



## Brief paper

# Bounds and invariant sets for a class of switching systems with delayed-state-dependent perturbations<sup>☆</sup>



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## ABSTRACT

We present a novel method to compute componentwise transient bounds, componentwise ultimate bounds, and invariant regions for a class of switching continuous-time linear systems with perturbation bounds that may depend nonlinearly on a delayed state. The main advantage of the method is its componentwise nature, i.e. the fact that it allows each component of the perturbation vector to have an independent bound and that the bounds and sets obtained are also given componentwise. This componentwise method does not employ a norm for bounding either the perturbation or state vectors, avoids the need for scaling the different state vector components in order to obtain useful results, and may also reduce conservativeness in some cases. The present paper builds upon and extends to switching systems with delayed-state-dependent perturbations previous results by the authors. In this sense, the contribution is three-fold: the derivation of the aforementioned extension; the elucidation of the precise relationship between the class of switching linear systems to which the proposed method can be applied and those that admit a common quadratic Lyapunov function (a question that was left open in our previous work); and the derivation of a technique to compute a common quadratic Lyapunov function for switching linear systems with perturbations bounded componentwise by affine functions of the absolute value of the state vector components. In this latter case, we also show how our componentwise method can be combined with standard techniques in order to derive bounds possibly tighter than those corresponding to either method applied individually.

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## 1. Introduction

Switched systems are dynamical systems that combine a finite number of subsystems by means of a switching rule (Liberzon, 2003; Lin & Antsaklis, 2009). The stability of switched systems has attracted considerable research attention in recent years (Decarlo, Branicky, Pettersson, & Lennartson, 2000; Liberzon & Morse, 1999; Lin & Antsaklis, 2009; Shorten, Wirth, Mason, Wulff, & King, 2007). In this paper we are concerned with stability under “arbitrary switching”, which refers to problems where the stability properties of interest hold for every admissible switching signal. In this context, we refer to a switched system

undergoing arbitrary switching as a *switching system*, and as a *switching linear system* if the individual subsystems have linear dynamics. Necessary and sufficient conditions for the asymptotic stability of the zero solution of a switching linear system were given in Blanchini (2000, Theorem 4.1 and Remark 4.1) and Molchanov and Pyatnitsky (1989, Theorem 3). In the present paper we will focus on the “practical stability” problem of analyzing the existence and computation of invariant sets and ultimate bounds for the switching system state trajectories. This type of stability is important in every practical setting where nonvanishing perturbations (also named persistent disturbances) may act on the system (Khalil, 2002, Chapter 9). We consider switching systems with a switching linear nominal (unperturbed) system affected by perturbations that may be nonvanishing and depend nonlinearly on a delayed state.

Standard methods for the computation of bounds and invariant sets are based on the use of a Lyapunov function (Khalil, 2002). Arguably, Lyapunov-function-based methods are the most powerful and widely applicable, although their inherent difficulty is the obtention of a suitable Lyapunov function. When the nominal system is linear, however, a quadratic Lyapunov function can easily be computed via solving a Lyapunov equation, but the bounds

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so obtained may be conservative, even for linear systems (see, e.g., Section 1 of Kofman, Haimovich, and Seron (2007)). State bounds computed by means of a quadratic Lyapunov function are given as a bound on the norm, usually the 2-norm, of the state vector and usually require a bound on the norm of the perturbation vector. The aforementioned conservativeness may be due to (a) the information on the different bounds for each component of the perturbation vector is lost when taking its norm and (b) the bounds corresponding to different state vector components are substantially different and hence its 2-norm is not the most suitable for bounding. Problem (b) may be ameliorated by properly scaling the state vector components. In order to avoid or at least reduce the effect of both problems (a) and (b), then Lyapunov functions of a form more complicated than quadratic may be employed. Likewise, for switching systems with a switching linear nominal system, a quadratic Lyapunov function common to all linear subsystems can be computed via linear matrix inequalities (LMIs) in case one exists (see, for example, Section 4.3 of Shorten et al. (2007) and the references therein). As in the non-switching case, the bounds thus obtained may be conservative in some cases.

The present paper follows a methodology which differs from the one just described in that the use of either a norm of the state or a Lyapunov function can be avoided. Moreover, this methodology can be easily combined with Lyapunov analysis in order to possibly improve on the results of either method applied individually. The methodology that we employ is based on componentwise analysis, avoids the need for scaling individual state components, and builds upon and extends to switching systems with delayed-state-dependent perturbations previous results of Haimovich and Seron (2009, 2010), Kofman et al. (2007) and Kofman, Seron, and Haimovich (2008). In Kofman et al. (2007), a method to compute componentwise ultimate bounds for perturbed (non-switching) linear systems is given. The perturbation bound is allowed to depend nonlinearly on the system state. Ultimate bounds are derived that are global (valid for every initial condition) when the perturbation bound is constant and local (valid only when the initial state is in a specific region) in the more general case of state-dependent perturbation bounds. Global componentwise ultimate bounds for perturbation bounds that have affine dependence on a delayed system state are derived in Section 3 of Kofman et al. (2008), jointly with a sufficient condition for practical stability. In Haimovich and Seron (2009, 2010), a method to derive global componentwise transient and ultimate bounds was proposed for a class of switching linear systems with constant perturbation bounds. It was shown in Haimovich and Seron (2010) that the proposed method can be applied when the switching linear system is close to being simultaneously triangularizable. In such a case, a common quadratic Lyapunov function (CQLF) exists for the switching system. However, the precise relationship between the class of switching linear systems to which the proposed method can be applied and those that admit a CQLF was left as an open question.

