Characterizations of reverse weighted inequalities for maximal operators in Orlicz spaces and Stein's result

Osvaldo Gorosito, Ana María Kanashiro, Gladis Pradolini *

Abstract

In this article we give a characterization of reverse type inequalities on weighted Orlicz spaces of the generalized maximal operator M_η, associated to a Young function η, in terms of an appropriated Dini type condition. Our result improves the one given in [3] and, as a consequence, Stein's result in [10] turns out to be true in more general contexts.

1 Introduction

In [2] the authors deal with the generalized maximal operator M_η associated to a Young function η. They study the boundedness of this operator between Orlicz spaces defined over spaces of homogeneous type and described in terms of an appropiated couple of functions ϕ and ψ. They prove that this behaviour is characterized by a Dini type condition involving the functions η, ϕ and ψ. We should also notice that, in proving this result, two main tools are required. The first one is given by a weak type inequality for M_η and the other one is a version of a reverse type inequality for the same operator.

In the sixties Stein proved in [10] that the local integrability of the Hardy-Littlewood maximal operator of a locally integrable function f is equivalent to the fact that f belongs to the class $L(\log^+ L)(\mathbb{R}^n)$, the space of functions whose smoothness is given in terms of the integrability of the function $|f| \log^+ |f|$. Again, the weak type inequality for M and the corresponding reverse type inequality play a central role in obtaining that characterization.

Revisiting the result in [2] we observe that, when the Orlicz spaces are defined over a cube Q_0 in $\mathbb{R}^n$, the local integrability of the function $\psi(|f|)$ implies the local integrability of $\phi(M_\eta f)$ by requiring the corresponding Dini type condition. When we consider $\eta(t) =$

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t, $\phi(t) = t$ and $\psi(t) = t\log^+ t$ the Dini condition is easily satisfied and one implication of Stein's result is straightforward, the one derived from the weak type inequality.

One of the purpose of this article is to obtain reverse inequalities for the maximal operator M_η in Orlicz spaces which allow us to derive Stein's result in more general contexts. In reaching this objective the reverse weak type inequality for M_η proved in [2] is essential.

On the other hand, in [3], the author proves that a reverse Dini type condition on certain functions turns out to be equivalent to a reverse weighted modular inequality in Orlicz spaces for the Hardy-Littlewood maximal operator. The weights involved belong to the class $A_1 \cap RH'_\infty$, (see definitions below).

In this article we improve the main result in [3] and the improvement is given in several ways. The first one is that the class of weights where our result holds is wider than the considered in [3]. Moreover we consider the generalized maximal operator M_η and the Orlicz spaces are defined not only over $\mathbb{R}^n$ but also over cubes of $\mathbb{R}^n$. We also obtain other reverse inequalities of the corresponding results given in [2]. A weighted reverse weak type inequality for M_η is also needed.

Finally, we should say that the interest in studying this type of maximal operators is due to the fact that they control a large class of operators in Harmonic Analysis such as, among others, commutators of singular and fractional integral operators of higher order, fractional and singular integral operators of convolution type with kernel satisfying certain Hormander conditions associated to Young functions and non linear commutators, (See [1], [4], [5], [7] and [8]).

2 Preliminaries

We begin by summarizing a few facts about Orlicz spaces. For more information see, for example, [9].

A non negative increasing function η, defined in $[0, \infty)$, is called a Young function if it is convex and satisfies $\eta(0) = 0$, $\lim_{s\to\infty} \eta(s) = \infty$. From this definition it immediately follows that $\eta(t)/t$ is an increasing function. A function η is called submultiplicative if it satisfies the inequality

$$\eta(ts) \leq \eta(t)\,\eta(s),$$

for all positive numbers t and s and it is said to be normalized if $\eta(1) = 1$.

Clearly, if η is a submultiplicative Young function, then it satisfies the inequality $\eta(2\,t) \leq C\eta(t)$ whenever t is a positive number. This inequality is usually referred to as a Δ_2-condition. Moreover, if η is submultiplicative there exist two positive constants c and C such that

$$c\,\eta(t) \leq t\,\eta'(t) \leq C\,\eta(t) \qquad (2.1)$$

for all $t > 0$. We briefly write $\eta'(t) \simeq \eta(t)/t$.

