SECOND-ORDER LOCAL MULTIPLIER ALGEBRAS OF CONTINUOUS TRACE C*-ALGEBRAS

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Abstract. We determine the injective envelope and local multiplier algebra of a continuous trace C*-algebra A that arises from a continuous Hilbert bundle over an arbitrary locally compact Hausdorff space. In addition, we show that the second-order local multiplier algebra $M_{\mathrm{loc}}^{[2]}(A)$ of any such algebra A is injective.

Introduction

The injective envelope $I(A)$ of a C*-algebra A [16] provides a useful ambient C*-algebra in which one can analyse the multipliers of essential ideals of A. In fact the local multiplier algebra $M_{\mathrm{loc}}(A)$ of A [2] can be obtained from the injective envelope of A by considering the C*-subalgebra of all $x \in I(A)$ for which x is a norm-limit of a sequence $x_n \in M(I_n)$ for various essential ideals I_n of A [14]. However, $I(A)$ and $M_{\mathrm{loc}}(A)$ are difficult to determine precisely, even if one has extensive knowledge about A itself. Indeed, on page 55 of [9], D. Blecher writes, "Thus the injective envelope is mostly useful as an abstract tool because of the properties it possesses; one cannot hope to concretely be able to say what it is." Anyone who has worked with injective envelopes will find this comment completely understandable. Nevertheless, in this paper we determine explicitly (Theorem 6.6) the injective envelope of a continuous trace C*-algebra A of the spatial type considered by Fell [12]. We then use the embedding of the local multiplier algebra of A into its injective envelope to prove that the second-order local multiplier algebras of such A are injective (Theorem 6.7). An immediate consequence of this last result is that the second-order local multiplier algebra of $C_0(T) \otimes \mathbb{K}$ is injective for every locally compact Hausdorff space T, a fact which was known previously to hold only under certain assumptions about the topology of T [5, 25]. The results of this paper complete a line of investigation that started with [6] and was continued in [7].

In the case of an abelian C*-algebra $A = C_0(T)$, to determine the local multiplier algebra and the injective envelope of A one must pass from T to a Stonean space obtained from T by performing an inverse limit $\Delta = \varprojlim \beta U$ of Stone–Čech compactifications βU of dense open subsets U of T. This passage from T to a Stonean space Δ cannot be avoided if one aims to compute explicitly the enveloping C*-algebras $M_{\mathrm{loc}}(A)$ and $I(A)$ in the case of arbitrary continuous trace C*-algebras A of the type studied by Fell. In [7] we determined $M_{\mathrm{loc}}(A)$ and $I(A)$ for those A in which the spectrum $T = \hat{A}$ was assumed to be a Stonean space. In this paper

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we make the passage from an arbitrary locally compact Hausdorff space T to its projective hull $\Delta = \lim\limits_{\leftarrow} \beta U$. Doing so entails the determination of a new continuous Hilbert bundle over Δ which is obtained as a direct limit of continuous Hilbert bundles over the compact spaces βU for all dense open subsets U of T. Because the essential ideals of A are parameterised by dense open subsets of T, this direct limit of Hilbert bundles also induces a direct limit A^{Δ} of continuous trace C*-algebras. It is through these limiting bundles and algebras that we obtain our main results.

The paper is organised as follows. In Section §1 the algebras and structures under study are introduced. The three subsequent sections treat the limiting processes of Hilbert bundles and C*-algebras that accompany the passage from T to $\Delta = \lim\limits_{\leftarrow} \beta U$. In particular, in Section §2 we note the bijective correspondence between essential ideals I of A and dense open subsets $X^I \subset T$ and represent each essential ideal I of A as an essential ideal of a spatial continuous trace C*-algebra A^I with spectrum βX^I. Section §3 constructs a continuous Hilbert bundle Ω^{Δ} over the Stonean space Δ by way of a direct limit of Hilbert bundles Ω^I over βX^I affiliated with each essential ideal I of A. In Section §4 we construct a direct limit C*-algebra $\lim\limits_{\rightarrow} A^I$ and in Section §5 we show that $A \subset \lim\limits_{\rightarrow} A^I \subset M_{\rm loc}(A)$. The main results concerning the determination of the injective envelope of A and the injectivity of the second-order local multiplier algebra $M_{\rm loc}^{[2]}(A)$ are given in Section §6.

The initial interest in local multiplier algebras originates in Pedersen's paper [21] on extending derivations of a C*-algebra A to its local mutliplier algebra $M_{\rm loc}(A)$. This work on derivations is one example of the role of local multiplier algebras in the theory of operators acting on C*-algebras (see [2] for a fuller account). There is one fairly substantial unresolved problem that dates back to 1978: if A is a separable C*-algebra, then is every derivation of $M_{\rm loc}(A)$ inner? One consequence of our main results is a related weaker assertion: for each spatial continuous trace C*-algebra A, every derivation of $M_{\rm loc}^{[2]}(A)$ is inner (Corollary 6.9).

1. Preliminaries

When referring to ideals of a C*-algebra, we shall always mean ideals which are closed in the norm topology. The term homomorphism is understood to be with respect to the category of C*-algebras, meaning that homomorphisms of C*-algebras are $*$-homomorphisms, and are unital homomorphisms if the algebras involved are unital.

Essential ideals and local multiplier algebras. Recall that an ideal K of a C*-algebra A is an *essential ideal* if $K \cap J \neq \{0\}$ for every nonzero ideal J of A.

Let $\mathcal{I}_{\rm ess}(A)$ be the set of all essential ideals of A, which we consider as a directed set under the partial order $\preccurlyeq$ defined by $J \preccurlyeq I$ if and only if $I \subset J$.

For each $I \in \mathcal{I}_{\rm ess}(A)$, let $M(I)$ denote its multiplier algebra. If $I, J \in \mathcal{I}_{\rm ess}(A)$ are such that $I \subset J$, then there is a unique monomorphism

$$\varrho_{JI} : M(J) \to M(I) \quad \text{such that} \quad \iota_I = \varrho_{JI} \circ \iota_{J|I}\,, \tag{1}$$

where $\iota_K : K \to M(K)$ denotes the canonical embedding of K into $M(K)$. Hence, $(\mathcal{I}_{\rm ess}(A), \{M(I)\}_I, \{\varrho_{JI}\}_{J\preccurlyeq I})$ is a direct system of C*-algebras and monomorphisms, and the direct limit C*-algebra of this system is denoted by

$$M_{\rm loc}(A) \;=\; \lim_{\rightarrow} M(I)\,.$$

The C*-algebra $M_{\mathrm{loc}}(A)$ is called the *local multiplier algebra* of A.

One can consider the local multiplier algebra of $M_{\mathrm{loc}}(A)$, and so forth, thereby yielding higher order local multiplier algebras. So we write

$$M_{\mathrm{loc}}^{[k]}(A) \;=\; M_{\mathrm{loc}}\left(M_{\mathrm{loc}}^{[k-1]}(A)\right)\,,\;\forall\, k\in\mathbb{N}\,,$$

where $M_{\mathrm{loc}}^{[0]}(A)$ is taken to be A. Although very little is known about the sequence $\{M_{\mathrm{loc}}^{[k]}(A)\}_{k\in\mathbb{N}}$, it is known that the sequence becomes constant if for some k_0 the C*-algebra $M_{\mathrm{loc}}^{[k_0]}(A)$ is an AW*-algebra—for in this case, $M_{\mathrm{loc}}^{[k]}(A) = M_{\mathrm{loc}}^{[k_0]}(A)$ for every $k\geq k_0$ [2, Theorem 2.3.8]. Only relatively recently has it been discovered [3, 4, 6] that $M_{\mathrm{loc}}^{[2]}(A)$ need not coincide with $M_{\mathrm{loc}}(A)$, and the reasons for this gap are just now starting to be understood [5].

Injective envelopes. An *injective C*-algebra* is a unital C*-algebra C with the property that, for any triple (B,D,κ) of unital C*-algebras B, D and unital completely isometric linear map $\kappa: B\rightarrow D$, every unital completely positive (ucp) linear map $\phi: B\rightarrow C$ extends to a ucp $\Phi: D\rightarrow C$ such that $\phi=\Phi\circ\kappa$ [8, §IV.2]. If A is an arbitrary C*-algebra, not necessarily unital, then an *injective envelope* of A is a pair (C,α) such that C is an injective C*-algebra, $\alpha: A\rightarrow C$ is a monomorphism which is assumed to be unital if A is unital, with the property that if $\tilde{C}$ is an injective C*-algebra with $\alpha(A)\subset\tilde{C}\subset C$, then $\tilde{C}=C$. Every C*-algebra has an injective envelope, and any two injective envelopes (C,α) and (C_1,α_1) of A are isomorphic by an isomorphism $\varphi: C\rightarrow C_1$ for which $\varphi\circ\alpha=\alpha_1$ [16].

Thus, we may refer generically to "the" injective envelope of A, which we denote by $I(A)$. The injective envelope of A and the local multiplier algebras of A are related by way of the C*-algebra inclusions

$$A\;\subset\; M_{\mathrm{loc}}^{[k]}(A)\;\subset\; M_{\mathrm{loc}}^{[k+1]}(A)\;\subset\; I(A)\,,\;\forall\, k\in\mathbb{N}\,, \tag{2}$$

where the inclusions are as unital C*-subalgebras, except for the first inclusion if A is nonunital. These inclusions are uniquely determined by the inclusion (embedding) $\alpha: A\rightarrow I(A)$ of A in $I(A)$. More explicitly, $M_{\mathrm{loc}}(A)$ is the closure in $I(A)$ of the union of all the idealizers in $I(A)$ of all essential ideals of A [14].

C-modules.* The Hilbert C*-modules [8, §II.7] that we use are left modules E over an abelian C*-algebra Z. Recall that $B(E)$ denotes the C*-algebra of bounded, adjointable endomorphisms of E and $K(E)$ denotes the set of compact elements of $B(E)$—namely, the norm closure of the linear space $\mathcal{F}(E)$ of all elements (called *finite-rank* endomorphisms) obtained through finite sums of endomorphisms of the form $\Theta_{\omega,\nu}$, where $\omega,\nu\in E$ and $\Theta_{\omega,\nu}\xi \;=\; \langle\xi,\nu\rangle\cdot\omega$, for all $\xi\in E$. The pertinent facts we require are: $K(E)$ is an essential ideal of $B(E)$ and $B(E)$ is the multiplier algebra of $K(E)$. We will also use the fact that $\mathcal{F}(E)$ is a left Z-module via $f\cdot\Theta_{\omega,\nu}=\Theta_{f\cdot\omega,\nu}$, for $f\in Z$.

Topology. Throughout we shall assume that T denotes a locally compact Hausdorff space. As usual, $C(T)$, $C_b(T)$, and $C_0(T)$ denote, respectively, the involutive algebras of all continuous complex-valued functions on T, all bounded $f\in C(T)$, and all $f\in C(T)$ that vanish at infinity respectively.

Vector and operator fields. Assume that $(T, \{H_t\}_{t\in T})$ and $(T, \{B(H_t)\}_{t\in T})$ are fibred spaces where each H_t is a Hilbert space. A cross section of $(T, \{H_t\}_{t\in T})$ is a vector field $\nu : T \to \bigsqcup_t H_t$ in which $\nu(t) \in H_t$, for every $t \in T$. Likewise, a cross section of $(T, \{B(H_t)\}_{t\in T})$ is an operator field $x : T \to \bigsqcup_t B(H_t)$ such that $x(t) \in B(H_t)$, for every $t \in T$.

For such cross sections ν, x, we define functions $\check{\nu}, \check{x} : T \to \mathbb{R}$ by

$$\check{\nu}\,(t) \;=\; \|\nu(t)\|\,, \quad \check{x}\,(t) \;=\; \|x(t)\|\,.$$

We say that ν is bounded if $\sup_{t\in T} \check{\nu}(t) \;<\; \infty$. The boundedness of x is defined analogously.

A *continuous Hilbert bundle* [10] is a triple $(T, \{H_t\}_{t\in T}, \Omega)$, where Ω is a set of vector fields on T with fibres H_t such that:

(I) Ω is a $C(T)$-module with the action $(f \cdot \omega)(t) = f(t)\omega(t)$;
(II) for each $t \in T$, $\{\omega(t) : \ \omega \in \Omega\} = H_t$;
(III) $\check{\omega} \in C(T)$, for all $\omega \in \Omega$;
(IV) Ω is closed under local uniform approximation—that is, if $\xi : T \to \bigsqcup_t H_t$ is any vector field such that for every $t_0 \in T$ and $\varepsilon > 0$ there is an open set $U \subset T$ containing t_0 and a $\omega \in \Omega$ with $\|\omega(t) - \xi(t)\| < \varepsilon$ for all $t \in U$, then necessarily $\xi \in \Omega$.

