Nonlinear Analysis 118 (2015) 130–143

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

The first non-zero Neumann p-fractional eigenvalue

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ARTICLE INFO

Article history:
Received 3 September 2014
Accepted 8 February 2015
Communicated by Enzo Mitidieri

Keywords:
Nonlinear fractional Laplacian
Neumann eigenvalues
Hölder infinity Laplacian

ABSTRACT

In this work we study the asymptotic behavior of the first non-zero Neumann p-fractional eigenvalue $\lambda_1(s, p)$ as $s \to 1^-$ and as $p \to \infty$. We show that there exists a constant $\mathcal{K}$ such that $\mathcal{K}(1-s)\lambda_1(s, p)$ goes to the first non-zero Neumann eigenvalue of the p-Laplacian. While in the limit case $p \to \infty$, we prove that $\lambda_1(1, s)^{1/p}$ goes to an eigenvalue of the Hölder ∞-Laplacian.

1. Introduction

In this paper we set out to study the following non-local Neumann eigenvalue problems in a smooth bounded domain $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$

$$\begin{cases} -\mathscr{L}_{s,p} u = \lambda |u|^{p-2} u & \text{in } \Omega, \\ u \in W^{s,p}(\Omega), \end{cases} \tag{1.1}$$

where $1 < p < \infty$ and $0 < s < 1$. Here λ stands for the eigenvalue and $\mathscr{L}_{s,p}$ is the regional fractional p-Laplacian, that is

$$\mathscr{L}_{s,p} u(x) := 2 \text{ p.v.} \int_\Omega \frac{|u(y) - u(x)|^{p-2}(u(y) - u(x))}{|x - y|^{n+sp}}\, dy,$$

where p.v. is a commonly used abbreviation for "in the principal value sense".

Observe that, in the case $p = 2$, $\mathscr{L}_{s,2}$ is the linear operator defined in [20], that is the regional fractional Laplacian.

The first non-zero eigenvalue of (1.1) can be characterized as

$$\lambda_1(s, p) := \inf \left\{ \frac{\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\, dx\, dy}{\int_\Omega |u(x)|^p\, dx} : u \in \mathcal{X}_{s,p} \right\},$$

where $\mathcal{X}_{s,p} = \left\{ v \in W^{s,p}(\Omega) : v \neq 0,\ \int_\Omega |v(x)|^{p-2} v(x)\, dx = 0 \right\}$. Here $W^{s,p}(\Omega)$ denotes a fractional Sobolev space (see Section 2).

Non-local eigenvalue problems were recently studied in several papers. In [4] it was analyzed the first Neumann eigenvalue of a non-local diffusion problem for some non-singular convolution type operators. In [3] this analysis was extended for non-local p-Laplacian type diffusion equations. Some properties about the first eigenvalue of the fractional

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http://dx.doi.org/10.1016/j.na.2015.02.006

Dirichlet p-Laplacian were established in [18,23] and up to our knowledge no investigations were made about fractional Neumann eigenvalues.

To be more concrete, we will study the asymptotic behavior of the first non-zero eigenvalue $\lambda_1(s,p)$ as $s\to 1^-$ and as $p\to\infty$.

In order to introduce our results, we need to mention the well-known result of Bourgain, Brézis and Mironescu [8]: for any smooth bounded domain $\Omega\subset\mathbb{R}^n$, $u\in W^{1,p}(\Omega)$ with $1<p<\infty$ there exists a constant $\mathcal{K}=\mathcal{K}(n,p,\Omega)$ such that

$$\lim_{s\to 1^-}\mathcal{K}(1-s)\int_\Omega\int_\Omega\frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dxdy=\int_\Omega|\nabla u|\,dx. \tag{1.2}$$

See Theorem 2.2 for more details.

Our first result is related to the limit as $s\to 1^-$ of $\lambda_1(s,p)$. We show that such that $\mathcal{K}(1-s)\lambda_1(s,p)$ goes to

$$\lambda_1(1,p):=\inf\left\{\frac{\|\nabla u\|^p_{L^p(\Omega)}}{\|u\|^p_{L^p(\Omega)}}: v\in\mathcal{X}_{1,p}\right\},$$

that is, the first non-zero eigenvalue of the p-Laplacian with Neumann boundary conditions, namely $\lambda_1(1,p)$ is the first non-zero eigenvalue of

$$\begin{cases}-\Delta_p u=\lambda|u|^{p-2}u & \text{in }\Omega,\\ \dfrac{\partial u}{\partial\nu}=0 & \text{on }\partial\Omega\end{cases}$$

where $\Delta_p u=\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the usual p-Laplacian and ν is the outer unit normal to $\partial\Omega$.

Theorem 1.1. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$, and $p\in(1,\infty)$. Then*

$$\lim_{s\to 1^-}\mathcal{K}(1-s)\lambda_1(s,p)=\lambda_1(1,p),$$

where $\mathcal{K}$ is the constant in (1.2).

Lastly we study the limit case $p\to\infty$. We show that

$$\lambda_1(s,\infty):=\lim_{p\to\infty}\lambda_1(s,p)^{\frac{1}{p}}=\frac{2}{\operatorname{diam}(\Omega)^s}.$$

Here $\operatorname{diam}(\Omega)$ denotes diameter of Ω, that is

$$\operatorname{diam}(\Omega)=\sup_{x,y\in\Omega}|x-y|.$$

This result is truly different than that obtained in the local case, in contrast with the Dirichlet p-fractional Laplacian. More precisely, in [28] the authors show that

$$\lambda_1(1,\infty)=\lim_{p\to\infty}\lambda_1(1,p)^{\frac{1}{p}}=\frac{2}{\operatorname{diam}_\Omega(\Omega)},$$

where

$$\lambda_1(1,\infty):=\inf\left\{\|\nabla u\|_{L^\infty(\Omega)}: u\in W^{1,\infty}(\Omega)\text{ s.t. }\max_\Omega u=-\min_\Omega u=1\right\},$$

and $\operatorname{diam}_\Omega(\Omega)$ is the intrinsic diameter of Ω, that is

$$\operatorname{diam}_\Omega(\Omega)=\sup_{x,y\in\Omega}d_\Omega(x,y)$$

with d_Ω denoting the geodesic distance in Ω. Moreover, they show that if u_p is a normalized minimizer of $\lambda_1(1,p)$, then up to a subsequence, u_p converges in $C(\overline{\Omega})$ to some minimizer $u\in W^{1,\infty}(\Omega)$ of $\lambda_1(1,\infty)$ which is a solution of

$$\begin{cases}\max\{\Delta_\infty u,-|\nabla u|+\lambda_1(1,\infty)u\} & \text{in }\{x\in\Omega: u(x)>0\},\\ \min\{\Delta_\infty u,|\nabla u|+\lambda_1(1,\infty)u\} & \text{in }\{x\in\Omega: u(x)<0\},\\ \Delta_\infty u=0 & \text{in }\{x\in\Omega: u(x)=0\},\\ \dfrac{\partial u}{\partial\nu}=0 & \text{on }\partial\Omega,\end{cases}$$

in the viscosity sense, where Δ_∞ is the ∞-Laplacian, that is

$$\Delta_\infty u=-\sum_{i,j=1}^N\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_j\partial x_i}\frac{\partial u}{\partial x_j}.$$

See also [17].