The present paper provides three contributions. The first contribution is to answer the aforementioned open question: the class of switching linear systems to which our componentwise bound and invariant set method can be applied is strictly contained in the class of switching linear systems that admit a CQLF, although the switching linear system need not be close to simultaneously triangularizable. This relationship was reported by Mori et al. in Mori, Mori, and Kuroe (2001) but the proof was not given. We provide a proof and, moreover, extend it so that it becomes useful in the derivation of our third contribution. The second contribution of the paper is to combine and extend the previous results in Haimovich and Seron (2009, 2010) and Kofman et al. (2007, 2008) by providing transient bounds, ultimate bounds, and invariant regions based on componentwise analysis for a class

of switching continuous-time linear systems with perturbation bounds that may depend nonlinearly on a delayed state. This kind of setting can describe, for example, switching linear systems with uncertainty in the state evolution matrix, switching linear systems with uncertain and varying time delay and, more generally, switching nonlinear systems expressed as their switching linear approximation perturbed by an additive disturbance with a bound depending nonlinearly on the system state. We derive conditions for the bounds to be of local or semi-global nature. We also address the particular case of perturbation bounds that have affine dependence on a delayed state. In this particular case, the bounds derived are shown to be of global nature and an extension of the sufficient condition for practical stability of Kofman et al. (2008, Section 3) is provided. The third contribution is to provide a technique to compute a CQLF for a class of switching linear systems with perturbations bounded componentwise by affine functions of the absolute value of the state vector components (provided no delays are present). The CQLF so derived can be used to compute ultimate bounds for this class of systems. Moreover, both the componentwise method and the Lyapunov technique can be combined to obtain tighter bounds than could be obtained by either methodology applied individually. The current paper subsumes all the aforementioned previous bound computation results (Haimovich & Seron, 2010; Kofman et al., 2007, 2008) for (switching and non-switching) continuous-time systems, in the sense that bounds for each of the cases considered in these results can be obtained by means of the current results (although the bounds obtained may not be identical). Although similar ideas are employed, the extension of the previous results to derive the ones presented in the current paper is not straightforward. Some of the results in the current paper have been presented in Haimovich and Seron (2011a,b). An extended version of the current paper including numerical examples and all proofs can be found in Haimovich and Seron (2012).

**Notation.**  $\mathbb{Z}$ ,  $\mathbb{R}$  and  $\mathbb{C}$  denote the sets of integer, real and complex numbers, and  $0$  denotes the zero scalar, vector or matrix, depending on the context.  $\mathbb{R}_+$  and  $\mathbb{R}_{+0}$  denote the positive and nonnegative real numbers, respectively, and similarly for  $\mathbb{Z}_+$  and  $\mathbb{Z}_{+0}$ . If  $M$  is a matrix, then  $M'$  denotes its transpose,  $M^*$  its conjugate transpose, and  $|M|$  is the matrix whose entries are the magnitude of the corresponding entries in  $M$ . If  $P$  is a square matrix, then  $\rho(P)$  denotes its spectral radius,  $\mathbf{a}(P)$  its spectral abscissa, and  $P > 0$  ( $P < 0$ ) means that  $P$  is positive (negative) definite. If  $x(t)$  is a vector-valued function, then  $\limsup_{t \rightarrow \infty} x(t)$  denotes the vector obtained by taking  $\limsup_{t \rightarrow \infty}$  of each component of  $x(t)$ . Similarly, 'lim' and 'max' denote componentwise operations on a vector or matrix. The expression  $x \leq y$  ( $x < y$ ) denotes the set of componentwise inequalities  $x_i \leq y_i$  ( $x_i < y_i$ ) between the elements of the real vectors  $x$  and  $y$ , and similarly for  $x \geq y$  ( $x > y$ ) and in the case when  $x$  and  $y$  are matrices. If  $T : \mathbb{R}_{+0}^n \rightarrow \mathbb{R}_{+0}^n$ , then  $T^k$  denotes the iteration of  $T$ , that is, the maps defined by  $T^1(x) = T(x)$  and  $T^{k+1}(x) = T(T^k(x))$ . The index set  $\{1, 2, \dots, N\}$  is denoted  $\underline{N}$  and  $\mathbf{i}$  denotes  $\sqrt{-1}$ . Employing this notation, note that  $P > 0$  means that every entry of  $P$  is positive and  $P > 0$  that  $P$  is positive definite.

## 2. Problem formulation

In this section, we formulate the problem to be addressed, followed by some preliminary definitions and properties.

### 2.1. Problem statement

We consider switching continuous-time perturbed systems of the form

$$\dot{x}(t) = A_{\sigma(t)}x(t) + H_{\sigma(t)}w_{\sigma(t)}(t), \quad (1)$$

where  $x(t) \in \mathbb{R}^n$  is the system state,  $\sigma(t) \in \underline{N} \triangleq \{1, 2, \dots, N\}$  is the switching function,  $A_i \in \mathbb{R}^{n \times n}$ ,  $H_i \in \mathbb{R}^{n \times k_i}$  for  $i \in \underline{N}$ , and the perturbation vectors  $w_i(t) \in \mathbb{R}^{k_i}$  satisfy the componentwise bound

$$|w_i(t)| \leq \delta_i(\theta(t)) \quad \text{for all } t \geq 0, \text{ for } i \in \underline{N}, \quad (2)$$

with continuous bounding functions  $\delta_i : \mathbb{R}_{+0}^n \rightarrow \mathbb{R}_{+0}^{k_i}$  and  $\theta(t) \in \mathbb{R}_{+0}^n$  defined as

$$\theta(t) \triangleq \max_{t-\bar{\tau} \leq \tau \leq t} |x(\tau)|, \quad (3)$$

where  $\bar{\tau} \geq 0$  and the maximum is taken componentwise.

Note that for each  $i \in \underline{N}$ , (2) expresses a bound for each one of the  $k_i$  components of the perturbation vector  $w_i(t)$ , and that the maximum in (3) denotes a componentwise operation.

