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On the equivalence problem for bracket-generating distributions

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Abstract. We describe Tanaka's version of Cartan Equivalence Method for bracket-generating distributions in a form suited to study the horizontal distributions on period domains and on their boundaries, and apply it to certain polycontact structures related to the latter.

1. Introduction

The theory of Exterior Differential Systems is one of the transcendental themes in Griffiths' work, starting with his early collaboration with Spencer and his papers on G-structures to his most recent contributions to Hodge Theory [**7**][**11**][**12**].

A main tool in the theory is Cartan's Equivalence Method, aimed to determine when two G-structures are equivalent under local diffeomorphisms of the underlying manifolds [**4**]; in other words, to find appropriate invariants, or normal forms, of a G-structure. The method is quite arduous to apply and various modifications and formalizations of Cartan's original idea have been introduced to simplify its use.

One version, due to N. Tanaka and not very well known, is especially adapted to completely non-integrable, or bracket-generating, distributions, a condition which is not very restrictive, particularly for the horizontal distribution in Mumford-Tate domains [**23**][**7**], and extends to Morimoto's filtered manifolds [**19**], of which the latter are examples. Using the notion of pseudo G-structure, it reduces many difficulties of the Equivalence Problem to questions about finite-dimensional graded Lie algebras.

In this article we explain Tanaka's method and apply it to the conformal geometry of certain polycontact systems.

First, we explain Tanaka's method. It applies to both smooth and analytic distributions, although for simplicity and in view of the application, we will consider the smooth real case. Let D be a vector distribution of rank d on a manifold M of dimension n. The standard way to proceed is to consider the bundle $P \to M$ of D-adapted coframes, or, as we will do here for reasons that will become evident,

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frames:

$$P = \{(x, \varphi) : \varphi : \mathbb{R}^n \to T_x(M) \text{ is a linear isomorphism and } \varphi(\mathbb{R}^d) = D(x)\}.$$

This is a G-structure, i.e., a principal bundle with structure group contained in $Gl(n, \mathbb{R})$, namely the stabilizer of $\mathbb{R}^d \subset \mathbb{R}^n$. Two distributions are (locally) equivalent if and only if the corresponding G-structures are equivalent. Additional structures supported by (M, D) correspond to reductions of P.

One can regard the φ's in P as comparing D at each point to the model integrable (flat) distribution of the same dimension and codimension, namely $\mathbb{R}^d \subset \mathbb{R}^n$. The *invariant torsion* of P measures how much D differs from the model. Cartan's Equivalence Method proceeds to normalize the torsion by successive reductions and prolongations of P, until a bundle $\tilde{P} \to M$ with a canonical coframe is obtained, so that two distributions are equivalent if and only if these coframes are equivalent, the Equivalence Problem for coframes having been solved by Cartan himself.

But for non-integrable distributions comparison with the integrable distribution is too coarse. A contact distribution, for example, should instead be compared to the standard non-integrable distribution on the Heisenberg group. In fact, any distribution satisfying certain regularity assumptions defines a bundle of graded nilpotent Lie algebras which are mutually isomorphic and therefore a corresponding "flat model" to which it can be compared, namely the corresponding left-invariant distribution on the corresponding simply connected Lie group. The Lie algebra is called the *symbol* or *nilpotentization* of the distribution and is defined as follows. Set

$$D^1(x) = D(x) \qquad D^{r+1}(x) = D^r(x) + [D, D^r](x)$$

where D is also used to denote the sections of D. Define

$$\mathfrak{n}_D^{-i}(x) = D^i(x)/D^{i-1}(x)$$

and let the symbol of D at x be

$$\mathfrak{n}_D(x) := \mathfrak{n}_D^{-k}(x) \oplus \cdots \oplus \mathfrak{n}_D^{-1}(x).$$

It is easy to see that the ordinary Lie bracket on M induces a structure of graded Lie algebra on $\mathfrak{n}_D(x)$ – which justifies using frames rather than coframes, and that equivalent distributions have isomorphic symbols.

We now assume that D is bracket-generating, i.e. $D^k(x) = T_x(M)$ for some k and all x, and that its symbols $\mathfrak{n}_D(x)$ are all isomorphic to a fixed graded nilpotent Lie algebra

$$\mathfrak{m} = \mathfrak{m}^{-k} \oplus \cdots \oplus \mathfrak{m}^{-1}$$

generated by $\mathfrak{m}^{-1}$. Let

$$P^0 = \{(x, \varphi) : \varphi : \mathfrak{m} \to \mathfrak{n}_D(x) \text{ is an isomorphism of graded Lie algebras}\}.$$

This is a principal fiber bundle over M, with the group $\mathrm{Aut}_{\mathrm{gr}}(\mathfrak{m})$ of graded automorphisms of $\mathfrak{m}$ as structure group. P^0 is not in general a G-structure, being associated to $\mathfrak{n}_D$ rather than to $T(M)$; it is an example of Tanaka's *pseudo* G-structure.

Now fix $\mathfrak{m}$ and a Lie subalgebra

$$\mathfrak{g}^0 \subset \mathrm{Der}_{\mathrm{gr}}(\mathfrak{m})$$

(as if defining a reduction of P^0), and consider the semidirect sum

$$\mathfrak{g}^0 \oplus_{\triangleleft} \mathfrak{m} \;\cong\; \mathfrak{m}^{-k} \oplus \cdots \oplus \mathfrak{m}^{-1} \oplus \mathfrak{g}^0.$$

This has a unique maximal extension

$$\mathfrak{m}^{-k} \oplus \cdots \oplus \mathfrak{m}^{-1} \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^1 \oplus \mathfrak{g}^2 \oplus \cdots$$

as a graded Lie algebra with the property that any non-zero $X \in \mathfrak{g}^1 \oplus \mathfrak{g}^2 \oplus \cdots$ satisfies

$$\ker(\mathrm{ad}(X)) \cap \mathfrak{m}^{-1} = 0.$$

The subspaces $\mathfrak{g}^i$ can be computed recursively and have the following geometric meaning.

