POINTWISE CONVERGENCE TO THE INITIAL DATA FOR NONLOCAL DYADIC DIFFUSIONS

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Abstract. In this paper we solve the initial value problem for the diffusion induced by dyadic fractional derivative s in $\mathbb{R}^+$. First we obtain the spectral analysis of the dyadic fractional derivative operator in terms of the Haar system, which unveils a structure for the underlying "heat kernel". We show that this kernel admits an integrable and decreasing majorant that involves the dyadic distance. This allows us to provide an estimate of the maximal operator of the diffusion by the Hardy-Littlewood dyadic maximal operator. As a consequence we obtain the pointwise convergence to the initial data.

Keywords: pointwise convergence, nonlocal diffusions, dyadic fractional derivatives, Haar basis.

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1. Introduction

The classical result of pointwise convergence to the initial data for the heat equation $\frac{\partial u}{\partial t} = \Delta u$, $u(x,0) = f(x)$ can naturally be extended to fractional diffusions of the type $\frac{\partial u}{\partial t} = -(-\Delta u)^s$, $u(x,0) = f(x)$, with $0 < s \leq 1$. This fact follows from the estimates for the Fourier transform of $e^{-|\xi|^{2s}}$ given by R. Blumenthal and R. Geetor in [2]. In fact when $s = 1$ we are in the classical local case. In this case the solution is given by convolution of the Weierstrass kernel with f. Hence its maximal function $\sup_{t>0} |u(x,t)|$ is bounded above by the Hardy-Littlewood maximal function $Mf(x)$. When $0 < s < 1$ the results in [2] prove that, even when the decay of the kernel is no longer gaussian, we have that it is bounded above by a constant times $(1+|x|)^{-n-2s}$. So that we still have the inequality

$$(1.1) \qquad \sup_{t>0} |u(x,t)| \leq CMf(x).$$

We can rephrase the above for $0 < \sigma < 2$ without the use of the Fourier transform by saying that (1.1) holds for the solution of

$$\begin{cases} \frac{\partial u}{\partial t} = -D^\sigma u, & x \in \mathbb{R}^n, t > 0, \\ u(x,0) = f(x), & x \in \mathbb{R}^n, \end{cases}$$

given by $k_t * f(x) = u(x,t)$ with $k(x)$ the inverse Fourier transform of $e^{-|\xi|^\sigma}$, where

$$D^\sigma g(x) = p.v. \int_{\mathbb{R}^n} \frac{g(x) - g(y)}{|x-y|^{n+\sigma}} \, dy.$$

It is well known that $D^\sigma = (-\Delta)^{\sigma/2}$, see for instance [3]. Let us remark that Fourier analysis is used here to obtain the right estimates for the kernel given by the Fourier transform of $e^{-|\xi|^s}$.

In this note we aim to prove pointwise convergence for the initial value problem

$$\begin{cases} \frac{\partial u}{\partial t} = -D^\sigma_{dy} u, & x \in \mathbb{R}^+, t > 0, \\ u(x,0) = f(x), & x \in \mathbb{R}^+, \end{cases}$$

where

$$D^\sigma_{dy} g(x) = \int_{R^+} \frac{g(x) - g(y)}{\delta(x,y)^{1+\sigma}} \, dy,$$

and $\delta(x,y)$ is the measure of the smallest dyadic interval containing both x and y. We say that δ is the dyadic distance in $\mathbb{R}+$. The dyadic operator D^σ_{dy} was introduce in [1]. We show that

$$\sup_{t>0} |u(x,t)| \le C M_{dy} f(x)$$

where M_{dy} denotes the dyadic Hardy-Littlewood maximal operator, i.e.

(1.2) $$M_{dy} f(x) = \sup_{x \in I \in \mathscr{D}} \frac{1}{|I|} \int_I |f|,$$

where the supremum is taken over the family of all dyadic intervals of $\mathbb{R}^+$ containing x. In doing that we unveil the generalized Fourier analysis involved in D^σ_{dy} ($0 < \sigma < 1$). The basic fact is that the Haar functions are eigenfunctions of D^σ_{dy}. This allows to provide a structure for the underlying "heat kernel". Finally we show that this kernel admits an integrable and decreasing majorant in terms of $\delta(x,y)$.

