The Maximum Number of Dominating Induced Matchings

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Abstract: A matching M of a graph G is a dominating induced matching (DIM) of G if every edge of G is either in M or adjacent with exactly one edge in M. We prove sharp upper bounds on the number $\mu(G)$ of DIMs of a graph G and characterize all extremal graphs. Our results imply that if G is a graph of order n, then $\mu(G) \leq 3^{\frac{n}{3}}$; $\mu(G) \leq 4^{\frac{n}{5}}$ provided G is triangle-free; and $\mu(G) \leq 4^{\frac{n-1}{5}}$ provided $n \geq 9$ and G is connected.
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1. INTRODUCTION

We consider finite, simple, and undirected graphs and use standard terminology. For a matching M of a graph G, let $V(M)$ denote the set of vertices of G incident with an edge in M. A matching M of G is a *dominating induced matching (DIM)* of G if every edge of G is either in M or adjacent with exactly one edge in M, that is, if $G[V(M)]$ is 1-regular and $V(G) \setminus V(M)$ is an independent set where $V(G)$ denotes the vertex set of G and for a set U of vertices of G, the subgraph of G induced by U is denoted by $G[U]$. For a graph G, let $\mu(G)$ denote the number of DIMs of G.

In this article, we give sharp upper bounds on the maximum possible value of $\mu(G)$ for a graph G that is either arbitrary, or triangle-free, or connected. Furthermore, we characterize all extremal graphs for our bounds.

The algorithmic questions related to DIMs have been studied in great detail. In [9], it is shown that $\mu(G)$ for a given arbitrary graph G of order n can be determined in time $O^*(1.1939^n)$. It is unlikely that there is a polynomial time algorithm computing $\mu(G)$, because it is already NP-complete to decide whether $\mu(G) = 0$ [7], even for planar bipartite graphs of maximum degree 3 [1], or regular graphs [4]. Dominating induced matchings have been the main subject of many recent papers [1–6,8,11,12]. Further studies about DIMs and some applications related to coding theory, network routing, and resource allocation can be found in [7,10].

Our results are as follows.

Theorem 1. *If G is a graph of order n, then $\mu(G) \leq f(n)$ where*

$$f(n) = \begin{cases} 1, & \text{if } n \leq 2, \\ 3^{\frac{n}{3}}, & \text{if } n \geq 3 \text{ and } n \equiv 0 \bmod 3, \\ 3^{\frac{n-1}{3}}, & \text{if } n \geq 4 \text{ and } n \equiv 1 \bmod 3, \text{ and} \\ 4 \cdot 3^{\frac{n-5}{3}}, & \text{if } n \geq 5 \text{ and } n \equiv 2 \bmod 3. \end{cases}$$

Furthermore, if the graph G of order n with $n \geq 3$ is such that $\mu(G) = f(n)$, then $G \in \mathcal{F}$ where

$$\mathcal{F} = \left\{\frac{n}{3}K_3 : n \geq 3 \text{ and } n \equiv 0 \bmod 3\right\}$$
$$\cup \left\{K_1 \cup \frac{n-1}{3}K_3 : n \geq 4 \text{ and } n \equiv 1 \bmod 3\right\}$$

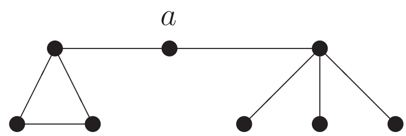

FIGURE 1. The graph H_8.

$$\cup \left\{K_{1,3} \cup \frac{n-4}{3}K_3 : n \geq 4 \textit{ and } n \equiv 1 \textit{ mod } 3\right\}$$
$$\cup \left\{K_{1,4} \cup \frac{n-5}{3}K_3 : n \geq 5 \textit{ and } n \equiv 2 \textit{ mod } 3\right\}.$$

Theorem 2. *If G is a triangle-free graph of order n, then $\mu(G) \leq g(n)$ where*

$$g(n) = \begin{cases} 1, & \textit{if } n = 1, \\ n-1, & \textit{if } n \in \{2,3,6,7\}, \\ 20, & \textit{if } n = 11, \textit{ and} \\ 3^t \cdot 4^{\frac{n-4t}{5}}, & \textit{if } n \geq 4t \textit{ and } n \equiv -t \textit{ mod } 5 \textit{ for some } t \in \{0,1,2,3,4\}. \end{cases}$$