Associated to $D, \mathfrak{m}, \mathfrak{g}_0$, there is a prolongation tower

$$\cdots \to P^2 \to P^1 \to P^0 \to M$$

of pseudo G-structures such that:

(a) $\mathfrak{g}^i$ is naturally isomorphic to the (abelian) Lie algebra of the structure group of $P^i \to P^{i-1}$, which is the kernel of a corresponding Spencer operator – just as in the case of prolongations of G-structures.

(b) Two distributions are equivalent iff their towers are isomorphic.

From this it follows that if $\mathfrak{g}^i = 0$ for some i, then P^i supports a canonical e-structure and, therefore, some P^j with $j \geq i$ carries a canonical frame preserving equivalence.

In the last two sections we apply Tanaka's method to structures on fat distributions. The latter are characterized by the property that for the associated Levi operator, or curvature,

$$K_x : \bigwedge^2 D(x) \to T_x(M)/D(x), \qquad K_x(X,Y) = [\tilde{X}, \tilde{Y}]_x + D(x),$$

where $\tilde{X}, \tilde{Y}$, are local extensions of X, Y, the map

$$Y \mapsto K(X,Y)_x$$

is onto for any x and any non-zero $X \in D(x)$.

One may consider the Equivalence Problem for subriemannian metrics supported by a given D, as in [**14**][**20**], but the most natural geometry is often the (sub)conformal analog, as in [**3**][**6**]. Here we will consider *compatible* subconformal structures D, i.e. those satisfying that for any non-zero $\lambda \in (T(M)/D)^*$ there is a metric g in the class of the conformal structure such that the endomorphisms J_λ of D defined by

$$g(J_\lambda x, y) = \lambda(K(X,Y))$$

satisfy $J_\lambda^2 = -I$. It is clear that the existence of a compatible subconformal structure on D implies that D is fat.

As an illustration, consider distributions of dimension 2 in 3-manifolds. Here fat means contact and the nilpotentization is the Heisenberg algebra. Without any additional structure, the resulting prolongation algebra $\mathfrak{g}$ is infinite-dimensional. With a subriemannian metric, $\mathfrak{g}$ is finite-dimensional and has $\mathfrak{g}^i = 0$ for all $i > 0$. Such distribution always admits compatible subconformal structures, for which the prolongation is still finite-dimensional, but with $\mathfrak{g}^1$ and $\mathfrak{g}^2$ are non-zero.

Fatness is equivalent to the nilpotentization

$$\mathfrak{n}_D(x) = \mathfrak{n}_D^{-2}(x) \oplus \mathfrak{n}_D^{-1}(x)$$

being a non-singular 2-step nilpotent Lie algebra, in the sense of [**10**]. The existence of a compatible subconformal structure supported on D is equivalent to the nilpotentization being of Heisenberg type, in the sense of [**15**]. These Lie algebras can be realized as

$$\mathfrak{n} = \mathbb{R}^n \oplus \mathbb{R}^m,$$

with the bracket such that the J_λ define a structure of Clifford module on $\mathbb{R}^n$ over the Clifford algebra $C(m)$. The left-invariant distributions defined by $\mathfrak{n}^{-1}$ on the simply connected Lie group of a Lie algebra of Heisenberg type $\mathfrak{n}$, are fat and support a compatible subconformal structure. But sometimes there inequivalent ones.

Since non-singular Lie algebras form wild sets (except in small dimensions) and those of Heisenberg type are countable, most fat distributions will not have compatible subconformal structures. However, these occur in the same dimensions and codimensions as do fat distributions (cf. [**18**][**22**]) and seem to be the most symmetric [**16**], hence likely models of Cartan geometries. In the last section we show that, indeed, the Tanaka prolongation of compatible subconformal structures on a fat distribution admit Cartan connections.

The nilpotent Lie algebras obtained from the Iwasawa decomposition of a semisimple real Lie algebra of split rank one, are of Heisenberg type, and will be said to be *of parabolic type*, since they give rise to parabolic geometries. These occur only in codimensions $m = 1, 3, 7$, while the $m \geq 1$ is arbitrary.

Subconformal structures on fat distributions on spheres arise as boundary values of complex, quaternionic and octonionic hyperbolic metrics, where they are parabolic and correspond to algebras of that type. Deforming them and solving a corresponding Dirichlet problem yields new Einstein metrics on the hyperbolic spaces [**3**]. An analogous "AdS-QFT correspondence" occurs for all compatible conformal structures on fat distributions, except that in those of non-parabolic type the distribution has a point singularity on the sphere at infinity. The symmetric metric is replaced by a harmonic one, which is still Einstein, on the corresponding Damek-Ricci space AN [**9**], where N is a group of Heisenberg type and A the group of graded dilations. Deforming the conformal structure – which involves some version of the Equivalence Problem, then leads formally to anisotropic, non-harmonic Einstein metrics in AN (cf. [**1**][**13**]).

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2. Distributions of constant type

In this and next section we follow Tanaka [**24**] and Zelenko [**27**], with some modifications and corrections.

A graded Lie algebra of the form

$$\mathfrak{m} = \bigoplus_{i=1}^{\mu} \mathfrak{g}^{-i}$$

is called *fundamental* if it generated by $\mathfrak{g}^{-1}$. A distribution D is called of *constant type* $\mathfrak{m}$ if its nilpotentizations are all isomorphic to $\mathfrak{m}$ as graded Lie algebras.

Fix a fundamental Lie algebra $\mathfrak{m}$ and let $G_\mathfrak{m}$ be the group of automorphisms of $\mathfrak{m}$ that preserve the grading. To every distribution of constant type $\mathfrak{m}$ one associates a principal $G_{\mathfrak{m}}$-bundle over M given by

$$P_{\mathfrak{m}} = \{(x,\varphi) : x \in M,\ \varphi : \mathfrak{m} \to \mathfrak{m}(x) \text{ is an isomorphism of graded Lie algebras}\}.$$

The right action of $G_{\mathfrak{m}}$ is

$$(x,\varphi)\cdot A = (x,\varphi\circ A)$$

for $A \in G_{\mathfrak{m}}$.

The assignment $D \to P_{\mathfrak{m}}$ is compatible with equivalence, i.e., two distributions are locally equivalent if and only if the respective $G_{\mathfrak{m}}$-bundles are locally isomorphic.