Given a continuous Hilbert bundle $(T, \{H_t\}_{t\in T}, \Omega)$, let

$$\Omega_b \;=\; \{\omega \in \Omega \,:\, \check{\omega} \in C_b(T)\} \quad \text{and} \quad \Omega_0 \;=\; \{\omega \in \Omega \,:\, \check{\omega} \in C_0(T)\}\,. \tag{3}$$

It is easy to see that Ω_b and Ω_0 are Hilbert C*-modules over $C_b(T)$ and $C_0(T)$ respectively, where the inner product $\langle\omega_1, \omega_2\rangle$ of $\omega_1, \omega_2 \in \Omega$ is the continuous function

$$\langle\omega_1, \omega_2\rangle\,(t) \;=\; \langle\omega_1(t), \omega_2(t)\rangle\,,\ t \in T\,.$$

Spatial continuous trace C-algebras.* We now describe the class of C*-algebras of interest in this paper.

Assume that $(T, \{H_t\}_{t\in T}, \Omega)$ is a continuous Hilbert bundle. An operator field a is *almost finite-dimensional* with respect to this bundle if for each $t_0 \in T$ and $\varepsilon > 0$ there exist an open set $U \subset T$ containing t_0 and $\omega_1, \dots, \omega_n \in \Omega$ such that

(a) $\omega_1(t), \dots, \omega_n(t)$ are linearly independent for every $t \in U$, and
(b) $\|p_t a(t) p_t - a(t)\| < \varepsilon$ for all $t \in U$, where $p_t \in B(H_t)$ is the projection with range Span $\{\omega_j(t) \,:\, 1 \le j \le n\}$.

Moreover, a is *weakly continuous* if the complex-valued function

$$t \;\mapsto\; \langle a(t)\omega_1(t), \omega_2(t)\rangle$$

is continuous for every $\omega_1, \omega_2 \in \Omega$.

We denote by $A = A(T, \{H_t\}_{t\in T}, \Omega)$ the C*-algebra, with respect to pointwise operations and norm $\|a\| = \max\{\|a(t)\| \,:\, t \in T\}$, of all weakly continuous almost finite-dimensional operator fields a for which $\check{a} \in C_0(T)$. Such C*-algebras A were studied by Fell [12], and he proved that each such A is a continuous trace C*-algebra with spectrum $\hat{A} \simeq T$ [12, Theorems 4.4, 4.5]. We call the algebra A the *Fell*, or the *spatial*, continuous trace C*-algebra associated with the Hilbert bundle $(T, \{H_t\}_{t\in T}, \Omega)$.

2. Extended Representations of Essential Ideals

Let $(T, \{H_t\}_{t\in T}, \Omega)$ be a continuous Hilbert bundle over a locally compact Hausdorff space T. Suppose that I is an arbitrary ideal of $A = A(T, \{H_t\}_{t\in T}, \Omega)$. In this section we shall construct a continuous Hilbert bundle $(\beta X^I, \{H_t^I\}_{t\in\beta X^I}, \Omega^I)$ over the Stone–Čech compactification βX^I of X^I. Moreover, if we let A^I be the Fell continuous C*-algebra associated with this bundle we shall show that I embeds into A^I as an essential ideal.

Let $Z^I \subset T$ denote the closed set

$$Z^I = \{t \in T : b(t) = 0, \forall\, b \in I\},$$

and let X^I be the open set $X^I = T \setminus Z^I$. The open set X^I is homeomorphic to both the primitive ideal space $\mathrm{Prim}\, I$ and to the spectrum $\hat{I}$ of I [22, Proposition A.27]. Moreover, I is an essential ideal of A if and only if X^I is dense in T.

Recall that $\Omega_b = \{\omega \in \Omega : \check{\omega} \in C_b(T)\}$ is a $C_b(T)$-module, and define a normed vector space $\Omega_b|_{X^I}$ of bounded restricted vector fields by

$$\Omega_b|_{X^I} = \{\omega|_{X^I} : \omega \in \Omega_b\}. \tag{4}$$

For any pair $\omega, \nu \in \Omega_b|_{X^I}$, let $\phi^I_{\omega,\nu} : X^I \to \mathbb{C}$ be given by

$$\phi^I_{\omega,\nu}(t) = \langle \omega(t), \nu(t)\rangle, \quad t \in X^I.$$

This map is continuous and bounded, and so $\phi^I_{\omega,\nu}$ extends to a unique continuous map $\tilde{\phi}^I_{\omega,\nu} : \beta X^I \to \mathbb{C}$. By uniqueness of this continuous extension, the form $\langle\cdot,\cdot\rangle^I_t$ on $\Omega_b|_{X^I}$ defined by

$$\langle \omega, \nu\rangle^I_t = \tilde{\phi}^I_{\omega,\nu}(t), \quad t \in \beta X^I,$$

is a pre-inner product on $\Omega_b|_{X^I}$ for each $t \in \beta X^I$. Let H^I_t denote the Hilbert space completion of $\Omega_b|_{X^I}/\mathcal{N}^I_t$, where

$$\mathcal{N}^I_t = \{\omega \in \Omega_b|_{X^I} : \tilde{\phi}^I_{\omega,\omega}(t) = 0\}.$$

If $\overline{\omega}^I(t)$ denotes the equivalence class of $\omega \in \Omega_b|_{X^I}$ in H^I_t, then for $t \in X^I$ the map $\overline{\omega}^I(t) \mapsto \omega(t)$ is well defined and is an isometric isomorphism from $\Omega_b|_{X^I}/\mathcal{N}^I_t$ onto H_t. Thus, we shall identify $H^I_t = H_t$ for every $t \in X^I$ so that, under this identification, we have $\overline{\omega}^I(t) = \omega(t)$. Hence, for every $\omega \in \Omega_b$ we have a bounded vector field

$$\overline{\omega}^I : \beta X^I \to \bigsqcup_{t\in\beta X^I} H^I_t$$

We shall consider

$$\mathcal{E}^I = \{\overline{\omega}^I : \omega \in \Omega_b|_{X^I}\}, \tag{5}$$

which is a vector space of bounded vector fields for which $t \mapsto \|\overline{\omega}^I(t)\|$ is continuous on βX^I.

Definition 2.1. *Let Ω^I denote the set of all vector fields $\nu : \beta X^I \to \bigsqcup_{t\in\beta X^I} H^I_t$ with the property that for every $t_0 \in \beta X^I$ and $\varepsilon > 0$ there is an open set $U \subset \beta X^I$ containing t_0 and a vector field $\overline{\omega}^I \in \mathcal{E}^I$ such that $\|\nu(t) - \overline{\omega}^I(t)\| < \varepsilon$ for every $t \in U$.*

We shall say that each $\nu \in \Omega^I$, as defined above, is a *local uniform limit* of vector fields in $\mathcal{E}^I$.

Proposition 2.2. *$(\beta X^I, \{H^I_t\}_{t\in\beta X^I}, \Omega^I)$ is a continuous Hilbert bundle.*

Proof. Because each $\nu \in \Omega^I$ is a local uniform limit of vector fields in $\mathcal{E}^I$, axiom (III) on the continuity of the map $\check{\nu}$ and axiom (IV) on the closure of Ω^I under local uniform limits are easily verified.

In order to prove axiom (I), let $f \in C(\beta X^I)$ and let $\nu \in \Omega^I$, and consider the bounded vector field $f \cdot \nu$ defined by $f \cdot \nu(t) = f(t)\nu(t)$, $t \in \beta X^I$. Assume $t_0 \in \beta X^I$ and let $\varepsilon > 0$ be given. By the continuity of f and the definition of Ω^I, there are an open neighbourhood U of t_0 in βX^I and a $\overline{\eta}^I \in \mathcal{E}^I$ such that, for all $t \in U$, $|f(t) - f(t_0)| < \epsilon/2$ and $\|\nu(t) - \overline{\eta}^I(t)\| < \epsilon/2$. Therefore,

$$\|f \cdot \nu(t) - f(t_0)\, \overline{\eta}^I(t)\| \;<\; \epsilon(\|\nu\| + \|f\|)\,, \quad \forall\, t \in U\,.$$

Thus, $f \cdot \nu$ is a local uniform limit of vector fields in $\mathcal{E}^I$—hence, an element of Ω^I. This proves that Ω^I is a $C(\beta X^I)$-module under the pointwise action.

That leaves axiom (II). However, in the presence of axioms (I), (III), and (IV), the axiom (II) is equivalent to the axiom that $\{\nu(t) : \nu \in \Omega^I\}$ be dense in H_t^I, for each $t \in \beta X^I$ [10]. This seemingly weaker axiom is satisfied by Ω^I because $\{\overline{\omega}^I(t) : \overline{\omega}^I \in \mathcal{E}^I\}$ is dense in H_t^I for each $t \in \beta X^I$. □

Definition 2.3. *If I is an ideal of A, we write $A^I = A(\beta X^I, \{H_t^I\}_{t\in\beta X^I}, \Omega^I)$ for the spatial continuous trace C^*-algebra associated with the continuous Hilbert bundle $(\beta X^I, \{H_t^I\}_{t\in\beta X^I}, \Omega^I)$ (see the last paragraph of section 1).*

Notational Convention. Assume that U is an open subset of T an let $f \in C(T)$. We shall write that $f \in C_0(U)$ whenever f is an element of the ideal $J = \{g \in C(T) : g(t) = 0,\ \forall\, t \in T \setminus U\}$. Conversely, note that every $h \in C_0(U)$ extends to a continuous function $h : T \to \mathbb{C}$ by defining $h(t) = 0$ for $t \in T \setminus U$. Thus, we shall sometimes consider h as an element of $C_b(T)$.

Lemma 2.4. *Let I be an essential ideal of A and suppose that $a \in A$. Then $a \in I$ if and only if $\check{a} \in C_0(X^I)$.*

Proof. For each $t \in T$ let $A_t = \{a(t) : a \in A\}$; by [12, Theorem 4.4], $A_t = K(H_t)$, the simple C*-algebra of compact operators acting on H_t. Next, let $I_t = \{b(t) : b \in I\} \subset A_t$. By [12, Lemma 1.8], if $a \in A$, then $a \in I$ if and only if $a \in I_t$ for all $t \in T$. Because I_t is an ideal of A_t, we conclude that $I_t = \{0\}$ for $t \in Z^I$ and $I_t = A_t$ for $t \in X^I$. Hence, a necessary and sufficient condition for a to belong to I is that $a(t) = 0$ for all $t \in Z^I$. That is, $a \in I$ if and only if $\check{a} \in C_0(X^I)$. □

Proposition 2.5. *There exists a monomorphism $\delta_I : I \to A^I$ such that*

- **(i)** *$\delta_I(I)$ is an essential ideal of A^I,*
- **(ii)** *$\delta_I(a)\,(t) \,=\, a(t)$, for all $a \in I$ and $t \in X^I$, and*
- **(iii)** *$\delta_I(a)\,(t) \,=\, 0$, for all $a \in I$ and $t \in \beta X^I \setminus X^I$*

Proof. The topological space X^I is regarded now as an open dense subset of βX^I; hence, $C_0(X^I)$ is an essential ideal of $C(\beta X^I)$.

For every $a \in I$, define an operator field $\mathfrak{a} : \beta X^I \to \bigsqcup_{t\in\beta X^I} K(H_t^I)$ by $\mathfrak{a}_{|X^I} = a_{|X^I}$ and $\mathfrak{a}(t) = 0$ for all $t \in \beta X^I \setminus X^I$. We show below that $\mathfrak{a} \in A^I$.

By Lemma 2.4, $\check{a} \in C_0(X^I)$. Thus, $\check{\mathfrak{a}}_{|X^I} \in C_0(X^I)$ and satisfies $\check{\mathfrak{a}}(t) = 0$ for all $t \in \beta X^I \setminus X^I$. Hence, $\check{\mathfrak{a}} \in C(\beta X^I)$.

To prove that $\mathfrak{a}$ is a weakly continuous operator field, it is sufficient to verify the weak continuity condition in vector fields in $\mathcal{E}^I$, as every $\nu \in \Omega^I$ is a local uniform limit of vector fields in $\mathcal{E}^I$. To this end, let $\omega, \eta \in \Omega_b$ and consider the function

$h(t) = \langle \mathfrak{a}(t)\,\overline{\omega}^I(t), \overline{\eta}^I(t)\rangle$, $t \in \beta X^I$. Restricted to X^I, h is continuous (since $a \in A$) and vanishes at infinity. As noted earlier, the facts $h_{|X^I} \in C_0(X)$ and $h(t) = 0$ for all $t \in \beta X^I \setminus X^I$ imply that $h \in C(\beta X^I)$. Thus, $\mathfrak{a}$ is a weakly continuous operator field.