Furthermore, if the triangle-free graph G of order n with $n \geq 2$ is such that $\mu(G) = g(n)$, then $G \in \mathcal{G}$ where

$$\mathcal{G} = \{K_{1,n-1} : 2 \leq n \leq 7\} \cup \{K_{1,2} \cup K_{1,3}, K_{1,4} \cup K_{1,5}\}$$
$$\cup \left\{tK_{1,3} \cup \frac{n-4t}{5}K_{1,4} : n \geq 4t \textit{ and } n \equiv -t \textit{ mod } 5 \textit{ for some } t \in \{0,1,2,3,4\}\right\}.$$

For an integer n with $n \geq 11$ and $n \equiv 1 \bmod 5$, let the graph H_n arise from $K_1 \cup \frac{n-1}{5}K_{1,4}$ by adding edges between the vertex of the K_1 and each center of the $\frac{n-1}{5}$ stars.

Let the graph H_8 of order 8 be as shown in Figure 1.

Theorem 3. *If G is a connected graph of order n, then $\mu(G) \leq h(n)$ where*

$$h(n) = \begin{cases} 1, & \textit{if } n \in \{1,2\}, \\ 3, & \textit{if } n = 3, \\ n-1, & \textit{if } 4 \leq n \leq 8, \textit{ and} \\ 4^{\frac{n-1}{5}}, & \textit{if } n \geq 9. \end{cases}$$

Furthermore, if the connected graph G of order n is such that $\mu(G) = h(n)$, then $G \in \mathcal{H}$ where

$$\mathcal{H} = \{K_1, K_2, K_3, K_{1,3}, K_{1,4}, K_{1,5}, K_{1,6}, K_{1,7}, H_8\} \cup \{H_n : n \geq 11 \textit{ and } n \equiv 1 \textit{ mod } 5\}.$$

The rest of the article is devoted to the proofs.

2. PROOFS

Before we proceed to the proofs of our theorems, we introduce some further notation and establish two preliminary results.

For a graph G and two disjoint subsets B and W of its vertex set $V(G)$, a DIM M of G is *compatible with* $(G; B, W)$ if $B \subseteq V(M)$ and $W \cap V(M) = \emptyset$. Let $\mu(G; B, W)$ denote the number of DIMs of G that are compatible with $(G; B, W)$. By the definition of DIMs, we have

$$\mu(G; B, W) > 0 \Rightarrow G[B] \text{ has maximum degree at most 1 and } W \text{ is independent.} \quad (1)$$

Note that if $V(G) \setminus (B \cup W)$ has at most n elements, then $\mu(G; B, W)$ is an integer at most 2^n. This implies that for a class $\mathcal{G}$ of graphs and a nonnegative integer n, the maximum

$$s_{\mathcal{G}}(n) = \max\{\mu(G; B, W) : \ G \text{ is a graph in } \mathcal{G}, B \text{ and } W \text{ are disjoint subsets of } V(G), \\ B \cup W \neq \emptyset, \text{ and } |V(G) \setminus (B \cup W)| \leq n\}$$

is well defined and finite, even though the maximum is possibly taken over infinitely many graphs. Note that $s_{\mathcal{G}}(0) \leq 1$. Furthermore, if $\mathcal{G}$ contains a nonempty graph that has a DIM, then $s_{\mathcal{G}}(n) \geq 1$.

Lemma 4. *If $\mathcal{C}$ is the class of connected $\{C_3, C_4\}$-free graphs of minimum degree at least 2, then $s_{\mathcal{C}}(n) = 1$ for $n \in \{0, 1, 2, 3\}$ and $s_{\mathcal{C}}(n) \leq s_{\mathcal{C}}(n-4) + s_{\mathcal{C}}(n-2)$ for every integer n with $n \geq 4$.*

Proof. We prove the statement by induction on n. For $n = 0$, the statement follows from the above observations using $\mu(C_6) > 0$ and $C_6 \in \mathcal{C}$. Now let $n \geq 1$. Clearly, $s_{\mathcal{C}}(n) \geq 1$ and $s_{\mathcal{C}}(n-1) \leq s_{\mathcal{C}}(n)$. Hence, in view of the desired statement, we may assume that $s_{\mathcal{C}}(n) \geq 2$. Let $(G; B, W)$ be a maximizer in the definition of $s_{\mathcal{C}}(n)$, that is, $s_{\mathcal{C}}(n) = \mu(G; B, W)$. Since $s_{\mathcal{C}}(n) \geq 2$, the set $B \cup W$ is a proper nonempty subset of $V(G)$. Since G is connected, there is an edge uv of G such that $u \in B \cup W$ and $v \in V(G) \setminus (B \cup W)$.