Fix the distribution D, a Lie subgroup $G^0 \subset G_{\mathfrak{m}}$ and a reduction

$$\pi : P^0 \to M$$

of $P_{\mathfrak{m}}$ with structure group G^0. The filtration $\{D^p\}_{p>0}$ of $T(M)$ lifts to a filtration $\mathfrak{D}^0 = \{D_0^p\}_{p>0}$ of $T_\varphi P^0$, where

$$D_0^p = \left\{v \in T_\varphi P^0 : d\pi(v) \in D^p(x)\right\}.$$

The *tautological form* of P^0 is the tuple $\Theta^0 = \{\theta_0^p\}_{p>0}$ where θ_0^p is the $\mathfrak{g}^{-p}$-valued linear form on D_0^p defined by:

$$\theta_0^p|_\varphi(Y) = \varphi^{-1}(d\pi(Y)_{(p)}).$$

where $d\pi(Y)_{(p)}$ denotes the equivalence class of $d\pi(Y)$ in $D^p(x)/D^{p-1}(x)$.

The system $(P^0,\mathfrak{D}^0,\Theta^0)$ is called a *pseudo-G^0-structure of type* $\mathfrak{m}$ on M. When $D(x) = T_x(M) = \mathfrak{m}(x)$ and $\mathfrak{m} = \mathbb{R}^n$ is abelian, P^0 is an ordinary G^0-structure and Θ^0 is the usual tautological form.

3. Algebraic prolongation

Given a fundamental Lie algebra $\mathfrak{m} = \mathfrak{g}^{-\mu}\oplus\cdots\oplus\mathfrak{g}^{-1}$, its *Tanaka prolongation* is the unique graded Lie algebra $\mathfrak{g}(\mathfrak{m}) = \sum_{k\in\mathbb{Z}}\mathfrak{g}^k(\mathfrak{m})$ satisfying:

(1) $\mathfrak{g}^{-i}(\mathfrak{m}) = \mathfrak{g}^{-i}$ for all $i > 0$;
(2) if $X \in \mathfrak{g}^i(\mathfrak{m})$ with $i > 0$ satisfies $[X,\mathfrak{g}^{-1}] = 0$, then $X = 0$;
(3) $\mathfrak{g}(\mathfrak{m})$ is the maximal graded Lie algebra satisfying 1 and 2.

The Lie algebra $\mathfrak{g}(\mathfrak{m})$ has the following explicit realization: the spaces $\mathfrak{g}^k(\mathfrak{m})$ for $k \geq 0$ are defined inductively by

$$\mathfrak{g}^k(\mathfrak{m}) = \{u \in \bigoplus_{p>0} Hom(\mathfrak{g}^{-p}(\mathfrak{m}),\mathfrak{g}^{-p+k}(\mathfrak{m})) \mid u([X,Y]) = [u(X),Y] + [X,u(Y)]\},$$

and the Lie bracket in $\mathfrak{g}(\mathfrak{m})$ is defined by the following conditions: for $X \in \mathfrak{m}$,

$[U,X] = U(X)$, for $U \in \mathfrak{g}^k(\mathfrak{m})$ with $k \geq 0$;

$[U,V](X) = [U,[V,X]] - [V,[U,X]]$, for $U \in \mathfrak{g}^k(\mathfrak{m})$ and $V \in \mathfrak{g}^l(\mathfrak{m})$ with $k,l \geq 0$ (so that $[U,V] \in \mathfrak{g}^{k+l}(\mathfrak{m})$).

In particular, $\mathfrak{g}^0(\mathfrak{m})$ is the Lie algebra of graded derivations of $\mathfrak{m}$, with

$$[U,V](X) = [U,[V,X]] - [V,[U,X]]$$

Fix a subalgebra $\mathfrak{g}^0 \subset \mathfrak{g}^0(\mathfrak{m})$. The *prolongation of the pair* $(\mathfrak{m},\mathfrak{g}^0)$ is the graded subalgebra

$$Prol(\mathfrak{m},\mathfrak{g}^0) = \mathfrak{m}\oplus\bigoplus_{k\geq 0}\mathfrak{g}^k \subset \mathfrak{g}(\mathfrak{m})$$

where for each $k \geq 1$, $\mathfrak{g}^k$ is the subspace of $\mathfrak{g}^k(\mathfrak{m})$ satisfying

$$[\mathfrak{g}^k, \mathfrak{g}^{-1}] \subset \mathfrak{g}^{k-1}.$$

The pair $(\mathfrak{m}, \mathfrak{g}^0)$ is called *of finite type* k if there exists an integer k so that $\mathfrak{g}^{k-1} \neq 0$ and $\mathfrak{g}^k = \{0\}$.

THEOREM 3.1. [**25**] $(\mathfrak{m}, \mathfrak{g}^0)$ *is of finite type if and only if the subspace*

$$\{X \in \mathfrak{g}^p : [X, \mathfrak{g}^{-d} + \cdots + \mathfrak{g}^{-2}] = 0\} \tag{3.1}$$

equals 0 *for some* $p \geq 1$.

In the semisimple case one has

THEOREM 3.2. [**21**] *Let* $\mathfrak{g} = \mathfrak{g}^{-d} \oplus \cdots \oplus \mathfrak{g}^{-1} \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^1 \oplus \cdots \oplus \mathfrak{g}^d$ *be a semisimple graded Lie algebra where the negative part* $\mathfrak{m} = \mathfrak{g}^{-d} \oplus \cdots \oplus \mathfrak{g}^{-1}$ *is fundamental of depth* $d \geq 2$. *Then the prolongation of the pair* $(\mathfrak{g}^{-d} \oplus \cdots \oplus \mathfrak{g}^{-1}, \mathfrak{g}^0)$ *is isomorphic to* $\mathfrak{g}$ *if and only if* $(\mathfrak{m}, \mathfrak{g}^0)$ *is of finite type.*