The paper is organized as follows. We introduce the general setting and we state the main results in Section 2. In Section 3 we obtain the spectral analysis of the operator D^σ_{dy} in terms of the Haar system and we prove Theorem 2.1. Section 4 is devoted to obtain the maximal estimate contained in statement (2.4) of Theorem 2.2 and as a consequence we demonstrate the pointwise convergence to the initial data.

2. Setting and statement of the main results

Let $\mathscr{D} = \bigcup_{j \in \mathbb{Z}} \mathscr{D}^j$ be the family of all dyadic intervals in $\mathbb{R}^+$ organized in generations $\mathscr{D}^j$. If I belongs to $\mathscr{D}^j$, then $I = I_k^j = [(k-1)2^{-j}, k2^{-j})$ for some $k \in \mathbb{Z}^+$ and $|I| = 2^{-j}$, where the vertical bars denote Lebesgue measure in $\mathbb{R}$.

For each $I \in \mathscr{D}^j$ there exists 2 disjoint intervals I^+ and I^- in $\mathscr{D}^{j+1}$ both contained in I, which are precisely the right and left halves of I, respectively. An "ancestor" of I is any $J \in \mathscr{D}$ such that $I \subseteq J$. Given I and I' in $\mathscr{D}$, we shall say that $J \in \mathscr{D}$ is the "first common ancestor" of I and I' if J is an ancestor of both I and I' and $J \subseteq J'$ for any common ancestor J' of I and I'.

The dyadic distance $\delta(x,y)$ from x to y, both in $\mathbb{R}^+$, is defined as the measure of the smallest dyadic interval $J \in \mathscr{D}$ containing both x and y and $\delta(x,x) = 0$. Notice that for any two points x and y in $\mathbb{R}^+$ $\delta(x,y)$ is well defined since for $|j|$ large enough and j negative the interval $[0, 2^{-j})$ is dyadic and contains x and y. As it is easy to see $|x - y| \le \delta(x,y)$ but $\frac{1}{\delta(x,y)}$ is still singular in the sense that $\int_{\mathbb{R}^+} \frac{dy}{\delta(x,y)} = +\infty$ even when $\int_{B_\delta(x,1)} \frac{dy}{\delta(x,y)^{1-\epsilon}}$ and $\int_{\mathbb{R}^+ \setminus B_\delta(x,1)} \frac{dy}{\delta(x,y)^{1+\epsilon}}$ are both finite for $\epsilon > 0$. See Lemma 3.1 in §3.

For $I \in \mathscr{D}$ we shall write h_I to denote the Haar function supported on I. In other words $h_I = |I|^{-\frac{1}{2}}(\chi_{I^-} - \chi_{I^+})$, where χ_E denotes the indicator function of the set E. The system $\mathcal{H} = \{h_I : I \in \mathscr{D}\}$, known as the Haar system, is an orthogonal basis for $L^2(\mathbb{R}^+)$ and an unconditional basis for $L^p(\mathbb{R}^+), 1 < p < \infty$ (see for example [4]). The span of the set of Haar functions will be indicated by $S(\mathcal{H})$, that is, if $f \in S(\mathcal{H})$ then there exists a finite subset $\mathcal{F}_n$ of $\mathscr{D}$ such that

$$f(x) = \sum_{I \in \mathcal{F}_n} \langle f, h_I \rangle h_I(x),$$

where $\langle f, h_I \rangle$ denotes the inner product $\int_{\mathbb{R}^+} f h_I \, dx$ as far as it is well defined.

The fractional dyadic derivative of order $\sigma \in (0,1)$ of a function f is defined by

$$D_{dy}^\sigma f(x) = \int_{\mathbb{R}^+} \frac{f(x) - f(y)}{\delta(x,y)^{1+\sigma}} \, dy,$$

provided that the integral is absolutely convergent. This is the case if for example f is a bounded Lipschitz function with respect to δ. In particular if f is Lipschitz in the classical sense, since $|x - y| \le \delta(x,y)$.

We are now in position to state our main results.