If $u \in W$, then $s_{\mathcal{C}}(n) = \mu(G; B, W) \overset{(1)}{=} \mu(G; B \cup \{v\}, W) \leq s_{\mathcal{C}}(n-1)$. If $u \in B$ and u has a neighbor in B, then $s_{\mathcal{C}}(n) = \mu(G; B, W) \overset{(1)}{=} \mu(G; B, W \cup \{v\}) \leq s_{\mathcal{C}}(n-1)$. If $u \in B$ and all neighbors of u distinct from v belong to W, then $s_{\mathcal{C}}(n) = \mu(G; B, W) \overset{(1)}{=} \mu(G; B \cup \{v\}, W) \leq s_{\mathcal{C}}(n-1)$. In all three cases, we obtain $s_{\mathcal{C}}(n) \leq s_{\mathcal{C}}(n-1)$. By induction, if $n-1 \leq 3$, then $1 \leq s_{\mathcal{C}}(n) \leq s_{\mathcal{C}}(n-1) = 1$, and if $n-1 \geq 4$, then $s_{\mathcal{C}}(n) \leq s_{\mathcal{C}}(n-1) \leq s_{\mathcal{C}}(n-5) + s_{\mathcal{C}}(n-3) \leq s_{\mathcal{C}}(n-4) + s_{\mathcal{C}}(n-2)$.

Hence, we may assume that u belongs to B and that u has a neighbor w in $V(G) \setminus (B \cup W)$ that is distinct from v. Since G is of minimum degree at least 2, the vertex v has a neighbor v' distinct from u and the vertex w has a neighbor w' distinct from u. Since G is $\{C_3, C_4\}$-free, the vertices v, v', w, and w' are all distinct.

If $v' \in W$, then $s_{\mathcal{C}}(n) = \mu(G; B, W) \overset{(1)}{=} \mu(G; B \cup \{v\}, W) \leq s_{\mathcal{C}}(n-1)$. If $v' \in B$, then $s_{\mathcal{C}}(n) = \mu(G; B, W) \overset{(1)}{=} \mu(G; B, W \cup \{v\}) \leq s_{\mathcal{C}}(n-1)$. Again, we obtain $s_{\mathcal{C}}(n) \leq s_{\mathcal{C}}(n-1)$ and can argue as above.

Hence, we may assume that v' and w' belong to $V(G) \setminus (B \cup W)$, which implies $n \geq 4$. Now

$$\begin{aligned} s_{\mathcal{C}}(n) &= \mu(G; B, W) \\ &= \mu(G; B \cup \{v\}, W) + \mu(G; B, W \cup \{v\}) \\ &\overset{(1)}{=} \mu(G; B \cup \{v, w'\}, W \cup \{v', w\}) + \mu(G; B \cup \{v'\}, W \cup \{v\}) \\ &\leq s_{\mathcal{C}}(n-4) + s_{\mathcal{C}}(n-2), \end{aligned}$$

which completes the proof. ■

If $F(n)$ denotes the n-th Fibonacci number, that is, $F(0) = 0$, $F(1) = 1$, and $F(n) = F(n-2) + F(n-1)$ for every integer n with $n \geq 2$, then Lemma 4 immediately implies

$$\max\{s_C(2n), s_C(2n+1)\} \leq F(n+1), \tag{2}$$

for every nonnegative integer n.

Lemma 5. *If G is a $\{C_3, C_4\}$-free graph of order n and minimum degree at least 2, then $\mu(G) < 0.928 \cdot \phi^{\frac{n}{2}}$ where $\phi = \frac{1+\sqrt{5}}{2}$.*

Proof. First, we assume that G is connected. Note that $n \geq 5$. If $n = 5$, then $G = C_5$ and $\mu(G) = 0$. Hence, let $n \geq 6$. Let u be a vertex of G. Since u has at least two neighbors, we obtain $\mu(G) = \mu(G; \emptyset, \{u\}) + \mu(G; \{u\}, \emptyset) \overset{(1)}{=} \mu(G; N_G(u), \{u\}) + \mu(G; \{u\}, \emptyset) \leq s_C(n-3) + s_C(n-1)$. If n is odd, then $\mu(G) \leq s_C(n-3) + s_C(n-1) \overset{(2)}{\leq} F\left(\frac{n-1}{2}\right) + F\left(\frac{n+1}{2}\right) = F\left(\frac{n+3}{2}\right)$. If n is even, then $\mu(G) \leq s_C(n-3) + s_C(n-1) \overset{(2)}{\leq} F\left(\frac{n-2}{2}\right) + F\left(\frac{n}{2}\right) = F\left(\frac{n+2}{2}\right)$. Using $\phi^{-2} + \phi^{-1} = 1$ and $\max\left\{F(4) \cdot \phi^{-\frac{6}{2}}, F(5) \cdot \phi^{-\frac{7}{2}}, F(6) \cdot \phi^{-\frac{9}{2}}\right\} < 0.928$, it follows easily by induction on n that for $n \geq 6$, we have