The relation with Spencer operators comes from the following. For $k \geq 0$ define the spaces:

$$S_k = \bigoplus_{i>0} Hom(\mathfrak{g}^{-i}, \mathfrak{g}^{-i+k+1}) \oplus \bigoplus_{i=0}^{k-1} Hom(\mathfrak{g}^i, \mathfrak{g}^k)$$

and

$$\begin{aligned} A_k &= \left(\bigoplus_{i>1} Hom(\mathfrak{g}^{-1} \otimes \mathfrak{g}^{-i}, \mathfrak{g}^{-i+k}) \right) \oplus Hom(\mathfrak{g}^{-1} \wedge \mathfrak{g}^{-1}, \mathfrak{g}^{k-1}) \\ &\oplus \left(\bigoplus_{i=0}^{k-1} Hom(\mathfrak{g}^{-1} \otimes \mathfrak{g}^i, \mathfrak{g}^{k-1}) \right), \end{aligned}$$

and the operator

$$\partial_k : S_k \to A_k$$

$$\partial_k f(v, w) = \begin{cases} [f(v), w] + [v, f(w)] - f([v, w]), & \text{if } v \in \mathfrak{g}^{-1},\ w \in \mathfrak{g}^{-i},\ i > 0; \\ [v, f(w)], & \text{if } v \in \mathfrak{g}^{-1},\ w \in \mathfrak{g}^i,\ 0 \leq i < k. \end{cases}$$

Then one has

THEOREM 3.3. *ker* $\partial_k = \mathfrak{g}^{k+1}$.

4. Geometric prolongation

Fix a pair $(\mathfrak{m}, \mathfrak{g}^0)$ and let $\mathfrak{g} = \sum_{i \in \mathbb{Z}} \mathfrak{g}^i$ be its prolongation. For every $k > 0$ let H^k be the subgroup of $GL(\sum_{i \leq k-1} \mathfrak{g}^i)$ consisting of elements A satisfying

(a) $(A - Id)|_{\mathfrak{g}^{k-1}} = 0$,
(b) $(A - Id)(\mathfrak{g}^p) \subset \mathfrak{g}^{k-1}$ for $p \geq 0$,
(c) $(A - Id)(\mathfrak{g}^{-p}) \subset \mathfrak{g}^{k-p} \oplus \ldots \oplus \mathfrak{g}^{k-1}$ for $p > 0$,

and N^k be the normal subgroup of H^k consisting of elements B satisfying:

(d) $(B - Id)(\mathfrak{g}^p) = 0$ for $p \geq 0$,
(e) $(B - Id)(\mathfrak{g}^{-p}) \subset \mathfrak{g}^{k-p+1} \oplus \ldots \oplus \mathfrak{g}^{k-1}$ for $p > 0$.

The Lie algebras of H^k and N^k are, respectively,

$$\mathfrak{s}^k = \bigoplus_{p>0} Hom(\mathfrak{g}^{-p}, \mathfrak{g}^{-p+k}) \oplus \bigoplus_{p=0}^{k-2} Hom(\mathfrak{g}^p, \mathfrak{g}^{k-1})$$

$$\mathfrak{n}^k = \bigoplus_{p<k} \bigoplus_{r<p-k} Hom(\mathfrak{g}^r, \mathfrak{g}^p).$$

Let

$$\mathfrak{h}^k = \mathfrak{s}^k \oplus \mathfrak{n}^k$$

and note the inclusion of graded Lie algebras $\mathfrak{g}^k \subset \mathfrak{s}^k \subset \mathfrak{h}^k$. The mapping

$$\mathfrak{i}^k : \mathfrak{h}^k \to H^k, \qquad \mathfrak{i}^k(A) = A + Id$$

is bijective and carries $\mathfrak{n}^k$ onto N^k. Define the Lie groups

$$G^k_\sharp = \mathfrak{i}^k(\mathfrak{g}^k + \mathfrak{n}^k) = \mathfrak{i}^k(\mathfrak{g}^k)N^k$$

$$G^k = G^k_\sharp / N^k$$

Observe that G^k is just the simply connected group with the abelian Lie algebra structure on $\mathfrak{g}^k$.

A *pseudo-G^k-structure* on a manifold M is a triple $(P^k, \mathfrak{D}^k, \Theta^k)$ where P^k is a principal fiber bundle over M with structure group G^k, $\mathfrak{D}^k = (D^p_k)_{p>0}$ is a family of distributions on P^k filtered as before, and $\Theta^k = (\theta^p_k)_{p>-k}$ is a family of forms, with θ^p_k a $\mathfrak{g}^{-p}$-valued 1-form on D^{p+k}_k, satisfying the following conditions:

– dim M = dim $\sum_{i\leq k-1} \mathfrak{g}^i$;

– dim D^p_k = dim $\sum_{i=-p}^{k} \mathfrak{g}^i$ for $p > 0$;

– D^0_k is the vertical space on P^k;

– the D^p_k's are invariant under the action of G^k on P^k;

– D^p_k is defined by the equations $\theta^i_k = 0$ for $i > -p$;

– D^0_k is defined by the equations $\theta^i_k = 0$ for $i > -k$;

For any $a \in G^k$ let $X \in \mathfrak{g}^k$ such that $exp\ X = a$ then,

– if $k = 0$, $R^*_a \theta^p_0 = a^{-1}\theta^p_0$;

– if $k > 0$, $R^*_a \theta^p_k = \theta^p_k - [X, \theta^{p+k}_k|_{D^{p+k}_k}]$ for $p > -k$.

The following results are due to Tanaka.

THEOREM 4.1. *Assume that G^0 is connected and for $k \geq 1$ fix a complementary subspace H_k of $\partial_k S_k$ in A_k. Then to every pseudo-G^0-structure $(P^0, \mathfrak{D}^0, \Theta^0)$ of type $\mathfrak{m}$ on a manifold M, there is associated a sequence*

$$(P^0, \mathfrak{D}^0, \Theta^0) \leftarrow \cdots \leftarrow (P^{k-1}, \mathfrak{D}^{k-1}, \Theta^{k-1}) \leftarrow (P^k, \mathfrak{D}^k, \Theta^k) \leftarrow \cdots$$

such that:

- *For each $k \geq 1$, $(P^k, \mathfrak{D}^k, \Theta^k)$ is a pseudo-G^k-structure of type $\mathfrak{m}$ on P^{k-1}.*
- *The assignment $(P^0, \mathfrak{D}^0, \Theta^0) \mapsto (P^k, \mathfrak{D}^k, \Theta^k)$ is compatible with the isomorphisms for all $k > 0$.*

A proof of this theorem can be found in [**24**]. A more amenable exposition of the construction of these pseudo-G^k-structures can be found in [**27**].