Let η be a Young function. The η-average of a function f over a cube Q in $\mathbb{R}^n$ with sides parallel to the coordinate axes and with Lebesgue measure $|Q|$ is the real number defined, in terms of the Luxemburg norm, by

$$\|f\|_{\eta,Q} = \inf\left\{\lambda > 0 : \frac{1}{|Q|}\int_Q \eta(|f(x)|/\lambda)\,dx \leq 1\right\}.$$

The generalized maximal operator associated to the function η is defined by

$$M_\eta f(x) = \sup_{Q \ni x} \|f\|_{\eta,Q},$$

where the supremum is taken over all cubes Q containing x. If $\eta(t) = t$, $M_\eta = M$ is the classical Hardy-Littlewood maximal function. When $\eta(t) = t^p$, $p > 1$ then $M_\eta(f) = M(f^p)^{1/p}$.

A non negative function w defined on $\mathbb{R}^n$ is said to be a weight if it is an almost everywhere positive locally integrable function. If E is a measurable set in $\mathbb{R}^n$ we write $w(E) = \int_E w(x)\,dx$. The classes of weights we shall be dealing with are defined below.

A weight w is said to belong to the A_1 Muckenhoupt class if there exists a positive constant C such that the inequality

$$\frac{w(Q)}{|Q|} \leq C \inf_{x \in Q} w(x)$$

holds for every cube Q in $\mathbb{R}^n$.

Let s be a positive integer. For every cube Q, $s\,Q$ will denote the cube with the same center as Q but with side-length s times the side-length of Q.

A weight w is said to belong to the $RH'_\infty(\mathbb{R}^n)$ class if there exists a positive constant C such that the inequality

$$\sup_{x \in Q} w(x) \leq C\,\frac{w(2\,Q)}{|Q|}$$

holds for every cube $Q \subset \mathbb{R}^n$.

Let w be a weight defined on a cube Q_o. We say that w belongs to the $RH'_\infty(Q_o)$ class if there exists a positive constant C such that the inequality

$$\sup_{x \in Q} w(x) \leq C\,\frac{w(2\,Q \cap Q_o)}{|Q|}$$

holds for every cube $Q \subset Q_o$.

Given a function $\phi : [0, \infty) \to [0, \infty)$ such that $\phi(0) = 0$ we consider the following set of measurable functions,

$$L^\phi(\mathbb{R}^n) = \left\{f : \int_{\mathbb{R}^n} \phi(|f(x)|/\lambda)\;dx < \infty,\ \text{for some positive } \lambda\right\}.$$

If $f \in L^\phi(\mathbb{R}^n)$, the real number

$$\|f\|_\phi = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \phi(|f(x)|/\lambda)\, dx \leq 1\right\}$$

is a Luxemburg-type norm and the space $(L^\phi(\mathbb{R}^n), \|\cdot\|_{\phi,})$ is a Banach space called an Orlicz space. Moreover, given a weight w we will be dealing with the following set of functions

$$L^\phi_w(\mathbb{R}^n) = \left\{f : \int_{\mathbb{R}^n} \phi(|f(x)|/\lambda)\, w(x)\ dx < \infty, \text{for some positive } \lambda\right\},$$

endowed with the Luxemburg-type norm defined by

$$\|f\|_{\phi,w} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \phi(|f(x)|/\lambda)\, w(x)\, dx \leq 1\right\}.$$

In order to describe the functional spaces over which the operator M_η will be acting, some additional definitions are required.

Let a and b be two positive continuous functions defined in $[0,\infty)$ satisfying $a(0) = b(0) = 0$. Let us also assume that b is non-decreasing and that $b(t) \to \infty$ as $t \to \infty$. We define

$$\phi(t) = \int_0^t a(s)\, ds \quad \text{and} \quad \psi(t) = \int_0^t b(s)\, ds, \tag{2.2}$$

for $t \geq 0$. Observe that ϕ is not necessarily a Young function.