For the local Dirichlet p-Laplacian eigenvalue problems the same limit was studied in [21,22], where the authors show that

$$\lim_{p\to\infty} \mu_1(1,p)^{\frac{1}{p}} = \frac{1}{R(\Omega)} = \mu_1(1,\infty) := \inf\left\{ \frac{\|\nabla u\|_{L^\infty(\Omega)}}{\|u\|_{L^\infty(\Omega)}} : u \in W_0^{1,\infty}(\Omega),\ u \neq 0 \right\}.$$

Here $R(\Omega)$ denotes the inradius (the radius of the largest ball contained in Ω) and $\mu_1(1,p)$ is the first eigenvalue of the Dirichlet p-Laplacian. In addition, they prove that the positive normalized eigenfunction v_p associated to $\mu(1,p)$ converges, up to a subsequence, to a positive function $v \in W_0^{1,\infty}(\Omega)$ which is a minimizer of $\mu(1,\infty)$ and is a viscosity solution of

$$\begin{cases} \min\{|Du| - \mu_1(1,\infty), \Delta_\infty u\} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Recently, the Dirichlet fractional p-Laplacian is considered, in [23] it was proved that

$$\lim_{p\to\infty} \mu_1(s,p)^{\frac{1}{p}} = \frac{1}{R(\Omega)^s} = \mu_1(s,\infty) := \inf\left\{ \frac{[\phi]_{W^{s,\infty}(\Omega)}}{\|\phi\|_{L^\infty(\Omega)}} : \phi \in C_0^\infty(\Omega),\ \phi \neq 0 \right\},$$

where $\mu_1(s,p)$ is the first eigenvalue of the non-local eigenvalue problems

$$\begin{cases} 2\displaystyle\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+sp}}\,dy + \lambda|u(x)|^{p-2}u(x) = 0 & \text{in } \Omega, \\ u \equiv 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

Moreover, they show that if w_p is a minimizer of $\mu_1(s,p)$, then there exists $w \in C_0(\overline{\Omega})$ such that, up to a subsequence $w_p \to w$ uniformly in $\mathbb{R}^n$ which is a minimizer of $\mu_1(s,\infty)$ and is a solution of

$$\begin{cases} \max\left\{\mathcal{L}_\infty u(x), \mathcal{L}_\infty^- u(x) + \mu_1(s,\infty)u(x)\right\} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

in the viscosity sense. Here

$$\mathcal{L}_\infty u(x) := \sup_{y\in\mathbb{R}^n} \frac{u(y)-u(x)}{|y-x|^s} + \inf_{y\in\mathbb{R}^n} \frac{u(y)-u(x)}{|y-x|^s},$$

and

$$\mathcal{L}_\infty^- u(x) := \inf_{y\in\mathbb{R}^n} \frac{u(y)-u(x)}{|y-x|^s}.$$

In this context, our result is the following.

Theorem 1.2. *Let Ω be a bounded open connected domain in $\mathbb{R}^n$ and $s \in (0,1)$. Then*

$$\lim_{p\to\infty} \lambda_1(s,p)^{\frac{1}{p}} = \frac{2}{\operatorname{diam}(\Omega)^s} = \lambda_1(s,\infty) := \inf\left\{ \frac{[u]_{W^{s,\infty}(\Omega)}}{\|u\|_{L^\infty(\Omega)}} : u \in \mathcal{A} \right\},$$

where $\mathcal{A} := \{u \in W^{s,\infty}(\Omega) : u \neq 0,\ \sup u + \inf u = 0\}$. Moreover, if u_p is the normalizer minimizer of $\lambda_1(s,p)$, then up to a subsequence u_p converges in $C(\overline{\Omega})$ to some minimizer $u_\infty \in W^{s,\infty}(\Omega)$ of $\lambda_1(s,\infty)$ which is a viscosity solution of

$$\begin{cases} \max\{\mathcal{L}_{s,\infty}u(x), \mathcal{L}_{s,\infty}^- u(x) + \lambda_1(s,\infty)u(x)\} = 0 & \text{when } u(x) > 0, \\ \mathcal{L}_{s,\infty}u(x) = 0 & \text{when } u(x) = 0, \\ \min\{\mathcal{L}_{s,\infty}u(x), \mathcal{L}_{s,\infty}^+ u(x) + \lambda_1(s,\infty)u(x)\} = 0 & \text{when } u(x) < 0, \end{cases} \tag{1.3}$$

where $\mathcal{L}_{s,\infty}u := \mathcal{L}_{s,\infty}^+ u + \mathcal{L}_{s,\infty}^- u$,

$$\mathcal{L}_{s,\infty}^+ u(x) := \sup_{y\in\overline{\Omega}, y\neq x} \frac{u(y)-u(x)}{|y-x|^s} \quad \text{and} \quad \mathcal{L}_{s,\infty}^- u(x) := \inf_{y\in\overline{\Omega}, y\neq x} \frac{u(y)-u(x)}{|y-x|^s}.$$

The operator $\mathcal{L}_{s,\infty}$ is the Hölder ∞-Laplacian, see [10].

Let us conclude the introduction with a brief comment on previous bibliography that concerns mostly the non-local operators.

One of the biggest interests in defining the operator $\mathcal{L}_{s,p}$ lies in its probabilistic interpretation in relation of a restricted type of Lévy processes. In [6], it was studied the s-stable processes, a particular kind of Lévy processes. For $s \in (0,1)$ and

$n \geq 1$ they proved that the Dirichlet form associated with a symmetric s-stable process in $\mathbb{R}^n$ is given by

$$\mathrm{E}(u, v) = C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} \, dx \, dy,$$

where u, v belong to $W^{s,2}(\mathbb{R}^n)$ and C is a constant depending on n and s. It is well known that E is related to the fractional Laplacian $(-\Delta)^s$, that is

$$(-\Delta)^s u(x) = C \text{ p.v.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} \, dy \quad \forall u \in W^{s,p}(\mathbb{R}^n)$$

where C is a constant depending on n and s, precisely given by

$$C = \left(\int_{\mathbb{R}^n} \frac{1 - \cos(\xi_1)}{|\xi|^{n+2s}} d\xi \right)^{-1},$$

see [14, Section 3].

Due to the action of the process in the whole space it was widely used to model systems of stochastic dynamics with applications in operation research, queuing theory, mathematical finance among others, see [2,5,9] for instance.

If one wished to restrict the action of a process to a bounded domain $\Omega \subset \mathbb{R}^n$, one could consider the so-called *s-stable process killed when leaving* Ω, in which the Dirichlet form still being the same, but the functions are taken with support in Ω, see [7].

Alternatively, another way is to study the so-called *censored stable process*, that is a stable process in which the jumps between Ω and its complement are forbidden. In this case, the functions are taken in the fractional Sobolev space $W^{s,2}(\Omega)$ and the correspondent Dirichlet form is given by

$$\mathcal{E}(u, v) = C \int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} \, dx \, dy.$$

This kind of processes are generated by

$$\Delta^s_{\Omega} u(x) = C \text{ p.v.} \int_{\Omega} \frac{u(x) - u(y)}{|x - y|^{n+2s}} \, dy$$

which is called *regional fractional Laplacian* in Ω. See [7,11,19,16,20] and references therein.

From a physical point of view, this operator describes a particle jumping from one point $x \in \Omega$ to another point $y \in \Omega$ with intensity proportional to $|x - y|^{-n-2s}$. Moreover, this kind of process can be used to describe some random flow in a closed domain with free action on the boundary, and they are always connected to the Neumann boundary problems. As it was pointed in [4,12] the idea of s-process in which its jumps from Ω to the complement of Ω are suppressed, are related to the Neumann non-local evolution equation

$$\begin{cases} u_t(x, t) = \Delta^s_{\Omega} u(x) \\ u \in W^{s,2}(\Omega) \end{cases}$$

since the individuals are "forced" to stay inside Ω. In contrast with the classical heat equation $u_t = \Delta u$, the diffusion of the density u at a point x and a time t depends not only on $u(x, t)$, but also on all values of u in a neighborhood of x.