**Remark 1.** The setting (1)–(3) can describe, inter-alia, the following situations:

- Uncertainty in the system evolution matrix, where  $\dot{x}(t)$  has the form  $(A_{\sigma(t)} + \Delta A_{\sigma(t)}(t))x(t)$ , and  $|\Delta A_i(t)| \leq \overline{\Delta A}_i$ , for all  $t \geq 0$  and  $i \in \underline{N}$ ; in this case, we can take  $H_i = I$  in (1),  $\delta_i(\theta) = \overline{\Delta A}_i \theta$  in (2), and  $\bar{\tau} = 0$  in (3).
- Uncertain time delays, where  $w_i(t) = F_i x(t - \tau_i)$ , and  $0 \leq \tau_i \leq \tau_{\max}$ ; in this case, we can take  $\delta_i(\theta) = |F_i| \theta$  in (2), and  $\bar{\tau} = \tau_{\max}$  in (3).
- Disturbances with constant bounds:  $\delta_i(\theta) = \mathbf{w}_i$  in (2).
- Switching nonlinear systems where  $\dot{x}(t)$  has the form  $f_{\sigma(t)}(x(t))$ ; in this case we may take  $A_i = \frac{\partial f_i}{\partial x}(x_0)$ ,  $H_i = I$ ,  $\bar{\tau} = 0$ ,  $\delta_i(\theta) = \max_{x: |x| \leq \theta} |f_i(x) - A_i x|$ .

The problem of interest is to derive transient bounds, ultimate bounds, and invariant sets for switching systems of the form (1) with perturbations bounded as in (2)–(3). This will be addressed in Section 3. In the next subsection, we give some definitions and preliminary results related to the concept of Metzler matrices and to a specific class of nonnegative functions.

## 2.2. Definitions and properties

**Definition 1** (Metzler). A matrix  $A \in \mathbb{R}^{n \times n}$  is Metzler if its off-diagonal entries are nonnegative.

Given an arbitrary matrix  $N \in \mathbb{C}^{n \times n}$ , we define  $\mathcal{M}(N) \in \mathbb{R}^{n \times n}$  as the matrix whose entries satisfy

$$[\mathcal{M}(N)]_{i,k} = \begin{cases} \operatorname{Re}\{N_{i,k}\} & \text{if } i = k, \\ |N_{i,k}| & \text{if } i \neq k. \end{cases} \quad (4)$$

Note that  $\mathcal{M}(N)$  is Metzler for every  $N \in \mathbb{C}^{n \times n}$ .

The following Lemma gives properties of Metzler matrices.

**Lemma 2.** Let  $A, M \in \mathbb{R}^{n \times n}$  and  $N \in \mathbb{C}^{n \times n}$ . Then,

- $A$  is Metzler if and only if  $e^{At} \geq 0$  for all  $t \geq 0$ .
- If  $A$  is Metzler, then it is Hurwitz if and only if  $-\Lambda^{-1} \geq 0$ .
- $A$  is Metzler and Hurwitz if and only if  $-\Lambda$  is an  $M$ -matrix.
- If  $M = M'$  and is Metzler, then  $x'Mx \leq |x|M|x|$  for all  $x \in \mathbb{R}^n$ .
- If  $N = N^*$ , then  $z^*Nz \leq |z|\mathcal{M}(N)|z|$  for all  $z \in \mathbb{C}^n$ .

Properties (a) and (b) can be found in Chapter 6 of Luenberger (1979); (c) follows from Definition 1 and the definition of an  $M$ -matrix (see, e.g., Chapter 6 of Berman and Plemmons (1994)); (d) and (e) are straightforward.

**Definition 3** (CNI). A nonnegative vector function  $f : \mathbb{R}_{+0}^n \rightarrow \mathbb{R}_{+0}^n$  is said to be Componentwise Non-Increasing (CNI) if, whenever  $x_1, x_2 \in \mathbb{R}_{+0}^n$  and  $x_1 \leq x_2$ , then  $f(x_1) \leq f(x_2)$ .

**Remark 2.** Every continuous function  $\hat{f} : \mathbb{R}_{+0}^n \rightarrow \mathbb{R}_{+0}^m$  can be overbounded by a continuous CNI function. In particular, the tightest continuous CNI overbound of  $\hat{f}$  is the function  $f : \mathbb{R}_{+0}^n \rightarrow \mathbb{R}_{+0}^m$  given by

$$f(x) = \max_{0 \leq y \leq x} \hat{f}(y). \quad (5)$$

## 3. Main results

In this section, we begin by briefly reviewing in Section 3.1 our previous result (Theorem 1 below) for switching linear systems with constant perturbation bounds (Haimovich & Seron, 2010). Section 3.2 provides the first contribution of the paper by establishing the link between the applicability of the previous results of Haimovich and Seron (2010) and that of the CQLF, a question that was left open in the latter reference. The proof of this result (Theorem 2) is given in the Appendix. The main results of the paper are given in Sections 3.3 and 3.4. In Section 3.3, we provide novel transient bounds, ultimate bounds and invariant sets for a class of switching continuous-time linear systems with perturbations bounded by a nonlinear function of a delayed state. In Section 3.4, we provide additional results for the special case of perturbation bounds having affine dependence on a delayed state and also show how to compute a CQLF when no delay is present. The corresponding proofs can be consulted in Haimovich and Seron (2012).

### 3.1. Previous results: constant perturbation bounds

The following is a minor modification of Theorem 1 of Haimovich and Seron (2010).