Corollary 4.2. *If the pair $(\mathfrak{m}, \mathfrak{g}^0)$ is of finite type with $\mathfrak{g}^l = \{0\}$. Then P^l has a canonical frame.*

Observe that the prolongation depends in each step of the choice of the complementary subspace H_k of $\partial_k S_k$ in A_k, so the canonical frames we get in P^l are not unique. We use the term canonical in the sense that it encodes all the invariants of the pseudo-G^0-structure for fixed H_k, so that the problem of equivalence of pseudo-G^0-structure is equivalent to the problem of equivalence of these frames.

Corollary 4.3. *Assume that the pair $(\mathfrak{m}, \mathfrak{g}^0)$ is of finite type, and let P^0 be a pseudo-G^0-structure of type $\mathfrak{m}$ on a connected manifold M. Then the Lie algebra of all infinitesimal automorphisms of P^0 is finite dimensional*

5. Conformal structures on fat distributions

Recall that the *curvature* [**18**], or *Levi operator* [**2**], of a distribution D at a point x is the linear map

$$K_x : \bigwedge^2 D(x) \to T_x(M)/D(x), \qquad K_x(X,Y) = [\tilde{X}, \tilde{Y}]_x + D(x)$$

where $\tilde{X}, \tilde{Y}$, are local extensions of X, Y. Every $\lambda \in (T(M)/D)^*$ defines a 2-form on D,

$$\omega_\lambda(X,Y) = \lambda(K(X,Y)).$$

A conformal structure on D will be called *compatible* if for any non-zero $\lambda \in Q^*$ there is a metric g in the class of the conformal structure such that the endomorphism J_λ of D defined by

$$g(J_\lambda x, y) = \omega_\lambda(X,Y)$$

satisfy $J_\lambda^2 = -I$.

This implies in particular that for any x and any non-zero $X \in D(x)$, the map $Y \mapsto K(X,Y)_x$ is onto, i.e., D is fat [**18**] or polycontact [**26**].

Any fat distribution it is locally of constant type, its symbol having the form

$$\mathfrak{n}_D(x) = \mathfrak{n}_D^{-2}(x) \oplus \mathfrak{n}_D^{-1}(x).$$

Fatness is equivalent to this Lie algebra being non-singular in the sense of [**10**]. As to the existence of conformal structure, let $\mathfrak{h}$ be a real 2-step nilpotent Lie algebra with an inner product. Then $\mathfrak{h} = \mathfrak{z} \oplus \mathfrak{v}$ where $\mathfrak{z}$ is the center of $\mathfrak{h}$ and $\mathfrak{v}$ its orthogonal complement. Let $J : \mathfrak{z} \to End(\mathfrak{v})$ be the linear mapping defined by

$$\langle J_z x, y \rangle = \langle z, [x,y] \rangle \quad z \in \mathfrak{z},\ x, y \in \mathfrak{v}. \tag{5.1}$$

One says that $\mathfrak{h}$ is of *Heisenberg type* (or of *type H*) if

$$J_z^2 = -|z|^2 Id \quad z \in \mathfrak{z}. \tag{5.2}$$

It follows that

Proposition 5.1. *A distribution has a compatible conformal structure if and only if its symbol is of Heisenberg type.*

The nilpotent algebra obtained from the Iwasawa decomposition of a semisimple real Lie algebra of split rank one, is of type H (of *parabolic type*).

It follows from (5.1) that

$$J_z^t = -J_z$$

Polarization of (5.2) leads to

$$J_z J_w + J_w J_z = -2\langle z, w\rangle Id$$

In particular, when z and w are orthogonal,

$$J_z J_w = -J_w J_z \tag{5.3}$$

These properties imply:

$$\begin{aligned} \langle J_z x, J_z y\rangle &= \langle z, z\rangle\langle x, y\rangle \\ \langle J_z x, J_w x\rangle &= \langle z, w\rangle\langle x, x\rangle \end{aligned}$$

$$[x, J_z x] = \langle x, x\rangle z$$

Let $\mathfrak{h}$ be an H-type algebra, $\mathfrak{d}$ the Lie algebra of antisymmetric derivations of $\mathfrak{h}$ and $\mathfrak{a}$ the one dimensional algebra generated by the dilation A, i.e. the derivation defined by $A|_{\mathfrak{v}} = Id$ and $A|_{\mathfrak{z}} = 2Id$. Then

$$\mathfrak{g}^0 = \mathfrak{d} \oplus \mathfrak{a} \tag{5.4}$$

is a subalgebra of $Der(\mathfrak{h})$ and $(\mathfrak{h}, \mathfrak{g}^0)$ is the pair associated to a fat distribution with a compatible conformal structure. To study the equivalence problem of this structures we consider the prolongation of the pair $(\mathfrak{h}, \mathfrak{g}^0)$, following the second author's thesis.

Let $W \in \mathfrak{g}^1$ and $Y \in \mathfrak{g}^{-1}$. Then

$$W(Y) = p(Y)A + D_Y^W \in \mathfrak{g}^0$$

with $D_Y^W \in \mathfrak{d}$ and $p \in (\mathfrak{g}^{-1})^*$. Write $p(Y) = \langle X_W, Y\rangle$ for some $X_W \in \mathfrak{g}^{-1}$. Then

$$\begin{aligned} \langle J_Z W(Z), Y\rangle = \langle [W(Z), Y], Z\rangle = \langle W(Y)(Z), Z\rangle = 2p(Y)\,|Z|^2 \\ = \left\langle 2\,|Z|^2\, X_W, Y\right\rangle. \end{aligned}$$

Therefore

$$W(Z) = -2J_Z X_W \qquad Z \in \mathfrak{g}^{-2} \tag{5.5}$$

and

$$W(Y) = \langle X_W, Y\rangle\, A + D_Y^W \qquad Y \in \mathfrak{g}^{-1}. \tag{5.6}$$