Lastly, we show that $\mathfrak{a}$ is approximately finite-dimensional with respect to Ω^I. Notice that a has this property (with respect to $\Omega_b|_{X^I}$) on X^I. Thus, at every point $t_0 \in X^I$ and for every $\varepsilon > 0$ there will be an open neighbourhood U of t_0 in X^I such that $\mathfrak{a}$ is approximately finite-dimensional with respect to Ω^I to within ε on U. Assume now $t_0 \in \beta X^I \setminus X^I$ and let $\varepsilon > 0$. Since $\check{\mathfrak{a}}(t_0) = 0$, there is an open set $U \subset \beta X^I$ containing t_0 such that $0 \le \check{\mathfrak{a}}(t) < \varepsilon$ for all $t \in U$. This shows that $\check{\mathfrak{a}}$ is approximately finite-dimensional with respect to Ω^I to within ε on U. This completes the proof that $\mathfrak{a} \in A^I$.

Now define $\delta_I : I \to A^I$ by $\delta_I(a) = \mathfrak{a}$. Clearly δ_I is a homomorphism. Because $a(s) = 0$ for all $s \in T \setminus X^I$, we have $\|a\| = \max_{t \in X^I} \|a(t)\|$. Thus,

$$\|\delta_I(a)\| \;=\; \max_{t\in\beta X^I} \|\mathfrak{a}(t)\| \;=\; \max_{t\in X^I} \|\mathfrak{a}(t)\| \;=\; \max_{t\in X^I} \|a(t)\| \;=\; \|a\|\,,$$

which shows that δ_I is a monomorphism.

It remains to prove that $\delta_I(I)$ is an essential ideal of A^I. Let

$$\mathfrak{I} \;=\; \{y \in A^I \mid y(t) = 0\,,\ \forall\, t \in \beta X^I \setminus X^I\}\,,$$

which is an essential ideal of A^I that contains $\delta_I(I)$. We claim that $\mathfrak{I} = \delta_I(I)$. Indeed, let $y \in \mathfrak{I}$. Thus, $\check{y} \in C_0(X^I)$, where X^I is viewed as an open dense subset of βX^I. Since $y \in A^I$ then $y|_{X^I}$ is an operator field which is almost finite-dimensional with respect to $\Omega_b|_{X^I}$ on X^I. If we now consider X^I as a dense open subset of T we conclude that $y_{|X^I}$ extends to an element $y_{\text{ext}} \in A$ such that $y_{\text{ext}}(s) = 0$ for all $s \in T \setminus X^I$. Moreover, by Lemma 2.4 $y_{\text{ext}} \in I$. It is clear that in this case $\delta_I(y_{\text{ext}}) = y$ which proves the previous claim. $\square$

3. A Direct-Limit Continuous Hilbert Bundle

For every essential ideal I of A, we have constructed in the previous section a continuous Hilbert bundle $(\beta X^I, \{H_t^I\}_{t\in\beta X^I}, \Omega^I)$ and considered the spatial continuous trace C*-algebra A^I associated with this Hilbert bundle. Our aim in this section is to use these constructions to pass to limiting objects:

$\Delta \;=\; \lim\limits_{\leftarrow} \beta X^I$	(a compact, extremely disconnected Hausdorff space)
$C(\Delta) \;=\; \lim\limits_{\rightarrow} C(\beta X^I)$	(an abelian AW*-algebra)
$H_s^\Delta \;=\; \lim\limits_{\rightarrow} H^I_{\Phi_I(s)}$	(a Hilbert space, for every $s \in \Delta$)
$\Omega^\Delta \;=\; \lim\limits_{\rightarrow} \Omega^I$	(a Banach space of vector fields)
$(\Delta, \{H_s\}_{s\in\Delta}, \Omega^\Delta)$	(a continuous Hilbert bundle).

We recall here for the reader's convenience the notions of inverse system and inverse limit of a family of sets $\{X_\alpha\}_{\alpha\in\Lambda}$, where Λ is a directed set. Assume $\mathcal{F}$ is a family of functions indexed by subsets of $\Lambda \times \Lambda$, whereby:

(i) $f_{\alpha\alpha} = \mathrm{id}_{X_\alpha}$;
(ii) if (α, β) satisfies $\alpha \le \beta$, then $f_{\alpha\,\beta} : X_\beta \to X_\alpha$;
(iii) if $\alpha \le \beta \le \gamma$, then $f_{\alpha\gamma} = f_{\alpha\beta} \circ f_{\beta\gamma}$.

The triple $(\Lambda, \{X_\alpha\}_{\alpha\in\Lambda}, \mathcal{F})$ is called an inverse system.

The inverse limit of the inverse system $(\Lambda, \{X_\alpha\}_{\alpha\in\Lambda}, \mathcal{F})$ is the set denoted by $\lim\limits_{\leftarrow} X_\alpha$ and defined to be the subset of the Cartesian product $\prod\limits_\alpha X_\alpha$ consisting of all $x = (x_\alpha)_\alpha$ with the property that $x_\alpha = f_{\alpha,\beta}(x_\beta)$ whenever $\alpha \le \beta$. If f_α denotes the projection of $\lim\limits_{\leftarrow} X_\alpha$ onto X_α, then $f_\alpha = f_{\alpha\beta} \circ f_\beta$ whenever $\alpha \le \beta$. We shall make use of the fact that the projections f_α are surjective if each $f_{\alpha\beta}$, $\alpha \le \beta$, is surjective.

If each X_α is a topological space, if $\prod\limits_\alpha X_\alpha$ has the product topology, and if the functions in $\mathcal{F}$ are continuous, then the functions $f_\alpha : \lim\limits_{\leftarrow} X_\alpha \to X_\alpha$ are continuous. When all X_α are compact, then so is $\lim\limits_{\leftarrow} X_\alpha$.

The dual notions of direct system and direct limit are familiar to operator algebraists, and so they will not be defined here.

Proposition 3.1. (Inverse and Direct Systems)

- **(i)** *There exists an inverse system* $(\mathcal{I}_{\rm ess}(A), \{\beta X^I\}_I, \{\Phi_{JI}\}_{J\preccurlyeq I})$ *of compact spaces and continuous surjections.*
- **(ii)** *There exists a direct system* $(\mathcal{I}_{\rm ess}(A), \{\Omega^I\}_I, \{\lambda_{JI}\}_{J\preccurlyeq I})$ *of bounded vector fields and linear isometries.*

Proof. By [11, Theorem VII.7.3], the continuous embedding of a locally compact Hausdoff space Y into its Stone–Čech compactification βY is an open map. Hence, assuming that $J \preccurlyeq I$, we have that $i_J(X^I)$ is open (and dense) in βX^J. By the Universal Property of the Stone–Čech compactification, there is a (unique) continuous $\Phi_{JI} : \beta X^I \to \beta X^J$ such that

$$\iota_{J|X^I} \;=\; \Phi_{JI} \circ \iota_I \,. \tag{6}$$

Moreover, Φ_{JI} is surjective because the open set $i_J(X^I)$ is dense in βX^J. Finally, it is evident that $\Phi_{II} = \mathrm{id}_{X^I}$ and that $K \preccurlyeq J \preccurlyeq I$ leads to $\Phi_{KI} = \Phi_{KJ} \circ \Phi_{JI}$. Hence, $(\mathcal{I}_{\rm ess}(A), \{\beta X^I\}_I, \{\Phi_{JI}\}_{J\preccurlyeq I})$ is an inverse system, proving the first statement.

The second assertion requires an intermediate step that we shall use later on. For every I, $J \in \mathcal{I}_{\rm ess}(A)$ for which $J \preccurlyeq I$ and every $t \in \beta X^I$ we shall define a unitary Ψ_{JIt}, such that

$$\left\{ \begin{array}{ll} \Psi_{JIt} : H^J_{\Phi_{JI}(t)} \to H^I_t, & \text{for } \; J \preccurlyeq I \\[2ex] \Psi_{KIt} = \Psi_{JIt} \circ \Psi_{KJ\Phi_{JI}(t)}, & \text{for } \; K \preccurlyeq J \preccurlyeq I \end{array} \right\} \tag{7}$$

To achieve this we fix $t \in \beta X^I$ and $I \in \mathcal{I}_{\rm ess}(A)$. Recall that for any $L \in \mathcal{I}_{\rm ess}(A)$ the linear space $\{\overline{\omega}^L(s) : \omega \in \Omega_{b|X^L})\}$ is dense in H^L_s. Hence, if $J \preccurlyeq I$, the map $\overline{\omega}^J\left(\Phi_{JI}(t)\right) \mapsto \overline{\omega}^I(t)$ is a well defined linear isometry, and so it extends to a unitary $\Psi_{JIt} : H^J_{\Phi_{JI}(t)} \to H^I_t$. Now if $K \preccurlyeq J \preccurlyeq I$, then $\Psi_{KIt} = \Psi_{JIt} \circ \Psi_{KJ\Phi_{JI}(t)}$ follows immediately from $\Phi_{KI}(t) = \Phi_{KJ} \circ \Phi_{JI}(t)$.

Now to prove our second assertion, assume $I, J \in \mathcal{I}_{\rm ess}(A)$ are such that $J \preccurlyeq I$. If $\nu \in \Omega^J$, then a vector field $\tilde{\nu} : \beta X^I \to \sqcup_{t\in\beta X^I} H^I_t$ is defined as follows:

$$\tilde{\nu}\,(t) \;=\; \Psi_{JIt} \circ \nu \circ \Phi_{JI}\,(t)\,, \quad t \in \beta X^I \,. \tag{8}$$

Observe that if $\nu \in \Omega_b$, then $(\tilde{\overline{\omega}^J}) = \overline{\omega}^I$. Let λ_{JI} be the function with domain Ω^J and defined by $\lambda_{IJ}\nu = \tilde{\nu}$. Note that λ_{JI} is a linear transformation and that

$$\sup_{t\in\beta X^I} \|\tilde{\nu}(t)\| \;=\; \sup_{t\in\beta X^I} \|\nu\left(\Phi_{JI}(t)\right)\| \;=\; \sup_{r\in\beta X^J} \|\nu(r)\|\,.$$

The first equality above is on account of the operator Ψ_{JIt} being an isometry, and the second is true because Φ_{JI} is a surjection. Hence, $\tilde{\nu}$ is a bounded vector field of norm $\|\tilde{\nu}\| = \|\nu\|$. Because $\lambda_{JI}\left(\mathcal{E}^J\right) = \mathcal{E}^I$ and every $\nu \in \Omega^J$ is a local uniform limit of vectors fields in $\mathcal{E}^J$, we conclude that $\lambda_{JI}\nu$ is a local uniform limit of vectors fields in $\mathcal{E}^I$, whence $\lambda_{JI}(\nu) \in \Omega^I$. Finally, by virtue of the properties of Ψ_{JIt} and Φ_{JI}, we obtain $\lambda_{II} = \mathrm{id}_{\Omega^I}$ and $\lambda_{KI} = \lambda_{JI} \circ \lambda_{KJ}$ whenever $K \preccurlyeq J \preccurlyeq I$. □

Notation. For the purposes of notational clarity, equation (8) is henceforth expressed more simply as

$$\lambda_{JI}\nu \;=\; \nu \circ \Phi_{JI}\,. \tag{9}$$

That is, (9) is shorthand for (8).

Denote the inverse limit of the inverse system $(\mathcal{I}_{\mathrm{ess}}(A), \{\beta X^I\}_I, \{\Phi_{JI}\}_{J\preccurlyeq I})$ by

$$\Delta \;=\; \lim_{\leftarrow} \beta X^I\,, \tag{10}$$

and let $\Phi_I : \Delta \to \beta X^I$ denote the continuous, surjective functions that satisfy $\Phi_J = \Phi_{JI} \circ \Phi_I$ whenever $J \preccurlyeq I$. The space Δ is compact and Hausdorff. We shall note below that Δ is also extremely disconnected; thus, it is a *Stonean* space.

If $J \preccurlyeq I$, then the continuous surjection $\Phi_{JI} : \beta X^I \to \beta X^J$ leads to a monomorphism $\rho_{JI} : C(\beta X^J) \to C(\beta X^I)$ defined by $\rho_{JI}(f) = f \circ \Phi_{JI}$ and in this way we produce a direct system of abelian C*-algebras and monomorphisms. By [23],

$$C(\Delta) \;=\; \lim_{\rightarrow} C(\beta X^I)\,, \tag{11}$$

the direct limit C*-algebra of the system $(\mathcal{I}_{\mathrm{ess}}(A), \{C(\beta X^I)\}_I, \{\rho_{JI}\}_{J\preccurlyeq I})$. Observe that (11) states that

$$M_{\mathrm{loc}}\left(C_0(T)\right) \;=\; C(\Delta)\,.$$

As the local multiplier algebra of an abelian C*-algebra is an abelian AW*-algebra [2, Proposition 3.4.5], the maximal ideal space of $M_{\mathrm{loc}}\left(C_0(T)\right)$ is extremely disconnected, which is why Δ is Stonean.