The fact that the inequality

$$\frac{1}{2}\, b\left(\frac{t}{2}\right) \leq \frac{\psi(t)}{t} \leq b(t) \tag{2.3}$$

holds for each positive real number t can easily be proved by applying definition (2.2) and the monotonicity property of b.

Throughout this paper, the letter C will denote a constant not necessarily the same at each occurrence.

3 Statement of the main results

We are now in a position of stating our main results related to weighted reverse type inequalities of M_η in Orlicz spaces.

(3.1) **Theorem:** *Let η be a submultiplicative Young function and a, b, ϕ and ψ defined as in (2.2). Let w be a weight such that $w \in RH'_\infty(\mathbb{R}^n)$. Then the following statements are equivalent:*

(3.2) *There exists a positive constant C such that the inequality*

$$\int_0^t \frac{a(s)}{s}\,\eta'(t/s)\,ds \geq Cb(Ct)$$

holds for every $t > 0$.

(3.3) *There exists a positive constant C such that the inequality*

$$\int_{\mathbb{R}^n} \phi(M_\eta f(x))\,w(x)\,dx \geq C \int_{\mathbb{R}^n} \psi(C|f(x)|)\,w(x)\,dx$$

holds for every positive function f.

(3.4) *There exists a positive constant C such that the inequality*

$$\int_{\mathbb{R}^n} \phi(M_\eta f(x))\,dx \geq C \int_{\mathbb{R}^n} \psi(C|f(x)|)\,dx$$

holds for every positive function f.

(3.5) *There exists a positive constant C such that the inequality*

$$\|M_\eta f\|_\phi \geq C\|f\|_\psi$$

holds for every positive function f.

(3.6) *There exists a positive constant C such that the inequality*

$$\|M_\eta f\|_{\phi,w} \geq C\|f\|_{\psi,w}$$

holds for every positive function f.

(3.7) **Remark:** For the case $\eta(t) = t$, the equivalence between the statements (3.2) and (3.3) was proved by Kita in [3], under more restrictive hypothesis on the spaces and the assumption that $\omega \in A_1 \cap RH'_{\infty},$. Our result improves Kita's not only in the sense that the class of weights we deal with is wider than his but also the generalized maximal operators involved include the classical Hardy-Littlewood maximal function. Moreover, other equivalent inequalities are obtained.

If the euclidean space is replaced by a fixed cube Q_o we obtain the following result.

(3.8) **Theorem:** *Let η be a submultiplicative Young function and a, b, ϕ and ψ defined as in (2.2). Let w be a weight defined on a cube Q_o such that $w \in RH_\infty(Q_o)'$. Then the following chains of implications hold:*

$$(3.9) \Rightarrow (3.10) \Rightarrow (3.13) \Rightarrow (3.12) \quad \text{and} \quad (3.10) \Rightarrow (3.11) \Rightarrow (3.12),$$

where the corresponding statements are as follows,

(3.9) *There exists a positive constant C such that the inequality*

$$\int_1^t \frac{a(s)}{s}\,\eta'(t/s)\,ds \geq Cb(Ct)$$

holds for every $t > 1$.

(3.10) *There exists a positive constant C such that the inequality*

$$C + C\int_{Q_o} \phi(M_\eta f(x))\, w(x)\, dx \geq C\int_{Q_o} \psi(C|f(x)|)\, w(x)\, dx$$

holds for every positive function f supported on Q_o.
(3.11) *There exists a positive constant C such that the inequality*

$$C + C\int_{Q_o} \phi(M_\eta f(x))\, dx \geq C\int_{Q_o} \psi(C|f(x)|)\, dx$$

holds for every positive function f supported on Q_o.
(3.12) *There exists a positive constant C such that the inequality*

$$\|M_\eta f\|_\phi \geq C\|f\|_\psi$$

holds for every positive function f supported on Q_o.
(3.13) *There exists a positive constant C such that the inequality*

$$\|M_\eta f\|_{\phi,w} \geq C\|f\|_{\psi,w}$$

holds for every positive function f supported on Q_o.