**Theorem 1** (Theorem 1 of Haimovich and Seron (2010)). Consider the switching system (1) with componentwise perturbation bound

$$|w_i(t)| \leq \mathbf{w}_i, \quad (6)$$

with  $\mathbf{w}_i \in \mathbb{R}_{+0}^{k_i}$ . Let  $V \in \mathbb{C}^{n \times n}$  be invertible and define

$$A_i \triangleq V^{-1}A_iV, \quad M_i \triangleq \mathcal{M}(A_i), \quad \Lambda \triangleq \max_{i \in \underline{N}} M_i \quad (7)$$

where  $\mathcal{M}(\cdot)$  is the operation defined in (4). Suppose that  $\Lambda$  is Hurwitz. Let  $\mathbf{z} \in \mathbb{R}_{+0}^n$  satisfy

$$\mathbf{z} \geq \max_{i \in \underline{N}} \left[ \max_{|w_i| \leq \mathbf{w}_i} |V^{-1}H_i w_i| \right], \quad (8)$$

and define

$$\eta \triangleq \max \{ |V^{-1}x(0)| + \Lambda^{-1}\mathbf{z}, 0 \}. \quad (9)$$

Then, the states of system (1) are bounded as

$$|V^{-1}x(t)| \leq -\Lambda^{-1}\mathbf{z} + e^{\Lambda t}\eta, \quad (10)$$

for all  $t \geq 0$ , and ultimately bounded as

$$\limsup_{t \rightarrow \infty} |V^{-1}x(t)| \leq -\Lambda^{-1}\mathbf{z}. \quad (11)$$

**Remark 3.** The main assumption that enables the application of Theorem 1 is the obtention of an invertible matrix  $V$  so that  $\Lambda$  in (7) be Hurwitz. In Haimovich and Seron (2010), an algorithm to seek such a matrix was provided. This algorithm searches over unitary matrices  $V$ . However, it may happen that even if a matrix  $V$  that makes  $\Lambda$  Hurwitz exists, no unitary matrix  $V$  ensuring such a condition exists. A general algorithm to seek the required matrix  $V$

is the following. Let  $\mathbf{a}(\Lambda)$  denote the spectral abscissa of  $\Lambda$ , i.e. the maximum over the real parts of the eigenvalues of  $\Lambda$ . We pose the following optimization problem:

Minimize  $\mathbf{a}(\Lambda)$  over  $V \in \mathbb{C}^{n \times n}$  invertible.

It is not necessary to find a global optimum of this nonconvex optimization problem: it suffices to find an invertible  $V$  such that  $\mathbf{a}(\Lambda) < 0$ , i.e. such that  $\Lambda$  is Hurwitz. Note that for every nonzero scalar  $\alpha \in \mathbb{C}$ , according to (7) the matrices  $V$  and  $\alpha V$  will produce the same  $\Lambda_i$  and hence the same  $\Lambda$ . Consequently, when searching for a suitable  $V$  according to the above optimization, the entries of  $V$  can be bounded a priori without affecting the success of the search.

**Remark 4.** A region of the form  $\{x \in \mathbb{R}^n : |V^{-1}x| \leq \bar{z}\}$ , with  $\bar{z} \geq 0$  as given by (10) and (11), has polyhedral shape if the entries of  $V$  are real, and a combined ellipsoidal/polyhedral shape if  $V$  has some complex entries (see Haimovich, Kofman, and Seron (2008) for more details). Every (componentwise) bound  $|V^{-1}x| \leq \bar{z}$  yields a corresponding componentwise bound  $|x| \leq |V|\bar{z}$ , since

$$|x| = |VV^{-1}x| \leq |V||V^{-1}x| \leq |V|\bar{z}. \quad (12)$$

### 3.2. Relationship to CQLF

The following result establishes the relationship between the existence of the matrix  $V$  required by Theorem 1 and the existence of a quadratic Lyapunov function. A similar result has been reported in Mori et al. (2001), where the class of systems for which the matrix  $V$  required by Theorem 1 exists was identified as a subclass of the switching systems that admit a CQLF. The result in Mori et al. (2001) was stated without proof, nor reference to another publication containing the proof. Here we provide a proof and, moreover, will present an extension (Theorem 4(e) in Section 3.4) where sufficient conditions for the existence of a CQLF guaranteeing practical stability are given for the case of perturbations bounded by an affine function of the (non-delayed) state.

**Theorem 2.** Let  $\Lambda \in \mathbb{R}^{n \times n}$  be Metzler and let  $\bar{\Lambda}$  be Hurwitz and satisfy  $\bar{\Lambda} \geq \Lambda$ . Then,

(a) there exists a diagonal and positive definite matrix  $D = \text{diag}(d_1, \dots, d_n) > 0$  satisfying

$$\bar{\Lambda}'D + D\bar{\Lambda} < 0; \quad (13)$$

(b)  $\Lambda$  is Hurwitz.

(c) If  $\Lambda$  satisfies (7) for some invertible  $V \in \mathbb{C}^{n \times n}$  and matrices  $A_i \in \mathbb{R}^{n \times n}$ , then for each  $D$  as in (a), the corresponding real symmetric and positive definite matrix  $P = \text{Re}\{(V^{-1})^*DV^{-1}\}$  satisfies

$$A_i'P + PA_i < 0, \quad \text{for all } i \in \underline{N}. \quad (14)$$

The following consequence of Theorem 2 constitutes an important fact regarding Metzler and Hurwitz matrices and the operation (4).

**Corollary 4.** Consider the switching system (1), let  $V \in \mathbb{C}^{n \times n}$  be invertible, and define  $\Lambda_i$  and  $\Lambda$  as in (7), where  $\mathcal{M}(\cdot)$  is the operation defined in (4). If  $\Lambda$  is Hurwitz, then  $\Lambda_i$  is Hurwitz for all  $i \in \underline{N}$ . Moreover, the  $A_i$  admit a common quadratic Lyapunov function.