Via the universal property, we deduce that the algebraic direct limit of the system $(\mathcal{I}_{\mathrm{ess}}(A), \{C(\beta X^I)\}_I, \{\rho_{JI}\}_{J\preccurlyeq I})$ is (identified with)

$$\mathrm{alg}\text{-}\lim_{\rightarrow} C(\beta X^I) \;=\; \{f \circ \Phi_I \,:\, I \in \mathcal{I}_{\mathrm{ess}}(A),\; f \in C(\beta X^I)\}\,, \tag{12}$$

which is uniformly dense in $C(\Delta)$.

To construct Hilbert spaces H_s^Δ, recall $(\mathcal{I}_{\mathrm{ess}}(A), \{H^I_{\Phi_I(s)}\}_I, \{\Psi_{JI\Phi_J(s)}\}_{J\preccurlyeq I})$ is a direct system of Hilbert spaces and unitaries, for each $s \in \Delta$, by (7). Thus, we consider the Hilbert space direct limit

$$H_s^\Delta \;=\; \lim_{\rightarrow} H^I_{\Phi_I(s)}\,. \tag{13}$$

(Note that for $J \preccurlyeq I$, $H^J_{\Phi_J(s)} = H^I_{\Phi_I(s)}$.) Hence, for every $I \in \mathcal{I}_{\mathrm{ess}}(A)$ there is a surjective linear isometry $\Psi_{I_s} : H^I_{\Phi_I(s)} \to H_s^\Delta$ such that

$$\Psi_{J_s} \;=\; \Psi_{I_s} \circ \Psi_{JI\Phi_J(s)}\,, \quad \forall\, J \preccurlyeq I\,.$$

Thus, the set

$$\{\Psi_{I_s}\nu\,(\Phi_I(s))\,:\,I\in\mathcal{I}_{\rm ess}(A),\ \nu\in\Omega^I\}\tag{14}$$

is dense in H_s^Δ. *For notational simplicity*, we write (14) as

$$\{\nu\circ\Phi_I(s)\,:\,I\in\mathcal{I}_{\rm ess}(A),\ \nu\in\Omega^I\}.\tag{15}$$

Observe that the inner product in H_s^Δ of any two such vectors $\nu_j\circ\Phi_{I_j}(s)$, $j=1,2$, is (well) defined by

$$\langle\nu_1\circ\Phi_{I_1}(s),\,\nu_2\circ\Phi_{I_2}(s)\rangle\;=\;\langle\nu_1\circ\Phi_J(s),\,\nu_2\circ\Phi_J(s)\rangle\;,$$

for any $J\in\mathcal{I}_{\rm ess}(A)$ with $J\preccurlyeq I_1$ and $J\preccurlyeq I_2$.

Likewise,

$$\operatorname{alg-}\varinjlim\,\Omega^I\;=\;\{\nu\circ\Phi_I\,:\,I\in\mathcal{I}_{\rm ess}(A),\ \nu\in\Omega^I\}\tag{16}$$

is an algebraic direct limit of vector spaces. Hence, every $\mu\in\operatorname{alg-}\varinjlim\,\Omega^I$ is a vector field $\Delta\to\bigsqcup_{s\in\Delta}H_s^\Delta$ via

$$\mu(s)\;=\;\nu\,(\Phi_I(s))\;\in\;H_s^\Delta\,,\ \text{for some } I\in\mathcal{I}_{\rm ess}(A)\text{ and }\nu\in\Omega^I\,.$$

Notational Summary. If $I,J\in\mathcal{I}_{\rm ess}(A)$ are such that $I\subset J$, and if $\nu\in\Omega^J$, then

$$\nu\circ\Phi_J\;=\;\nu'\circ\Phi_I\,,\quad\text{where}\quad\nu'=\lambda_{JI}\nu=\nu\circ\Phi_{JI}\,.\tag{17}$$

Our aim below is to complete $\operatorname{alg-}\varinjlim\,\Omega^I$ in a manner that will give it the structure of a continuous Hilbert bundle over Δ. Not only should this completion be closed under local uniform limits, but it should be a $C(\Delta)$-module as well.

In what follows, let

$$\mathcal{E}\;=\;\operatorname{alg-}\varinjlim\,\Omega^I\;=\;\{\nu\circ\Phi_I\,:\,I\in\mathcal{I}_{\rm ess}(A),\ \nu\in\Omega^I\}\,.\tag{18}$$

Definition 3.2. Ω^Δ *is the set of all bounded vector fields* $\nu:\Delta\to\bigsqcup_{s\in\Delta}H_s^\Delta$ *with the property that for each* $s_0\in\Delta$ *and* $\varepsilon>0$ *there exist an open set* $U\subset\Delta$ *containing* s_0 *and* $\omega\in\mathcal{E}$ *such that* $\|\nu(s)-\omega(s)\|<\varepsilon$ *for all* $s\in U$.

Proposition 3.3. $(\Delta,\{H_s^\Delta\}_{s\in\Delta},\Omega^\Delta)$ *is a continuous Hilbert bundle.*

Proof. Of the axioms to be satisfied, the only one that is not immediate is axiom (I): that Ω^Δ is a $C(\Delta)$-module. To prove this, let $\xi\in\Omega^\Delta$ and $f\in C(\Delta)$. Choose $s_0\in\Delta$ and $\varepsilon>0$. By the continuity of f, there is an open neighbourhood $U_1\subset\Delta$ of s_0 such that $|f(s)-f(s_0)|<\frac{\varepsilon}{2\|\xi\|}$, for all $s\in U_1$. By definition of Ω^Δ, there exist an open neighbourhood $U_2\subset\Delta$ of s_0, an $I\in\mathcal{I}_{\rm ess}(A)$, and a $\nu\in\Omega^I$ such that $\|\xi(s)-\nu\circ\Phi_I(s)\|<\frac{\varepsilon}{2\|f\|}$, for all $s\in U_2$. Let $U=U_1\cap U_2$ to obtain

$$\|f\cdot\xi(s)-f(s_0)\nu\circ\Phi_I(s)\|\;<\;\varepsilon\,,\quad\text{for all } s\in U\,.$$

Now as $f(s_0)\nu\circ\Phi_I\in\Omega^I$, the inequality above implies that $f\cdot\xi$ is a local uniform limit of elements of $\operatorname{alg-}\varinjlim\,\Omega^I$. Hence, $f\cdot\xi\in\Omega^\Delta$. □

We call $(\Delta,\{H_s\}_{s\in\Delta},\Omega^\Delta)$ the *direct limit continuous Hilbert bundle* of the system described in item (ii) of Proposition 3.1.

The next result shows that elements of Ω^Δ are not just local uniform limits of elements of $\operatorname{alg-}\varinjlim\,\Omega^I$, but rather each $\xi\in\Omega^\Delta$ is a global uniform limit of elements of $\operatorname{alg-}\varinjlim\,\Omega^I$.

Theorem 3.4. $\Omega^{\Delta} = \lim\limits_{\to} \Omega^I$ *as a Banach space.*

Lemma 3.5. *Assume $\xi \in \Omega^{\Delta}$. For every $s_0 \in \Delta$ and $\varepsilon > 0$ there exist $I \in \mathcal{I}_{\text{ess}}(A)$, $\nu \in \Omega^I$, and an open set $V \subset \beta X^I$ such that the open set $U = \Phi_I^{-1}(V) \subset \Delta$ contains s_0 and $\|\xi(s) - \nu \circ \Phi_I(s)\| < \varepsilon$ for all $s \in U$.*

Proof. Let $s_0 \in \Delta$. By definition, there are $J \in \mathcal{I}_{\text{ess}}(A)$, $\nu' \in \Omega^J$, and $U_1 \subset \Delta$ such that U_1 is an open neighbourhood of s_0 and $\|\xi(s) - \nu' \circ \Phi_J(s)\| < \varepsilon$ for all $s \in U_1$. Inside U_1 there is an open set U containing s_0 such that U has the form $U = \Phi_K^{-1}(W)$, for some $K \in \mathcal{I}_{\text{ess}}(A)$ and open set $W \subset \beta X^K$ [11, Proposition 2.3 in Appendix Two]. Consider the essential ideal $I = J \cap K$; thus, $J \preccurlyeq I$ and $K \preccurlyeq I$, and so we consider the continuous functions $\Phi_{JI} : \beta X^I \to \beta X^J$ and $\Phi_{KI} : \beta X^I \to \beta X^K$. Let $V = \Phi_{KI}^{-1}(W) \subset \beta X^I$ and $U = \Phi_I^{-1}(V) \subset \Delta$. The relation $\Phi_K = \Phi_{KI} \circ \Phi_I$ implies that $\Phi_K^{-1}(W) = \Phi_I^{-1}\left(\Phi_{KI}^{-1}(W)\right)$. Thus, $s_0 \in U = \Phi_K^{-1}(W) \subset U_1$. Now let $\nu \in \Omega^I$ be given by $\nu = \nu' \circ \Phi_{JI}$. Thus, for any $s \in U$, we have that

$$\|\xi(s) - \nu \circ \Phi_I(s)\| = \|\xi(s) - \nu'\left(\Phi_{JI} \circ \Phi_I(s)\right)\| = \|\xi(s) - \nu' \circ \Phi_J(s)\| < \varepsilon ,$$

which completes the proof. $\square$

We now prove Theorem 3.4.

Proof. What we aim to prove is: for each $\xi \in \Omega^{\Delta}$ and $\varepsilon > 0$ there exist $I \in \mathcal{I}_{\text{ess}}(A)$ and $\nu \in \Omega^I$ such that $\|\xi(s) - \nu \circ \Phi_I(s)\| < \varepsilon$ for every $s \in \Delta$.

Fix $\varepsilon > 0$. Lemma 3.5 provides us with an open cover of Δ of a specific type. Let $U_1, \dots, U_n$ be a finite subcover. Thus, for each $1 \le i \le n$ there are $I_i \in \mathcal{I}_{\text{ess}}(A)$, $\nu_i \in \Omega^{I_i}$, and open sets $V_i \subset \beta X^{I_i}$ such that $U_i = \Phi_{I_i}^{-1}(V_i) \subset \Delta$ and $\|\xi(s) - \nu_i \circ \Phi_{I_i}(s)\| < \varepsilon$ for all $s \in U_i$. Suppose that $\{\varphi_i\}_{i=1}^n$ is a partition of unity subordinate to the open cover $\{U_i\}_{i=1}^n$. By properties of the inverse limit [23], for each i there exists $J_i \in \mathcal{I}_{\text{ess}}(A)$ and $\psi_i \in C(\beta X^{J_i})$ such that $\|\varphi_i - \psi_i \circ \Phi_{J_i}\| < \frac{\varepsilon}{n}$. Let $I = \bigcap_{i=1}^n (I_i \cap J_i) \in \mathcal{I}_{\text{ess}}(A)$ and let $X^I \subset T$ be the open set corresponding to I. Let $\psi_i' \in C(\beta X^I)$ denote $\psi_i' = \psi_i \circ \Phi_{J_i I}$; hence,

$$\|\varphi_i - \psi_i' \circ \Phi_I\| = \|\varphi_i - \psi_i \circ \Phi_{J_i I} \circ \Phi_I\| = \|\varphi_i - \psi_i \circ \Phi_{J_i}\| < \frac{\varepsilon}{n} .$$

Consider now the following element $\nu \in \Omega^I$:

$$\nu = \sum_{i=1}^n (\psi_i' \circ \Phi_I) \cdot \lambda_{I_i I} \nu_i .$$

Then, for every $s \in \Delta$,

$$\begin{aligned}
\|\xi(s) - \nu \circ \Phi_I(s)\| &= \left\| \sum_{i=1}^n \varphi_i(s)\xi(s) - \sum_{i=1}^n \psi_i'(\Phi_I(s))\ (\nu_i \circ \Phi_{I_i}(s)) \right\| \\
&\le \sum_{i=1}^n \varphi_i(s)\ \|\xi(s) - \nu_i \circ \Phi_{I_i}(s)\| \\
&\quad + \sum_{i=1}^n |\varphi_i(s) - \psi_i' \circ \Phi_I(s)|\ \|\nu_i \circ \Phi_{I_i}(s)\| \\
&< \varepsilon + \varepsilon(\|\xi(s)\| + \varepsilon) .
\end{aligned}$$

Hence, $\|\xi - \nu \circ \Phi_I\| < \varepsilon + \varepsilon(\|\xi\| + \varepsilon)$. □

4. A Direct Limit C*-Algebra

In this section we keep the notation from the previous sections. In particular, we use the maps δ_I from Proposition 2.5.

