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On the Isometry Groups of Invariant Lorentzian Metrics on the Heisenberg Group

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Abstract. This work concerns the invariant Lorentzian metrics on the Heisenberg Lie group of dimension three $H_3(\mathbb{R})$ and the bi-invariant metrics on the solvable Lie groups of dimension four. We start with the indecomposable Lie groups of dimension four admitting bi-invariant metrics and which act on $H_3(\mathbb{R})$ by isometries and we study some geometrical features on these spaces. On $H_3(\mathbb{R})$, we prove that the property of the metric being proper naturally reductive is equivalent to the property of the center being non-degenerate. These metrics are Lorentzian algebraic Ricci solitons.

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1. Introduction

Homogeneous manifolds constitute the goal of several modern researches in pseudo-Riemannian geometry, for instance Lorentzian spaces for which all null geodesics are homogeneous became relevant in physics [14,19]. This fact motivated several studies on g.o. spaces in the last years, see for instance [7–9,12] and their references. Symmetric pseudo-Riemannian spaces and three-dimensional Lie groups equipped with a left-invariant Lorentzian metric include all the possible connected, simply connected, complete homogeneous Lorentzian manifolds [7].

In the case of the Heisenberg Lie group of dimension three $H_3(\mathbb{R})$, it was proved in [25] that there are three classes of left-invariant Lorentzian metrics, and only one of them is flat (see also [20]), which is characterized by the property of the center being degenerate.

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In this work, we concentrate the attention to the non-flat metrics on $\mathrm{H}_3(\mathbb{R})$ and their isometry groups. According to [23] any left-invariant metric on a Heisenberg Lie group, for which the center is non-degenerate, is naturally reductive, so these spaces are geodesically complete and non-flat. Here, we prove a partial converse to that result: *Any naturally reductive Lorentzian metric on* $\mathrm{H}_3(\mathbb{R})$ *admitting an action by isometric isomorphisms of a one-dimensional group, restricts to a metric on the center.*

Thus, for any left-invariant Lorentzian metric on $\mathrm{H}_3(\mathbb{R})$, the following statements are equivalent:

(i) non-flat metric,
(ii) non-degenerate center,
(iii) proper naturally reductive metric,

where proper means non-symmetric. The first equivalence (i) $\Leftrightarrow$ (ii) follows from Theorem 1 in [16]. On the other hand, $\mathrm{H}_3(\mathbb{R})$ equipped with the left-invariant metric which is flat is a space form hence isometric to $\mathbb{R}^3_1$ [21]. The statement above does not hold in higher dimensions: a non-flat left-invariant Lorentzian metric with degenerate center on $\mathbb{R} \times \mathrm{H}_3(\mathbb{R})$ is proved to be naturally reductive in [24]. Properties of flat or Ricci-flat Lorentzian metrics were investigated for instance in [1,2,16] and references therein. Here, we also compute the corresponding isometry groups following results on naturally reductive metrics in [23] (comparing with [6]) and we see that the non-flat metrics are algebraic Ricci solitons (see Ricci solitons on Lorentzian Lie groups of dimension three in [5]).

The study of these naturally reductive non-flat metrics on $\mathrm{H}_3(\mathbb{R})$ is motivated by the results on [22], which state that a naturally reductive pseudo-Riemannian space admits a transitive action by isometries of a Lie group equipped with a bi-invariant metric. Hence, we start with the classification of all Lie algebras up to dimension four admitting an ad-invariant metric. It is important to remark that the method used here is constructive and independent of the classification of low-dimensional Lie algebras.

So a naturally reductive Lorentzian metric on $\mathrm{H}_3(\mathbb{R})$ admits an action by isometries of a Lie group G with a bi-invariant metric. If G has dimension four, it corresponds to one of the Lie algebras obtained before. This is a key point in the proof of the equivalence stated above.

Finally, we complete the work by investigating the geometry of the bi-invariant metrics of the solvable Lie groups G_0 and G_1, which are associated to the non-flat metrics on $\mathrm{H}_3(\mathbb{R})$. We compute the isometry groups $\mathsf{I}(G_0)$ and $\mathsf{I}(G_1)$ in the aim of establishing a relationship between them and G_0 and G_1 as isometry groups of $\mathrm{H}_3(\mathbb{R})$. Also geodesics are described.