In the course of the writing of this paper, the authors in [15] introduced a new Neumann problem for the fractional Laplacian by considering the non-local prescription

$$\text{p.v.} \int_{\Omega} \frac{u(x) - u(y)}{|x - y|^{n+2s}} \, dy = 0$$

for $x \in \mathbb{R}^n \setminus \Omega$ as a generalization of the classical Neumann condition $\partial_\nu u = 0$ on $\partial\Omega$.

The paper is organized as follows: in Section 2 we collect some preliminaries; in Section 3 we deal with the first non-zero eigenvalue; in Section 4 we prove Theorem 1.1; in Section 5 we prove Theorem 1.2, while in the final section we give an example of nonlinear non-local operator such that its first non-zero eigenvalue $\mu(s, p)$ has the following property: $\mu(s, p)^{1/p} \to 2/\mathrm{diam}_{\Omega}(\Omega)$ as $p \to \infty$.

2. Preliminaries

We begin by recalling some results concerning the fractional Sobolev spaces.

Let Ω be an open set in $\mathbb{R}^n$, $s \in (0, 1)$ and $p \in [1, \infty)$. The fractional Sobolev spaces are defined as

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) \colon \frac{|u(x) - u(y)|}{|x - y|^{n/p+s}} \in L^p(\Omega \times \Omega) \right\},$$

which endowed with the norm

$$\|u\|^p_{W^{s,p}(\Omega)} := \|u\|^p_{L^p(\Omega)} + \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\, dx\, dy,$$

is a separable Banach space. Moreover, if $p \in (1,\infty)$ then $W^{s,p}(\Omega)$ is reflexive.

The fractional space $W^{s,\infty}(\Omega)$ is defined as the space of functions

$$W^{s,\infty}(\Omega) := \left\{ u \in L^\infty(\Omega) \colon \frac{u(x)-u(y)}{|x-y|^s} \in L^\infty(\Omega \times \Omega) \right\}$$

with the norm

$$\|u\|_{W^{s,\infty}(\Omega)} := \|u\|_{L^\infty(\Omega)} + \left\| \frac{u(x)-u(y)}{|x-y|^s} \right\|_{L^\infty(\Omega\times\Omega)}.$$

Throughout the paper $[u]_{W^{s,p}(\Omega)}$ denotes the so-called Gagliardo seminorm

$$[u]_{W^{s,p}(\Omega)} := \begin{cases} \left(\displaystyle\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\, dx\, dy \right)^{\frac{1}{p}}, & \text{if } 1 \le p < \infty, \\ \left\| \dfrac{u(x)-u(y)}{|x-y|^s} \right\|_{L^\infty(\Omega\times\Omega)} & \text{if } p = \infty. \end{cases}$$

For more details related to these spaces and their properties, see, for instance, [1,13,14].

The proof of the following lemma can be found in [13].

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set of class C^1. Then $C^1(\overline{\Omega})$ is dense in $W^{s,p}(\Omega)$.*

The next results are established in [8, Corollaries 2 and 7].

Theorem 2.2. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$, and $p \in (1,\infty)$. Assume $u \in L^p(\Omega)$, then*

$$\lim_{s\to 1^-} \mathcal{K}(1-s)[u]^p_{W^{s,p}(\Omega)} = [u]^p_{W^{1,p}(\Omega)}$$

with

$$[u]^p_{W^{1,p}(\Omega)} = \begin{cases} \displaystyle\int_\Omega |\nabla u|^p\, dx, & \text{if } u \in W^{1,p}(\Omega), \\ \infty & \text{if } u \notin W^{1,p}(\Omega). \end{cases}$$

Here $\mathcal{K}$ depends only on the p and Ω.

This result was later completed in [24], where the authors show that for $u \in \bigcup_{s\in(0,1)} W^{s,p}_0(\mathbb{R}^n)$ with $1 \le p < \infty$, we have that

$$\lim_{s\to 0^+} \frac{sp}{2\omega_{n-1}} [u]^p_{W^{s,p}(\mathbb{R}^n)} = \|u\|^p_{L^p(\mathbb{R}^n)}.$$

Here the space $W^{s,p}_0(\mathbb{R}^n)$ is the closure of $C^\infty_0(\mathbb{R}^n)$ in the norm $[u]_{W^{s,p}(\mathbb{R}^n)}$ and ω_{n-1} is the $(n-1)$-dimensional Hausdorff measure of the unit sphere S^{n-1}.

Finally in [25], the author shows that the above two results can be viewed as consequences of continuity principles for real interpolation scales.

Theorem 2.3. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$, and $p \in (1,\infty)$. Let $\{u_s\}_{s\in(0,1)}$ be a subset of $L^p(\Omega)$ such that for any $s \in (0,1)$ we have that $u_s \in W^{s,p}(\Omega)$ and*

$$(1-s)[u_s]_{W^{s,p}(\Omega)} \le C.$$

Then, there exist $u \in W^{1,p}(\Omega)$ and a subsequence $\{u_{s_k}\}_{k\in\mathbb{N}}$ such that

$$u_{s_k} \to u \quad \text{strongly in } L^p(\Omega),$$
$$u_{s_k} \rightharpoonup u \quad \text{weakly in } W^{1-\varepsilon,p}(\Omega),$$

for all $\varepsilon > 0$.

Remark 2.4. In [8] some inequalities involving fractional integrals are established. A careful computation allows us to compute explicitly the constant in [8, Lemma 2]. By means of the Chebyshev inequality together with Lemma 2 from [8], in

Eq. (36) from [8] it is obtained that

$$\varepsilon[u_\varepsilon]^p_{W^{1-\varepsilon,p}(\Omega)} \geq 2^{-p\delta}\delta[u_\varepsilon]^p_{W^{1-\delta,p}(\Omega)},$$

where $0 < \varepsilon < \delta$.

Denoting $s := 1-\varepsilon$ and $t := 1-\delta$, last inequality is equivalent to

$$(1-t)[u_s]^p_{W^{t,p}(\Omega)} \leq 2^{p(1-t)}(1-s)[u_s]^p_{W^{s,p}(\Omega)}, \tag{2.1}$$

where $0 < t < s < 1$.

For any $s \in (0, 1)$ and any $p \in [1, \infty)$, we say that an open set $\Omega \subset \mathbb{R}^n$ admits an (s, p)-extension domain if there exists a positive constant $C = C(n, p, s, \Omega)$ such that: for every function $u \in W^{s,p}(\Omega)$ there exists $\tilde{u} \in W^{s,p}(\mathbb{R}^n)$ with $\tilde{u}(x) = u(x)$ for all $x \in \Omega$ and $\|\tilde{u}\|_{W^{s,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{s,p}(\Omega)}$. For example, any Lipschitz open set Ω admits a (s, p)-extension, see [13, Proposition 4.43].

A useful result to be used is the fractional compact embeddings. For the proof see [14, Corollary 7.2] and [13, Theorem 4.54].