In the next result we prove the equivalence between modular and strong inequalities for M_η in Orlicz spaces defined over cubes with a Dini type condition.

(3.14) **Theorem:** *Let η be a submultipicative Young function and a, b, ϕ and ψ defined as in* (2.2)*. Let w be a weight defined on a cube Q_o. Then the following statements are equivalent:*
(3.15) *There exists a positive constant C such that the inequality*

$$\int_1^t \frac{a(s)}{s}\, \eta'(t/s)\, ds \leq Cb(Ct)$$

holds for every $t > 1$.
(3.16) *There exists a positive constant C such that the inequality*

$$\int_{Q_o} \phi(M_\eta f(x))\, w(x)\, dx \leq C + C\int_{Q_o} \psi(C|f(x)|)\, Mw(x)\, dx$$

holds for every positive function f supported on Q_o.
(3.17) *There exists a positive constant C such that the inequality*

$$\|M_\eta f\|_{\phi,w} \leq C\|f\|_{\psi,Mw}$$

holds for every positive function f supported on Q_o.

(3.18) *There exists a positive constant C such that the inequality*

$$\int_{Q_o} \phi(M_\eta f(x))\, dx \le C + C\int_{Q_o} \psi(C|f(x)|)\, dx$$

holds for every positive function f supported on Q_o.
(3.19) *There exists a positive constant C such that the inequality*

$$\|M_\eta f\|_\phi \le C\|f\|_\psi$$

holds for every positive function f supported on Q_o.

(3.20) **Remark:** In [2] the authors proved that (3.19) is equivalent to the following statement:

(3.21) *There exists a positive constant C such that the inequality*

$$\int_0^t \frac{a(s)}{s}\, \eta'\left(\frac{t}{s}\right)\; ds \le C\; b(C\,t),$$

holds for every $t > 0$.

In fact, a straightforward consequence of the result above is that both Dini type conditions (3.15) and (3.21) are equivalent.

As a consequence of theorems 3.8 and 3.14 we obtain the following unweighted result, interesting in itself.

(3.22) **Theorem:** *Let η be a submultiplicative Young function and a, b, ϕ and ψ defined as in* (2.2). *Suppose that there exist two positive constants C_1 and C_2 such that*

$$C_1 b(C_1 t) \le \int_1^t \frac{a(s)}{s}\, \eta'(t/s)\, ds \le C_2 b(C_2 t) \tag{3.23}$$

holds for every $t > 1$. Then, there exist two positive constants c and C such that

$$c\|f\|_\psi \le \|M_\eta f\|_\phi \le C\|f\|_\psi.$$

(3.24) **Remark:** Let us observe that condition (3.23) implies the following statement

$$f \in L^\phi(Q_0) \Leftrightarrow M_\eta f \in L^\psi(Q_0).$$

Particularly if $\eta(t) = t$, $\phi(t) = t$ and $\psi(t) = t\log^+ t$ then, it is easy to check that (3.23) holds, for example, with $C_1 = 1/e$ and $C_2 = 1$. Thus Stein's result in [10] is recovered.

4 Some technical lemmas

The following two lemmas give us weak-type inequalities for the operator M_η, which can be obtained from the corresponding results proved in [2] in the context of spaces of homogeneous type.

(4.1) **Lemma:** *Let η be a submultiplicative Young function. Then there exists a positive constant C such that the inequality*

$$|\{x \in \mathbb{R}^n : M_\eta f(x) > 2\,t\}| \leq C \int_{\{x \in \mathbb{R}^n : f(x) > t\}} \eta\left(\frac{f(x)}{t}\right)\, dx$$

holds for every positive real number t and for every nonnegative function f.

(4.2) **Lemma:** *Let η be a Young function and w be a weight. Then, the following estimate holds*

$$w(\{x \in \mathbb{R}^n : M_\eta f(x) > \lambda\}) \leq C \int_{1/4}^{\infty} Mw(\{x \in \mathbb{R}^n : |f(x)|/\lambda > s\})\eta'(s)\, ds.$$

The following covering lemma can be found in [6].