PROPOSITION 5.2. *If $\mathfrak{h}$ is an H-type algebra, $(\mathfrak{h}, \mathfrak{g}^0)$ is of finite type.*

PROOF. Let $W \in \mathfrak{h}^1$. Then $W(Z) = -2J_Z X_W = 0$ for all $Z \in \mathfrak{g}^{-2}$, so $X_W = 0$. It follows that $W(X) \in \mathfrak{d}$ for all $X \in \mathfrak{g}^{-1}$, and

$$0 = W([X, Y]) = W(X)(Y) - W(Y)(X).$$

If $\{X_i\}$ is a basis of $\mathfrak{g}^{-1}$, the coefficients $c_{ijk} = \langle W(X_i)(X_j), X_k\rangle$ verify $c_{ijk} = c_{jik}$ and $c_{ijk} = -c_{ikj}$. By the S_3-lemma [**18**], $c_{ijk} = 0$, thus $W = 0$. □

$(\mathfrak{h}, \mathfrak{d})$ is the pair associated to a fat distribution with a compatible subriemannian structure. As a consequence of this proposition we have that $Prol(\mathfrak{h}, \mathfrak{d}) = \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \oplus \mathfrak{g}^0$ for any H-type algebra. This is actually true for the symbol of any subriemannian structure [**20**].

Theorem 5.3. *Let $\mathfrak{h} = \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1}$ an H-type algebra and let $\mathfrak{g}^0$ be as in (5.4). Then:*

(i) *If $\mathfrak{h}$ is of parabolic type with associated semisimple algebra $\mathfrak{g}$ then $Prol(\mathfrak{h}, \mathfrak{g}^0) = \mathfrak{g}$;*
(ii) *If $\mathfrak{h}$ is not of parabolic type, then $Prol(\mathfrak{h}, \mathfrak{g}^0) = \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \oplus \mathfrak{g}^0$.*

Proof. (i) is immediate from proposition 5.2 and theorem 3.2

For (ii), suppose that $\mathfrak{g}^1 \neq \{0\}$. Consider

$$\Theta_1 : \mathfrak{g}^1 \to \mathfrak{g}^{-1},$$

defined by

$$\Psi_1(W) = X_W.$$

As in the proof of 5.2, we get that Ψ_1 is one to one.

To prove that Ψ_1 is actually an isomorphism, begin by proving that if $W(Y) = 0$ with $W \in \mathfrak{g}^1$ and $0 \neq Y \in \mathfrak{g}^{-1}$, then $W = 0$. Indeed,

$$W(X_W)(Y) = W(X_W)(Y) - W(Y)(X_W) = W([X_W, Y]) = -2J_{[X_W,Y]}X_W$$

$$|X_W|^2\, Y + D^W_{X_W}(Y) = -2J_{[X_W,Y]}X_W$$

$$|X_W|^2\, |Y|^2 + \left\langle D^W_{X_W}(Y), Y\right\rangle = \left\langle -2J_{[X_W,Y]}X_W, Y\right\rangle$$

$$|X_W|^2\, |Y|^2 = -2\, |[X_W, Y]|^2 = 0$$

Since $Y \neq 0$, $X_W = 0$ and therefore $W = 0$ as claimed.

By hypothesis, there exists $0 \neq W \in \mathfrak{g}^1$. We define the linear

$$\Phi : \mathfrak{g}^{-1} \to \mathfrak{g}^1$$

by $\Phi(Y) = [W, W(Y)]$. Φ is injective, because $\Phi(Y) = 0 \Rightarrow [W, W(Y)]\,(Y) = 0 \Rightarrow W(W(Y)(Y)) - [W(Y), W(Y)] = 0 \Rightarrow W(W(Y)(Y)) = 0$.

Since $W \neq 0$, we have proved that W is one to one in $\mathfrak{g}^{-1}$, so $W(Y)(Y) = 0$ and therefore

$$\langle X_W, Y\rangle\, Y + D^W_Y(Y) = 0.$$

The two terms of the last sum are orthogonal, so $\langle X_W, Y\rangle\, Y = 0$ and $D^W_Y(Y) = 0$. On the other hand,

$$W(X_W)(Y) - W(Y)(X_W) = W([X_W, Y]) = -2J_{[X_W,Y]}X_W,$$

which implies the identities

$$\langle X_W, X_W\rangle\, Y + D^W_{X_W}(Y) - \langle X_W, Y\rangle\, X_W - D^W_Y(X_W) = -2J_{[X_W,Y]}X_W$$

$$|X_W|^2\, |Y|^2 - \langle X_W, Y\rangle^2 - \left\langle D^W_Y(X_W), Y\right\rangle = -2\left\langle J_{[X_W,Y]}X_W, Y\right\rangle = -2\, |[X_W, Y]|^2$$

$$|X_W|^2\, |Y|^2 - \langle X_W, Y\rangle^2 + \left\langle X_W, D^W_Y(Y)\right\rangle = -2\, |[X_W, Y]|^2$$

Since $D^W_Y(Y) = 0$, $\langle X_W, Y\rangle\, Y = 0$ and $X_W \neq 0$, one concludes $Y = 0$, proving Φ is injective.

This implies $\dim \mathfrak{g}^{-1} \leq \dim \mathfrak{g}^1$ and, therefore, $\Psi_1 : \mathfrak{g}^1 \to \mathfrak{g}^{-1}$ is an isomorphism.