Theorem 2.5. *Let $s \in (0, 1)$, $p \in (1, \infty)$ and $\Omega \subset \mathbb{R}^n$ be a bounded open set that admits an (s, p)-extension. If $sp < n$ then we have the following compact embeddings*

$$W^{s,p}(\Omega) \hookrightarrow L^q(\Omega) \quad \text{for all } q \in [1, p_s^\star).$$

In addition, if Ω has a Lipschitz boundary and $sp \geq n$ then we have the following compact embeddings:

$$W^{s,p}(\Omega) \hookrightarrow L^q(\Omega) \quad \text{for all } q \in [1, p_s^\star), \text{ if } sp = n;$$

$$W^{s,p}(\Omega) \hookrightarrow C_b^{0,\lambda}(\Omega) \quad \text{for all } \lambda < s - n/p, \text{ if } sp > n.$$

Here $p_s^\star$ is the fractional critical Sobolev exponent, that is

$$p_s^\star := \begin{cases} \dfrac{np}{n-sp}, & \text{if } sp < n, \\ \infty, & \text{if } sp \geq n. \end{cases}$$

3. The first non-zero eigenvalue

Now we will show that $\lambda_1(s, p)$ is the first non-zero eigenvalue of (1.1).

We say that the value $\lambda \in \mathbb{R}$ is an eigenvalue of problem (1.1) if there exists $u \in W^{s,p}(\Omega) \setminus \{0\}$ such that

$$\mathcal{E}(u, \phi) = \lambda \int_\Omega |u|^{p-2}(x)u(x)\phi(x)\,dx \quad \forall \phi \in C^1(\overline{\Omega}), \tag{3.1}$$

where

$$\mathcal{E}(u, \phi) := \int_\Omega \int_\Omega \frac{|u(y)-u(x)|^{p-2}(u(y)-u(x))(\phi(y)-\phi(x))}{|x-y|^{n+sp}}\,dx\,dy.$$

In which case, we say that u is an eigenfunction associated to λ.

Of course $\lambda = 0$ is an eigenvalue and it is isolated and simple. Moreover, if $\lambda > 0$ is an eigenvalue and u is an eigenfunction associated to λ, then, taking $\phi \equiv 1$ as a test function in (3.1), we have

$$\int_\Omega |u(x)|^{p-2}u(x)\,dx = 0.$$

Thus, the existence of the first non-zero eigenvalue $\lambda_1(s, p)$ of (1.1) is related to the problem of minimizing the following non-local quotient

$$\frac{[v]^p_{W^{s,p}(\Omega)}}{\|v\|^p_{L^p(\Omega)}}$$

among all functions $v \in W^{s,p}(\Omega) \setminus \{0\}$ such that $\int_\Omega |v(x)|^{p-2}v(x)\,dx = 0$.

We begin establishing the following result.

Theorem 3.1. *Let Ω be an open set of class C^1, $s \in (0, 1)$ and $p \in (1, \infty)$. Then*

$$\lambda_1(s, p) = \inf\left\{\frac{[v]^p_{W^{s,p}(\Omega)}}{\|v\|^p_{L^p(\Omega)}} : v \in W^{s,p}(\Omega),\ v \neq 0,\ \int_\Omega |v(x)|^{p-2}v(x)\,dx = 0\right\} \tag{3.2}$$

is the first non-zero eigenvalue of (1.1).

Proof. Let $\{u_j\}_{j\in\mathbb{N}} \subset W^{s,p}(\Omega)$ be a minimizing sequence for $\lambda_1(s,p)$ such that $\|u_j\|_{L^p(\Omega)} = 1$ for all $j \in \mathbb{N}$. Then there exists a constant C such that

$$[u_j]_{W^{s,p}(\Omega)} \le C.$$

Therefore $\{u_j\}_{j\in\mathbb{N}}$ is bounded in $W^{s,p}(\Omega)$. Then, by Theorem 2.5, there exists a function $u \in W^{s,p}(\Omega)$ such that, up to a subsequence that we still call $\{u_j\}_{j\in\mathbb{N}}$,

$$\begin{aligned} &u_j \rightharpoonup u \quad \text{weakly in } W^{s,p}(\Omega),\\ &u_j \to u \quad \text{strongly in } L^p(\Omega). \end{aligned}$$

Hence $\|u\|_{L^p(\Omega)} = 1$, $|u_j(x)|^{p-2}u_j(x) \to |u(x)|^{p-2}u(x)$ a.e. in Ω, and

$$\| |u_j|^{p-2}u_j\|_{L^{p/(p-1)}(\Omega)} \to \| |u|^{p-2}u\|_{L^{p/(p-1)}(\Omega)}.$$

Then, by [27, Theorem 12], $|u_j|^{p-2}u_j \to |u|^{p-2}u$ strongly in $L^{p/(p-1)}(\Omega)$. Therefore, since $\int_\Omega |u_j(x)|^{p-2}u_j(x)\,dx = 0$ for all $j \in \mathbb{N}$, we have that $\int_\Omega |u(x)|^{p-2}u(x)\,dx = 0$. Then u is not constant.

On the other hand, since $u_j \rightharpoonup u$ weakly in $W^{s,p}(\Omega)$,

$$[u]^p_{W^{s,p}(\Omega)} \le \liminf_{j\to\infty} [u_j]^p_{W^{s,p}(\Omega)} = \lim_{j\to\infty} [u_j]^p_{W^{s,p}(\Omega)} = \lambda_1(s,p).$$

Then, by (3.2), we have that

$$[u]^p_{W^{s,p}(\Omega)} = \lambda_1(s,p).$$

Observe that $\lambda_1(s,p) > 0$ due to u is not constant. In addition, $\lambda_1(s,p)$ is attained in

$$\left\{ v \in W^{s,p}(\Omega) \colon \int_\Omega |v(x)|^{p-2}v(x)\,dx = 0 \text{ and } \|v\|_{L^p(\Omega)} = 1 \right\}.$$

Then, proceeding as in the proof of Theorem 4.3.77 in [26], we have that $\lambda_1(s,p)$ is the first non-zero eigenvalue of (1.1). □

Finally we show that if an eigenfunction belongs to $C(\overline{\Omega})$ then it is a viscosity solution of

$$-\mathscr{L}_{s,p}u = \lambda_1(s,p)|u|^{p-2}u \tag{3.3}$$

in the following sense.

Definition 3.2. Suppose that $u \in C(\overline{\Omega})$. We say that u is a *viscosity super-solution* (resp. *viscosity sub-solution*) in Ω of Eq. (3.3) if the following holds: whenever $x_0 \in \Omega$ and $\varphi \in C^1(\overline{\Omega})$ are such that

$$\varphi(x_0) = u(x_0) \quad \text{and} \quad \varphi(x) \le u(x) \quad (\text{resp. } \varphi(x) \ge u(x)) \text{ for all } x \in \mathbb{R}^n$$

then we have

$$\mathscr{L}_{s,p}\varphi(x_0) + \lambda_1(s,p)|\varphi(x_0)|^{p-2}\varphi(x_0) \le 0 \quad (\text{resp. } \ge 0).$$

A *viscosity solution* is defined as being both a viscosity super-solution and a viscosity sub-solution.

For the proof of the following theorem, see [23, Proposition 11].

Theorem 3.3. *Let $s \in (0,1)$ and $p \in (1,\infty)$ such that $s < 1 - 1/p$. An eigenfunction $u \in C(\overline{\Omega})$ associated to $\lambda_1(s,p)$ is a viscosity solution of (3.3).*

4. The limit as $s \to 1^-$

In this section, our main aim is to prove that

$$\mathcal{K}(1-s)\lambda_1(s,p) \to \lambda_1(1,p) \quad \text{as } s \to 1^-,$$

where $\mathcal{K}$ is the constant of Theorem 2.2.

Before we prove Theorem 1.1, we need to show the following technical lemma.