$$\left.\begin{array}{ll} F\left(\dfrac{n+3}{2}\right), & \text{if } n \text{ is odd and} \\ F\left(\dfrac{n+2}{2}\right), & \text{if } n \text{ is even} \end{array}\right\} < 0.928 \cdot \phi^{\frac{n}{2}}$$

and hence $\mu(G) < 0.928 \cdot \phi^{\frac{n}{2}}$.

If G has components $G_1, \ldots, G_k$ of orders $n_1, \ldots, n_k$, respectively, then $\mu(G) \leq \prod_{i=1}^{k} \mu(G_i) < 0.928^k \cdot \phi^{\frac{n_1+\ldots+n_k}{2}} \leq 0.928 \cdot \phi^{\frac{n}{2}}$, which completes the proof. ∎

We proceed to the proofs of our theorems. The general structure of all three proofs is very similar.

A. Proof of Theorem 1

Let G be a graph of order n and size m. We prove, by induction on $n+m$, that $\mu(G) \leq f(n)$ and, for $n \geq 3$, $\mu(G) = f(n)$ if and only if G belongs to $\mathcal{F}$. Since the result is easily verified for $n \leq 5$, we assume now that $n \geq 6$. We establish a series of claims concerning properties that G can be assumed to have.

Claim 1. *Every edge of G belongs to some DIM of G.*

Proof of Claim 1. If G contains an edge e such that no DIM of G contains e, then every DIM of G is a DIM of $G - e$ and, by induction, $\mu(G) \leq \mu(G-e) \leq f(n)$. If $\mu(G) = f(n)$, then $\mu(G-e) = f(n)$ and hence, by induction, $G - e \in \mathcal{F}$. It is easily verified that adding any edge to a graph H in $\mathcal{F}$ results in a graph with strictly less DIMs than H. Therefore, $\mu(G) < \mu(G-e)$, which is the contradiction $\mu(G) < f(n)$. ∎

Since no DIM of G can contain an edge that belongs to a cycle of length 4, Claim 1 implies that G has no such cycle.

Claim 2. *The graph G is triangle-free.*

Proof of Claim 2. Let $T : xyzx$ be a triangle in G. Since every DIM of G contains exactly one of the three edges of T, no DIM of G contains an edge between a vertex in $V(T)$ and a vertex in $V(G) \setminus V(T)$. By Claim 1, this implies that T is a component of G. Now, by induction, $\mu(G) = 3 \cdot \mu(G - V(T)) \leq 3 \cdot f(n-3) = f(n)$. Furthermore, if $\mu(G) = f(n)$, then $\mu(G - V(T)) = f(n-3)$ and hence, by induction, $G - V(T) \in \mathcal{F}$. Since G is the disjoint union of a triangle and $G - V(T)$, we obtain $G \in \mathcal{F}$. ■

Claim 3. *The graph G has no isolated vertex.*

Proof of Claim 3. If u is an isolated vertex of G, then every DIM of G is a DIM of $G - u$. Therefore, by induction, $\mu(G) \leq \mu(G-u) \leq f(n-1) \leq f(n)$. If $\mu(G) = f(n)$, then $f(n-1) = f(n)$, which implies that $n \equiv 1 \bmod 3$. Furthermore, $\mu(G-u) = f(n-1)$ and hence, by induction, $G - u = \frac{n-1}{3}K_3$. Now $G = K_1 \cup \frac{n-1}{3}K_3 \in \mathcal{F}$. ■

Claim 4. *The graph G has minimum degree at least* 2.

Proof of Claim 4. By Claim 3, the graph G has no isolated vertex. If u is a vertex of degree 1 and v is the unique neighbor of u in G, then every DIM of G contains an edge incident with v. Hence, no DIM contains an edge between a vertex in $N_G[v]$ and $V(G) \setminus N_G[v]$. By Claims 1 and 2, the closed neighborhood $N_G[v]$ of v in G is the vertex set of a component of G and induces a star $K_{1,d}$ where $d = d_G(v) \geq 1$. Now, by induction, $\mu(G) = d \cdot \mu(G - N_G[v]) \leq d \cdot f(n - (d+1))$.