Write $\Psi_{-1} = \Psi_1^{-1}$. Fix $X \in \mathfrak{g}^{-1}$ with $|X| = 1$ and define the linear

$$\Psi_{-2} : \mathfrak{g}^{-2} \to \mathfrak{g}^2$$

by

$$\Psi_{-2}(Z) = [\Psi_{-1}(X), \Psi_{-1}(J_Z X)]\,.$$

Then Ψ_{-2} is one to one by the following calculation:

$$\begin{aligned}
\Psi_{-2}(Z)(Z') &= [\Psi_{-1}(X), \Psi_{-1}(J_Z X)](Z') \\
&= \Psi_{-1}(X)\Psi_{-1}(J_Z X)(Z') - \Psi_{-1}(J_Z X)\Psi_{-1}(X)(Z') \\
&= -2\Psi_{-1}(X)(J_{Z'}J_Z X) + 2\Psi_{-1}(J_Z X)(J_{Z'}X) \\
&= -2\langle X, J_{Z'}J_Z X\rangle A - 2D^X_{J_{Z'}J_Z X} + 2\langle J_Z X, J_{Z'}X\rangle A + 2D^{J_Z X}_{J_{Z'}X} \\
&= 4\langle J_Z X, J_{Z'}X\rangle A - 2D^X_{J_{Z'}J_Z X} + 2D^{J_Z X}_{J_{Z'}X} \\
&= 4\langle Z, Z'\rangle A - 2D^X_{J_{Z'}J_Z X} + 2D^{J_Z X}_{J_{Z'}X}
\end{aligned}$$

On the other hand, if $U \in \mathfrak{g}^2$ and $Z \in \mathfrak{g}^{-2}$, then

$$U(Z) = t(Z)A + E^U_Z$$

with $E^U_Z \in \mathfrak{d}$ and $t : \mathfrak{g}^{-2} \to \mathbb{R}$ linear. As before, $t(Z) = \langle Z_U, Y\rangle$ for some $Z_U \in \mathfrak{g}^{-2}$. Define a linear map

$$\Psi_2 : \mathfrak{g}^2 \to \mathfrak{g}^{-2}$$

by $\Psi_2(U) = Z_U$. We prove below that this map is also an isomorphism.

If $\Psi_2(U) = Z_U = 0$, then $U(Z) \in \mathfrak{d}$ for all $Z \in \mathfrak{g}^{-2}$. Since $U(X) \in \mathfrak{g}^1$, for $X \in \mathfrak{g}^{-1}$, $U(X) = \Psi_{-1}(\varphi(X))$, for some linear endomorphism $\varphi\mathfrak{g}^{-1} \to \mathfrak{g}^{-1}$.

$$\begin{aligned}
U([X,Y]) &= U(X)(Y) - U(Y)(X) \\
&= \langle\varphi(X), Y\rangle .A + D^{U(X)}_Y - \langle\varphi(Y), X\rangle .A + D^{U(Y)}_X
\end{aligned}$$

Since $U([X,Y]) \in \mathfrak{d}$ we have that:

$$\langle\varphi(X), Y\rangle = \langle\varphi(Y), X\rangle$$

i.e. φ is symmetric, and

$$\langle U(X)(Z), Y\rangle = \langle U(Z)(X), Y\rangle = -\langle U(Z)(Y), X\rangle = -\langle U(Y)(Z), X\rangle .$$

From the definition of φ and (5.5) it follows that

$$\langle J_Z\varphi(X), Y\rangle = -\langle J_Z\varphi(Y), X\rangle ,$$

and since J_Z is antisymmetric and φ is symmetric, this implies

$$J_Z\varphi = \varphi J_Z$$

Let $\{Z_i\}$ be a basis of $\mathfrak{g}^{-2}$. By the S_3-lemma, $\langle U(Z_i)(Z_j), Z_k\rangle = 0 \ \forall\ i,j,k$. Then $U(Z)(Z') = 0 \ \forall\ Z,\ Z' \in \mathfrak{g}^{-2}$, while

$$\begin{aligned}
[U(Z)(X), Y] &= [U(Z)(Y), X] + U(Z)([X,Y]) \\
[U(X)(Z), Y] &= [U(Y)(Z), X] \\
[J_Z\varphi X, Y] &= [J_Z\varphi Y, X] \\
\langle J_{Z'}J_Z\varphi X, Y\rangle &= \langle J_{Z'}J_Z\varphi Y, X\rangle \\
\langle X, \varphi J_Z J_{Z'} Y\rangle &= \langle J_{Z'}J_Z\varphi Y, X\rangle \\
\langle J_Z J_{Z'}\varphi Y, X\rangle &= \langle J_{Z'}J_Z\varphi Y, X\rangle
\end{aligned}$$

for any $X,\ Y \in \mathfrak{g}^{-1}$ and $Z,\ Z' \in \mathfrak{g}^{-2}$.

We already know that $\mathfrak{h}$ is of parabolic type when $\dim \mathfrak{g}^{-2} = 1$, so we suppose that $\dim \mathfrak{g}^{-2} \geq 2$. We can take Z and Z' nonzero orthogonal elements of $\mathfrak{g}^{-2}$ and from (5.3) we get

$$J_Z J_{Z'}\varphi = J_{Z'}J_Z\varphi = -J_Z J_{Z'}\varphi$$

Since J_Z and $J_{Z'}$ are invertible, $\varphi \equiv 0$. So $U(X) = 0$ for all $X \in \mathfrak{g}^{-1}$, and $U \equiv 0$.

We conclude that Ψ_2 y Ψ_{-2} are isomorphisms. It is easy to verify that $\Psi_{-2} = \Psi_2^{-1}$, which implies in particular that Ψ_{-2} is independent from the choice of the initial X.

Next, we prove that $\mathfrak{g}^3 = \{0\}$. Let $V \in \mathfrak{g}^3$. Since dim $\mathfrak{g}^2 =$ dim $\mathfrak{g}^{-2} <$ dim $\mathfrak{g}^{-1}$, there exists $0 \neq Y \in \mathfrak{g}^{-1}$ such that $V(Y) = 0$ and therefore,

$$V(Z)(Y) = V(Y)(Z) = 0 \qquad \forall Z \in \mathfrak{g}^{-2}.$$

Since $V(Z) \in \mathfrak{g}^1$, $V(Z)$ must be injective or zero in $\mathfrak{g}^{-1}$. Hence, $V(Z) = 0\, \forall Z \in \mathfrak{g}^{-2}$.