Theorem 2.1. *Let $0 < \sigma < 1$ be given. Then,*

(1) *for each $h_I \in \mathcal{H}$ we have*

$$(2.1) \qquad\qquad D_{dy}^\sigma h_I(x) = b_\sigma |I|^{-\sigma} h_I(x),$$

with $b_\sigma = 1 + [2(2^\sigma - 1)]^{-1}$;

(2) for $f \in S(\mathcal{H})$ the function u defined in $\mathbb{R}^+ \times \mathbb{R}^+$ by

$$u(x,t) = \sum_{I \in \mathscr{D}} e^{-b_\sigma |I|^{-\sigma} t} \langle f, h_I \rangle h_I(x)$$

solves the problem

(2.2)
$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = -D_{dy}^\sigma u(x,t), & x \in \mathbb{R}^+, t > 0, \\ u(x,0) = f(x), & x \in \mathbb{R}^+; \end{cases}$$

(3) $u(x,t)$ can be written as an integral operator with the positive and finite kernel

$$k_t(x,y) = \frac{1}{t^{1/\sigma}} \varphi\left(\frac{\delta(x,y)}{t^{1/\sigma}}\right),$$

where

$$\varphi(s) = \frac{1}{s}\left[-e^{-b_\sigma s^{-\sigma}} + \sum_{j \geq 1} 2^{-j} e^{-b_\sigma (2^j s)^{-\sigma}}\right],$$

for $t > 0$. In other words

(2.3)
$$u(x,t) = \int_{\mathbb{R}+} k_t(x,y) f(y) \, dy.$$

Theorem 2.2. *Let $0 < \sigma < 1$, $1 < p < \infty$ and $f \in L^p(\mathbb{R}^+)$ be given. Then, the integral in (2.3) is absolutely convergent for almost every $x \in \mathbb{R}^+$ and $u(\cdot, t)$ belongs to $L^p(\mathbb{R}^+)$ for each $t > 0$. Moreover*

(1) *the function $u(x,t)$ satisfies that*

(2.4)
$$\sup_{t>0} |u(x,t)| \leq CM_{dy}f(x),$$

for some constant $C > 0$;

(2) *$u(\cdot, t)$ converges to f in L^p as t tends to 0;*

(3) *$\lim_{t \to 0^+} u(x,t) = f(x)$ for almost every $x \in \mathbb{R}^+$.*

3. The dyadic fractional differential operator and the proof of Theorem 2.1

The first result in this section is an elementary lemma which reflects the one dimensional character of $\mathbb{R}^+$ equipped with the distance δ.

Lemma 3.1. *Let $0 < \epsilon < 1$, and let I be a given dyadic interval in $\mathbb{R}^+$. Then, for $x \in I$, we have*

$$\int_I \frac{dy}{\delta(x,y)^{1-\epsilon}} = c_\epsilon |I|^\epsilon$$

and

$$\int_{\mathbb{R}^+ \setminus I} \frac{dy}{\delta(x,y)^{1+\epsilon}} = C_\epsilon |I|^{-\epsilon},$$

where $c_\epsilon = 2^{\epsilon-1}(2^\epsilon - 1)^{-1}$ and $C_\epsilon = [2(2^\epsilon - 1)]^{-1}$.

Proof. Observe that the open δ ball $B_\delta(x,r)$ is the largest dyadic interval I containing x with length less than r. Then, for $I \in \mathscr{D}^j$ and $x \in I$ we have

$$\int_I \frac{dy}{\delta(x,y)^{1-\epsilon}} = \int_{B_\delta(x,2^{-j+1})} \frac{dy}{\delta(x,y)^{1-\epsilon}}$$

$$= \sum_{k=j-1}^\infty \int_{\{y:\, 2^{-k-1} \leq \delta(x,y) < 2^{-k}\}} \frac{dy}{\delta(x,y)^{1-\epsilon}}$$

$$= \sum_{k=j-1}^\infty |\{y:\, \delta(x,y) = 2^{-k-1}\}| 2^{-(k+1)(\epsilon-1)}$$

$$= \frac{1}{2} \sum_{k=j-1}^\infty 2^{-(k+1)\epsilon} = \frac{2^{\epsilon-1}}{2^\epsilon - 1} |I|^\epsilon.$$