Lemma 4.1. *Let $\{s_j\}_{j\in\mathbb{N}} \subset (0,1)$ and $\{u_j\}_{j\in\mathbb{N}} \subset L^p(\Omega)$ such that $s_j \to 1^-$ as $j \to \infty$, $u_j \in W^{s_j,p}(\Omega)$,*

$$\mathcal{K}(1-s_j)[u_j]^p_{W^{s_j,p}(\Omega)} = 1 \quad \text{and} \quad \int_\Omega |u_j(x)|^{p-2}u_j(x)\,dx = 0 \tag{4.1}$$

for all $j \in N$. Then there exist subsequences $\{s_{j_k}\}_{k\in N}$ and $\{u_{j_k}\}_{k\in N}$, and a function $u \in W^{1,p}(\Omega)$ such that

$$u_{j_k} \to u \quad \text{strongly in } L^p(\Omega)$$

and

$$[u]^p_{W^{1,p}(\Omega)} \le \liminf_{k\to\infty} \mathcal{K}(1-s_{j_k})[u_{s_{j_k}}]^p_{W^{s_{j_k},p}(\Omega)}$$

with $\int_\Omega |u(x)|^{p-2}u(x)\,dx = 0$.

Proof. For any $t \in (0,1)$, there exists $j_0 \in \mathbb{N}$ such that $0 < t < s_j < 1$ for all $j \ge j_0$. By (2.1) and (4.1) it follows that

$$\mathcal{K}(1-t)[u_j]^p_{W^{t,p}(\Omega)} \le 2^{p(1-t)}\mathcal{K}(1-s_j)[u_j]^p_{W^{s_j,p}(\Omega)} \le 2^{p(1-t)} \quad \forall j \ge j_0. \tag{4.2}$$

Then, by Theorem 2.5, there exist a subsequence $\{u_{j_k}\}_{k\in\mathbb{N}}$, and a function $u \in W^{1,p}(\Omega)$ such that

$$u_{j_k} \to u \quad \text{strongly in } L^p(\Omega),$$
$$u_{j_k} \rightharpoonup u \quad \text{weakly in } W^{t,p}(\Omega).$$

Using (4.2), we have

$$\begin{aligned}\mathcal{K}(1-t)[u]^p_{W^{t,p}(\Omega)} &\le \liminf_{k\to\infty} \mathcal{K}(1-s_{j_k})[u_{j_k}]^p_{W^{t,p}(\Omega)}\\ &\le 2^{p(1-t)} \liminf_{k\to\infty} \mathcal{K}(1-s_{j_k})[u_{j_k}]^p_{W^{s_{j_k},p}(\Omega)}.\end{aligned}$$

On the other hand, by Theorem 2.2, we get

$$[u]^p_{W^{1,p}(\Omega)} = \lim_{t\to 1^-} \mathcal{K}(1-t)[u]^p_{W^{t,p}(\Omega)} \le \liminf_{k\to\infty} \mathcal{K}(1-s_{j_k})[u_{j_k}]^p_{W^{s_{j_k},p}(\Omega)}.$$

Finally, we show that $\int_\Omega |u(x)|^{p-2}u(x)\,dx = 0$. We have that $|u_{j_k}(x)|^{p-2}u_{j_k}(x) \to |u(x)|^{p-2}u(x)$ a.e. in Ω, and

$$\|\,|u_{j_k}|^{p-2}u_{j_k}\|_{L^{p/(p-1)}(\Omega)} \to \|\,|u|^{p-2}u\|_{L^{p/(p-1)}(\Omega)},$$

due to $u_{j_k} \to u$ strongly in $L^p(\Omega)$. Then, by [27, Theorem 12], $|u_{j_k}|^{p-2}u_{j_k} \to |u|^{p-2}u$ strongly in $L^{p/(p-1)}(\Omega)$. Therefore, since $\int_\Omega |u_{j_k}(x)|^{p-2}u_{j_k}(x)\,dx = 0$ for all k, we have that $\int_\Omega |u(x)|^{p-2}u(x)\,dx = 0$. □

We finish this section by proving Theorem 1.1.

Proof of Theorem 1.1. Let $u \in W^{1,p}(\Omega)$ be an eigenfunction associated to $\lambda_1(1,p)$. Since $W^{1,p}(\Omega) \subset W^{s,p}(\Omega)$ for all $s \in (0,1)$ and $\int_\Omega |u(x)|^{p-2}u(x)\,dx = 0$, u is an admissible function in the variational characterization of $\lambda_1(s,p)$ for all $s \in (0,1)$. Then,

$$\mathcal{K}(1-s)\lambda_1(s,p) \le \mathcal{K}(1-s)\frac{[u]^p_{W^{s,p}(\Omega)}}{\|u\|^p_{L^p(\Omega)}}.$$

Therefore, by Theorem 2.2, we get that

$$\limsup_{s\to 1^-} \mathcal{K}(1-s)\lambda_1(s,p) \le \lim_{s\to 1^-} \mathcal{K}(1-s)\frac{[u]^p_{W^{s,p}(\Omega)}}{\|u\|^p_{L^p(\Omega)}} = \frac{[u]^p_{W^{1,p}(\Omega)}}{\|u\|^p_{L^p(\Omega)}} = \lambda_1(1,p). \tag{4.3}$$

On the other hand, let $\{s_j\}_{j\in\mathbb{N}}$ be a sequence in $(0,1)$ such that $s_j \to 1^-$ as $j \to \infty$ and

$$\lim_{j\to\infty} \mathcal{K}(1-s_j)\lambda_1(s_j,p) = \liminf_{s\to 1^-} \mathcal{K}(1-s_j)\lambda_1(s,p). \tag{4.4}$$

For $j \in \mathbb{N}$, let us choose $u_j \in W^{s,p}(\Omega)$ such that

$$\mathcal{K}(1-s_j)[u_j]^p_{W^{s_j,p}(\Omega)} = 1, \qquad \int_\Omega |u_j(x)|^{p-2}u_j(x)\,dx = 0,$$

and

$$\mathcal{K}(1-s_j)[u_j]^p_{W^{s_j,p}(\Omega)} = \mathcal{K}(1-s_j)\lambda_1(s_j,p)\|u_{s_j}\|^p_{L^p(\Omega)}.$$

By Lemma 4.1, there exist a subsequence, still denoted by $\{u_j\}_{j\in\mathbb{N}}$, and a function $u \in W^{1,p}(\Omega)$ such that

$$u_j \to u \quad \text{strongly in } L^p(\Omega), \qquad \int_\Omega |u(x)|^{p-2}u(x)\,dx = 0,$$

and

$$[u]^p_{W^{1,p}(\Omega)} \leq \liminf_{j\to\infty} \mathcal{K}(1-s_j)[u_j]^p_{W^{s_j,p}(\Omega)}.$$

Therefore, $[u]^p_{W^{1,p}(\Omega)} \leq 1$. Moreover, since

$$1 = \mathcal{K}(1-s_j)[u_j]^p_{W^{s_j,p}(\Omega)} = \mathcal{K}(1-s_j)\lambda_1(s_j,p)\|u_j\|^p_{L^p(\Omega)}$$

for all $j \in \mathbb{N}$ and $u_j \to u$ strongly in $L^p(\Omega)$, by (4.4), we have

$$1 = \liminf_{s\to 1^-} \mathcal{K}(1-s)\lambda_1(s,p)\|u\|^p_{L^p(\Omega)}. \tag{4.5}$$

Thus, u is an admissible function in the variational characterization of $\lambda_1(1,p)$. Then, using that $[u]^p_{W^{1,p}(\Omega)} \leq 1$ and (4.5), we have that

$$\lambda_1(1,p) \leq \liminf_{s\to 1^-} \mathcal{K}(1-s)\lambda_1(s,p). \tag{4.6}$$

From (4.3) and (4.6) the result follows. $\square$

5. The limit as $p \to \infty$

The goal of this section is to study the limit as $p \to \infty$ of the first non-zero eigenvalue $\lambda_1(s,p)$. Before beginning, we need to establish the following lemma.