2. Lie algebras with ad-invariant metrics up to dimension four

In this section, we revisit the Lie algebras of dimension $d \leq 4$ that can be furnished with an ad-invariant metric. The proofs given here are constructive and they do not make use of the double extension procedure [4,13,18].

Let $\mathfrak{g}$ be a real Lie algebra. A symmetric bilinear form $\langle\, ,\, \rangle$ on $\mathfrak{g}$ is called *ad-invariant* if the following condition holds:

$$\langle \operatorname{ad}_X Y, Z\rangle + \langle Y, \operatorname{ad}_X Z\rangle = 0 \qquad \text{for all } X, Y, Z \in \mathfrak{g}.$$

Whenever $\langle\, ,\, \rangle$ is non-degenerate, the symmetric bilinear form is just called a *metric.*

Example 2.1. The Killing form is an ad-invariant symmetric bilinear form on any Lie algebra $\mathfrak{g}$, which is non-degenerate if $\mathfrak{g}$ is semisimple. Moreover, if $\mathfrak{g}$ is simple any ad-invariant metric on $\mathfrak{g}$ is a non-zero multiple of the Killing form.

Recall that the central descending series $\{C^r(\mathfrak{g})\}$ and central ascending series $\{C_r(\mathfrak{g})\}$ of a Lie algebra $\mathfrak{g}$, are for $r \geq 0$, respectively, given by the ideals

$$\begin{array}{ll} C^0(\mathfrak{g}) = \mathfrak{g} & C_0(\mathfrak{g}) = 0 \\ C^r(\mathfrak{g}) = [\mathfrak{g}, C^{r-1}(\mathfrak{g})] & C_r(\mathfrak{g}) = \{X \in \mathfrak{g} : [X, \mathfrak{g}] \subseteq C_{r-1}(\mathfrak{g})\}. \end{array}$$

Fixing a subspace $\mathfrak{m}$ of $\mathfrak{g}$, its orthogonal subspace is defined as usual by

$$\mathfrak{m}^{\perp} = \{X \in \mathfrak{g} : \langle X, Y\rangle = 0, \ \forall\, Y \in \mathfrak{m}\}.$$

The next result follows by applying the definitions above and an inductive procedure.

Lemma 2.2. *Let* $(\mathfrak{g}, \langle\, ,\, \rangle)$ *denote a Lie algebra endowed with an ad-invariant metric.*

1. *If* $\mathfrak{h}$ *is an ideal in* $\mathfrak{g}$ *then* $\mathfrak{h}^{\perp}$ *is also an ideal of* $\mathfrak{g}$*.*
2. $C^r(\mathfrak{g}) = (C_r(\mathfrak{g}))^{\perp}$ *for all* $r \geq 0$*.*

Notice that if the metric is indefinite, for any subspace $\mathfrak{m}$ the decomposition $\mathfrak{m} + \mathfrak{m}^{\perp}$ is not necessarily a direct sum. Nevertheless, the next formula holds

$$\dim \mathfrak{g} = \dim C^r(\mathfrak{g}) + \dim C_r(\mathfrak{g}) \quad \forall r \geq 0 \tag{1}$$

and, in particular,

$$\dim \mathfrak{g} = \dim C^1(\mathfrak{g}) + \dim \mathfrak{z}(\mathfrak{g}) \tag{2}$$

where $\mathfrak{z}(\mathfrak{g})$ denotes the center of $\mathfrak{g}$. Moreover,

- if $\mathfrak{m} \subseteq C^1(\mathfrak{g})$ is a vector subspace such that $C^1(\mathfrak{g}) = (\mathfrak{z}(\mathfrak{g}) \cap C^1(\mathfrak{g})) \oplus \mathfrak{m}$, then $\mathfrak{m}$ is non-degenerate;
- if $\mathfrak{m}' \subseteq \mathfrak{z}(\mathfrak{g})$ is a vector subspace such that $\mathfrak{z}(\mathfrak{g}) = (\mathfrak{z}(\mathfrak{g}) \cap C^1(\mathfrak{g})) \oplus \mathfrak{m}'$, then $\mathfrak{m}'$ is non-degenerate.