If $d \in \{1, 2\}$ or $d \geq 5$, then it is easily verified that $d \cdot f(n-(d+1)) < f(n)$ and hence $\mu(G) < f(n)$ in these cases.

If $d = 3$, then $d \cdot f(n-(d+1)) \leq f(n)$ with equality if and only if $n \equiv 1 \bmod 3$. Hence, $\mu(G) \leq f(n)$. Furthermore, if $\mu(G) = f(n)$, then $\mu(G - N_G[v]) = f(n-(d+1))$ and hence, by induction, $G - N_G[v] = \frac{n-4}{3}K_3$. Now $G = K_{1,3} \cup \frac{n-4}{3}K_3 \in \mathcal{F}$.

If $d = 4$, then $d \cdot f(n-(d+1)) \leq f(n)$ with equality if and only if $n \equiv 2 \bmod 3$. Hence, $\mu(G) \leq f(n)$. Furthermore, if $\mu(G) = f(n)$, then $\mu(G - N_G[v]) = f(n-(d+1))$ and hence, by induction, $G - N_G[v] = \frac{n-5}{3}K_3$. Now $G = K_{1,4} \cup \frac{n-5}{3}K_3 \in \mathcal{F}$. ■

By Claims 1–4, the graph G is a $\{C_3, C_4\}$-free graph of minimum degree at least 2. Since $f(n) \geq 4 \cdot 3^{-\frac{5}{3}} \cdot 3^{\frac{n}{3}}$ and $0.928 \cdot \phi^{\frac{n}{2}} < 4 \cdot 3^{-\frac{5}{3}} \cdot 3^{\frac{n}{3}}$ for $n \geq 6$, Lemma 5 implies $\mu(G) < f(n)$, which completes the proof. ■

B. Proof of Theorem 2

Let G be a triangle-free graph of order n and size m. We prove, by induction on $n + m$, that $\mu(G) \leq g(n)$ and, for $n \geq 2$, $\mu(G) = g(n)$ if and only if G belongs to $\mathcal{G}$. Since the result is easily verified for $n \leq 8$, we assume now that $n \geq 9$. We establish a series of claims concerning properties that G can be assumed to have.

Claim 5. *Every edge of G belongs to some DIM of G.*

Proof of Claim 5. This can be proved exactly as Claim 1. ■

Claim 5 implies that G is $\{C_3, C_4\}$-free.

Claim 6. *The graph G has no isolated vertex.*

Proof of Claim 6. Note that unlike the function f from Theorem 1, the function g is strictly increasing for $n \geq 3$. Using this fact, this claim can be proved as Claim 3. ■

Claim 7. *The graph G has minimum degree at least* 2.

Proof of Claim 7. By Claim 6, the graph G has no isolated vertex. Let u be a vertex of degree 1 and let v be the unique neighbor of u in G. Arguing as in the proof of Claim 4, we obtain that the closed neighborhood $N_G[v]$ of v in G is the vertex set of a component of G and induces a star $K_{1,d}$ where $d = d_G(v) \geq 1$. Now, by induction, $\mu(G) = d \cdot \mu(G - N_G[v]) \leq d \cdot g(n - (d + 1))$.

It is easy to verify $d \cdot g(n - (d + 1)) \leq g(n)$ for every $n \geq 9$ with equality if and only if

- either $d = 3$, $n \bmod 5 \neq 0$ and $n \neq 11$,
- or $d = 4$ and $n \notin \{12, 16\}$,
- or $d = 5$ and $n = 11$.

The proof can now be completed similarly as the proof of Claim 4. We give details only for $d = 3$.

Let $d = 3$. We obtain $\mu(G) = d \cdot \mu(G - N_G[v]) \leq d \cdot g(n - (d + 1)) \leq g(n)$. If $\mu(G) = g(n)$, then $d \cdot g(n - (d + 1)) = g(n)$, which implies $n \bmod 5 \neq 0$ and $n \neq 11$. Furthermore, $\mu(G - N_G[v]) = g(n - (d + 1))$, which implies, by induction, that $G - N[v] \in \mathcal{G}$. Since for every graph H in $\mathcal{G}$ of order $n' = n - 4$ with $n' \geq 5$, $n' \bmod 5 \neq 1$, and $n' \neq 7$, we have $K_{1,3} \cup H \in \mathcal{G}$, we obtain $G \in \mathcal{G}$. ■