The proof of the second identity follows the same lines. $\qquad\square$

Let us notice that the indicator function of a dyadic interval $I \in \mathscr{D}$ is a Lipschitz function with respect to the distance δ. In fact $|\chi_I(x) - \chi_I(y)| \leq \delta(x,y)|I|^{-1}$. Hence for $0 < \sigma < 1$, the integral

$$\int_{\mathbb{R}^+} \frac{\chi_I(x) - \chi_I(y)}{\delta(x,y)^{1+\sigma}} dy$$

is absolutely convergent since by Lemma 3.1 for every $x \in \mathbb{R}^+$ and for any dyadic interval J containing x we have that

$$\left| \int_{\mathbb{R}^+} \frac{\chi_I(x) - \chi_I(y)}{\delta(x,y)^{1+\sigma}} dy \right| \leq \int_J \frac{|\chi_I(x) - \chi_I(y)|}{\delta(x,y)^{1+\sigma}} dy + \int_{J^c} \frac{|\chi_I(x) - \chi_I(y)|}{\delta(x,y)^{1+\sigma}} dy$$

$$\leq \frac{1}{|I|} \int_J \frac{dy}{\delta(x,y)^\sigma} + 2 \int_{J^c} \frac{dy}{\delta(x,y)^{1+\sigma}}$$

$$\leq \frac{1}{|I|} c_{1-\sigma} |J|^{1-\sigma} + 2C_\sigma |J|^{-\sigma}.$$

Now, for $0 < \sigma < 1$ we are in position to define the operator D_{dy}^σ on the linear span $S(\mathcal{H})$ of the Haar system $\mathcal{H}$, by

(3.1) $$D_{dy}^\sigma f(x) = \int_{\mathbb{R}^+} \frac{f(x) - f(y)}{\delta(x,y)^{1+\sigma}} dy.$$

Although in [1] the authors prove that Haar functions are eigenfunctions of D_{dy}^σ, we will give a simpler alternative proof.

Proof of Theorem 2.1. Notice that for $I, I' \in \mathscr{D}$, with $I \cap I' = \emptyset$, we have that

$$(3.2) \qquad\qquad \delta(x, y) = C, \qquad \text{for all } x \in I \text{ and all } y \in I'.$$

Moreover, $C = |J|$, where J is the first common ancestor of I and I'.

Take $h_I \in \mathcal{H}$. Suppose first that $x \notin I$. Since h_I is supported on I, then $h_I(x) = 0$. Hence

$$\int \frac{h_I(x) - h_I(y)}{\delta(x,y)^{1+\sigma}} dy = \int_{\mathbb{R}^+ \backslash I} \frac{-h_I(y)}{\delta(x,y)^{1+\sigma}} dy + \int_I \frac{-h_I(y)}{\delta(x,y)^{1+\sigma}} dy,$$

The first integral on the right hand side is zero since $h_I(y) \equiv 0$ for all $y \in \mathbb{R}^+ \backslash I$. For the second integral, since $x \notin I$ and $y \in I$, we apply (3.2) to obtain

$$\int_I \frac{-h_I(y)}{\delta(x,y)^{1+\sigma}} dy = -C^{-1-\sigma} \int_I h_I(y) dy = 0$$

Therefore, we have proved (2.1) for $x \notin I$.

Suppose now that $x \in I$. Let us denote with I^* the child of I which contains x. Then

$$\int_I \frac{h_I(x) - h_I(y)}{\delta(x,y)^{1+\sigma}} dy = \int_{I^*} \frac{h_I(x) - h_I(y)}{\delta(x,y)^{1+\sigma}} dy + \int_{I \backslash I^*} \frac{h_I(x) - h_I(y)}{\delta(x,y)^{1+\sigma}} dy.$$

Since h_I is constant in each child of I, then the integral over I^* vanishes. Note that in the integral over $I \backslash I^*$ we have $\delta(x, y) = |I|$, then

$$\begin{aligned}
\int_I \frac{h_I(x) - h_I(y)}{\delta(x,y)^{1+\sigma}} dy &= |I|^{-1-\sigma} \int_{I \backslash I^*} h_I(x) - h_I(y) dy \\
&= |I|^{-1-\sigma} \int_I h_I(x) - h_I(y) dy \\
&= |I|^{-1-\sigma} \left[\int_I h_I(x) dy - \int_I h_I(y) dy \right] \\
&= |I|^{-1-\sigma} h_I(x) |I| \\
(3.3) \qquad\qquad &= |I|^{-\sigma} h_I(x).
\end{aligned}$$

Finally, applying Lemma 3.1, we have that

$$\begin{aligned}
\int_{\mathbb{R}^+ \backslash I} \frac{h_I(x) - h_I(y)}{\delta(x,y)^{1+\sigma}} dy &= h_I(x) \int_{\mathbb{R}^+ \backslash I} \delta(x,y)^{-1-\sigma} dy \\
(3.4) \qquad\qquad &= h_I(x) C_\sigma |I|^{-\sigma}.
\end{aligned}$$