Proposition 4.1. *There exists a direct system $(\mathcal{I}_{\text{ess}}(A), \{A^I\}_I, \{\pi_{JI}\}_{J\preccurlyeq I})$ of C*-algebras and monomorphisms such that, for all $J \preccurlyeq I$, $\delta_I = \pi_{JI} \circ \delta_{J|I}$.*

Proof. Assume that $J \preccurlyeq I$. For each $a \in A^J$, consider the bounded cross section $\tilde{a}$ of the fibred space $(\beta X^I, \{B(H_t^I)\}_{t\in\beta X^I})$ that is defined by

$$\tilde{a}(t) \;=\; [\Psi_{JIt}]\,[a\,(\Phi_{JI}\,(t))]\,[\Psi_{JIt}]^{-1}\,, \quad t \in \beta X^I\,. \tag{19}$$

As before, we simplify the notation so that

$$\tilde{a} \;=\; a \circ \Phi_{JI} \tag{20}$$

is now a shorthand expression of (19).

Let us now show that $\tilde{a} \in A^I$. Continuity of $\check{\tilde{a}}$ follows from $\check{\tilde{a}} = \check{a} \circ \Phi_{JI}$. To show that $\tilde{a}$ is weakly continuous, it is sufficient to use vector fields from $\mathcal{E}^I$. To this end, let $\omega_1, \omega_2 \in \Omega_b$. Then

$$\langle \tilde{a}(t)\,\overline{\omega}_1^I(t), \overline{\omega}_2^I(t)\rangle \;=\; \left\langle a(\Phi_{JI}(t))\,\overline{\omega}_1^J(\Phi_{JI}(t)), \overline{\omega}_2^J(\Phi_{JI}(t))\right\rangle\,,$$

which is continuous as a function of $t \in \beta X^I$.

To show that $\tilde{a}$ is approximately finite-dimensional, select $t_0 \in \beta X^I$ and $\varepsilon > 0$. Consider $r_0 = \Phi_{JI}(t_0) \in \beta X^J$. Because $a \in A^J$, there is an open set $V \subset \beta X^J$ and $\nu_1, \dots, \nu_n \in \Omega^J$ such that, for every $r \in U$, Span $\{\nu_1(r), \dots, \nu_n(r)\}$ is n-dimensional and $\|a(r) - p_r a(r) p_r\| < \varepsilon$, where $p_r \in B(H_r^J)$ is the projection with range spanned by $\nu_1(r)$,...,$\nu_n(r)$. Let $U = \Phi_{JI}^{-1}(V)$, an open neighbourhood of t_0. Consider the vector fields $\lambda_{JI}\nu_\ell \in \Omega^I$, $1 \le \ell \le n$. Because $\lambda_{JI}\nu_\ell(t) = \Psi_{JIt}\,(\nu(\Phi_{JI}(t)))$,

$$\text{Span}\,\{\lambda_{JI}\nu_1(t), \dots, \lambda_{JI}\nu_n(t)\}$$

is an n-dimensional subspace for all $t \in U$. Let $q_t \in B(H_t^I)$ denote the projection onto this subspace, for each $t \in U$. Then $q_t = \Psi_{JIt} p_{\Phi_{JI}(t)} \Psi_{JIt}^{-1}$, which yields $\|\tilde{a}(t) - q_t \tilde{a}(t) q_t\| < \varepsilon$ for all $t \in U$.

Define $\pi_{JI} : A^J \to A^I$ by $\pi_{JI}(a) = a \circ \Phi_{JI}$. It is now straightforward to verify that π_{JI} is a homomorphism, that π_{JI} is isometric (since Φ_{JI} is surjective), and that $(\mathcal{I}_{\text{ess}}(A), \{A^I\}_I, \{\pi_{JI}\}_{J\preccurlyeq I})$ is a direct system of C*-algebras and monomorphisms.

To prove that $\delta_I = \pi_{JI} \circ \delta_{J|I}$, assume that $a \in I$. Thus, $\delta_I(a)$ is an operator field on βX^I that vanishes on $\beta X^I \setminus X^I$ and agrees with a on X^I. Thinking now of I sitting inside J, $\delta_J(a)$ is an operator field on βX^J that vanishes on $\beta X^J \setminus X^J$. Therefore the operator field $\pi_{JI}\delta_J(a)$ on βX^I vanishes on $\beta X^I \setminus X^I$ because Φ_{JI} maps $\beta X^I \setminus X^I$ into $\beta X^J \setminus X^J$ [15, Theorem 6.12]. Hence, $\pi_{JI} \circ \delta_J(a) \in \delta_I(I)$. It is now straightforward to verify that $\delta_I = \pi_{JI} \circ \delta_{J|I}$. □

Notational Summary. If $I, J \in \mathcal{I}_{\text{ess}}(A)$ are such that $J \preccurlyeq I$ and if $a \in A^J$, then

$$a \circ \Phi_J \;=\; a' \circ \Phi_I\,, \;\text{ where }\; a' = \pi_{JI}(a) = a \circ \Phi_{JI}\,. \tag{21}$$

Therefore, if $a \in A^I$, then $a \circ \Phi_I : \Delta \to \bigsqcup_{s\in\Delta} B(H_s^\Delta)$, which induces a C*-embedding of A^I into $\bigsqcup_{s\in\Delta} B(H_s^\Delta)$. Moreover, these embeddings are compatible with the direct system structure of $(\mathcal{I}_{\rm ess}(A), \{A^I\}_I, \{\pi_{JI}\}_{J\preccurlyeq I})$ of Proposition 4.1. Therefore if A_Δ denotes the norm-closure

$$A_\Delta := \left(\bigcup_{I\in\mathcal{I}_{\rm ess}(A)} \{a\circ\Phi_I \,:\, a\in A^I\} \right)^{-\,\|\cdot\|},$$

then $A_\Delta = \lim\limits_{\rightarrow} A^I$; that is, A_Δ is a concrete realisation of a C*-limit of the directed system $\{A^I\}_{I\in\mathcal{I}_{\rm ess}(A)}$.

Proposition 4.2 below identifies the algebras A^Δ and $K(\Omega^\Delta)$, which were studied as separate entities in the prequel [7].

Proposition 4.2. *Let $A^\Delta = A\left(\Delta, \{H_s\}_{s\in\Delta}, \Omega^\Delta\right)$ be the continuous trace C*-algebra associated with the continuous Hilbert bundle $(\Delta, \{H_s\}_{s\in\Delta}, \Omega^\Delta)$. Then*

$$K(\Omega^\Delta) \,=\, A_\Delta \,=\, A^\Delta\,.$$

Proof. We first show that $A_\Delta\subset A^\Delta$; to so, it is sufficient to prove that $\{a\circ\Phi_I \,:\, a\in A^I\}\subset A^\Delta$, for every $I\in\mathcal{I}_{\rm ess}(A)$. Suppose that $a\circ\Phi_I$ and that $\omega_1,\omega_2\in\Omega^\Delta$ are of the form $\omega_i=\omega_i'\circ\Phi_{J_i}$ for some $J_1,J_2\in\mathcal{I}_{\rm ess}(A)$ and $\omega_i'\in\Omega^{J_i}$. Let $K=I\cap J_1\cap J_2$, an essential ideal of A such that $I\preccurlyeq K$ and $J_i\preccurlyeq K$. Because $a\circ\Phi_I=(a\circ\Phi_{IK})\circ\Phi_K$ and $\omega_i'\circ\Phi_{J_i}=(\omega_i'\circ\Phi_{J_iK})\circ\Phi_K$, the continuity of the map $s\mapsto\langle a\,(\Phi_I(s))\,\omega_1(s),\omega_2(s)\rangle$ is immediate. As vector fields of the form $\omega=\omega'\circ\Phi_J$ are uniformly dense in Ω^Δ, the operator field $a\circ\Phi_I$ is weakly continuous.

To show that $a\circ\Phi_I$ is almost finite-dimensional, assume $s_0\in\Delta$ and $\varepsilon>0$. Let $t_0=\Phi_I(s_0)$. As a is almost finite-dimensional, there are an open set $V\in\beta X^I$ containing t_0 and $\omega_j\in\Omega^I$, $1\le j\le n$, such that, for all $t\in V$, $\{\omega_j(t)\}_{j=1}^n$ is a set of linearly independent vectors and $\|a'(t)-p_ta'(t)p_t\|<\varepsilon$, where p_t is the projection onto the span of $\{\omega_j(t)\}_{j=1}^n$. Pull back to Δ using the open neighbourhood $U=\Phi_I^{-1}(V)$ of s_0 and rank-n projections $q_s=p_{\Phi_I(s)}$ onto the span $\{\omega_j(\Phi_I(s))\}_{j=1}^n$ to obtain $\|a(s)-q_sa(s)q_s\|<\varepsilon$ for all $s\in U$. This completes the proof that $\{a\circ\Phi_I \,:\, a\in A^I\}\subset A^\Delta$, thereby establishing the inclusion $A_\Delta\subset A^\Delta$.

By Theorem 3.4 any $\xi\in\Omega^\Delta$ is uniformly approximated to within ε on Δ by some $\omega=\nu\circ\Phi_J$ of norm within ε of $\|\xi\|$; therefore, we can conclude that

$$\|\Theta_{\xi,\xi}\,-\,\Theta_{\omega,\omega}\| \;\le\; \|\check\xi-\check\omega\|(\|\check\xi\|+\|\check\omega\|) \;\le\; C\varepsilon\,,$$

where C is a constant depending on $\|\xi\|$. As the set of all finite sums of the form $\Theta_{\xi,\xi}$ is dense in the positive cone of $K(\Omega^\Delta)$ [7, Lemma 4.2], we deduce that $K(\Omega^\Delta)\subset A_\Delta$.

To conclude, we now prove that $A^\Delta\subset K(\Omega^\Delta)$. Select $a\in A^\Delta$ and $\varepsilon>0$. For every $s_0\in\Delta$ there are an open set $U_{s_0}\subset\Delta$ containing s_0 and vector fields $\omega_1,\dots,\omega_n\in\Omega^\Delta$ such that $\|a(s)-p_s\,a(s)\,p_s\|<\varepsilon$, for all $s\in U_{s_0}$, where p_s is the orthogonal projection onto $\mathrm{Span}\{\omega_j(s) :\; 1\le j\le n\}$. It turns out that $h_{s_0}:=p_s\,a(s)\,p_s\in\mathcal{F}(\Omega^\Delta)$ (see the proof of Lemma 5.1). Therefore, $\{U_{s_0}\}_{s_0\in\Delta}$ is an open cover of Δ from which a finite subcovering $U_1,\dots,U_n$ exists; let $h_j\in\mathcal{F}(\Omega^\Delta)$ denote the local approximant of a on U_j, for each $1\le j\le n$, and let $\{\varphi_j\}_{1\le j\le n}\subset C(\Delta)$ be a partition of unity subordinate to $\{U_j\}_{1\le j\le n}$. Because $\mathcal{F}(\Omega^\Delta)$ is a $C(\Delta)$-module we have $h=\sum_{j=1}^n\varphi_i\cdot h_j\in\mathcal{F}(\Omega^\Delta)$. Therefore, for every $s\in\Delta$,

$$\|a(s)-h(s)\| \;\le\; \sum_{j=1}^n\varphi_i(s)\,\|a(s)-h_j(s)\| \;<\; \varepsilon\,.$$

Hence, $\|a-h\|<\varepsilon$ and so $a\in K(\Omega^\Delta)$. ☐

5. A Chain of Inclusions of C*-Algebras

In this section we prove that $A\subset K(\Omega^\Delta)\subset M_{\rm loc}(A)$, an inclusion as C*-subalgebras, where $A\subset M_{\rm loc}(A)$ is the canonical embedding of A into its local multiplier algebra. To do so, an alternate description of A [1] is useful.

Consider Ω_0 as a Hilbert C*-module over $C_0(T)$. Every $\kappa\in\mathcal{F}(\Omega_0)$ is a cross section of the fibred space $(T,\{K(H_t)\}_{t\in T})$, and the set $\mathcal{F}(\Omega_0)$ has the following properties: (i) $\mathcal{F}(\Omega_0)$ is a $*$-algebra with respect to pointwise operations; (ii) $\{\kappa(t):\kappa\in\mathcal{F}(\Omega_0)\}$ is dense in $K(H_t)$ for all $t\in T$; and (iii) $\check{k}\in C_0(T)$, for each $\kappa\in\mathcal{F}(\Omega_0)$.