By Claims 5–7, the graph G is a $\{C_3, C_4\}$-free graph of minimum degree at least 2. Clearly, $g(n) > 0.928 \cdot \phi^{\frac{n}{2}}$ for $n \in \{9, 10, 11\}$. Furthermore, for $n \geq 12$, we have $g(n) \geq 81 \cdot 4^{\frac{n-16}{5}} > 0.956 \cdot 1.319^n > 0.928 \cdot \phi^{\frac{n}{2}}$, Lemma 5 implies $\mu(G) < g(n)$, which completes the proof. ■

C. Proof of Theorem 3

Let G be a connected graph of order n and size m. We prove the statement by induction on $n + m$. For $n \leq 8$, the result is easily verified. Note that

$$\text{for every positive integer } p, \text{ we have } p \cdot 4^{\frac{-(p+1)}{5}} \leq 1 \text{ with equality if and only if } p = 4. \quad (3)$$

This implies that if $n \leq 8$ and G is neither a star nor a triangle nor H_8, then $n \geq 4$ and $\mu(G) \leq h(n) - 1 = n - 2 \leq 4^{\frac{n-1}{5}}$.

We assume now that $n \geq 9$. Note that if G' is a graph of order n' less than n and no component of G' is a star or a triangle or H_8, then, by induction, every component K of G of order $n(K)$ satisfies $\mu(K) \leq 4^{\frac{n(K)-1}{5}}$, which implies $\mu(G') \leq 4^{\frac{n'-1}{5}}$. We establish a series of claims concerning properties that G can be assumed to have.

Claim 8. *Every edge of G that does not belong to some DIM of G is a bridge.*

Proof of Claim 8. If G contains an edge e such that no DIM of G contains e and e is not a bridge of G, then every DIM of G is a DIM of the connected graph $G - e$ and, by induction, $\mu(G) \leq \mu(G - e) \leq h(n)$. If $\mu(G) = h(n)$, then $\mu(G - e) = h(n)$ and hence, by induction, $G - e \in \mathcal{H}$. It is easily verified that adding any edge to a graph H in $\mathcal{H}$ results in a graph with strictly less DIMs than H. Therefore, $\mu(G) < \mu(G - e) = h(n)$, which is a contradiction. ■

By Claim 8, the graph G has no cycle of length 4.

Claim 9. *No edge of G that does not belong to some DIM of G is incident with a vertex of degree* 1.

Proof of Claim 9. If uv is an edge of G that does not belong to some DIM of G such that u has degree 1, then $\mu(G) \leq \mu(G-u) \leq h(n-1) < h(n)$. ■

Claim 10. *The graph G is triangle-free.*

Proof of Claim 10. Let $T : xyzx$ be a triangle in G. Since G is connected, we may assume that z has a neighbor z' that does not lie on T.

First, we assume that y has a neighbor y' that does not lie on T. Since G has no cycle of length 4, the vertices y' and z' are distinct. For every DIM M of G, the set M contains an edge of T and $M \setminus E(T)$ is a DIM of $G - V(T)$. This implies, by induction,

$$\begin{aligned}
\mu(G) &= \mu(G; \{x, y\}, \emptyset) + \mu(G; \{x, z\}, \emptyset) + \mu(G; \{y, z\}, \emptyset)\\
&\overset{(1)}{=} \mu(G; \{x, y, z'\}, \{y'\}) + \mu(G; \{x, z, y'\}, \{z'\}) + \mu(G; \{y, z\}, \{y', z'\})\\
&\leq \mu(G - V(T); \{z'\}, \{y'\}) + \mu(G - V(T); \{y'\}, \{z'\}) + \mu(G - V(T); \emptyset, \{y', z'\})\\
&\leq \mu(G - V(T))\\
&\leq h(n-3)\\
&< h(n).
\end{aligned}$$

Hence, we may assume that for every triangle $\tilde{T}$ of G, exactly one vertex of $\tilde{T}$ has degree at least 3.

Next, we assume that no component of $G - V(T)$ is either a star or a triangle or H_8. By induction, this implies that $\mu(G - V(T)) \leq 4^{\frac{(n-3)-1}{5}}$. Now

$$\begin{aligned}
\mu(G) &= \mu(G; \{x, y\}, \emptyset) + \mu(G; \{x, z\}, \emptyset) + \mu(G; \{y, z\}, \emptyset)\\
&\overset{(1)}{=} \mu(G; \{x, y, z'\}, \emptyset) + \mu(G; \{x, z\}, \{z'\}) + \mu(G; \{y, z\}, \{z'\})\\
&\leq \mu(G - V(T); \{z'\}, \emptyset) + 2 \cdot \mu(G - V(T); \emptyset, \{z'\})\\
&\leq 2 \cdot \mu(G - V(T))\\
&\leq 2 \cdot 4^{\frac{(n-3)-1}{5}}\\
&< 4^{\frac{n-1}{5}}.
\end{aligned}$$

Hence, we may assume that for every triangle $\tilde{T}$ of G, some component of $G - V(\tilde{T})$ is either a star or a triangle or H_8.