**Proof.** Just apply Theorem 2(c) with  $\bar{\Lambda} = \Lambda$ .  $\square$

The above theorem and corollary establish that the class of switching systems considered in this paper, that is, those for which the matrix  $V$  required by Theorem 1 exists, admit a common quadratic Lyapunov function. This closes a problem left open in our

previous paper (Haimovich & Seron, 2010). As shown previously in Haimovich and Seron (2010) and Mori et al. (2001), the class of switching systems considered in the present paper contains the class of systems that can be simultaneously triangularized by means of a common transformation. Moreover, the class of switching systems considered is not a trivial extension of the class of switching systems admitting simultaneous triangularization. To illustrate this point, we revisit the example presented in Dayawansa and Martin (1999) consisting of system (1) with no disturbance,  $\sigma(t) \in \{1, 2\}$  and

$$A_1 = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & -a \\ 1/a & -1 \end{bmatrix}.$$

Note that for every value of  $a$ , the eigenvalues of  $A_2$  are  $-1 \pm \mathbf{i}$ , identical to those of  $A_1$ , and hence both  $A_1$  and  $A_2$  are Hurwitz. However, the eigenvectors of  $A_1$  are  $[1, \pm \mathbf{i}]'$  and those of  $A_2$  are  $[1, \pm a\mathbf{i}]'$ . In order to be simultaneously triangularizable, it is necessary that both  $A_1$  and  $A_2$  have a common eigenvector. Consequently, loosely speaking we may say that this switching system is farther away from simultaneous triangularization as  $a$  is varied farther away from 1. It was shown in Dayawansa and Martin (1999) that for  $a > 3 + \sqrt{8}$  the above switching system does not admit a CQLF. For  $a = 3 + \sqrt{8} - 10^{-3}$ , which corresponds to a switching system with stable subsystems but so far from simultaneous triangularization that it is at the verge of not admitting a CQLF, searching for a unitary  $V$  by means of the algorithm in Haimovich and Seron (2010) yields a solution for which  $\Lambda$  is not Hurwitz. However, searching for an arbitrary  $V$  by means of the optimization proposed in Remark 3, we are able to obtain the feasible solution

$$V = \begin{bmatrix} -6.0069 & 5.5729 \\ -0.3554 & -1.0843 \end{bmatrix} + \begin{bmatrix} 0.8605 & -2.6151 \\ -2.4885 & -2.3081 \end{bmatrix} \mathbf{i},$$

for which the corresponding  $\Lambda$  is Hurwitz.

In addition, the class of switching systems considered in this paper is strictly contained in the class of switching linear systems that admit a CQLF, i.e., some switching systems may admit a CQLF but the matrix  $V$  required by Theorem 1 may not exist. To see this, consider Example 4.1 of Shorten and Narendra (2000), which consists of system (1) with no disturbance,  $\sigma(t) \in \{1, 2, 3\}$  and

$$A_1 = \begin{bmatrix} 0 & 5 \\ -30 & -1.4 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 5 \\ -26 & -1 \end{bmatrix}, \\ A_3 = \begin{bmatrix} -6 & 27 \\ -150 & -1 \end{bmatrix}.$$

This switching system admits a CQLF but the search for  $V$  outlined in Remark 3 does not give a useful solution, even when the optimization is run over 1000 times from different arbitrary initial conditions.

### 3.3. Nonlinear perturbation bounds

Theorem 3 below establishes local transient and ultimate bounds for system (1) with perturbation bounds of the form (2)–(3). The theorem is followed by the derivation of invariant regions (Corollary 5) and of conditions for the bounds to be of semi-global nature (Corollary 6). See Haimovich and Seron (2012) for proofs.

**Theorem 3.** Consider the switching system (1) with perturbation bound of the form (2)–(3), where the bounding functions  $\delta_i$  are CNI. Let  $V \in \mathbb{C}^{n \times n}$  be invertible and define  $\Lambda_i$  for  $i \in \underline{N}$  and  $\Lambda$  as in (7), where  $\mathcal{M}(\cdot)$  is the operation defined in (4). Suppose that  $\Lambda$  is Hurwitz. Let  $\psi : \mathbb{R}_{+0}^n \rightarrow \mathbb{R}_{+0}^n$  be defined as in (15), let  $\delta : \mathbb{R}_{+0}^n \rightarrow \mathbb{R}_{+0}^n$



be continuous, CNI and satisfy (16), and for every  $\gamma \in \mathbb{R}_{+0}^n$  consider  $T_\gamma : \mathbb{R}_{+0}^n \rightarrow \mathbb{R}_{+0}^n$  defined in (17).

$$\psi(x) = \max_{i \in \mathbb{N}} \left[ \max_{|w_i| \leq \delta_i(|Vx|)} |V^{-1}H_i w_i| \right], \tag{15}$$

$$\delta(x) \geq \psi(x), \quad \text{for all } x \in \mathbb{R}_{+0}^n, \tag{16}$$

$$T_\gamma(x) = -\Lambda^{-1}\delta(x) + \gamma. \tag{17}$$

Suppose that there exists  $\beta \in \mathbb{R}_{+0}^n$  satisfying  $T_0(\beta) < \beta$ . Then,