Hence, from (3.3) and (3.4) we obtain

$$D_{dy}^\sigma h_I(x) = \int_I \frac{h_I(x) - h_I(y)}{\delta(x,y)^{1+\sigma}} dy + \int_{\mathbb{R}^+\setminus I} \frac{h_I(x) - h_I(y)}{\delta(x,y)^{1+\sigma}} dy$$
$$= |I|^{-\sigma} h_I(x) + C_\sigma |I|^{-\sigma} h_I(x)$$
$$= (1 + C_\sigma)|I|^{-\sigma} h_I(x).$$

Then we have proved (2.1) for $x \in I$, and the proof of 1 is complete.

In order to show 2, notice first that, from 1, the identity

$$D_{dy}^\sigma f(x) = \sum_{I \in \mathscr{D}} b_\sigma |I|^{-\sigma} \langle f, h_I \rangle h_I(x)$$

holds for every $f \in S(\mathcal{H})$. Hence for such an f, the function

$$(3.5) \qquad u(x,t) = \sum_{I \in \mathscr{D}} e^{-b_\sigma |I|^{-\sigma} t} \langle f, h_I \rangle h_I(x).$$

solves problem (2.2). In fact, the orthonormality of the Haar system shows that the series defining u has only a finite number of non vanishing terms, then

$$\frac{\partial u}{\partial t}(x,t) = \sum_{I \in \mathscr{D}} -b_\sigma |I|^{-\sigma} e^{-b_\sigma |I|^{-\sigma} t} \langle f, h_I \rangle h_I(x)$$
$$= -\sum_{I \in \mathscr{D}} e^{-b_\sigma |I|^{-\sigma} t} \langle f, h_I \rangle D_{dy}^\sigma h_I(x)$$
$$= -D_{dy}^\sigma \left(\sum_{I \in \mathscr{D}} e^{-b_\sigma |I|^{-\sigma} t} \langle f, h_I \rangle h_I \right)(x)$$
$$= -D_{dy}^\sigma u(x,t).$$

On the other hand, is immediate that $u(x,0) = f(x)$.

Finally, to prove 3, set for $t > 0$

$$(3.6) \qquad k_t(x,y) = \sum_{I \in \mathscr{D}} e^{-b_\sigma |I|^{-\sigma} t} h_I(y) h_I(x).$$

Notice that for fixed positive x, $k_t(x,y)$ can be regarded as the function of y whose Haar coefficients are given by $c_I(x,t) = e^{-b_\sigma |I|^{-\sigma} t} h_I(x)$. This function of y belongs to $L^2(\mathbb{R}^+)$ since $|c_I(x,t)|^2 = e^{-2b_\sigma |I|^{-\sigma} t} |I|^{-1}$, whenever $x \in I$. Hence

$$\sum_{I \in \mathscr{D}} |c_I(x,t)|^2 = \sum_{\substack{I \in \mathscr{D} \\ I \ni x}} e^{-2b_\sigma |I|^{-\sigma} t} |I|^{-1} = \sum_{j \in \mathbb{Z}} = e^{-2b_\sigma 2^{j\sigma} t} 2^j,$$

which is finite. Then for $f \in S(\mathcal{H})$ the integral $\int_{\mathbb{R}+} k_t(x,y)f(y)dy$ is absolutely convergent. Therefore writing the integral as an inner product and noticing that

$$\left\langle f, \sum_{I \in \mathscr{D}} c_I(x,t)h_I \right\rangle = \sum_{I \in \mathscr{D}} c_I(x,t) \langle f, h_I \rangle,$$

we obtain from (3.5) that

$$\int_{\mathbb{R}+} k_t(x,y)f(y)dy = u(x,t).$$

Notice first that for fixed x and y in $\mathbb{R}$, only contribute to in (3.6) the terms in which I contains both x and y. We shall denote I^0 the first common ancestor of x and y, and let ℓ be such that $I^0 \in \mathscr{D}^\ell$. Also we shall denote I^j the dyadic interval in $\mathscr{D}^{\ell-j}$ containing I^0. Then

$$k_t(x,y) = \sum_{j \geq 0} e^{-b_\sigma |I^j|^{-\sigma}t} h_{I^j}(y)h_{I^j}(x)$$

$$= e^{-b_\sigma |I^0|^{-\sigma}t} h_{I^0}(y)h_{I^0}(x)$$
$$+ \sum_{j \geq 1} e^{-b_\sigma |I^j|^{-\sigma}t} h_{I^j}(y)h_{I^j}(x)$$