Lemma 5.1. *Let Ω be a bounded open and connected domain in $\mathbb{R}^n$, $s \in (0,1)$, $x_0 \in \Omega$ and $c \in \mathbb{R}$. The function $w(x) = |x-x_0| - c$ belongs to $W^{1,\infty}(\Omega)$ and*

$$[w]_{W^{s,p}(\Omega)} \leq \frac{\kappa_n^{\frac{1}{p}} \operatorname{diam}(\Omega)^{1-s}|\Omega|^{\frac{1}{p}}}{(p(1-s))^{\frac{1}{p}}} \quad \forall p \in (1,\infty)$$

where κ_n is the measure of unit ball and $|\Omega|$ is the measure of Ω.

Proof. We start the proof recalling that

$$w \in W^{s,\infty}(\Omega) \quad \text{and} \quad |w|_{W^{s,\infty}(\Omega)} = \operatorname{diam}(\Omega)^{1-s} \quad \text{a.e. in } \Omega.$$

Then, we have that $w \in W^{s,p}(\Omega)$ for all $p \in (1,\infty)$.

On the other hand

$$\begin{aligned}
[w]^p_{W^{s,p}(\Omega)} &= \int_\Omega\int_\Omega \frac{|w(x)-w(y)|^p}{|x-y|^{n+ps}}\,dxdy \\
&= \int_\Omega\int_\Omega \frac{|\,|x-x_0|-|y-x_0|\,|^p}{|x-y|^{n+ps}}\,dxdy \\
&\leq \int_\Omega\int_\Omega |x-y|^{p(1-s)-n}\,dxdy \\
&\leq \frac{\kappa_n \operatorname{diam}(\Omega)^{p(1-s)}|\Omega|}{p(1-s)}.
\end{aligned}$$

This proves the lemma. $\square$

We carry out the proof of Theorem 1.2 in the two following lemmas.

Lemma 5.2. *Let Ω be a bounded open and connected domain in $\mathbb{R}^n$ and $s \in (0,1)$. Then*

$$\lim_{p\to\infty} \lambda_1(s,p)^{\frac{1}{p}} = \frac{2}{\operatorname{diam}(\Omega)^s} = \lambda_1(s,\infty) := \inf\left\{\frac{[u]_{W^{s,\infty}(\Omega)}}{\|u\|_{L^\infty(\Omega)}} : u \in \mathcal{A}\right\},$$

where $\mathcal{A} := \{u \in W^{s,\infty}(\Omega) : u \neq 0,\ \sup u + \inf u = 0\}$. Moreover, if u_p is the normalizer minimizer of $\lambda_1(1,p)$, then up to a subsequence, u_p converges in $C(\overline{\Omega})$ to some minimizer $u_\infty \in W^{s,\infty}(\Omega)$ of $\lambda_1(1,\infty)$.

Proof. We split the proof in three steps.

Step 1. Let us prove that

$$\limsup_{p\to\infty} \lambda_1(s,p)^{\frac{1}{p}} \leq \frac{2}{\operatorname{diam}(\Omega)^s}. \tag{5.1}$$

Let $x_0 \in \Omega$. We choose $c_p \in \mathbb{R}$ such that the function

$$w_p(x) = |x - x_0| - c_p$$

satisfies that

$$\int_\Omega |w_p(x)|^{p-2} w_p(x)\, dx = 0.$$

We can also observe that $w_p \in W^{s,p}(\Omega)$ for all $p \in (1, \infty)$. Then, by Lemma 5.1, for any $p \in (1, \infty)$ we have that

$$\lambda_1(s,p) \leq \frac{\int_\Omega \int_\Omega \frac{|w_p(x)-w_p(y)|^p}{|x-y|^{n+sp}}\, dxdy}{\int_\Omega |w_p(x)|^p\, dx} \leq \frac{\kappa_n^{\frac{1}{p}} \operatorname{diam}(\Omega)^{1-s} |\Omega|^{\frac{1}{p}}}{(p(1-s))^{\frac{1}{p}} \int_\Omega |w_p(x)|^p\, dx}.$$

Then

$$\limsup_{p\to\infty} \lambda_1(s,p)^{\frac{1}{p}} \leq \frac{\operatorname{diam}(\Omega)^{1-s}}{\liminf_{p\to\infty} \left(\int_\Omega |w_p(x)|^p\, dx\right)^{\frac{1}{p}}}. \tag{5.2}$$

On the other hand, proceeding as in the proof of Lemma 1 in [17], we have that

$$\liminf_{p\to\infty} \left(\int_\Omega |w_p(x)|^p\, dx\right)^{\frac{1}{p}} \geq \frac{\operatorname{diam}(\Omega)}{2}. \tag{5.3}$$

Thus, by (5.2) and (5.3), we have that (5.1) holds.

Step 2. Let us prove that

$$\inf\left\{ \frac{[u]_{W^{s,\infty}(\Omega)}}{\|u\|_{L^\infty(\Omega)}} : u \in \mathcal{A} \right\} \leq \liminf_{p\to\infty} \lambda_1(s,p)^{\frac{1}{p}}.$$

Let $\{p_j\}_{j\in\mathbb{N}}$ be an increasing sequence in $(1, \infty)$ and $\{u_j\}_{j\in\mathbb{N}}$ be a sequence of measurable functions such that $p_j \to \infty$ as $j \to \infty$,

$$\lim_{j\to\infty} \lambda_1(s,p_j)^{\frac{1}{p_j}} = \liminf_{p\to\infty} \lambda_1(s,p)^{\frac{1}{p}}, \tag{5.4}$$

and for any $j \in \mathbb{N}$ $u_j \in W^{s,p_j}(\Omega)$,

$$\|u_j\|_{L^{p_j}(\Omega)} = 1, \qquad \int_\Omega |u_j(x)|^{p_j-2} u_j(x)\, dx = 0,$$

and

$$\lambda_1(s,p_j) = \int_\Omega \int_\Omega \frac{|u_j(y) - u_j(x)|^{p_j}}{|x-y|^{n+sp_j}}\, dx\, dy. \tag{5.5}$$

Then, there exists a constant C independent of j such that

$$[u_j]_{W^{s,p_j}(\Omega)} \leq C \tag{5.6}$$

for all $j \in \mathbb{N}$.