Next, we assume that some component S of $G - V(T)$ is a star of order s. Since the edge xy belongs to some DIM of G, we obtain that $s \geq 2$. Since the edge xz belongs to some DIM of G, we obtain that $s \geq 3$ and that z is adjacent to a leaf z' of S. If z has degree 3, then the graph is completely determined. Note that the structure of G is similar to H_8 in this case. Using $n \geq 9$, it is easy to verify that $\mu(G) < h(n)$. Hence, we may assume that z has degree at least 4. If $G - V(S)$ is H_8, then the graph is completely determined. Again, it is easy to verify that $\mu(G) < h(n)$. Hence, no component of $G - V(S)$ is either a star or a triangle or H_8, which implies, by induction, $\mu(G - V(S)) \leq 4^{\frac{(n-s)-1}{5}}$. Since every DIM of G contains an edge of S, we obtain, by induction,

$$\begin{aligned}
\mu(G) &= \mu(G; \{z'\}, \emptyset) + \mu(G; \emptyset, \{z'\})\\
&\overset{(1)}{=} \mu(G; \{z'\}, \{z\}) + \mu(G; \{z\}, \{z'\})\\
&\leq \mu(G - V(S); \emptyset, \{z\}) + (s-2) \cdot \mu(G - V(S); \{z\}, \emptyset)
\end{aligned}$$

$$\begin{aligned}&\leq (s-2)\cdot \mu(G-V(S))\\&\leq (s-2)\cdot 4^{\frac{(n-s)-1}{5}}\\&\overset{(3)}{<} 4^{\frac{n-1}{5}}.\end{aligned}$$

Hence, we may assume that for every triangle $\tilde{T}$ of G, no component of $G-V(\tilde{T})$ is a star.

Next, we assume that some component T' of $G-V(T)$ is a triangle. Since $n\geq 9$, the degree of z is at least 4. This implies that the connected graph $G-V(T')$ is H_8. Now the graph is completely determined, $n=11$, and $\mu(G)=8<h(n)$. Hence, we may assume that for every triangle $\tilde{T}$ of G, some component of $G-V(\tilde{T})$ is H_8. Now the graph G arises from $T\cup H_8$ by adding an edge between z and the vertex a in H_8 (see Fig. 1). This implies $n=11$ and $\mu(G)=13<h(n)$, which completes the proof of the claim. ■

Claims 8 and 10 imply that G is $\{C_3,C_4\}$-free. By assumption, G has no isolated vertex.

Claim 11. *The graph G has minimum degree at least* 2.

Proof of Claim 11. Let v be a vertex of G of degree $p+q$ such that v has $p\geq 1$ neighbors $u_1,\ldots,u_p$ of degree 1 and q neighbors $w_1,\ldots,w_q$ of degree at least 2. If G is a star, then the theorem is easily verified. Hence, we may assume that $q\geq 1$. Since every DIM of G contains an edge incident with v, every edge between a vertex in $N_G[v]$ and $V(G)\setminus N_G[v]$ is a bridge. Since G is triangle-free, this implies that every edge incident with a vertex in $N_G[v]$ is a bridge and that $N_G[v]$ induces a star S. For $j\in[q]$, let z_j denote a neighbor of w_j that is distinct from v. Let $Z=\{z_1,\ldots,z_q\}$.

We have

$$\begin{aligned}\mu(G)&=\sum_{i=1}^{p}\mu(G;\{v,u_i\},\emptyset)+\sum_{j=1}^{q}\mu(G;\{v,w_j\},\emptyset)\\&\overset{(1)}{=}\sum_{i=1}^{p}\mu(G;\{v,u_i\}\cup Z,\emptyset)+\sum_{j=1}^{q}\mu(G;\{v,w_j\}\cup(Z\setminus\{z_j\}),\{z_j\})\\&\leq\sum_{i=1}^{p}\mu(G-V(S);Z,\emptyset)+\sum_{j=1}^{q}\mu(G-V(S);Z\setminus\{z_j\},\{z_j\})\\&=p\cdot\mu(G-V(S);Z,\emptyset)+\sum_{j=1}^{q}\mu(G-V(S);Z\setminus\{z_j\},\{z_j\})\\&\leq p\cdot\mu(G-V(S)).\end{aligned}$$

If $q\geq 2$ and some component S' of $G-V(S)$ is a star, then Claim 9 implies that S' has order at least 2 and, by exchanging the roles of S and S', we may assume that $q=1$. Hence, we may assume that

- either $q\geq 2$ and no component of $G-V(S)$ is a star,
- or $q=1$.