Remark 1. Suppose $\mathfrak{g}$ admits an ad-invariant metric and $\mathfrak{z}(\mathfrak{g}) \neq 0$. Then as said above any complementary space $\tilde{\mathfrak{z}}$ such that $\mathfrak{z}(\mathfrak{g}) = \tilde{\mathfrak{z}} \oplus \big(\mathfrak{z}(\mathfrak{g}) \cap C^1(\mathfrak{g})\big)$ is non-degenerate. It follows that $\mathfrak{g} = \tilde{\mathfrak{z}} \oplus \tilde{\mathfrak{g}}$ is a direct sum of non-degenerate ideals where $\tilde{\mathfrak{g}} = \tilde{\mathfrak{z}}^{\perp}$ each of them having ad-invariant metrics. In addition $\mathfrak{z}(\tilde{\mathfrak{g}}) = \mathfrak{z}(\mathfrak{g}) \cap C^1(\mathfrak{g})$.

Now suppose $\mathfrak{g}$ is solvable. Then by (2) it has non-trivial center. If moreover $\mathfrak{g}$ is non-abelian then both $C^1(\mathfrak{g})$ and $\mathfrak{z}(\mathfrak{g})$ are non-trivial and $C^1(\mathfrak{g}) \cap \mathfrak{z}(\mathfrak{g}) \neq 0$. In fact, using the decomposition described above $\mathfrak{g} = \tilde{\mathfrak{z}} \oplus \tilde{\mathfrak{g}}$

where $\tilde{\mathfrak{g}}$ turns to be a solvable Lie algebra with an ad-invariant metric. Then its center $\mathfrak{z}(\tilde{\mathfrak{g}}) = \mathfrak{z}(\mathfrak{g}) \cap C^1(\mathfrak{g})$ is not trivial.

Proposition 2.3. *Let $\mathfrak{g}$ denote a real Lie algebra of dimension two or three. If it can be endowed with an ad-invariant metric, then*

- *in dimension two $\mathfrak{g}$ is abelian and*
- *in dimension three $\mathfrak{g}$ is abelian or simple.*

Proof. Assume first that $\mathfrak{g}$ has dimension two. Then it is either abelian or isomorphic to the solvable Lie algebra spanned by the vectors X, Y with $[X, Y] = Y$. Since the center of this solvable Lie algebra is trivial, it cannot be equipped with an ad-invariant metric.

Assume now that $\mathfrak{g}$ has dimension 3. It is well known that it must be either solvable or simple. If it is abelian or simple, it admits an ad-invariant metric (see Example 2.1).

Suppose now $\mathfrak{g}$ is a non-abelian solvable Lie algebra equipped with an ad-invariant bilinear form $\langle\, ,\, \rangle$. Since $\mathfrak{z}(\mathfrak{g}) \cap C^1(\mathfrak{g})$ is non-trivial (see Remark 1), there exist $X, Y \in \mathfrak{g}$ such that $[X, Y] = Z \in C^1(\mathfrak{g}) \cap \mathfrak{z}(\mathfrak{g})$. It is not difficult to see that the vectors X, Y, Z form a basis of $\mathfrak{g}$. Since $Z \in C^1(\mathfrak{g}) \cap \big(C^1(\mathfrak{g})\big)^{\perp}$ then $\langle Z, Z\rangle = 0$. Furthermore,

$$\langle Z, X\rangle = \langle [X, Y], X\rangle = -\langle Y, [X, X]\rangle = 0$$

and in the same way one gets $\langle Z, Y\rangle = 0$. Thus, any ad-invariant bilinear form on $\mathfrak{g}$ must be degenerate. □

A Lie algebra $(\mathfrak{g}, \langle\, ,\, \rangle)$ is called *indecomposable* if it has no non-degenerate ideals.