(4.3) **Lemma:** *Let G be an open set in $\mathbb{R}^n$ and let Q be a closed cube such that $G \subset 3\,Q$. Then there exists a sequence $\{Q_j\}$ of closed cubes satisfying the following properties*
a) $Q_j \subset Q$,
b) $G \cap Q^\circ \subset \cup_j\, Q_j$,
c) $Q_i^\circ \cap Q_j^\circ = \emptyset$, if $i \neq j$,
d) diam$(Q_j) \leq$ dist$(Q_j, G^c) \leq 4$ diam(Q_j),
e) $\sum_j \chi_{2Q_j}(x) \leq (33\sqrt{n})^n \chi_G(x)$.

The next lemma gives a reverse type inequality for the generalized maximal operator M_η. The corresponding result for the Hardy-Littlewood maximal operator can be found in [6].

(4.4) **Lemma:** *Let w be a nonnegative function defined on a cube Q_o such that w belongs to $RH_\infty(Q_o)'$. Then the inequality*

$$w(Q_o \cap \{x : M_\eta f(x) > \lambda\}) \geq C \int_{\{x \in \mathbb{R}^n : \eta(f(x)/\lambda) > 1\}} \eta\left(\frac{f(x)}{\lambda}\right) w(x)\, dx \tag{4.5}$$

holds for every real number $\lambda \geq \|f\|_{\eta, Q_o}$ and for every function f supported on Q_o.

Proof: Let G be the open set defined by $G = \{x : M_\eta f(x) > \lambda\}$. Thus $G \subset 3\,Q_o$. In fact, let us suppose that $x \notin 3\,Q_o$. In this case, since every cube Q containing x and intersecting Q_0 has Lebesgue measure greater or equal than $|Q_0|$ we have that

$$\frac{1}{|Q|} \int_{Q \cap Q_0} \eta\left(\frac{|f(x)|}{\mu}\right)\, dx \leq \frac{1}{|Q_0|} \int_{Q_0} \eta\left(\frac{|f(x)|}{\mu}\right)\, dx$$

for any $\mu > 0$. Then

$$\|f\|_{\eta, Q} \leq \|f\|_{\eta, Q_0} \leq \lambda$$

and thus $M_\eta f(x) \leq \lambda$, which says that $x \notin G$.
By applying lemma 4.3 and taking into account the hypothesis on w we obtain that

$$\begin{aligned}
w(Q_o \cap G) &= \int_{Q_o} \chi_G(x)\ w(x)\ dx \\
&\geq C \sum_j \int_{Q_o} \chi_{2Q_j}(x)\ w(x)\ dx \\
&= C \sum_j w(Q_o \cap 2Q_j) \\
&\geq C \sum_j |Q_j| \ \sup_{z\in Q_j} w(z).
\end{aligned}$$

For $j \geq 1$, if c_j is the center of Q_j, then, by virtue of item d) of lemma 4.3, there exists a point $x \in G^c$ such that $|x - c_j| \leq (9/2)\,\mathrm{diam}(Q_j)$. Moreover it is easy to prove that $x \in \tilde{Q} = 9\sqrt{n}\ Q_j$. As $M_\eta f(x) \leq \lambda$, then $\|f\|_{\eta,\tilde{Q}} \leq \lambda$ and the following inequality

$$\frac{1}{|\,9\sqrt{n}\ Q_j\,|} \int_{Q_j} \eta\left(\frac{f(x)}{\lambda}\right) \leq \frac{1}{|\,9\sqrt{n}\ Q_j\,|} \int_{9\sqrt{n}\ Q_j} \eta\left(\frac{f(x)}{\lambda}\right) \leq 1$$

holds.
Then

$$|Q_j| \geq \frac{1}{(9\sqrt{n})^n} \int_{Q_j} \eta\left(\frac{f(x)}{\lambda}\right)\ dx$$

and we finally have

$$\begin{aligned}
w(Q_o \cap G) &\geq C \sum_j \frac{1}{(9\sqrt{n})^n} \int_{Q_j} \eta\left(\frac{f(x)}{\lambda}\right)\ \sup_{z\in Q_j} w(z)\ dx \\
&\geq C \int_{Q_o\cap G} \eta\left(\frac{f(x)}{\lambda}\right)\ w(x)\ dx \\
&\geq C \int_{\{x\in\mathbb{R}^n:\ \eta(f(x)/\lambda)>1\}} \eta\left(\frac{f(x)}{\lambda}\right)\ w(x)\ dx.
\end{aligned}$$

□

The following inequality is obtained by a straightforward application of the result above.