A cross section a of the fibred space $(T,\{K(H_t)\}_{t\in T})$ is said to be *continuous with respect to* $\mathcal{F}(\Omega_0)$ if for each $t_0\in T$ and $\varepsilon>0$ there exist $\kappa\in\mathcal{F}(\Omega_0)$ and an open set $U\subset T$ containing t_0 such that $\|a(t)-\kappa(t)\|<\varepsilon$ for every $t\in U$. (The terms "continuous with respect to" and "local uniform limit of" have the same meaning; however, as the former terminology is used in the paper [1], we adopt this phrase here.)

Let $C=C_0(T,\{K(H_t)\}_{t\in T},\mathcal{F}(\Omega_0))$ be the set of all cross sections a of the fibred space $(T,\{K(H_t)\}_{t\in T})$ that are continuous with respect to $\mathcal{F}(\Omega_0)$ and satisfy $\check{a}\in C_0(T)$. With respect to pointwise operations and the supremum norm, C is a C*-algebra.

Lemma 5.1. $A(T,\{H_t\}_{t\in T},\Omega)\;=\;C_0(T,\{K(H_t)\}_{t\in T},\mathcal{F}(\Omega_0))$.

Proof. By construction, $\mathcal{F}(\Omega_0)\subset A$. Therefore, since A is closed under local uniform approximation, $C\subset A$. Conversely, assume $a\in A$. Let $t\in T$ and $\varepsilon>0$. Thus, there exist an open set $V\subset T$ containing t and $\omega_i\in\Omega$, $1\le i\le n$, such that, for every $s\in V$, the set of vectors $\{\omega_i(s)\}_{i=1}^n$ is a linearly independent set and $\|a(s)-p_s\,a(s)\,p_s\|<\varepsilon$, where $p_s\in B(H_s)$ denotes the projection onto Span$\{\omega_i(s):1\le i\le n\}$. Via the Gram–Schmidt process [12, Lemma 4.2], we may assume that the vectors $\omega_i(s)$, $1\le i\le n$, are pairwise orthogonal for every s in some open set $U\subset V$ containing t. By Urysohn's Lemma, we can also assume that each $\omega_i\in\Omega_0$.

For each $1\le i,j\le n$, let $f_{ij}(t)=\langle a(t)\omega_i(t),\omega_j(t)\rangle$, $t\in T$. Thus, $f_{ij}\in C_0(T)$ and so $f_{ij}\cdot\omega_i\in\Omega_0$ for all i,j. Now note that

$$p_s\,a(s)\,p_s\;=\;\sum_{i,\,j=1}^n\langle a(s)\,\omega_j(s),\omega_i(s)\rangle\;\Theta_{\omega_i,\,\omega_j}(s)\;\in\;\mathcal{F}(\Omega_0)\,,\quad\forall\,s\in U\,.$$

Hence, a is continuous with respect to $\mathcal{F}(\Omega_0)$, which proves that $A\subset C$. ☐

The previous result implies the following convenient description of the multiplier algebras of essential ideals. If I is an ideal of A, then by Lemma 5.1, A^I is given by $A^I=C_0(\beta X^I,\{K(H_t^I)\}_{t\in\beta X^I},\mathcal{F}(\Omega^I))$. In viewing A^I in this way, the ideal $\delta_I(I)$ is given by

$$I\;\cong\;\delta_I(I)\;=\;C_0\left(X^I,\{K(H_t^I)\}_{t\in X^I},\mathcal{F}(({\Omega^I}_{|X^I})_0)\right)\,.$$

In this framework, $x\in M(I)$ if and only if x is a bounded cross section of the fibred space $(X^I,\{B(H_t^I)\}_{t\in X^I})$ for which x is *strictly continuous* with respect to $\mathcal{F}(({\Omega^I}_{|X^I})_0)$ [1, Theorem 3.3]. That is, for each $t_0\in X^I$, $a\in\mathcal{F}(({\Omega^I}_{|X^I})_0)$, and

$\varepsilon > 0$ there are an open set $U \subset X^I$ containing t_0 and $b \in \mathcal{F}((\Omega^I{}_{|X^I})_0)$ such that

$$\| (x(t) - b(t))\, a(t)\| + \|a(t)\, (x(t) - b(t))\| < \varepsilon, \quad \text{for all } t \in U.$$

We summarise this fact in the next proposition.

Proposition 5.2. *If I is an essential ideal of A, then $x \in M(I)$ if and only if x is a bounded cross section of the fibred space $(X^I, \{B(H_t^I)\}_{t\in X^I})$ such that x is strictly continuous with respect to $\mathcal{F}((\Omega^I{}_{|X^I})_0)$.*

Theorem 5.3. *There exists a monomorphism $\gamma : K(\Omega^\Delta) \to M_{\mathrm{loc}}(A)$.*

Proof. We shall exploit the fact that $A_\Delta = K(\Omega^\Delta)$ (Proposition 4.2). Fix $J \in \mathcal{I}_{\mathrm{ess}}(A)$ and let $\gamma_J : A^J \to M(J)$ be the canonical embedding of A^J into $M(J)$, using the fact that $J \cong \delta_J(J)$ is an essential ideal of A^J. Recall that, by Lemma 5.1,

$$J \cong \delta_J(J) = C_0\left(X^J, \{K(H_t^J)\}_{t\in X^J}, \mathcal{F}((\Omega^J{}_{|X^J})_0)\right)$$

and, by Proposition 5.2, $x \in M(J)$ if and only if x is a bounded cross section of the fibred space $(X^J, \{B(H_t^J)\}_{t\in X^J})$ which is strictly continuous with respect to $\mathcal{F}((\Omega^J{}_{|X^J})_0)$.

Suppose now that $J \preccurlyeq I$ and let $x \in M(J)$. Because $\Phi_{JI} \circ \iota_I = \iota_{J|X^I}$, $x \circ \Phi_{JI|X^I}$ is a well defined bounded section $\tilde{x}$ of the fibred space $(X^I, \{B(H_t^I)\}_{t\in X^I})$. Select $t_0 \in X^I$ and $\varepsilon > 0$. Let $s_0 \in \Phi_{JI}(t_0) \in X^J$ and choose $a \in \mathcal{F}((\Omega^J{}_{|X^J})_0)$. Because $x \in M(J)$, [1, Theorem 3.3] asserts that there are an open set $V \subset X^J$ containing s_0 and $b \in \mathcal{F}((\Omega^J{}_{|X^J})_0)$ such that

$$\| (x(s) - b(s))\, a(s)\| + \|a(s)\, (x(s) - b(s))\| < \varepsilon, \quad \text{for all } s \in V.$$

Let $U = \Phi_{JI}^{-1}(V)$ and observe that $\pi_{JI}(a), \pi_{JI}(b) \in \mathcal{F}((\Omega^I J_{|X^I})_0)$. Hence, the pull back to X^I of the inequality above holds for $\tilde{x}$ in U and, thus, $\tilde{x} \in M(I)$.

Define $\tilde{\pi}_{JI} : M(J) \to M(I)$ by $\tilde{\pi}_{JI}(x) = x \circ \Phi_{JI|X^I}$. Thus $\tilde{\pi}_{JI}$ is a homomorphism and satisfies the commutative diagram

$$\begin{array}{ccc} A^J & \xrightarrow{\pi_{JI}} & A^I \\ {\scriptstyle\gamma_J}\big\downarrow & & \big\downarrow{\scriptstyle\gamma_I} \\ M(J) & \xrightarrow[\tilde{\pi}_{JI}]{} & M(I) \end{array} \tag{22}$$

If $\tilde{\pi}_{JI}(x) = 0$, then $x(s) = 0$ for all $s \in \Phi_{JI}(X^I) = X^J$, whence $x = 0$. Therefore, $\tilde{\pi}_{JI}$ is a monomorphism. Hence, $(\mathcal{I}_{\mathrm{ess}}(A), \{A^I\}_I, \{\pi_{JI}\}_{J\preccurlyeq I})$ is a subsystem of the direct system $(\mathcal{I}_{\mathrm{ess}}(A), \{M(I)\}_I, \{\tilde{\pi}_{JI}\}_{J\preccurlyeq I})$. Let N denote the direct limit of the direct system $(\mathcal{I}_{\mathrm{ess}}(A), \{M(I)\}_I, \{\tilde{\pi}_{JI}\}_{J\preccurlyeq I})$. The previous facts imply that there is a monomorphism $\gamma : A_\Delta \to N$ such that

$$\begin{array}{ccc} A^I & \xrightarrow{\pi_I} & A_\Delta \\ {\scriptstyle\gamma_I}\big\downarrow & & \big\downarrow{\scriptstyle\gamma} \\ M(I) & \xrightarrow[\tilde{\pi}_I]{} & N \end{array}$$

is a commutative diagram for all $I \in \mathcal{I}_{\mathrm{ess}}(A)$, where π_I and $\tilde{\pi}_I$ are the embeddings of A^I and $M(I)$ into their respective direct limits which satisfy $\pi_J = \pi_I \circ \pi_{JI}$ and $\tilde{\pi}_J = \tilde{\pi}_I \circ \tilde{\pi}_{JI}$ for all $J \preccurlyeq I$.

On the other hand, since $\delta_I = \pi_{JI} \circ \delta_{J|I}$ (Proposition 4.1), the commutativity of the previous diagram implies that $\tilde{\pi}_{JI}$ is the unique monomorphism induced by the inclusion of essential ideals $\delta_J(I) \subset \delta_J(J)$, by the Universal Property of Multiplier Algebras. Therefore, N and $M_{\mathrm{loc}}(A)$ are canonically isomorphic and, thus, we may identify them. $\square$

Theorem 5.4. *There exists a monomorphism $\beta : A \to K(\Omega^\Delta)$ such that*

$$A \xrightarrow{\ \beta\ } K(\Omega^\Delta) \xrightarrow{\ \gamma\ } M_{\mathrm{loc}}(A)$$

is the canonical embedding of A into its local multiplier algebra.

Proof. Let $j : A \to M(A)$ denote the canonical embedding of A into $M(A)$. Because δ_A embeds A as an essential ideal of A^A, the Universal Property of Multiplier Algebras tells us that the homomorphism $\gamma_A : A^A \to M(A)$ in Theorem 5.3 is the unique embedding for which

$$\begin{array}{ccc} A & \xrightarrow{\ \delta_A\ } & A^A \\ {\scriptstyle j}\big\downarrow & & \big\downarrow{\scriptstyle \gamma_A} \\ M(A) & \xrightarrow[\ \mathrm{id}\ ]{} & M(A) \end{array}$$

is a commutative diagram. Therefore, by Proposition 4.2 and Theorem 5.3, the diagram

$$\begin{array}{ccccc} A & \xrightarrow{\ \delta_A\ } & A^A & \xrightarrow{\ \pi_A\ } & A_\Delta = K(\Omega^\Delta) \\ {\scriptstyle j}\big\downarrow & & {\scriptstyle \gamma_A}\big\downarrow & & \big\downarrow{\scriptstyle \gamma} \\ M(A) & \xrightarrow[\ \mathrm{id}\ ]{} & M(A) & \xrightarrow[\ \tilde{\pi}_A\ ]{} & M_{\mathrm{loc}}(A) . \end{array}$$

is commutative. Thus,

$$A \xrightarrow{\ \delta_A\ } A^A \xrightarrow{\ \gamma_A\ } M(A) \xrightarrow{\ \tilde{\pi}_A\ } M_{\mathrm{loc}}(A)$$

is a canonical embedding of A into $M_{\mathrm{loc}}(A)$. Therefore, if $\beta = \pi_A \circ \delta_A$, then

$$A \xrightarrow{\ \beta\ } K(\Omega^\Delta) \xrightarrow{\ \gamma\ } M_{\mathrm{loc}}(A)$$

is also a canonical embedding of A into $M_{\mathrm{loc}}(A)$. $\square$

6. Main Results

6.1. **Determination of the injective envelope.** To this point our analysis has made extensive use of continuous Hilbert bundles for the study of A and its essential ideals, but for the determination of the injective envelope and local multiplier algebras of A, a larger class of vector fields is required. We shall now draw upon our work in the prequel [7] to the present paper.

Definition 6.1. ([7]) *A vector field $\mu : \Delta \to \bigsqcup_{s\in\Delta}, H_s^\Delta$ is said to be* weakly continuous *with respect to the continuous Hilbert bundle $(\Delta, \{H_s^\Delta\}_{s\in\Delta}, \Omega^\Delta)$ if the function*

$$s \mapsto \langle \mu(s), \xi(s) \rangle$$

is continuous for all $\xi \in \Omega^\Delta$.