Observe that if a Lie algebra $\mathfrak{g}$ with an ad-invariant metric admits a non-degenerate ideal $\mathfrak{j}$, then $\mathfrak{j}^{\perp}$ is also a non-degenerate ideal and so $\mathfrak{g} = \mathfrak{j} \oplus \mathfrak{j}^{\perp}$.

Remark 2. By Remark 1 if $(\mathfrak{g}, \langle\, ,\, \rangle)$ is indecomposable and with non-trivial center, then the center is contained in the commutator $\mathfrak{z}(\mathfrak{g}) \subseteq C^1(\mathfrak{g})$.

Lemma 2.4. *Let $\mathfrak{g}$ denote a Lie algebra of dimension four furnished with an ad-invariant metric. If it is non-solvable then it is decomposable.*

Proof. Let $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$ be a Levi decomposition of $\mathfrak{g}$, where $\mathfrak{r}$ denotes the radical. Since $\mathfrak{g}$ is not solvable $\dim \mathfrak{r} < 4$. Moreover, since there are no simple Lie algebras of dimension one or two, it holds $\dim \mathfrak{r} = 1$ and $\mathfrak{s}$ is either $\mathfrak{sl}(2)$ or $\mathfrak{so}(3)$. In every case, the action $\mathfrak{s} \to Der(\mathfrak{r})$ is trivial. In fact, let $\mathfrak{r} = \mathbb{R}e_0$ and $\mathfrak{s} = span\{e_1, e_2, e_3\}$.

Assume $[e_i, e_0] = \lambda_i e_0$. For all $i, j = 1, 2, 3$ there exist $\xi_{ij} \in \mathbb{R} - \{0\}$ such that $[e_i, e_j] = \xi_{ij} e_k$ for some $k = 1, 2, 3$ (see the Lie brackets in $\mathfrak{sl}(2)$ or $\mathfrak{so}(3)$) and where $\xi_{ij} \neq 0$ for all i, j. Since $[\mathfrak{s}, \mathfrak{s}] = \mathfrak{s}$ from $\operatorname{ad}([e_i, e_j])e_0 = \xi_{ij} \operatorname{ad}(e_k)e_0$ one gets $\lambda_k = 0$ for all k.

Let $\langle\, ,\, \rangle$ denote an ad-invariant metric on $\mathfrak{g}$ and denote $\mu_k = \langle e_0, e_k\rangle$. So

$$\xi_{ij}\mu_k = \xi_{ij}\langle e_0, e_k\rangle = \langle e_0, [e_i, e_j]\rangle = \langle [e_j, e_0], e_i\rangle = 0$$

and since $\xi_{ij} \neq 0$ it must hold $\mu_k = 0$ for all k. Since $\langle \, , \, \rangle$ is non-degenerate, it then follows $\langle e_0, e_0 \rangle \neq 0$, so that $\mathfrak{r}$ is a non-degenerate ideal and the proof is finished. $\square$

To complete the description of all the Lie algebras of dimension four admitting ad-invariant metrics, we have the following result.

Proposition 2.5. *Let $\mathfrak{g}$ denote a real Lie algebra of dimension four which can be endowed with an ad-invariant metric. Then $\mathfrak{g} = span\{e_0, e_1, e_2, e_3\}$ is isomorphic to one of the following Lie algebras:*

- $\mathbb{R}^4$
- $\mathbb{R} \oplus \mathfrak{sl}(2, \mathbb{R})$
- $\mathbb{R} \oplus \mathfrak{so}(3, \mathbb{R})$
- *the oscillator Lie algebra $\mathfrak{g}_0 = span\{e_0, \cdots e_3\}$ with the non-zero Lie brackets:*
$$[e_0, e_1] = e_2 \quad [e_0, e_2] = -e_1 \quad [e_1, e_2] = e_3 \tag{3}$$
- *$\mathfrak{g}_1 = span\{e_0, \cdots, e_3\}$ with the non-zero Lie brackets:*
$$[e_0, e_1] = e_1 \quad [e_0, e_2] = -e_2 \quad [e_1, e_2] = e_3. \tag{4}$$

Proof. Let $\mathfrak{g}$ be a Lie algebra equipped with an ad-invariant metric $\langle \, , \, \rangle$. If $\mathfrak{g}$ is decomposable then $\mathfrak{g}$ corresponds to one of the following Lie algebras: $\mathbb{R}^4$, $\mathfrak{g} = \mathbb{R} \oplus \mathfrak{sl}(2, \mathbb{R})$, $\mathfrak{g} = \mathbb{R} \oplus \mathfrak{so}(3, \mathbb{R})$ (by Proposition 2.3).