(4.6) **Corollary:** *If w is a weight belonging to the $RH'_\infty(\mathbb{R}^n)$ class, then there exists a positive constant C such that the inequality*

$$w(\{x : M_\eta f(x) > \lambda\}) \geq C \int_{\{x\in\mathbb{R}^n:\,\eta(f(x)/\lambda)>1\}} \eta\left(\frac{f(x)}{\lambda}\right) w(x)\,dx$$

holds for every positive real number λ and for every measurable function f.

5 Proof of the main result

Proof of theorem 3.1: We begin by proving (3.2) $\Rightarrow$ (3.3) $\Rightarrow$ (3.6) $\Rightarrow$ (3.5) $\Rightarrow$ (3.2). Let us first prove that (3.2) $\Rightarrow$ (3.3). Let C be the constant in (3.2). From the hypothesis and corollary 4.6 we obtain that

$$\begin{aligned}
\int_{\mathbb{R}^n} \psi(Cf(x))\, w(x)\, dx &= C\int_0^\infty w(\{x\in\mathbb{R}^n : f(x)>t\})\, b(Ct)\, dt \\
&\leq \int_0^\infty w(\{x\in\mathbb{R}^n : f(x)>t\}) \left(\int_0^t \frac{a(s)}{s}\ \eta'(t/s)\, ds\right)\, dt \\
&= \int_0^\infty \left(\int_1^\infty w(\{f(x)>s\,u\})\, \eta'(u)\ du\right) a(s)\, ds \\
&\leq \int_0^\infty \left(\int_0^\infty w(\{f(x)>s\,u \text{ and } f(x)>s\})\, \eta'(u)\ du\right) a(s)\, ds \\
&= \int_0^\infty \left(\int_{\{x\,\in\mathbb{R}^n : f(x)>s\}} \eta\left(\frac{f(x)}{s}\right)\ w(x)\ dx\right) a(s)\, ds \\
&= \int_0^\infty \left(\int_{\{x\in\mathbb{R}^n :\, \eta(f(x)/s)>1\}} \eta\left(\frac{f(x)}{s}\right)\ w(x)\, dx\right) a(s)\, ds \\
&\leq C\int_0^\infty w(\{x\in\mathbb{R}^n : M_\eta f(x)>s\})\, a(s)\, ds \\
&= C\int_{\mathbb{R}^n} \phi(M_\eta f)(x)\, w(x)\, dx,
\end{aligned}$$

and then (3.3) is proved.

In order to prove (3.3) $\Rightarrow$ (3.6) let C be the constant in (3.3), then

$$C\int_{\mathbb{R}^n} \psi\left(\frac{C|f(x)|}{\|M_\eta f\|_{\phi,w}}\right) w(x)\ dx \leq \int_{\mathbb{R}^n} \phi\left(\frac{M_\eta f(x)}{\|M_\eta f\|_{\phi,w}}\right) w(x)\ dx \leq 1.$$

Thus, if $C\geq 1$ we obtain that

$$C\|f\|_{\psi,w} \leq \|M_\eta f\|_{\phi,w}.$$

On the other hand, if $C<1$, from the hypothesis on ψ

$$\int_{\mathbb{R}^n} \psi\left(\frac{C^2|f(x)|}{\|M_\eta f\|_{\phi,w}}\right) w(x)\ dx \leq C\int_{\mathbb{R}^n} \psi\left(\frac{C|f(x)|}{\|M_\eta f\|_{\phi,w}}\right) w(x)\ dx \leq 1,$$

which implies that

$$C^2\|f\|_{\psi,w} \leq \|M_\eta f\|_{\phi,w}.$$

Thus, in both cases (3.6) is proved.