For any $Z \in \mathfrak{g}^{-2}$, $X \in \mathfrak{g}^{-1}$,

$$0 = V(Z)(X) = V(X)(Z) = 4\,\langle \Psi_2(V(X)), Z\rangle\, A + E_Z^{V(X)}.$$

Thus $V(X) = 0\ \forall X \in \mathfrak{g}^{-1}$. But $\mathfrak{g}^{-1}$ generates $\mathfrak{h}$, so $V = 0$. We conclude that

$$\mathfrak{g} := Prol(\mathfrak{h}, \mathfrak{g}^0) = \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^1 \oplus \mathfrak{g}^2$$

where

$$\mathfrak{g}^0 = \mathfrak{d} \oplus \mathfrak{a}, \qquad \mathfrak{h} = \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1}.$$

To prove that $\mathfrak{g}$ is semisimple, suppose that it has an abelian ideal $\mathfrak{b}$ containing a non-zero element $L \in \mathfrak{b}$. We get a contradiction considering the following four cases.

1. $L = X + Z$ with $0 \neq X \in \mathfrak{g}^{-1}$ and $Z \in \mathfrak{g}^{-2}$. Then

$$[\Psi_1(X), [X + Z, J_{Z'}X]] = |X|^2\, \Psi_1(X)(Z') = -2\,|X|^2\, J_{Z'}X \in \mathfrak{b}$$

for $0 \neq Z' \in \mathfrak{g}^{-2}$. Thus $J_{Z'}X \in b$ and, since b is abelian, $[X, J_{Z'}X] = |X|^2\, Z' = 0$, a contradiction. Similarly, if $Z \neq 0$,

$$0 \neq \Psi_1(X)(Z) = -2J_Z X \in \mathfrak{b},$$

also a contradiction.

2. $L = T + X + Z$ with $0 \neq T \in \mathfrak{g}^0$, $X \in \mathfrak{g}^{-1}$ y $Z \in \mathfrak{g}^{-2}$. Since $T \neq 0$, exists $Y \in \mathfrak{g}^{-1}$ such that $T(Y) \neq 0$. Then,

$$[T + X + Z, Y] = T(Y) + [X, Y] \in \mathfrak{b}.$$

From case 1, we also get a contradiction.

3. $L = U + T + X + Z$ with $0 \neq U \in \mathfrak{g}^1$, $T \in \mathfrak{g}^0$, $X \in \mathfrak{g}^{-1}$ y $Z \in \mathfrak{g}^{-2}$. Given $0 \neq Y \in g^{-1}$, we already proved that $U(Y) \neq 0$.

$$[U + T + X + Z, Y] = U(Y) + T(Y) + [X, Y] \in \mathfrak{b}$$

as in case 2.

4. $L = V + U + T + X + Z$ with $0 \neq V \in \mathfrak{g}^2$, $U \in \mathfrak{g}^1$, $T \in \mathfrak{g}^0$, $X \in \mathfrak{g}^{-1}$ and $Z \in \mathfrak{g}^{-2}$. There exist $0 \neq Z' \in g^{-2}$ such that $V = \Psi_{-2}(Z')$, thus $V(Z') \neq 0$.

$$[V + U + T + X + Z, Z'] = V(Z') + U(Z') + T(Z') \in \mathfrak{b},$$

again reducing to case 3.

This shows that $\mathfrak{g}$ has no non-zero abelian ideals, i.e. it is semisimple. Let $B = -A \in \mathfrak{a}$, $\mathfrak{g}^i$ is the eigenspace of $ad\, B$ corresponding to the eigenvalue i. This is clear for $i = -2, -1, 0$. If $W \in \mathfrak{g}^1$ and $X \in \mathfrak{g}^{-1}$,

$$[B, W](X) = [B, W(X)] - W(B(X)) = W(X).$$

Then $[B, W] = W$. Analogously $[B, U] = 2U$ for $U \in \mathfrak{g}^2$. Since $\mathfrak{g}$ has split rank one, $\mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1}$ is an Iwasawa subalgebra of $\mathfrak{g}$, i.e. of parabolic type.

This finishes the proof of the Theorem. □

One concludes

THEOREM 5.4. *Let $\mathfrak{h}$ an H-type algebra of non-parabolic type and G^0 the group of conformal automorphisms of $\mathfrak{h}$. Then any pseudo-G^0-structure of constant type $\mathfrak{h}$ has a canonical frame.*

6. Cartan connections

In some cases we can construct some special canonical frames by the prolongation method. Recall that given a manifold M, a Lie group G and a Lie subgroup $H \subset G$, a *Cartan geometry of type* (G, H) on M is a principal fiber bundle $P \to M$ with structure group H, which is endowed with a $\mathfrak{g}$-valued one-form ω – the *Cartan connection*, satisfying

- $\omega(u) : T_uP \to \mathfrak{g}$ is a linear isomorphism for all $u \in P$;
- $(R_h)^*\omega = Ad(h^{-1})\omega$ for all $h \in H$;
- $\omega(\widetilde{X}(u)) = X$ for each $X \in \mathfrak{h}$.

In the terminology of Theorem 4.1, one has

THEOREM 6.1. *Suppose that $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{g}^0$ is the prolongation of the pair $(\mathfrak{m}, \mathfrak{g}^0)$, and let G^0 be the connected Lie group of graded automorphism of $\mathfrak{m}$ with Lie algebra $\mathfrak{g}^0$. If we choose the complementary subspace H_k of $\partial_k S_k$ in A_k invariant under the natural action of G^0 in A_k, then the canonical frame obtained by the prolongation procedure is a Cartan connection of type $(\mathfrak{g}, G^0)$.*

This proposition follows from a theorem of Morimoto on Cartan connections on Filtered Manifolds [**19**]. For an application of this to subriemannian distributions with a constant subriemannian symbol see [**20**].

In our case, since the inner product in $\mathfrak{m} = \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1}$ induces an inner product in A_k for every k, and the orthogonal complement of $\partial_k S_k$ is invariant by the action of G^0 (the dilations act as scalars), we get

COROLLARY 6.2. *If $\mathfrak{h}$ is an H-type algebra of non-parabolic type and G^0 the group of conformal automorphisms of $\mathfrak{h}$. Then any distribution of type $\mathfrak{h}$ with a compatible conformal metric has asociated to it a canonical Cartan connection of type $(\mathfrak{g}, G^0)$.*

The existence and construction of canonical Cartan connections for the parabolic type were studied by Tanaka using the structure of semisimple Lie algebras [**25**] (see also [**5**] [**19**]).

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