- (a) For every  $k \in \mathbb{Z}_+$ ,  $T_0^{k+1}(\beta) \leq T_0^k(\beta)$  and  $\lim_{k \rightarrow \infty} T_0^k(\beta) = b \geq 0$ .
- (b) Transient bounds. For every  $\gamma \in \mathbb{R}_{+0}^n$  such that  $-\Lambda^{-1}[\delta(\beta) + \max\{-\Lambda\gamma, 0\}] < \beta$ , it happens that if  $|V^{-1}x(t)| \leq T_\gamma(\beta)$  for all  $-\bar{\tau} \leq t \leq 0$ , then  $|V^{-1}x(t)| \leq \beta$  for all  $t \geq -\bar{\tau}$ .
- (c) Selection of  $\gamma \in \mathbb{R}_+^n$  for transient bounds. For every positive vector  $c \in \mathbb{R}_+^n$ , let  $p(c)$  denote the vector in  $\mathbb{R}_{+0}^n$  whose components satisfy

$$[p(c)]_j = \begin{cases} (-\Lambda c)_j & \text{if } (-\Lambda c)_j > 0, \\ 0 & \text{if } (-\Lambda c)_j \leq 0, \end{cases} \tag{18}$$

for  $j = 1, \dots, n$ . Then,  $p(c) \neq 0$  and for every  $\epsilon$  satisfying  $0 < \epsilon < \bar{\epsilon}$ , where

$$\bar{\epsilon} \triangleq \min_{j: [-\Lambda^{-1}p(c)]_j \neq 0} \frac{[\beta + \Lambda^{-1}\delta(\beta)]_j}{[-\Lambda^{-1}p(c)]_j} > 0, \tag{19}$$

it happens that  $-\Lambda^{-1}[\delta(\beta) + \max\{-\Lambda c \epsilon, 0\}] < \beta$ .

- (d) Ultimate bounds. If  $|V^{-1}x(t)| \leq \beta$  for all  $t \geq -\bar{\tau}$ , then  $\limsup_{t \rightarrow \infty} |V^{-1}x(t)| \leq b$ .

In addition to the obtention of  $V$  such that  $\Lambda$  is Hurwitz, whose computation is explained in Remark 3, Theorem 3 requires a nonnegative vector  $\beta$  satisfying  $T_0(\beta) < \beta$ . If such a vector exists, then it can be computed by means of Algorithm 1 and Theorem 3 of Kofman et al. (2007).

Theorem 3(a) establishes a monotonicity property of the sequence of vectors obtained by iterating the map  $T_0$  on the vector  $\beta$ . This property is useful to ensure the existence of the limiting vector  $b$ , which constitutes the smallest componentwise ultimate bound that can be obtained for  $|V^{-1}x(t)|$  by direct application of this theorem for the given vector  $\beta$  (Theorem 3(d)).

Theorem 3(b) provides bounds for each of the components of  $|V^{-1}x(t)|$  that are valid at every time instant, provided the initial condition  $|V^{-1}x(t)|$ ,  $-\bar{\tau} \leq t \leq 0$ , is bounded by  $T_\gamma(\beta)$ . For the bounds provided by Theorem 3(b) to be valid, the existence of  $\gamma \in \mathbb{R}_{+0}^n$  so that  $-\Lambda^{-1}[\delta(\beta) + \max\{-\Lambda\gamma, 0\}] < \beta$  is required. Note that substituting 0 for  $\gamma$  into the latter condition, and recalling (17), yields  $T_0(\beta) < \beta$ , which holds by assumption. Therefore, such a condition always holds for  $\gamma = 0$ , and by continuity, it will also hold for every  $\gamma \in \mathbb{R}_{+0}^n$  with small enough components. The advantage of employing  $\gamma$  with greater components is a larger set of initial conditions for which the bound given by Theorem 3(b) is valid.

Theorem 3(c) shows how the aforementioned vector  $\gamma$  can be computed so that all of its components are not only nonnegative but also positive. Specifically, Theorem 3(c) establishes that if an arbitrary positive vector  $c$  is selected,  $\gamma = \epsilon c$  will satisfy the requirement in Theorem 3(b) for every positive scalar  $\epsilon$  satisfying  $\epsilon < \bar{\epsilon}$  with  $\bar{\epsilon}$  as in (19). Note that there is ample leeway in the selection of  $\gamma$ , since the vector  $c$  is positive but otherwise arbitrary.

Theorem 3(d) provides componentwise ultimate bounds whenever the state remains within the bound given by Theorem 3(b) at all times. The combination of parts (b) and (d) of Theorem 3 gives local ultimate bounds, i.e., ultimate bounds that are guaranteed to hold for initial conditions within a certain set.

**Corollary 5 (Invariance).** In addition to the hypotheses of Theorem 3, suppose that for every  $\epsilon \in \mathbb{R}_+^n$ , there exists  $\beta_\epsilon$  such that  $b \leq \beta_\epsilon \leq b + \epsilon$ , and  $T_0(\beta_\epsilon) < \beta_\epsilon$ . Then, if  $|V^{-1}x(t)| \leq b$  for all  $-\bar{\tau} \leq t \leq 0$ , then  $|V^{-1}x(t)| \leq b$  for all  $t \geq -\bar{\tau}$ .

**Corollary 6 (Semi-Global Ultimate Bounds).** In addition to the hypotheses of Theorem 3, suppose that for every  $\xi \in \mathbb{R}_{+0}^n$  there exist  $\beta, \gamma \in \mathbb{R}_{+0}^n$  satisfying

$$\xi \leq T_\gamma(\beta), \quad \text{and} \tag{20}$$

$$-\Lambda^{-1}[\delta(\beta) + \max\{-\Lambda\gamma, 0\}] < \beta. \tag{21}$$

Then,  $\limsup_{t \rightarrow \infty} |V^{-1}x(t)| \leq \lim_{k \rightarrow \infty} T_0^k(\beta)$ , with  $\beta$  as above for  $\xi = \max_{-\bar{\tau} \leq t \leq 0} |V^{-1}x(t)|$ .

The ultimate bounds provided by Corollary 6 are semi-global because every initial condition has an associated ultimate bound but different initial conditions may produce different ultimate bounds.

### 3.4. Special case: affine perturbation bounds

In this subsection, we analyze a specific form of the bounding function  $\delta$  for which global ultimate bounds can be obtained under a simple sufficient condition. All proofs can be consulted in Haimovich and Seron (2012). We require the following preliminary lemma.