Now (3.5) follows from (3.6) by taking $w=1$. Let us now prove (3.5) $\Rightarrow$ (3.2). By hypothesis

$$1 \leq \left\|\frac{CM_\eta f}{\|f\|_\psi}\right\|_\phi$$

Thus, by changing variables and applying lemma 4.1

$$\begin{aligned}
1 &\leq \int_{\mathbb{R}^n} \phi\left(\frac{CM_\eta f(x)}{\|f\|_\psi}\right) dx \\
&= \int_0^\infty a(\lambda) \left|\left\{x \in \mathbb{R}^n : M_\eta f(x) > \frac{\|f\|_\psi}{C}\lambda\right\}\right| d\lambda \\
&\leq \frac{C}{\|f\|_\psi} \int_0^\infty a\left(\frac{Cs}{\|f\|_\psi}\right) |\{x \in \mathbb{R}^n : M_\eta f(x) > 2s\}| \, ds \\
&\leq \frac{C}{\|f\|_\psi} \int_0^\infty a\left(\frac{Cs}{\|f\|_\psi}\right) \int_{\{x\in\mathbb{R}^n : f(x)>s\}} \eta\left(\frac{f(x)}{s}\right) dx\, ds
\end{aligned}$$

Particularly, if $t > 0$ and $f_t = t\chi_Q$, the inequality above looks like

$$1 \leq \frac{C|Q|}{t}\psi^{-1}(1/|Q|) \int_0^t a\left(\frac{C\psi^{-1}(1/|Q|)}{t}s\right) \eta\left(\frac{t}{s}\right) ds,$$

where we have used the fact that if $s \geq t$, then $\{x \in \mathbb{R}^n : f_t(x) > s\} = \emptyset$. By changing variables we obtain

$$\begin{aligned}
1 &\leq |Q| \int_0^{C\psi^{-1}(1/|Q|)} a(\lambda)\, \eta\left(\frac{C\psi^{-1}(1/|Q|)}{\lambda}\right) d\lambda \\
&= \frac{1}{\psi(Cr)} \int_0^r a(\lambda)\, \eta\left(\frac{r}{\lambda}\right) d\lambda, \\
&\leq C\frac{r}{\psi(Cr)} \int_0^r \frac{a(\lambda)}{\lambda}\, \eta'\left(\frac{r}{\lambda}\right) d\lambda
\end{aligned}$$

where $r = C\psi^{-1}(1/|Q|)$. By (2.3) we obtain that

$$1 \leq \frac{C}{b(Cr)} \int_0^r \frac{a(\lambda)}{\lambda}\, \eta'\left(\frac{r}{\lambda}\right) d\lambda,$$

which is the desired reverse Dini condition.

We now prove that (3.2) $\Leftrightarrow$ (3.4). The implication (3.2) $\Rightarrow$ (3.4) follows the same argument as in the proof of (3.2) $\Rightarrow$ (3.3) by taking $w = 1$.

Finally, let us prove that (3.4) $\Rightarrow$ (3.2). Let Q be a fixed cube in $\mathbb{R}^n$ and, for each positive real number t, let us define $f_t = t\chi_Q$. Applying the hypothesis to this function and using lemma 4.1 and the fact that η is submultiplicative, we obtain that

$$\begin{aligned}
|Q|\psi(Ct) &\leq \int_0^\infty a(\lambda)\, |\{x \in Q : M_\eta f(x) > \lambda\}|\, d\lambda \\
&\leq C \int_0^\infty \int_{\{x\in Q:\, t\,\chi_Q(x)>\lambda/2\}} \eta\left(\frac{t}{\lambda}\right) a(\lambda)\, dx\, d\lambda \\
&\leq C|Q| \int_0^{2t} \eta\left(\frac{t}{\lambda}\right) a(\lambda)\, d\lambda
\end{aligned}$$