Let us observe that, for every $j \geq 1$, x and y belong to the same child of I^j, so that $h_{I^j}(y) = h_{I^j}(x)$. Moreover,

$$h_{I^j}(y)h_{I^j}(x) = \left|I^j\right|^{-1}.$$

Hence,

$$k_t(x,y) = e^{-b_\sigma |I^0|^{-\sigma}t} h_{I^0}(y)h_{I^0}(x) + \sum_{j \geq 1} \frac{e^{-b_\sigma |I^j|^{-\sigma}t}}{|I^j|}.$$

Now, notice that $\delta(x,y) = |I^0|$ and that $|I^j| = 2^j|I^0|$. Also, since x and y belong to different children of I^0, we have that $h_{I^0}(y)h_{I^0}(x) = -|I^0|^{-1}$. Then, we obtain that

$$k_t(x,y) = -e^{-b_\sigma \delta(x,y)^{-\sigma}t} \delta(x,y)^{-1} + \sum_{j \geq 1} \frac{e^{-b_\sigma (2^j \delta(x,y))^{-\sigma}t}}{2^j \delta(x,y)}$$

$$= \frac{1}{\delta(x,y)} \left[-e^{-b_\sigma \delta(x,y)^{-\sigma}t} + \sum_{j \geq 1} 2^{-j} e^{-b_\sigma (2^j \delta(x,y))^{-\sigma}t} \right].$$

Hence, defining $\varphi : \mathbb{R}^+ \to \mathbb{R}$ by

$$\varphi(s) = \frac{1}{s}\left[-e^{-b_\sigma s^{-\sigma}} + \sum_{j \geq 1} 2^{-j} e^{-b_\sigma (2^j s)^{-\sigma}}\right],$$

we have that

$$k_t(x, y) = \frac{1}{t^{1/\sigma}}\varphi\left(\frac{\delta(x, y)}{t^{1/\sigma}}\right).$$

Notice that the series defining $\varphi(s)$ converges for every positive s. On the other hand, since $e^{-b_\sigma s^{-\sigma}} < e^{-b_\sigma (2^j s)^{-\sigma}}$ for every $j \geq 1$, then $\varphi(s)$ is strictly positive for every $s > 0$. Therefore the proof of 3 is complete. $\qquad\square$

4. Maximal function estimates and the proof of Theorem 2.2

To start with the analysis of the way in which the initial condition is attained, in this section we shall denote K_t the operator with kernel k_t, i.e.

$$K_t f(x) := \int_{\mathbb{R}^+} k_t(x, y) f(y) dy = u(x, t).$$

In this section we aim to prove that the maximal operator K^* associated to $u(x, t)$ satisfies that

$$K^* f(x) := \sup_{t>0} |K_t f(x)| = \sup_{t>0} |u(x, t)| \leq C M_{dy} f(x),$$

for every $f \in L^p(\mathbb{R}^+)$, where M_{dy} denotes the dyadic Hardy-Littlewood maximal operator defined in (1.2). In order to do this, we shall construct a decreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\psi \in L^1(0, \infty)$ with

$$|k_t(x, y)| \leq \frac{1}{t^{1/\sigma}}\psi\left(\frac{\delta(x, y)}{t^{1/\sigma}}\right).$$

Proof of Theorem 2.2. For $s > 0$, since $\sum_{j \geq 1} 2^{-j} = 1$ and $|e^{-x}| \leq 1$ for $x \in \mathbb{R}^+$, we have that

$$\varphi(s) \leq \frac{1}{s}\sum_{j \geq 1} 2^{-j}\left[1 - e^{-b_\sigma s^{-\sigma}}\right].$$

Then using the Taylor expansion for the exponential function we obtain

(4.1) $$\varphi(s) \leq \frac{1}{s}\sum_{j \geq 1} 2^{-j}\left[\frac{b_\sigma}{s^\sigma}\right] = \frac{b_\sigma}{s^{1+\sigma}}.$$