Let us fix $q \in (1, \infty)$ such that $sq > 2n$. There exists $j_0 \in \mathbb{N}$ such that $p_j \geq q$ for all $j \geq j_0$. Then by Hölder's Inequality, we have that

$$\|u_j\|_{L^q(\Omega)} \leq |\Omega|^{\frac{1}{q}-\frac{1}{p_j}} \|u_j\|_{L^{p_j}(\Omega)} \leq |\Omega|^{\frac{1}{q}-\frac{1}{p_j}} \quad \forall j \geq j_0, \tag{5.7}$$

and taking $r = s - n/q \in (0, 1)$, again by Hölder's Inequality, we get

$$\begin{aligned}\int_\Omega\int_\Omega \frac{|u_j(x)-u_j(y)|^q}{|x-y|^{n+rq}}\,dxdy &= \int_\Omega\int_\Omega \frac{|u_j(x)-u_j(y)|^q}{|x-y|^{sq}}\,dxdy \\ &\le |\Omega|^{2(1-\frac{q}{p_j})}\left(\int_\Omega\int_\Omega \frac{|u_j(x)-u_j(y)|^{p_j}}{|x-y|^{sp_j}}\,dxdy\right)^{\frac{q}{p_j}} \\ &\le \operatorname{diam}(\Omega)^{\frac{nq}{p_j}}|\Omega|^{2(1-\frac{q}{p_j})}[u_j]^q_{W^{s,p_j}(\Omega)}. \end{aligned} \tag{5.8}$$

Then, by (5.6),

$$\int_\Omega\int_\Omega \frac{|u_j(x)-u_j(y)|^q}{|x-y|^{n+rq}}\,dxdy \le \operatorname{diam}(\Omega)^{\frac{nq}{p_j}}|\Omega|^{2(1-\frac{q}{p_j})}C^q \quad \forall j \ge j_0,$$

where C is a constant independent of j. Hence $\{u_j\}_{j\ge j_0}$ is a bounded sequence in $W^{r,q}(\Omega)$. Then, since $rq = sq - n > n$, by Theorem 2.5, there exist a subsequence of $\{u_j\}_{j\ge j_0}$, which we still denoted by $\{u_j\}_{j\ge j_0}$, and a function $u_\infty \in C(\overline{\Omega})$ such that

$$\begin{aligned} &u_j \to u_\infty \quad \text{uniformly in } \overline{\Omega}, \\ &u_j \rightharpoonup u_\infty \quad \text{weakly in } W^{r,q}(\Omega). \end{aligned}$$

Then, by (5.7), $\|u_\infty\|_{L^q(\Omega)} \le |\Omega|^{\frac{1}{q}}$, and by (5.4), (5.5) and (5.8), we get

$$\begin{aligned} [u_\infty]_{W^{r,q}(\Omega)} &\le \liminf_{j\to\infty}[u_j]_{W^{r,q}(\Omega)} \\ &\le \liminf_{j\to\infty}\operatorname{diam}(\Omega)^{\frac{n}{p_j}}|\Omega|^{2\left(\frac{1}{q}-\frac{1}{p_j}\right)}[u_j]_{W^{s,p_j}(\Omega)} \\ &\le |\Omega|^{\frac{2}{q}}\liminf_{p\to\infty}\lambda_1(s,p)^{\frac{1}{p}}. \end{aligned}$$

Letting $q \to \infty$, we get $\|u_\infty\|_{L^\infty(\Omega)} \le 1$ and

$$[u_\infty]_{W^{s,\infty}(\Omega)} \le \liminf_{p\to\infty}\lambda_1(s,p)^{\frac{1}{p}}. \tag{5.9}$$

On the other hand,

$$1 = \|u_j\|_{L^{p_j}(\Omega)} \le |\Omega|^{\frac{1}{p_j}}\|u_j\|_{L^\infty(\Omega)} \quad \forall j \ge j_0$$

then $1 \le \|u_\infty\|_{L^\infty(\Omega)}$. Hence $\|u_\infty\|_{L^\infty(\Omega)} = 1$ and by (5.9) we get

$$\frac{[u_\infty]_{W^{s,\infty}(\Omega)}}{\|u_\infty\|_{L^\infty(\Omega)}} \le \liminf_{p\to\infty}\lambda_1(s,p)^{\frac{1}{p}}. \tag{5.10}$$

Finally, in [17] it was proved that the condition $\int_\Omega |u_j(x)|^{p_j-2}u_j(x)\,dx = 0$ leads to $\sup u_\infty + \inf u_\infty = 0$. Then, using (5.10), we get

$$\inf\left\{\frac{[u]_{W^{s,\infty}(\Omega)}}{\|u\|_{L^\infty(\Omega)}} : u \in \mathcal{A}\right\} \le \liminf_{p\to\infty}\lambda_1(s,p)^{\frac{1}{p}}.$$

Step 3. Finally, we prove that

$$\frac{2}{\operatorname{diam}(\Omega)^s} \le \inf\left\{\frac{[u]_{W^{s,\infty}(\Omega)}}{\|u\|_{L^\infty(\Omega)}} : u \in \mathcal{A}\right\}. \tag{5.11}$$

For any $u \in \mathcal{A}$, we have

$$\begin{aligned} 2\|u\|_{L^\infty(\Omega)} &= \sup u - \inf u \\ &= \sup\{|u(x)-u(y)| : x, y \in \Omega\} \\ &= \sup\left\{|x-y|^s\frac{|u(x)-u(y)|}{|x-y|^s} : x, y \in \Omega\right\} \\ &\le \operatorname{diam}(\Omega)^s[u]_{W^{s,\infty}(\Omega)}. \end{aligned}$$

Thus

$$\frac{2}{\mathrm{diam}(\Omega)^s} \le \frac{[u]_{W^{s,\infty}(\Omega)}}{\|u\|_{L^\infty(\Omega)}}$$

for all $u \in \mathcal{A}$. Hence (5.11) holds.

Then, by steps 1–3, we get

$$\begin{aligned}\frac{2}{\mathrm{diam}(\Omega)^s} &\le \inf\left\{\frac{[u]_{W^{s,\infty}(\Omega)}}{\|u\|_{L^\infty(\Omega)}} : u \in \mathcal{A}\right\}\\ &\le \liminf_{p\to\infty} \lambda_1(s,p)^{\frac{1}{p}}\\ &\le \limsup_{p\to\infty} \lambda_1(s,p)^{\frac{1}{p}}\\ &\le \frac{2}{\mathrm{diam}(\Omega)^s},\end{aligned}$$

that is

$$\lim_{p\to\infty} \lambda_1(s,p)^{\frac{1}{p}} = \frac{2}{\mathrm{diam}(\Omega)^s} = \inf\left\{\frac{[u]_{W^{s,\infty}(\Omega)}}{\|u\|_{L^\infty(\Omega)}} : u \in \mathcal{A}\right\}.$$

In addition, by (5.10), we have that u_∞ is a minimizer of $\lambda_1(1,\infty)$ which proves the lemma. □

Our last aim is to show that u_∞ is a viscosity solution of (1.3). We start by intruding the definition of viscosity solution.

Definition 5.3. Suppose that $u \in C(\Omega)$. We say that u is a *viscosity super-solution* (resp. *viscosity sub-solution*) in Ω of Eq. (1.3) if the following holds: whenever $x_0 \in \Omega$ and $\varphi \in C^1(\overline{\Omega})$ are such that

$$\varphi(x_0) = u(x_0) \quad \text{and} \quad \varphi(x) \le u(x) \quad (\text{resp. } \varphi(x) \ge u(x)) \text{ for all } x \in \mathbb{R}^n$$

then we have

$$\begin{cases} \max\{\mathcal{L}_{s,\infty}\varphi(x_0), \mathcal{L}^-_{s,\infty}\varphi(x_0) + \lambda_1(1,\infty)\varphi(x_0)\} \le 0 \text{ (resp. } \ge 0) & \text{if } \varphi(x_0) > 0\\ \mathcal{L}_{s,\infty}\varphi(x_0) \le 0 \text{ (resp. } \ge 0) & \text{if } \varphi(x_0) = 0\\ \min\{\mathcal{L}_{s,\infty}\varphi(x_0), \mathcal{L}^+_{s,\infty}\varphi(x) + \lambda_1(1,\infty)\varphi(x_0)\} \le 0 \text{ (resp. } \ge 0) & \text{if } \varphi(x_0) < 0.\end{cases}$$

A *viscosity solution* is defined as being both a viscosity super-solution and a viscosity sub-solution.