Thus, by (2.3) and (2.1) we obtain that

$$C\ b(Ct) \leq C\frac{\psi(Ct)}{t} \leq \int_0^{2t} \frac{\lambda}{t}\,\eta\left(\frac{t}{\lambda}\right)\frac{a(\lambda)}{\lambda}\,d\lambda \ \leq\ C\int_0^{2t} \eta'\left(\frac{t}{\lambda}\right)\frac{a(\lambda)}{\lambda}\,d\lambda,$$

which is equivalent to (3.2) by the submultiplicative property of η. □

Proof of theorem 3.8: We have to prove (3.9) $\Rightarrow$ (3.10) $\Rightarrow$ (3.13) $\Rightarrow$ (3.12) and (3.9) $\Rightarrow$ (3.11). Let us first prove that (3.9) $\Rightarrow$ (3.10). Let C be the constant in (3.9), then

$$\begin{aligned}\int_{Q_o} \psi(Cf(x))\,w(x)\,dx &= C\int_0^\infty w(\{x\in Q_o : f(x)>t\})\,b(Ct)\,dt,\\ &\leq I_1 + I_2\end{aligned}$$

where

$$I_1 = \int_0^1 w(\{x\in Q_o : f(x)>t\})\,b(Ct)\,dt,$$

and

$$I_2 = \int_1^\infty w(\{x\in Q_o : f(x)>t\})\,b(Ct)\,dt.$$

Clearly

$$\begin{aligned}I_1 &\leq w(Q_o)\int_0^1 b(Ct)\,dt\\ &\leq C\end{aligned}$$

On the other hand, from the hypothesis and lemma 4.4 we obtain that

$$\begin{aligned}I_2 &\leq \int_1^\infty w(\{x\in Q_o : f(x)>t\})\left(\int_1^t \frac{a(s)}{s}\eta'(t/s)\,ds\right)\,dt,\\ &\leq \int_1^\infty\left(\int_1^\infty w(\{x\in Q_o : f(x)>su\})\,\eta'(u)du\right)a(s)\,ds,\\ &\leq \int_0^\infty\left(\int_{\{x\in Q_o:\,\eta(f(x)/s)>1\}} \eta\left(\frac{f(x)}{s}\right)\,w(x)\,dx\right)a(s)\,ds,\\ &\leq C\int_0^\infty w(\{x\in Q_o : M_\eta f(x)>s\})\,a(s)\,ds,\\ &= C\int_{Q_o} \phi(M_\eta f)(x)\,w(x)\,dx,\end{aligned}$$

and then (3.10) is proved. The other implications can be proved by following similar lines as in theorem 3.1. □

Proof of theorem 3.14: In view of remark 3.20, in order to get our result, we only need to prove the implications

$$(3.15) \Rightarrow (3.16) \Rightarrow (3.17) \Rightarrow (3.18) \Rightarrow (3.19),$$

since (3.21) $\Rightarrow$ (3.15) is trivial.

Let us first prove that (3.15) $\Rightarrow$ (3.16). From lemma 4.2 and the hypothesis we obtain

$$\begin{aligned}\int_{Q_o} \phi(M_\eta f(x))\, w(x)\, dx &= \int_0^\infty a(\lambda) w(\{x \in Q_o : M_\eta f(x) > \lambda\})\, d\lambda \\ &\leq C + \int_1^\infty a(\lambda) w(\{x \in Q_o : M_\eta f(x) > \lambda\})\, d\lambda \\ &\leq C + \int_1^\infty a(\lambda) \int_{1/4}^\infty Mw(\{x \in Q_o : |f(x)|/\lambda > s\}) \eta'(s)\, ds\, d\lambda \\ &\leq C + \int_1^\infty Mw(\{x \in Q_o : |f(x)| > s\}) \left(\int_1^{4s} \frac{a(\lambda)}{\lambda} \eta'(s/\lambda)\, d\lambda \right) ds \\ &\leq C + \int_1^\infty b(Cs) Mw(\{x \in Q_o : |f(x)| > s\})\, ds \\ &\leq C + \int_{Q_o} \psi(C|f(x)|) Mw(x)\, d\mu(x),\end{aligned}$$

which is (3.16). The other implications follow similar arguments as in [2]. □

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