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For $0 < s < 1$, fix $0 < \epsilon < 1$ and define $n(s) := \lfloor -\epsilon \log_2 s \rfloor$, where $\lfloor \cdot \rfloor$ denotes the floor function. Then we can split the series defining φ in the following way

$$(4.2) \qquad \varphi(s) = \frac{-e^{-b_\sigma s^{-\sigma}}}{s} + \frac{1}{s} \sum_{j=1}^{n(s)} 2^{-j} e^{-b_\sigma (2^j s)^{-\sigma}} + \frac{1}{s} \sum_{j>n(s)} 2^{-j} e^{-b_\sigma (2^j s)^{-\sigma}}.$$

The absolute value of the first term of the right hand side of (4.2) is clearly bounded. The second term is also bounded since for $j \leq n(s)$ we have that $e^{-b_\sigma (2^j s)^{-\sigma}} \leq e^{-b_\sigma s^{-(1-\epsilon)\sigma}}$. Then

$$\frac{1}{s} \sum_{j=1}^{n(s)} 2^{-j} e^{-b_\sigma (2^j s)^{-\sigma}} \leq \frac{e^{-b_\sigma s^{-(1-\epsilon)\sigma}}}{s} \sum_{j=1}^{n(s)} 2^{-j} \leq \frac{e^{-b_\sigma s^{-(1-\epsilon)\sigma}}}{s}.$$

For the third term we can see that

$$\frac{1}{s} \sum_{j>n(s)} 2^{-j} e^{-b_\sigma (2^j s)^{-\sigma}} \leq \frac{1}{s} \sum_{j>n(s)} 2^{-j} \leq \frac{1}{s} 2^{-n(s)} \leq \frac{2}{s} s^\epsilon = \frac{2}{s^{1-\epsilon}}.$$

Therefore, for $0 < s < 1$ we have that

$$(4.3) \qquad\qquad\qquad \varphi(s) \leq \frac{C}{s^{1-\epsilon}}.$$

So that from (4.1) and (4.3), $\varphi(s) \leq \psi(s)$ for every $s \in \mathbb{R}^+$ with

$$\psi(s) = C \begin{cases} 1/s^{1-\epsilon}, & \text{if } 0 < s < 1, \\ 1/s^{1+\sigma}, & \text{if } s \geq 1, \end{cases}$$

for some positive constant C. Hence,

$$\begin{aligned}
|K_t f(x)| &\leq \int_{\mathbb{R}^+} |k_t(x,y)||f(y)|\, dy \\
&\leq \int_{\mathbb{R}^+} \frac{1}{t^{1/\sigma}} \psi\left(\frac{\delta(x,y)}{t^{1/\sigma}}\right) |f(y)|\, dy \\
&= \sum_{j=-\infty}^{\infty} \frac{1}{t^{1/\sigma}} \int_{\{y : t^{1/\sigma} 2^j \leq \delta(x,y) < t^{1/\sigma} 2^{j+1}\}} \psi\left(\frac{\delta(x,y)}{t^{1/\sigma}}\right) |f(y)|\, dy \\
&\leq \sum_{j=-\infty}^{\infty} 2^{j+1} \psi(2^j) \frac{1}{t^{1/\sigma} 2^{j+1}} \int_{B_\delta(x, t^{1/\sigma} 2^{j+1})} |f(y)|\, dy.
\end{aligned}$$

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Since $|B_\delta(x, r)| < r$ and each B_δ is a dyadic interval, we have

$$
\begin{aligned}
|K_t f(x)| &\leq \sum_{j=-\infty}^{\infty} 2^{j+1} \psi(2^j) \frac{1}{|B_\delta(x, t^{1/\sigma} 2^{j+1})|} \int_{B_\delta(x, t^{1/\sigma} 2^{j+1})} |f(y)| \, dy \\
&\leq \sum_{j=-\infty}^{\infty} 2^{j+1} \psi(2^j) M_{dy} f(x) \\
&= 4 M_{dy} f(x) \sum_{j=-\infty}^{\infty} \int_{\{y : 2^{j-1} \leq y < 2^j\}} \psi(2^j) \, dy \\
&\leq 4 M_{dy} f(x) \int_{\mathbb{R}^+} \psi(y) \, dy \\
&\leq 4 \|\psi\|_{L^1} M_{dy} f(x).
\end{aligned}
$$

Therefore, taking supremum in t we obtain

$$
\sup_{t>0} |K_t f(x)| \leq 4 \|\psi\|_{L^1} M_{dy} f(x),
$$

which completes the proof of 1.