For the proof of the following lemma we borrow ideas from [23, Theorem 23].

Lemma 5.4. *Let Ω be a bounded open connected domain in $\mathbb{R}^n$ and $s \in (0,1)$. Then u_∞ is a solution of* (1.3) *in the viscosity sense.*

Proof. We begin by observing that, by Lemma 5.2, u_∞ is a minimizer of $\lambda_1(1,\infty)$ and there exists a sequence $\{p_j\}_{j\in\mathbb{N}}$ such that $p_j \to \infty$ and $u_j \to u_\infty$ uniformly in $\overline{\Omega}$ as $j \to \infty$, where u_j is an eigenfunction associated to $\lambda_1(s,p_j)$. Without loss of generality, we can assume that $p_j s > n$ for all $j \in \mathbb{N}$. Then $u_j \in C(\overline{\Omega})$ for all $j \in \mathbb{N}$.

We only verify that u_∞ is a viscosity super-solution of (1.3). The proof that u_∞ is also a sub-solution is similar. Let us fix some point $x_0 \in \Omega$. We assume that φ is a test function touching u_∞ from below at a point x_0, and we may assume that the touching is strict by considering $\varphi(x) - |x|^2\eta(x)$, where $\eta = 1$ in a neighborhood of x_0 and $\eta \ge 0$. It follows that $u_j - \varphi$ attains its minimum at points $x_j \to x_0$. By adding a suitable constant c_j we can arrange it so that $\varphi + c_j$ touches u_j from below at the point x_j.

By Theorem 3.3, an eigenfunction is a viscosity solution of (3.3), then we have

$$\mathcal{L}_{s,p_j}\varphi(x_j) + \lambda_1(s,p_j)u_j^{p_j-1}(x_j) \le 0.$$

We write the last inequality as

$$A_j^{p_j-1} - B_j^{p_j-1} + C_j^{p_j-1} - D_j^{p_j-1} \le 0$$

where

$$A_j^{p_j-1} = 2\int_\Omega \frac{|\varphi(y)-\varphi(x_j)|^{p_j-2}(\varphi(y)-\varphi(x_j))^+}{|y-x_j|^{n+sp_j}}\,dy,$$

$$B_j^{p_j-1} = 2\int_\Omega \frac{|\varphi(y)-\varphi(x_j)|^{p_j-2}(\varphi(y)-\varphi(x_j))^-}{|y-x_j|^{n+sp_j}}\,dy,$$

$$C_j^{p_j-1} = \lambda_1(s, p_j)(u_j^+(x_j))^{p_j-1},$$
$$D_j^{p_j-1} = \lambda_1(s, p_j)(u_j^-(x_j))^{p_j-1}.$$

In [10, Lemma 6.5], it is proved that

$$A_j \to \mathscr{L}^+_{s,\infty}\varphi(x_0), \qquad B_j \to -\mathscr{L}^-_{s,\infty}\varphi(x_0),$$

as $j \to \infty$. In addition, by Lemma 5.2, we have

$$C_j \to \lambda_1(s,\infty)\varphi(x_0)^+, \qquad D_j \to \lambda_1(s,\infty)\varphi(x_0)^-.$$

On the other hand, if $u_\infty(x_0) > 0$ we get

$$A_j^{p_j-1} + C_j^{p_j-1} \le B_j^{p_j-1},$$

and by dropping either $A_j^{p_j-1}$ or $C_j^{p_j-1}$, and sending $j \to \infty$ we see that

$$\mathscr{L}^+_{s,\infty}\varphi(x_0) \le -\mathscr{L}^-_{s,\infty}\varphi(x_0) \quad \text{and} \quad \lambda_1(s,\infty)\varphi(x_0)^+ \le -\mathscr{L}^-_{s,\infty}\varphi(x_0),$$

which leads to

$$\mathscr{L}_{s,\infty}\varphi(x_0) \le 0 \quad \text{and} \quad \mathscr{L}^-_{s,\infty}\varphi(x_0) + \lambda_1(s,\infty)\varphi(x_0)^+ \le 0,$$

and we can write

$$\max\{\mathscr{L}_{s,\infty}\varphi(x_0), \mathscr{L}^-_{s,\infty}\varphi(x_0) + \lambda_1(s,\infty)\varphi(x_0)^+\} \le 0.$$

If $u_\infty(x_0) < 0$ we obtain that

$$A_j^{p_j-1} \le D_j^{p_j-1} + B_j^{p_j-1} \le 2\max\{B_j^{p_j-1}, D_j^{p_j-1}\},$$

that is

$$A_j \le 2^{\frac{1}{p_j-1}} \max\{B_j, D_j\}.$$

Then, sending $j \to \infty$, we get

$$\mathscr{L}_{s,\infty}\varphi(x_0) \le 0 \quad \text{or} \quad \mathscr{L}^+_{s,\infty}\varphi(x_0) - \lambda_1(s,\infty)\varphi(x_0)^- \le 0,$$

which can be written as

$$\min\{\mathscr{L}_{s,\infty}\varphi(x_0),\ \mathscr{L}^+_{s,\infty}\varphi(x_0) - \lambda_1(s,\infty)\varphi(x_0)^-\} \le 0.$$

Finally if $u_\infty(x_0) = 0$, it follows that $\mathscr{L}_{s,\infty}\varphi(x_0) \le 0$. This proves that u_∞ is a viscosity super-solution of Eq. (1.3). $\square$

6. Comments

Let $d(\cdot,\cdot)$ be a distance equivalent to the usual distance. If we take the following nonlinear non-local operator

$$\mathfrak{L}_{s,p}u(x) := 2\,\text{p.v.} \int_\Omega \frac{|u(y)-u(x)|^{p-2}(u(y)-u(x))}{d(x,y)^{n+sp}}\,dy,$$

in place of $\mathscr{L}_{s,p}$, following what was done in the previous section, we can see that the first non-zero eigenvalue of

$$\begin{cases} -\mathfrak{L}_{s,p}u = \lambda|u|^{p-2}u & \text{in } \Omega, \\ u \in W^{s,p}(\Omega), \end{cases}$$

is

$$\lambda_1^d(s,p) := \inf\left\{ \frac{\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{d(x,y)^{n+sp}}\,dx\,dy}{\int_\Omega |u(x)|^p\,dx} : u \in \mathfrak{X}_{s,p} \right\}.$$

Moreover

$$\lim_{p\to\infty}\left(\lambda_1^d(s,p)\right)^{\frac{1}{p}} = \frac{2}{\operatorname{diam}_d(\Omega)^s} = \lambda_1^d(s,\infty) := \inf\left\{ \frac{[u]_{d,W^{s,\infty}(\Omega)}}{\|u\|_{L^\infty(\Omega)}} : u \in \mathcal{A} \right\}$$

where

$$[u]_{d,W^{s,\infty}(\Omega)} = \sup\left\{ \frac{|u(x)-u(y)|}{d_\Omega(x,y)^s} : x, y \in \Omega \right\}$$

and $\operatorname{diam}_d(\Omega) = \sup\{d(x,y) : x, y \in \Omega\}$.

Finally, observe that if d is the geodesic distance inside Ω then $\operatorname{diam}_d(\Omega)$ is the intrinsic diameter as in the local case.

Acknowledgments

We want to thank Prof. Nicolas Saintier and Prof. Julio Rossi for their comments that helped us to improve Section 5.

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