If $\Omega^{\Delta}_{\mathrm{wk}}$ is the vector space of all weakly continuous vector fields with respect to the bundle $(\Delta, \{H^{\Delta}_s\}_{s\in\Delta}, \Omega^{\Delta})$, then the quadruple $(\Delta, \{H^{\Delta}_s\}_{s\in\Delta}, \Omega^{\Delta}, \Omega^{\Delta}_{\mathrm{wk}})$ is called a weakly continuous Hilbert bundle*.*

Definition 6.2. ([20]) *A Hilbert C*-module E over an abelian AW*-algebra Z is called a* Kaplansky–Hilbert module *if the following three properties hold:*

- **(i)** *if $c_i \cdot \nu = 0$ for some family $\{c_i\}_i \subset Z$ of pairwise-orthogonal projections and $\nu \in E$, then also $c \cdot \nu = 0$, where $c = \sup_i c_i$;*
- **(ii)** *if $\{c_i\}_i \subset Z$ is a family of pairwise-orthogonal projections such that $1 = \sup_i c_i$, and if $\{\nu_i\}_i \subset E$ is a bounded family, then there is a $\nu \in E$ such that $c_i \cdot \nu = c_i \cdot \nu_i$ for all i;*
- **(iii)** *if $\nu \in E$, then $g \cdot \nu = 0$ for all $g \in Z$ only if $\nu = 0$.*

The element $\nu \in E$ described in (ii) will be denoted by

$$\nu \;=\; \sum_i c_i \cdot \nu_i\,. \tag{23}$$

Theorem 6.3. ([7]) *The vector space $\Omega^{\Delta}_{\mathrm{wk}}$ is a Kaplansky–Hilbert module over the abelian AW*-algebra $C(\Delta)$, where the $C(\Delta)$-valued inner product $\langle\cdot,\cdot\rangle$ on $\Omega^{\Delta}_{\mathrm{wk}}$ has the property that for every pair $\xi, \eta \in \Omega^{\Delta}_{\mathrm{wk}}$ there is a meagre subset $M_{\xi,\eta} \subset \Delta$ such that*

$$\langle \xi, \eta\rangle\,(s) \;=\; \langle \xi(s), \eta(s)\rangle\,, \quad \textit{for all } s \in \Delta \setminus M_{\xi,\eta}\,.$$

Kaplansky [20] proved that the C*-algebra of bounded adjointable endomorphisms of a Kaplansky–Hilbert module is an AW*-algebra of type I and Hamana [18] proved that every type I AW*-algebra is injective. Thus:

Corollary 6.4. *The C*-algebra of $B(\Omega^{\Delta}_{\mathrm{wk}})$ of all bounded adjointable endomorphisms of $\Omega^{\Delta}_{\mathrm{wk}}$ is an injective AW*-algebra of type I.*

To determine the injective envelope of A we use the following criterion. Recall that an embedding or inclusion of a C*-algebra B into an injective C*-algebra C is said be *rigid* if the only unital completely positive linear map $\varphi : C \to C$ that is the identity on B is the map $\varphi = \mathrm{id}_C$. In [17] Hamana shows that a necessary and sufficient condition for an injective C*-algebra C to be an injective envelope of one its C*-subalgebras B is that the inclusion $B \subset C$ be rigid.

Theorem 6.5. ([7]) *There exists a monomorphism $\alpha : A^{\Delta} \to B(\Omega^{\Delta}_{\mathrm{wk}})$ such that:*

- **(i)** *$\alpha(a)\nu\,(s) = a(s)\nu(s)$, for every $a \in A^{\Delta}$, $\nu \in \Omega^{\Delta}_{\mathrm{wk}}$, $s \in \Delta$; and*
- **(ii)** *$\alpha(A^{\Delta})$ is a rigid C*-subalgebra of $B(\Omega^{\Delta}_{\mathrm{wk}})$.*

That is, $(B(\Omega^{\Delta}_{\mathrm{wk}}), \alpha)$ is an injective envelope of A^{Δ}.

We now arrive at the first main result of the present paper. Recall, from Theorem 5.4, that there is a monomorphism $\beta : A \to K(\Omega^{\Delta}) = A^{\Delta}$.

Theorem 6.6. *$\big(B(\Omega^{\Delta}_{\mathrm{wk}}), \alpha\circ\beta\big)$ is an injective envelope for A.*

Proof. Theorem 5.4 asserts that

$$A \xrightarrow{\;\beta\;} K(\Omega^{\Delta}) \xrightarrow{\;\gamma\;} M_{\mathrm{loc}}\,(A)$$

is a canonical embedding of A into its local multiplier algebra. Let $\iota_{\mathrm{mloc}} : M_{\mathrm{loc}}\,(A) \to I\,(M_{\mathrm{loc}}\,(A))$ denote the canonical embedding of $M_{\mathrm{loc}}\,(A)$ into its injective envelope.

By [13, Theorem 5], $(I(M_{\mathrm{loc}}(A)), \iota_{\mathrm{mloc}} \circ \gamma \circ \beta)$ is an injective envelope of A. Hence, by writing $I(A) = I(M_{\mathrm{loc}}(A))$, there exist embeddings

$$A \subset K(\Omega^{\Delta}) \subset M_{\mathrm{loc}}(A) \subset I(A)\,, \tag{24}$$

where the inclusions of A into $M_{\mathrm{loc}}(A)$ and $I(A)$ are the canonical inclusions. Moreover, the inclusion of $K(\Omega^{\Delta})$ into $I(A)$ is rigid because $K(\Omega^{\Delta})$ contains A. Hence, $(I(A), \kappa)$ is an injective envelope of $K(\Omega^{\Delta})$, where $\kappa = \iota_{\mathrm{mloc}} \circ \gamma$.

If, for a given C*-algebra B, (C, κ) and $(\tilde{C}, \tilde{\kappa})$ are two injective envelopes of B, then there is an isomorphism $\varphi : C \to \tilde{C}$ such that $\varphi \circ \kappa = \tilde{\kappa}$ [16, Theorem 4.1]. Theorem 6.5 asserts that $\big(B(\Omega^{\Delta}_{\mathrm{wk}}), \alpha\big)$ is an injective envelope of $K(\Omega^{\Delta})$. Hence,

$$\begin{array}{ccccc}
A & \xrightarrow{\beta} & K(\Omega^{\Delta}) & \xrightarrow{\iota_{\mathrm{mloc}}\circ\gamma} & I(A) \\
 & & \Big\| & & \Big\downarrow{\scriptstyle\varphi} \\
 & & K(\Omega^{\Delta}) & \xrightarrow[\alpha]{} & B(\Omega^{\Delta}_{\mathrm{wk}})
\end{array}$$

for some isomorphism φ, which proves that $\big(B(\Omega^{\Delta}_{\mathrm{wk}}), \alpha \circ \beta\big)$ is an injective envelope for A. □

6.2. **The second order local multiplier algebra.**

Theorem 6.7. *$M^{[2]}_{\mathrm{loc}}(A) = M^{[2+k]}_{\mathrm{loc}}(A) = I(A)$ for all $k \in \mathbb{N}$.*

Proof. The injective algebra $I(A) = B(\Omega^{\Delta}_{\mathrm{wk}})$ is a type I AW*-algebra and the ideal generated by the abelian projections of $I(A)$ is $K(\Omega^{\Delta}_{\mathrm{wk}})$ [7, Proposition 3.8]. We will prove below that $e \in M_{\mathrm{loc}}(A)$, for every abelian projection $e \in I(A)$. Assuming this statement holds, we therefore conclude that $K(\Omega^{\Delta}_{\mathrm{wk}}) \subset M_{\mathrm{loc}}(A)$. But $K(\Omega^{\Delta}_{\mathrm{wk}})$ is an essential ideal of $I(A)$, and hence it is also an essential ideal of $M_{\mathrm{loc}}(A)$. Therefore, $K(\Omega^{\Delta}_{\mathrm{wk}})$ and $M_{\mathrm{loc}}(A)$ have the same local multiplier algebras, which yields $M^{[2]}_{\mathrm{loc}}(A) = I(A)$ because of

$$B(\Omega^{\Delta}_{\mathrm{wk}}) \supset M^{[2]}_{\mathrm{loc}}(A) = M_{\mathrm{loc}}(K(\Omega^{\Delta}_{\mathrm{wk}})) \supset M(K(\Omega^{\Delta}_{\mathrm{wk}})) = B(\Omega^{\Delta}_{\mathrm{wk}})\,.$$

Hence, $M^{[2]}_{\mathrm{loc}}(A) = M^{[2+k]}_{\mathrm{loc}}(A) = I(A)$, for all $k \in \mathbb{N}$.

Therefore, to complete the proof assume that $e \in I(A)$ and $\varepsilon > 0$. Recall that $e = \Theta_{\nu,\nu}$ for some $\nu \in \Omega^{\Delta}_{\mathrm{wk}}$ for which $\langle \nu, \nu \rangle$ is a projection in $C(\Delta)$ [20, Lemma 13]. Because $\nu \in \Omega^{\Delta}_{\mathrm{wk}}$, there are a family $\{c_i\}_i$ of pairwise orthogonal projections in $C(\Delta)$ with supremum $1 \in C(\Delta)$ and a bounded family $\{\omega_i\}_i \subset \Omega^{\Delta}$ such that $\|\nu - \xi\| < \varepsilon$ [7, Proposition 4.4], where $\xi = \sum_i c_i \cdot \omega_i$ is in the sense of (23) and $\|\xi\| < 1 + \varepsilon$.

By (24), $K(\Omega^{\Delta}) \subset M_{\mathrm{loc}}(A) \subset B(\Omega^{\Delta}_{\mathrm{wk}})$. Therefore, the centre of $B(\Omega^{\Delta}_{\mathrm{wk}})$, namely $\{f \cdot 1 : f \in C(\Delta)\}$, is contained in the centre of $M_{\mathrm{loc}}(A)$ and $\Theta_{\omega_i,\omega_i} \in M_{\mathrm{loc}}(A)$ for all i. Thus, by [2, Lemma 3.3.6] (see also [24, Lemma 2.3]),

$$M_{\mathrm{loc}}(A) = \prod_i c_i M_{\mathrm{loc}}(A)\,,$$

and under this isomorphism, $(c_i \cdot \Theta_{\omega_i,\omega_i})_i$ determines a hermitian element $x \in M_{\mathrm{loc}}(A)$. Hence,

$$M_{\mathrm{loc}}(A) = \prod_i c_i M_{\mathrm{loc}}(A) \subset \prod_i c_i I(A) = I(A)\,,$$

where the last equality is a fact about AW*-algebras [19, Lemma 2.7]. As $e \in I(A)$ is identified with $(c_i \cdot e)_i$ under this isomorphism, we obtain

$$\|e - x\| \;=\; \sup_i \|c_i\,(\Theta_{\nu,\nu} - \Theta_{\omega_i,\omega_i})\,\| \;\le\; \|\nu - \xi\|\,(\|\nu\| + \|\xi\|) \;<\; \varepsilon(1+\varepsilon)\,.$$

Since $\varepsilon > 0$ is arbitrary, e is a limit of elements $x \in M_{\mathrm{loc}}(A)$. □

Theorem 6.7 demonstrates that the injectivity of $M_{\mathrm{loc}}^{[2]}(A)$, which was proved to hold for separable type I C*-algebras [25, Theorem 2.7] (see [5, Theorem 3.2] also), can hold as well for certain nonseparable type I C*-algebras. In particular, the following special case of Theorem 6.7 is new at this level of generality.