**Lemma 7.** Let  $\Lambda \in \mathbb{R}^{n \times n}$  be Metzler, let  $\bar{F} \in \mathbb{R}_{+0}^{n \times n}$ , and consider

$$R \triangleq -\Lambda^{-1}\bar{F}. \tag{22}$$

Then,

- (a) If  $\rho(R) < 1$  and  $\Lambda$  is Hurwitz, then  $\Lambda + \bar{F}$  is Hurwitz.
- (b) If  $\Lambda + \bar{F}$  is Hurwitz, then  $\Lambda$  is Hurwitz and  $\rho(R) < 1$ .

The main result for the case of affine perturbation bounds is the following.

**Theorem 4.** Consider a switching system (1) with perturbation bound of the form (2)–(3), where the bounding functions  $\delta_i$  are CNI. Let  $V \in \mathbb{C}^{n \times n}$  be invertible, define  $\Lambda_i$  and  $\Lambda$  as in (7), and suppose that  $\Lambda$  is Hurwitz. Consider  $\psi : \mathbb{R}_{+0}^n \rightarrow \mathbb{R}_{+0}^n$  as defined in (15) and suppose that there exists

$$\tilde{\delta}(x) \triangleq \bar{F}x + \bar{w}, \tag{23}$$

for some  $\bar{F} \in \mathbb{R}_{+0}^{n \times n}$  and  $\bar{w} \in \mathbb{R}_{+0}^n$ , satisfying  $\tilde{\delta}(x) \geq \psi(x)$  for all  $x \in \mathbb{R}_{+0}^n$ , and such that  $\rho(R) < 1$  with  $R$  as in (22). Define

$$\tilde{b} \triangleq (I - R)^{-1}(-\Lambda^{-1})\bar{w}. \tag{24}$$

Then,

- (a) Invariance. If  $|V^{-1}x(t)| \leq \tilde{b}$  for  $-\bar{\tau} \leq t \leq 0$ , then  $|V^{-1}x(t)| \leq \tilde{b}$  for all  $t \geq -\bar{\tau}$ .
- (b) Global ultimate bounds.  $\limsup_{t \rightarrow \infty} |V^{-1}x(t)| \leq \tilde{b}$ .
- (c) Tighter global ultimate bounds. Suppose that there exists a continuous and CNI  $\delta : \mathbb{R}_{+0}^n \rightarrow \mathbb{R}_{+0}^n$  satisfying

$$\psi(x) \leq \delta(x) \leq \tilde{\delta}(x), \quad \text{for all } x \in \mathbb{R}_{+0}^n. \tag{25}$$

Define  $T_0 : \mathbb{R}_{+0}^n \rightarrow \mathbb{R}_{+0}^n$  as  $T_0(x) = -\Lambda^{-1}\delta(x)$ . Then,  $\limsup_{t \rightarrow \infty} |V^{-1}x(t)| \leq \lim_{k \rightarrow \infty} T_0^k(\tilde{b}) \leq \tilde{b}$ .

- (d) There exists  $D$  diagonal and positive definite such that

$$(\Lambda + \bar{F})'D + D(\Lambda + \bar{F}) < 0. \tag{26}$$

- (e) *Ultimate bounds via standard Lyapunov techniques.* If, in addition,  $\bar{\tau} = 0$  (no delay), then for each  $D$  as in (d) above, the derivative<sup>2</sup> of the function  $L(x) \triangleq x'Px$  with  $P = \mathbb{R}e\{(V^{-1})^*DV^{-1}\}$  along any trajectory of (1) satisfies  $\dot{L}(t, x) < 0$  for all  $t$  and all  $x$  such that  $\|x\|$  is big enough.

**Theorem 4** gives an invariant region and global ultimate bounds for the case when the perturbation bound  $\bar{\delta}$  has affine form (see (23)). The main additional assumption required by this theorem is that the matrix  $R$  constructed from the system matrix  $A$  and the perturbation bound matrix  $\bar{F}$  (see (22)) has spectral radius less than 1. According to Lemma 7(b), we may seek  $V$  causing both  $A$  to be Hurwitz and  $\rho(R) < 1$  by means of the following optimization problem, similar to that in Remark 3:

minimize  $\mathbf{a}(A + \bar{F})$  subject to  $V \in \mathbb{C}^{n \times n}$  invertible,

where it is sufficient to find  $V$  so that  $\mathbf{a}(A + \bar{F}) < 0$ . Note also that, according to the hypotheses of Theorem 4 and Lemma 7(a), and since the matrix  $A$  from (7) is Metzler for every  $V \in \mathbb{C}^{n \times n}$  invertible, seeking  $V$  in the proposed manner does not incur any loss of generality. An illustration of this procedure can be found in Section 4 of Haimovich and Seron (2012).

The main advantage of the affine form of the perturbation bound is that an invariant region (Theorem 4(a)) and global ultimate bound (Theorem 4(b)) can be straightforwardly computed, without having to iterate a map or to search for a vector  $\beta$  such that  $T_0(\beta) < \beta$  as was required in Theorem 3: the quantity  $\bar{b}$  is guaranteed to exist (under the assumption that  $\rho(R) < 1$ ), and can be computed directly from the expression (24).

Theorem 4(c) deals with the case when the perturbation can be overbounded with affine  $\bar{\delta}$  but a tighter CNI perturbation bound  $\delta$  exists which is not of affine form. In this case, Theorem 4(c) avoids the need to search for a vector  $\beta$  such that  $T_0(\beta) < \beta$  as in Theorem 3 and shows that a global ultimate bound possibly tighter than that provided by the quantity  $\bar{b}$  in Theorem 4(b) can be obtained by iterating the map  $T_0$  on  $\bar{b}$ .

Theorem 4(d)–(e) provide a way of computing a quadratic function so that ultimate bounds can be obtained via standard Lyapunov techniques, in the case when no delay is present. Note that how to compute such a suitable quadratic function is not evident due to the componentwise absolute value in the form of the perturbation bound (2)–(3).

