\hat{\xi}} = \frac{e}{x}(1+ah)^{\hat{\xi}-\rho}. \tag{5.5}$$

The first addend of (5.4) is

$$\sum_{\xi=0}^{\hat{\xi}-1} y_h^a(\xi)a(1-3h)^{\xi} = \sum_{\xi=0}^{\hat{\xi}-1} xa(1+ah)^{\xi}(1-3h)^{\xi} = ax\frac{1-[(1+ah)(1-3h)]^{\hat{\xi}}}{1-(1+ah)(1-3h)},$$

while the second is

$$\sum_{\xi=\hat{\xi}}^{\infty} ea(1-3h)^{\xi} = \frac{ea}{3h}(1-3h)^{\hat{\xi}},$$

so, from (5.3) and (5.5),

$$\begin{aligned} u^h(x,a) &= \frac{ax}{3-a+3ah} - \frac{ax(1+ah)^{\hat{\xi}}(1-3h)^{\hat{\xi}}}{3-a+3ah} + \frac{ea}{3}(1-3h)^{\hat{\xi}} \\ &= \frac{ax}{3-a+3ah} + \frac{3[1-(1+ah)^{\hat{\xi}-\rho}]-a(1-3h)}{9-3a+9ah}ea(1-3h)^{\hat{\xi}} \end{aligned} \tag{5.6}$$

2. If $x \le e$, we have:

$$u^h(x,a) = hea\sum_{\xi=0}^{\infty}(1-3h)^{\xi} = hea\frac{1}{3h} = \frac{ae}{3}.$$

Therefore

$$u^h(x,a) = \begin{cases} 0 & \text{if } a = 0, \\ \dfrac{ax}{3-a+3ah} - \dfrac{3[(1+ah)^{\hat{\xi}-\rho}-1]+a(1-3h)}{9-3a+9ah}ea(1-3h)^{\hat{\xi}} & \text{if } a > 0, x < e, \\ \dfrac{ea}{3} & \text{if } a > 0, x \ge e. \end{cases}$$

It is clear that $u = u^h$ whenever $a = 0$ or $a > 0$ and $x \ge e$. We consider then $a > 0$ and $x < e$. It is easy to prove the following facts: as $h \longrightarrow 0^+$ we have, for $a \in (0,1]$,

$$0 \le \frac{ax}{3-a-3ah} - \frac{ax}{3-a} = \frac{3a^2x}{(3-a)^2-(3-a)(3ah)}h \le \frac{3x}{4}h, \tag{5.7}$$

$$0 \le (1+ah)^{\hat{\xi}-\rho} - 1 \le ah \le h, \qquad 1-3h \le (1-3h)^{\hat{\xi}-\rho} \le 1, \tag{5.8}$$

$$0 \le (1-3h)^{\hat{\xi}} \le 1, \qquad 0 \le 1-(1-3h)^{\hat{\xi}-\rho+1} \le 6h, \tag{5.9}$$

$$(1-3h)^{\rho} = (1-3h)^{\frac{1-\log x}{\log(1+ah)}} = \left(\frac{e}{x}\right)^{\frac{\log(1-3h)}{\log(1+ah)}} \longrightarrow \left(\frac{e}{x}\right)^{-\frac{3}{a}}, \tag{5.10}$$

where the expressions in (5.10) are bounded by 1 and vanish when $a \to 0$.

Now, from (5.7), (5.8), (5.9) and (5.10),

$$\begin{aligned} |u(x,a)-u^h(x,a)| &\le \frac{3x}{4}h + \frac{(1+ah)^{\hat{\xi}-\rho}-1}{3-a+3ah}ea(1-3h)^{\hat{\xi}} \\ &\quad + \left|\frac{a(1-3h)}{9-3a+9ah}ea(1-3h)^{\hat{\xi}} - \frac{a^2e}{3(3-a)}\left(\frac{e}{x}\right)^{-\frac{3}{a}}\right| \\ &\le \frac{3x}{4}h + \frac{ea}{3-a}h + \frac{ea^2(1-3h)^{\hat{\xi}-\rho+1}}{9-3a+9ah}\left|(1-3h)^{\rho} - \left(\frac{e}{x}\right)^{-\frac{3}{a}}\right| \\ &\quad + \frac{ea^2}{(3-a)(9-3a+3ah)}\left(\frac{e}{x}\right)^{-\frac{3}{a}}\left((3-a)\left|(1-3h)^{\hat{\xi}-\rho+1}-1\right| + |3ah|\right) \\ &\le \frac{3x}{4}h + \frac{e}{2}h + \frac{e}{6}\left|\left(\frac{e}{x}\right)^{\frac{\log(1-3h)}{\log(1+ah)}} - \left(\frac{e}{x}\right)^{-\frac{3}{a}}\right| + eh + \frac{e}{4}h \\ &= \left(\frac{3x}{4}+\frac{7e}{4}\right)h + \frac{e}{6}\left|\left(\frac{e}{x}\right)^{\frac{\log(1-3h)}{\log(1+ah)}} - \left(\frac{e}{x}\right)^{-\frac{3}{a}}\right|. \end{aligned} \tag{5.11}$$

By applying successively the L'Hopital rule, and using the fact that $y < e^y$ for every y, we obtain

$$\lim_{h\to 0^+}\frac{1}{h}\left(\left(\frac{e}{x}\right)^{-\frac{3}{a}} - \left(\frac{e}{x}\right)^{\frac{\log(1-3h)}{\log(1+ah)}}\right) = \frac{3+a}{2}\frac{3}{a}\left(\frac{e}{x}\right)^{-\frac{3}{a}} < \frac{2}{1-\log x},$$

so (5.11) gives

$$|u(x, a) - u^h(x, a)| \leq \mathcal{C}h,$$

for $\mathcal{C}$ large enough, independent of a, as expected.

6. Conclussions. For the monotone optimal control problem with infinite horizon, we considered a time discretization scheme. We defined two discrete time schemes for the auxiliar finite horizon monotone control problem and proved some useful properties of their solutions and their h-equivalence. Without semiconcavity-like assumptions we improved the convergence rate of the classical control problem seizing the good properties of monotone controls.

In the aim to obtain computational implementable algorithms, we shall consider on future works a fully discrete monotone control problem concerning the space-state variable discretization via finite element methods.

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