Assume now $\mathfrak{g}$ is indecomposable. From Lemma 2.4, the Lie algebra $\mathfrak{g}$ is solvable and hence $C^1(\mathfrak{g}) \neq \mathfrak{g}$. By Remark 2, $\mathfrak{z}(\mathfrak{g}) \subseteq C^1(\mathfrak{g})$ and $4 = \dim \mathfrak{z}(\mathfrak{g}) + \dim C^1(\mathfrak{g}) \leq 2 \dim C^1(\mathfrak{g})$. It follows that $\dim \mathfrak{z}(\mathfrak{g}) = 1$ or $\dim \mathfrak{z}(\mathfrak{g}) = 2$. But since we cannot have $\mathfrak{z}(\mathfrak{g}) = C^1(\mathfrak{g})$ (in dimension four), it should be $\dim \mathfrak{z}(\mathfrak{g}) = 1$ and $\dim C^1(\mathfrak{g}) = 3$.

Let e_3 be a generator of $\mathfrak{z}(\mathfrak{g})$ and let $e_0 \in \mathfrak{g} - C^1(\mathfrak{g})$ such that $\langle e_0, e_3 \rangle = 1$. Denote by $\mathfrak{m} = span\{e_0, e_3\}^\perp$. Then $\mathfrak{m} \subseteq \mathfrak{z}(\mathfrak{g})^\perp = C^1(\mathfrak{g})$, $\mathfrak{m}$ is non-degenerate and it is not difficult to see that $C^1(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{m}$. Then there exists a basis $\{e_1, e_2\}$ of $\mathfrak{m}$ such that the matrix of the metric in this basis takes one of the following forms

$$B^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad B^{1,1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad -B^0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Thus, $C^1(\mathfrak{g}) = span\{e_1, e_2, e_3\}$ and e_0 acts on $C^1(\mathfrak{g})$ by the adjoint action. Due to the ad-invariance property of $\langle \, , \, \rangle$ it follows that $\mathrm{ad}(e_0)\mathfrak{m} \subseteq \mathfrak{m}$.

Assume that $\mathfrak{m}$ has the metric given by B^0, hence $\mathrm{ad}(e_0) \in \mathfrak{so}(2)$ for B^0, implying that

$$\mathrm{ad}(e_0) = \begin{pmatrix} 0 & -\lambda \\ \lambda & 0 \end{pmatrix} \tag{5}$$

for some $\lambda \neq 0$. In the case that the metric is given by $-B^0$ the same matrix is obtained for $\mathrm{ad}(e_0)$. Similarly $\mathrm{ad}(e_0) \in \mathfrak{so}(1, 1)$ for $B^{1,1}$, implying that

$$\mathrm{ad}(e_0) = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \tag{6}$$

for some $\lambda \neq 0$.

In either case, since $\langle [e_0, e_1], e_2\rangle = \langle e_0, [e_1, e_2]\rangle$ one gets that $[e_1, e_2] = \lambda e_3$.

In the basis $\{\frac{1}{\lambda}e_0, e_1, e_2, \lambda e_3\}$, the action of $\mathrm{ad}(\frac{1}{\lambda}e_0)$ on $\mathfrak{m}$ is as in (5) taking $\lambda = 1$ while the metric obeys the rules

$$1 = \left\langle \frac{1}{\lambda}e_0, \lambda e_3 \right\rangle = \langle e_1, e_1\rangle = \langle e_2, e_2\rangle \qquad \langle e_0, e_0\rangle = \mu \in \mathbb{R} \tag{7}$$

and this is for $\mathfrak{g}_0$. In fact, in this basis the relations of (3) are verified.