Results similar to those of Theorem 4(b) were given in Theorem 3.1 of Kofman et al. (2008) (for non-switching systems). However, the bounds in the latter reference require the matrix  $V$  to yield the similarity transformation that takes the system  $A$  matrix into Jordan canonical form. Note that requesting such a condition for  $V$  in the current switching case is usually impossible since not all the different  $A_i$  will be taken to their Jordan canonical form by the same transformation. In addition, the bounds in Theorem 3.1 of Kofman et al. (2008) are derived directly on the components of  $|x(t)|$  whereas those in Theorem 4(b) above correspond to  $|V^{-1}x(t)|$ . This difference makes possible the extension of the ultimate bound results in order to obtain tighter bounds in Theorem 4(c) and to derive the relationship with CQLF in Theorem 4(d)–(e).

**Remark 5.** If the constant part  $\bar{w}$  of the affine bound (23) is zero, then  $\bar{b} = 0$  in (24) and Theorem 4(b) implies that  $\lim_{t \rightarrow \infty} x(t) = 0$ . Consequently, the condition  $\rho(R) < 1$  in Theorem 4 or, equivalently according to Lemma 7, the condition  $A + \bar{F}$  Hurwitz, is a

sufficient condition for the uniform stability of a switching system with a perturbation bound depending linearly on the componentwise absolute value of a delayed state.

## 4. Conclusions

We have proposed a method to compute componentwise transient bounds, componentwise ultimate bounds, and invariant regions for a class of switching continuous-time linear systems with perturbation bounds that may depend nonlinearly on a delayed state. We have provided conditions for the bounds to be of local or semi-global nature. We have also addressed the particular case of perturbation bounds that have affine dependence on a delayed state, for which the bounds derived are shown to be of global nature and a novel sufficient condition for practical stability was provided. Another contribution of the paper was to establish that the class of switching linear systems to which our componentwise bound and invariant set method can be applied is strictly contained in the class of switching linear systems that admit a CQLF, although the switching linear system need not be close to simultaneously triangularizable. This closes a problem left open in our previous paper (Haimovich & Seron, 2010). A third contribution was to provide a technique to compute a CQLF for switching linear systems with perturbations bounded componentwise by affine functions of the absolute value of the state vector components (when no delay is present). Future work may focus on switched systems where either the switching signal or a continuous control input can be designed in order to ensure a given ultimate bound (cf. Kofman et al. (2008)) and on the extension and application of the current results to networked control systems (cf. Haimovich, Kofman, and Seron (2007)) and to switching systems with mixed continuous- and discrete-time dynamics.

## Appendix. Proof of Theorem 2

(a) Since  $A$  is Metzler and  $\bar{\Lambda} \geq A$ , then  $\bar{\Lambda}$  also is Metzler. Since  $\bar{\Lambda}$  is then Metzler and Hurwitz, it admits a diagonal Lyapunov function (see, e.g. Berman and Plemmons (1994, Chapter 6)).

(b) Since  $\bar{\Lambda}$  is Metzler and  $D$  is diagonal with positive main-diagonal entries, then  $\bar{\Lambda}'D + D\bar{\Lambda}$  is Metzler and symmetric. Combining the latter fact with Lemma 2(d) and (13), then

$$x'(\bar{\Lambda}'D + D\bar{\Lambda})x \leq |x'|(\bar{\Lambda}'D + D\bar{\Lambda})|x| < 0 \quad (\text{A.1})$$

for all nonzero  $x \in \mathbb{R}^n$ . Since  $A \leq \bar{\Lambda}$ , then  $A'D + DA \leq \bar{\Lambda}'D + D\bar{\Lambda}$  and hence

$$|x'|(\bar{\Lambda}'D + D\bar{\Lambda})|x| \leq |x'|(\bar{\Lambda}'D + D\bar{\Lambda})|x| \quad (\text{A.2})$$

for all  $x \in \mathbb{R}^n$ . Combining (A.1)–(A.2) and Lemma 2(d), then

$$A'D + DA < 0. \quad (\text{A.3})$$

This establishes that  $A$  is Hurwitz.

(c) Since  $A$  satisfies (A.3) and by (7)  $M_i \leq A$  and are Metzler, arguments identical to those in the proof of part (b) above show that

$$M_i'D + DM_i < 0, \quad \text{for all } i \in \underline{N}. \quad (\text{A.4})$$

By (7) and since  $D$  is diagonal with positive main-diagonal entries, then  $\mathcal{M}(A_i^*D + DA_i) \leq M_i'D + DM_i$ . The latter fact implies that

$$|z'| \mathcal{M}(A_i^*D + DA_i) |z| \leq |z'| (M_i'D + DM_i) |z| \quad (\text{A.5})$$

for all  $z \in \mathbb{C}^n$ . By Lemma 2(e) and combining with (A.4)–(A.5), it follows that

$$z^*(A_i^*D + DA_i)z < 0 \quad (\text{A.6})$$

for all nonzero  $z \in \mathbb{C}^n$ . Therefore  $A_i^*D + DA_i < 0$  and hence, using (7), then  $V^*A_i'(V^{-1})^*D + DV^{-1}A_iV < 0$ . Left-multiplying by  $(V^{-1})^*$  and right-multiplying by  $V^{-1}$  yields  $A_i'(V^{-1})^*DV^{-1} + (V^{-1})^*DV^{-1}A_i < 0$ , whence  $A_i'P + PA_i < 0$ .  $\square$

<sup>2</sup> Strictly mathematically speaking, this derivative may not exist at switching instants. This problem can be avoided by requiring the switching function to be right-continuous and to have a finite number of discontinuities in every bounded interval, and by defining  $L(t, x)$  as an upper Dini derivative. We do not delve into these technicalities here.

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