If $q\geq 2$ and no component of $G-V(S)$ is a star, then, by induction, $\mu(G-V(S))\leq 4^{\frac{(n-|V(S)|)-1}{5}}=4^{\frac{(n-(p+q+1))-1}{5}}\leq 4^{\frac{(n-(p+3))-1}{5}}$ and we obtain $\mu(G)\leq p\cdot\mu(G-V(S))\leq p\cdot 4^{\frac{(n-(p+3))-1}{5}}\overset{(3)}{<}4^{\frac{n-1}{5}}$. Hence, we may assume now that $q=1$.

First, we assume that the edge vw_1 does not belong to any DIM of G. In this case, $\mu(G) \leq p \cdot \mu(G - \{u_1, \ldots, u_p, v\})$. If the connected graph $G - \{u_1, \ldots, u_p, v\}$ is a star, then the result is easily verified. Hence, we may assume that $G - \{u_1, \ldots, u_p, v\}$ is not a star. By induction, this implies $\mu(G - \{u_1, \ldots, u_p, v\}) \leq 4^{\frac{(n-(p+1))-1}{5}}$ and hence $\mu(G) \leq p \cdot \mu(G - \{u_1, \ldots, u_p, v\}) \leq p \cdot 4^{\frac{(n-(p+1))-1}{5}} \overset{(3)}{\leq} 4^{\frac{n-1}{5}}$. Furthermore, if $\mu(G) = 4^{\frac{n-1}{5}}$, then, by (3), we have $p = 4$, $n \equiv 1 \bmod 5$, and $\mu(G - \{u_1, \ldots, u_p, v\}) = 4^{\frac{(n-(p+1))-1}{5}}$. By induction, this implies $G - \{u_1, \ldots, u_p, v\} = H_{n-5}$, which easily implies $G = H_n \in \mathcal{H}$. Hence, we may assume that the edge vw_1 belongs to some DIM of G.

Next, we assume that some component S' of $G - V(S)$ is a star of order s'. Since, by Claim 9, the edge u_1v belongs to some DIM of G, we obtain that $s \geq 2$. Since the edge vw_1 belongs to some DIM of G, we obtain that $s \geq 3$ and that w_1 is adjacent to a leaf z' of S'. If w_1 has degree 2, then the graph is completely determined and it is easy to verify that $\mu(G) < h(n)$. Hence, we may assume that w_1 has degree at least 3. Since vw_1 belongs to some DIM of G, the graph $G - V(S')$ does not belong to $\mathcal{H}$ and $n - s' \geq 6$. By induction, this implies $\mu(G - V(S')) \leq 4^{\frac{(n-s')-1}{5}}$. Since every DIM of G contains an edge of S', we obtain

$$\begin{aligned}
\mu(G) &= \mu(G; \{z'\}, \emptyset) + \mu(G; \emptyset, \{z'\}) \\
&\overset{(1)}{=} \mu(G; \{z'\}, \{w_1\}) + \mu(G; \{w_1\}, \{z'\}) \\
&\leq \mu(G - V(S'); \emptyset, \{w_1\}) + (s' - 2) \cdot \mu(G - V(S'); \{w_1\}, \emptyset) \\
&\leq (s' - 2) \cdot \mu(G - V(S')) \\
&\leq (s' - 2) \cdot 4^{\frac{(n-s')-1}{5}} \\
&\overset{(3)}{<} 4^{\frac{n-1}{5}}.
\end{aligned}$$

Hence, we may assume that no component of $G - V(S)$ is a star. By induction, this implies $\mu(G - V(S)) \leq 4^{\frac{(n-(p+2))-1}{5}}$ and we obtain $\mu(G) \leq p \cdot \mu(G - V(S)) \leq p \cdot 4^{\frac{(n-(p+2))-1}{5}} \overset{(3)}{<} 4^{\frac{n-1}{5}}$, which completes the proof of the claim. ■

By Claims 8–11, the graph G is a $\{C_3, C_4\}$-free graph of minimum degree at least 2. Since $0.928 \cdot \phi^{\frac{n}{2}} < 4^{\frac{n-1}{5}}$ for $n \geq 9$, Lemma 5 implies $\mu(G) < h(n)$, which completes the proof. ■

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