In the other case, a similar reasoning gives the results of the statement, that is, one gets the basis $\{e_1, e_2, e_3\}$ for the action (6) and proceeding as above one gets the Lie algebra $\mathfrak{g}_1$ together with the ad-invariant metric given by:

$$1 = \left\langle \frac{1}{\lambda}e_0, \lambda e_3 \right\rangle = \langle e_1, e_2\rangle \qquad \langle e_0, e_0\rangle = \mu \in \mathbb{R}. \tag{8}$$

□

Remark 3. The ad-invariant metric on the Lie algebra $\mathfrak{g}_0$ (resp. $\mathfrak{g}_1$) can be taken with $\mu = 0$. In fact, it suffices to change e_0 by $\sqrt{\frac{2}{\mu}}e_0 - e_3$ whenever $\mu > 0$ and by $\sqrt{\frac{2}{-\mu}}e_0 + e_3$ if $\mu < 0$. This gives the following matrices for the ad-invariant metrics

$$\mathfrak{g}_0 : \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad \mathfrak{g}_1 : \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{9}$$

which will be used from now on.

3. Naturally reductive metrics on the Heisenberg Lie group

Let G denote a Lie group with Lie algebra $\mathfrak{g}$ and let $H < G$ be a closed Lie subgroup of G whose Lie algebra is denoted by $\mathfrak{h}$. A homogeneous pseudo-Riemannian manifold $(M = G/H, \langle\, , \rangle)$ is said to be *naturally reductive* if it is reductive, i.e. there is a reductive decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \qquad \text{with} \qquad \mathrm{Ad}(H)\mathfrak{m} \subseteq \mathfrak{m}$$

and

$$\langle [x, y]_{\mathfrak{m}}, z\rangle + \langle y, [x, z]_{\mathfrak{m}}\rangle = 0 \qquad \text{for all} \quad x, y, z \in \mathfrak{m}.$$

We shall say that a metric on M is naturally reductive if the conditions above are satisfied for some pair (G, H). If M is naturally reductive, the geodesics passing through the point $o \in M$ are

$$\gamma(t) = \exp tx \cdot o \qquad \text{for some } x \in \mathfrak{m},$$

which implies that these spaces are geodesically complete. For the Heisenberg Lie group of dimension $2n+1, \mathrm{H}_{2n+1}(\mathbb{R})$, one has the next result.

Theorem [23]. *If* $\mathrm{H}_{2n+1}(\mathbb{R})$ *is endowed with a left-invariant pseudo-Riemannian metric for which the center is non-degenerate, then this metric is naturally reductive.*

Our aim here is to characterize the Lorentzian naturally reductive metrics on the Heisenberg Lie group of dimension three. We shall prove a converse of the result above.

Theorem 3.1. *If* $\mathrm{H}_3(\mathbb{R})$ *is endowed with a naturally reductive pseudo-Riemannian left-invariant metric with pair* $(G, \mathbb{R})$ *where* G *has dimension four and* $\mathbb{R} < G$ *acts by isometric automorphisms on* $\mathrm{H}_3(\mathbb{R})$*, then the center of* $\mathrm{H}_3(\mathbb{R})$ *is non-degenerate.*

Thus, the property of the center being non-degenerate characterizes the naturally reductive metrics on $\mathrm{H}_3(\mathbb{R})$ whenever the isometries fixing a point act by isometric isomorphisms.

As known there is a one-to-one correspondence between left-invariant pseudo-Riemannian metrics on $\mathrm{H}_3(\mathbb{R})$ and metrics on the corresponding Lie algebra $\mathfrak{h}_3$, which is generated by e_1, e_2, e_3 obeying the non-trivial Lie bracket relation $[e_1, e_2] = e_3$. To prove the theorem above we start with the next result, which does not make use of any metric.