In order to prove 2, notice first that for $g \in S(\mathcal{H})$ and $1 < p < \infty$ we have the L^p convergence of $K_t g$ to g. Take $f \in L^p$, then for a function g in $S(\mathcal{H})$ we have that

$$
\begin{aligned}
\|K_t f - f\|_{L^p} &\leq \|K_t(f - g)\|_{L^p} + \|K_t g - g\|_{L^p} + \|g - f\|_{L^p} \\
&\leq C \|M_{dy}(f - g)\|_{L^p} + \|K_t g - g\|_{L^p} + \|g - f\|_{L^p} \\
&\leq \widetilde{C} \|f - g\|_{L^p} + \|K_t g - g\|_{L^p}.
\end{aligned}
$$

So that from the above remark and the density of $S(\mathcal{H})$ in L^p we obtain the desired result.

Finally, as usual, the pointwise convergence to the initial data in 3 is an immediate consequence of the boundedness properties of the maximal operator associated to u and the pointwise convergence in a dense subset of L^p ($1 < p < \infty$). We will sketch a brief proof for the sake of completeness.

Since we already know that $K_t f \to f$ in the L^p sense as $t \to 0^+$, in order to prove the pointwise convergence, define

$$
E = \{f \in L^p : \lim_{t \to 0^+} K_t f \text{ exists for almost every } x \in \mathbb{R}^+\}.
$$

Notice that $S(\mathcal{H}) \subseteq E \subseteq L^p$. Since $S(\mathcal{H})$ is dense in L^p, then we only need to prove that E is a closed subset of L^p. Let $\{f_n\}$ be a sequence contained in E such that f_n

converges in L^p to a function f. To see that $f \in E$ it is enough to prove that for all $\epsilon > 0$ we have

(4.4) $$|E_\epsilon| := \left| \left\{ x : \limsup_{t \to 0^+} K_t f(x) - \liminf_{t \to 0^+} K_t f(x) > \epsilon \right\} \right| = 0.$$

For every n we can write

$$|E_\epsilon| \leq \left| \left\{ x : \limsup_{t \to 0^+} K_t f_n(x) - \liminf_{t \to 0^+} K_t f_n(x) > \frac{\epsilon}{3} \right\} \right|$$
$$+ \left| \left\{ x : \limsup_{t \to 0^+} K_t(f - f_n)(x) > \frac{\epsilon}{3} \right\} \right| + \left| \left\{ x : \liminf_{t \to 0^+} K_t(f - f_n)(x) < -\frac{\epsilon}{3} \right\} \right|.$$

The first term is zero since $f_n \in E$. For the other two terms we use the weak type inequality boundedness on L^p of the maximal operator K^* which follows from the item 1. Notice that for every function g we have that

$$\left| \liminf_{t \to 0^+} K_t g(x) \right| \leq K^* g(x) \quad \text{and} \quad \left| \limsup_{t \to 0^+} K_t g(x) \right| \leq K^* g(x).$$

Then, since K^* is weakly bounded on L^p, we obtain

$$\left| \left\{ x : \limsup_{t \to 0^+} K_t(f - f_n)(x) > \frac{\epsilon}{3} \right\} \right| \leq \frac{C}{\epsilon^p} \|f_n - f\|_{L^p}^p$$

and

$$\left| \left\{ x : \liminf_{t \to 0^+} K_t(f - f_n)(x) < -\frac{\epsilon}{3} \right\} \right| \leq \frac{C}{\epsilon^p} \|f_n - f\|_{L^p}^p.$$

Hence,

$$|E_\epsilon| \leq \frac{C}{\epsilon^p} \|f_n - f\|_{L^p}^p.$$

When n tends to infinity we have (4.4). Then E is closed and therefore $E = L^p$. This means that for every $f \in L^p$ we have that

$$\lim_{t \to 0^+} u(x, t) = \lim_{t \to 0^+} K_t f \quad \text{exists for a.e. } x \in \mathbb{R}^+.$$

But we already know that $u(x, t) \to f(x)$ when $t \to 0^+$ in L^p, then 3 follows, which completes the proof. $\qquad \square$

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