Corollary 6.8. *For every locally compact Hausdorff space T, $M_{\mathrm{loc}}^{[2]}(C_0(T) \otimes \mathbb{K})$ is injective and therefore $M_{\mathrm{loc}}^{[2]}(C_0(T) \otimes \mathbb{K}) = M_{\mathrm{loc}}^{[2+k]}(C_0(T) \otimes \mathbb{K})$, for all $k \in \mathbb{N}$.*

Corollary 6.9. *Every derivation $D : M_{\mathrm{loc}}^{[2]}(A) \to M_{\mathrm{loc}}^{[2]}(A)$ is inner, and so for every derivation $d : A \to A$ there is an inner derivation $D : M_{\mathrm{loc}}^{[2]}(A) \to M_{\mathrm{loc}}^{[2]}(A)$ such that $D_{|A} = d$.*

Proof. Every derivation of a C*-algebra extends to a derivation of its local multiplier algebra [2, Chapter 4], [21]. Applying this argument to A and then to $M_{\mathrm{loc}}(A)$, we have that every derivation d of A extends to a derivation of $M_{\mathrm{loc}}^{[2]}(A)$. On the other hand, if D is an arbitrary derivation of $M_{\mathrm{loc}}^{[2]}(A)$, then D is inner because every derivation of an AW*-algebra is inner. □

6.3. **A refinement of the chain of inclusions.**

Theorem 6.10. *There exist monomorphisms through which the following inclusions are as C*-subalgebras:*

$$A \subset K(\Omega^{\Delta}) \subset M_{\mathrm{loc}}(A) \subset M_{\mathrm{loc}}(K(\Omega^{\Delta})) \subset M_{\mathrm{loc}}^{[2]}(A) = M_{\mathrm{loc}}^{[2]}(K(\Omega^{\Delta}))\,.$$

Lemma 6.11. *Let $\nu \in \mathcal{E}^I$ be such that $\check{\nu} \in C_0(X^I)$. If $x \in M(I)$ is considered as strictly continuous bounded operator field on X^I, then there exists $\omega \in \Omega^I$ such that $x(t)\nu(t) = \omega(t)$ for every $t \in X^I$.*

Proof. Let $\omega : \beta X^I \to \sqcup_{t\in\beta X^I} H_t^I$ be defined by $\omega(t) = 0$ for $t \in \beta X^I \setminus X^I$ and $\omega(t) = x(t)\nu(t)$ for $t \in X^I$. We will show that for every $t_0 \in \beta X^I$ and $\varepsilon > 0$ there are an open set $U \subset \beta X^I$ containing t_0 and a $\mu \in \Omega^I$ such that $\|\omega(t) - \mu(t)\| < \varepsilon$ for all $t \in U$. Because Ω^I is closed under local uniform approximation, this will imply that $\omega \in \Omega^I$, thereby completing the proof.

Assume $t_0 \in \beta X^I$ and let $\varepsilon > 0$. Notice that $\|x\| = \sup_{t\in X^I}\|x(t)\| < \infty$ and $\|\omega(t)\| \le \|x\|\,\check{\nu}(t)$ for $t \in X^I$. Thus, if $t_0 \in \beta X^I \setminus X^I$, then there exists an open set $t_0 \in U \subset \beta X^I$ such that $\|\omega(t)\| \le \epsilon$ for $t \in U$, since $\check{\nu} \in C_0(X^I)$.

Assume now that $t_0 \in X^I$. Choose a bounded vector field $\eta \in \Omega^I$ such that there exists an open set $t_0 \in W \subset X^I$ such that $\|\eta(t)\| = 1$ for all $t \in W$.

Let $a = \Theta_{\nu,\eta}|_{X^I}$, which is an element of $\mathcal{F}((\Omega^I_{|X^I})_0)$. Because x is strictly continuous with respect to $\mathcal{F}((\Omega^I_{|X^I})_0)$, there are an open set $U \subset W$ containing t_0 and $b \in \mathcal{F}((\Omega^I_{|X^I})_0)$ such that $\|\,(x(t) - b(t))\,a(t)\| < \varepsilon$ for all $t \in U$. Note that we have

$$a(t)\eta(t) \;=\; \langle \eta(t), \eta(t)\rangle\,\nu(t) \;=\; \nu(t)\,,\ \forall\, t \in U\,.$$

Let $\mu = b\nu \in \Omega^I$. Hence, for any $t \in U$,

$$\|\omega(t) - \mu(t)\| \;=\; \|x(t)\nu(t) - b(t)\nu(t)\| \;=\; \|\left(x(t) - b(t)\right)a(t)\eta(t)\| \;<\; \varepsilon\,.$$

$\square$

Proof of Theorem 6.10. By Theorems 5.4 and 6.7 we are left to show that there is a monomorphism $\rho : M_{\mathrm{loc}}(A) \to M_{\mathrm{loc}}(A^\Delta)$, since $A^\Delta = K(\Omega^\Delta)$ by Proposition 4.2.

To that end, let $I \in \mathcal{I}_{\mathrm{ess}}(A)$ and consider the set $Y^I = \Phi_I^{-1}(X^I) \subset \Delta$ which is open and dense [6, Lemma 1.1]. Because $\Phi_I : \Delta \to \beta X^I$ is a (continuous) surjection Φ_I must map Y^I onto X^I. The open dense set Y^I determines an essential ideal of A^Δ that we denote by $\mathfrak{h}(I)$. Thus, $\mathfrak{h} : \mathcal{I}_{\mathrm{ess}}(A) \to \mathcal{I}_{\mathrm{ess}}(A^\Delta)$ is a well defined function. Note that if $K \in \mathcal{I}_{\mathrm{ess}}(A)$ is such that $K \preccurlyeq I$, then (6) states that $X^I \subset \Phi_{KI}^{-1}(X^K)$ (because Φ_{KI} maps $\beta X^I \setminus X^I$ into $\beta X^K \setminus X^K$ [15, Theorem 6.12]). Thus,

$$Y^K \;=\; \Phi_K^{-1}(X^K) \;=\; \Phi_I^{-1}\left(\Phi_{KI}^{-1}(X^K)\right) \;\supset\; Y^I\,, \tag{25}$$

and so $\mathfrak{h}$ preserves order; i.e., $K \preccurlyeq I \;\Rightarrow\; \mathfrak{h}(K) \preccurlyeq \mathfrak{h}(I)$.

Fix $I \in \mathcal{I}_{\mathrm{ess}}(A)$ and let $x \in M(I)$. Thus, by Proposition 5.2, x is a bounded cross section of $(X^I, \{B(H_t^I)\}_{t\in X^I})$ which is strictly continuous with respect to $\mathcal{F}((\Omega^I{}_{|X^I})_0)$. Consider the bounded section $\tilde{x} = x \circ \Phi_{I|Y^I}$ of the fibred space $(Y^I, \{B(H_s^\Delta)\}_{s\in Y^I})$. We aim to show that $\tilde{x}$ is strictly continuous with respect to $\mathcal{F}((\Omega^\Delta{}_{|Y^I})_0)$, as this is sufficient (and necessary) for $\tilde{x} \in M(\mathfrak{h}(I))$ by Proposition 5.2. To this end, let $s_0 \in Y^I$, $\varepsilon > 0$, and $a \in \mathcal{F}((\Omega^\Delta{}_{|Y^I})_0)$. Recall that $\Delta = \varprojlim \beta X^K$ and, by Theorem 3.4, $\Omega^\Delta \;=\; \varinjlim \Omega^K$. Thus, without loss of generality we can assume that there are an essential ideal $K \subset A$ with $K \subset I$, an open set $U \subset \beta X^K$ with $s_0 \in \Phi_K^{-1}(U) \subset Y^I$ and $\omega_j, \eta_j \in \mathcal{E}^K$ such that

$$a(s) = \sum_{j=1}^{n} \Theta_{\omega_j,\eta_j} \circ \Phi_K(s) \;\text{ for }\; s \in \Phi_K^{-1}(U)\,,$$

since Ω^K consists of all vector fields $\nu : \beta X^K \to \bigsqcup_{t\in\beta X^K} H_t^K$ that are local uniform limits of $\mathcal{E}^K$. Again, since the strict continuity is a local property, we can further assume that $\check{\omega_j}$, $\check{\eta_j} \in C_0(U)$.

Within the open subset $U \subset \beta X^K$, apply the Gram–Schmidt orthogonalisation procedure [12, Lemma 4.2] to the vector fields $\omega_j, \eta_j \in \mathcal{E}^K$ to obtain vector fields $\nu_1, \ldots, \nu_N \in \mathcal{E}^K$ that are pairwise orthogonal in an open set $t_0 = \Phi_K(s_0) \in U_0 \subset U$ and are such that each $\omega_j(t)$ and $\eta_j(t)$ are in the linear span of $\nu_1(t), \ldots, \nu_N(t)$ for every $t \in U_0$, $1 \le j \le n$. (Notice that $\mathcal{E}^K$ is a $C_b(T)$-module via the natural monomorphism from $C_b(T)$ into $C(\beta X^K)$; this is all that is needed for the Gram–Schmidt process.) Relabel so that U now has the property of U_0.

Because $I \preccurlyeq K$, the proof of Proposition 3.1 demonstrates that the map $\omega \mapsto \omega\circ\Phi_{IK}$ is a linear isomorphism $\mathcal{E}^K \to \mathcal{E}^I$, allowing one to go back and forth between $\mathcal{E}^K$ and $\mathcal{E}^I$. Hence, we may further assume that the vector fields $\omega_j, \eta_j, \nu_\ell \in \mathcal{E}^K$ are contained in $\mathcal{E}^I$ and defined on βX^I and are such that $\check{\omega_j}$, $\check{\eta_j} \in C_0(U) \subset C_0(X^I)$ (since $U \subset X^K \subset X^I$). Now let $p = \sum_{i=1}^N \Theta_{\nu_i,\nu_i} \in A^I$. By Lemma 6.11, each of

$$px = \sum_{i=1}^{N} \Theta_{\nu_i,\, x^*\nu_i} \;,\; xp = \sum_{i=1}^{N} \Theta_{x\nu_i,\, \nu_i} \;,\; \text{and } pxp = \sum_{i=1}^{N} \Theta_{x\nu_i,\, x^*\nu_i}$$

can naturally be regarded as an element of A^I. Notice that $p(t)$ is the orthogonal projection onto the span of $\{\nu_1(t), \dots, \nu_N(t)\}$ for every $t \in U$. Let $c = (px + xp - pxp) \circ \Phi_{IK} \in A^K$ and let $d = \sum_{i=1}^n \Theta_{\omega_i,\eta_i}$ whereby $a = d \circ \Phi_K$. Hence $d(x-c)(t) = dx(t) - dx(t) = 0$ for $t \in U$, since

$$dc(t) = dpx(t) + dxp(t) - dpxp(t) = dx(t) + dpxp(t) - dpxp(t)\,,$$

because $d(t) = dp(t) = pd(t)$ for $t \in U$. Similarly $(x-c)d(t) = xd(t) - xd(t) = 0$ for $t \in U$.

If we now let $b = c \circ \Phi_K$ then $b \in \mathcal{F}((\Omega^\Delta|_{Y^I})_0)$ - since $\check{c} \in C_0(U)$ - is such that

$$\|\,(\tilde{x}(s) - b(s))\,a(s)\| \,+\, \|a(s)\,(\tilde{x}(s) - b(s))\,\| \,<\, \varepsilon\,, \quad \text{for all} \quad s \in \Phi_K^{-1}(U) \subset Y^I\,.$$

This proves that $\tilde{x} \in M(\mathfrak{I}(I))$. The map $\zeta_I : M(I) \to M(\mathfrak{h}(I))$ given by $\zeta_I(x) = \tilde{x}$ is evidently a homomorphism. If $\zeta_I(x) = 0$, then $x(t) = 0$ for all $t \in \Phi_I(Y^I) = X^I$, and so $x = 0$. Therefore, ζ_I is a monomorphism. Let $\alpha_{\mathfrak{h}(I)} : M(\mathfrak{h}(I)) \to M_{\mathrm{loc}}\,(A^\Delta)$ be the unique monomorphism that embeds $M(\mathfrak{h}(I))$ into the local multiplier algebra of A^Δ and, for $J \preccurlyeq I$, let $\alpha_{\mathfrak{h}(J)\mathfrak{h}(I)} : M(\mathfrak{h}(J)) \to M(\mathfrak{h}(I))$ be the connecting monomorphisms induced by $\mathfrak{h}(J) \preccurlyeq \mathfrak{h}(I)$. For each $I \in \mathcal{I}_{\mathrm{ess}}(A)$, let $\rho_I : M(I) \to M_{\mathrm{loc}}\,(A^\Delta)$ be the monomorphism $\rho_I = \alpha_{\mathfrak{h}(I)} \circ \zeta_I$. Because $\tilde{\pi}_{JI} = \alpha_{\mathfrak{h}(J)\mathfrak{h}(I)} \circ \zeta_J$ (where $\tilde{\pi}_{JI}$ is as in (22)) we conclude that the following diagram

$$\begin{array}{ccc} M(J) & \xrightarrow{\tilde{\pi}_{JI}} & M(I) \\ {\scriptstyle \zeta_J}\big\downarrow & & \big\downarrow{\scriptstyle \zeta_I} \\ M(\mathfrak{h}(J)) & \xrightarrow[\alpha_{\mathfrak{h}(J)\mathfrak{h}(I)}]{} & M(\mathfrak{h}(I)) \end{array}$$

is commutative. Therefore, there exists a monomorphism $\rho : M_{\mathrm{loc}}\,(A) \to M_{\mathrm{loc}}\,(A^\Delta)$. $\square$

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