Lemma 3.2. *Let* $\mathfrak{g} = \mathbb{R}e_0 \oplus \mathfrak{h}_3$ *where the commutator* $C^1(\mathfrak{g}) \subseteq \mathfrak{h}_3$ *and the restriction of* $\mathrm{ad}(e_0)$ *to* $\mathfrak{v} = span\{e_1, e_2\}$ *is non-singular. If* $\mathfrak{m} \subset \mathfrak{g}$ *is a Lie subalgebra of* $\mathfrak{g}$ *which is isomorphic to* $\mathfrak{h}_3$ *then* $\mathfrak{m} = \mathfrak{h}_3 = span\{e_1, e_2, e_3\}$.

Proof. Let $\mathfrak{m}$ denote a subalgebra of $\mathfrak{g}$ such that $\mathfrak{m} = span\{v_1, v_2, v_3\}$ with $[v_1, v_2] = v_3$ and $[v_i, v_3] = 0$ for $i = 1, 2$. Take

$$v_1 = a_0 e_0 + w_1 + a_3 e_3 \qquad v_2 = b_0 e_0 + w_2 + b_3 e_3 \qquad v_3 = c_0 e_0 + w_3 + c_3 e_3$$

where $w_i \in span\{e_1, e_2\}$ for all $i = 1, 2, 3$. Since $C^1(\mathfrak{g}) \subseteq span\{e_1, e_2, e_3\}$ it follows that $c_0 = 0$. Let A denote the restriction of $\mathrm{ad}(e_0)$ to $\mathfrak{v}$, thus, we have the following equations

$$\begin{aligned} v_3 &= [v_1, v_2] = A(a_0 w_2 - b_0 w_1) + \omega(w_1, w_2) e_3 \\ 0 &= [v_1, v_3] = a_0 A w_3 + \omega(w_1, w_3) e_3 \\ 0 &= b_0 A w_3 + \omega(w_2, w_3) e_3. \end{aligned}$$

If either a_0 or b_0 is different from zero, then $w_3 = 0$ and so $v_3 = c_3 e_3$. Therefore $a_0 w_2 - b_0 w_1 = 0$ and so we can write w_2 in terms of w_1 or w_1 in terms of w_2 depending on whether $a_0 \neq 0$ or $b_0 \neq 0$, respectively. It is not hard to see that putting these conditions in v_1, v_2, v_3 then one gets that the set v_1, v_2, v_3 is linearly dependent which is a contradiction. So $a_0 = b_0 = 0$ and $\mathfrak{m} = span\{e_1, e_2, e_3\}$. □

Let $\mathrm{H}_3(\mathbb{R})$ denote the Heisenberg Lie group equipped with a left-invariant Lorentzian metric with non-degenerate center. Now if G is a Lie group acting by isometries on $\mathrm{H}_3(\mathbb{R})$ which is naturally reductive with pair $(G, \mathbb{R})$, then G is a semidirect extension of $\mathrm{H}_3(\mathbb{R})$ and $\mathbb{R}$ [10,11] and G admits a bi-invariant metric (according to Theorem 2.2 in [22]). Hence, the Lie algebra of G should be a solvable Lie algebra of dimension four admitting an ad-invariant metric, therefore either $\mathfrak{g}_0$ or $\mathfrak{g}_1$ of the previous section. Thus, Theorem 3.1 follows from the next result and the previous lemma.

Lemma 3.3. *Let $\mathfrak{h}_3$ denote the Heisenberg Lie algebra of dimension three equipped with a naturally reductive metric with pair $(\mathfrak{g}_i, \mathbb{R})$ i=0,1 where $\mathbb{R} \simeq \mathfrak{g}_i/\mathfrak{h}_3$ acts by skew adjoint derivations on $\mathfrak{h}_3$. Then the center of $\mathfrak{h}_3$ is non-degenerate.*

Proof. Let $v \in \mathfrak{g}_i$ be an element which is not in $span\{e_1, e_2, e_3\}$. Thus, $\mathfrak{g}_i = \mathbb{R}v \oplus \mathfrak{h}_3$ and we may assume $v = e_0 + \alpha e_1 + \beta e_2 + \gamma e_3$ and $[v, \mathfrak{h}_3] \subseteq \mathfrak{h}_3$.

For $\mathfrak{g}_0$ the action of $\mathrm{ad}(v)$ is given by

$$\mathrm{ad}(v)e_1 = e_2 - \beta e_3 \qquad \mathrm{ad}(v)e_2 = -e_1 + \alpha e_3 \qquad \mathrm{ad}(v)e_3 = 0.$$

Let Q denote a metric on $\mathfrak{h}_3$ such that $b_{ij} = Q(e_i, e_j)$ and for which $\mathrm{ad}(v)$ is skew adjoint. The condition $Q(\mathrm{ad}(v)x, y) = -Q(x, \mathrm{ad}(v)y)$ for all $x, y \in \mathfrak{h}_3$ gives rise to a system of equations on the coefficients b_{ij}:

$$\begin{array}{lll} b_{12} - \beta b_{13} = 0 & b_{22} - \beta b_{13} = b_{11} - \alpha b_{13} & b_{23} - \beta b_{33} = 0 \\ b_{12} - \alpha b_{23} = 0 & b_{13} - \alpha b_{33} = 0. & \end{array}$$

It is not hard to see that if we write $B = (b_{ij})$ then $\det B \neq 0$ implies $b_{33} \neq 0$, that is Q non-degenerate implies the center of $\mathfrak{h}_3$ non-degenerate.

This also applies for $\mathfrak{g}_1$. One writes down the action of $\mathrm{ad}(v)$ and from $Q(\mathrm{ad}(v)x, y) = -Q(x, \mathrm{ad}(v)y)$ the equations follow

$$\begin{array}{lll} b_{11} - \beta b_{13} = 0 & b_{12} - \beta b_{23} = b_{12} - \alpha b_{13} & b_{13} - \beta b_{33} = 0 \\ b_{22} - \alpha b_{23} = 0 & b_{23} - \alpha b_{33} = 0. & \end{array}$$

In this case also $b_{33} \neq 0$ says that the center of $\mathfrak{h}_3$ must be non-degenerate. □

The simply connected Lie group $\mathrm{H}_3(\mathbb{R})$ with Lie algebra $\mathfrak{h}_3$ can be realized on the usual differentiable structure of $\mathbb{R}^3$ together with the next multiplication

$$(v, z) \cdot (v', z') = \left(v + v', z + z' + \frac{1}{2} v^T J v'\right),$$

where $v, v' \in \mathbb{R}^2, v^T$ denotes the transpose matrix of the 2×1 matrix v, and J denotes the matrix given by

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

A basis of left-invariant vector fields at every point $(x, y, z) \in \mathbb{R}^3$ satisfying the non-trivial Lie bracket relation $[X_1, X_2] = X_3$ is given by

$$\begin{aligned} X_1 &= \partial_x - \frac{y}{2}\partial_z \\ X_2 &= \partial_y + \frac{x}{2}\partial_z \\ X_3 &= \partial_z. \end{aligned}$$

Two non-isometric Lorentzian metrics on $\mathrm{H}_3(\mathbb{R})$ can be taken by defining

$$1 = \langle X_1, X_1\rangle = \langle X_2, X_2\rangle = -\langle X_3, X_3\rangle \tag{10}$$

$$1 = \langle X_1, X_2\rangle = \langle X_3, X_3\rangle \tag{11}$$

and the other relations are zero. Each of them is a naturally reductive pseudo-Riemannian metric on $\mathrm{H}_3(\mathbb{R})$ with the following expression in the usual coordinates of $\mathbb{R}^3$:

$$h_1 = \Big(1 - \frac{y^2}{4}\Big)dx^2 + \Big(1 - \frac{x^2}{4}\Big)dy^2 - dz^2 + \tfrac{1}{4}xy\,dxdy - \frac{y}{2}\,dxdz + \frac{x}{2}\,dydz$$

$$h_2 = \frac{y^2}{4}\,dx^2 + \frac{x^2}{4}\,dy^2 + dz^2 + \frac{1}{4}xy\,dxdy + \frac{y}{2}\,dxdz - \frac{x}{2}\,dydz.$$

Making use of this information one can compute several geometrical features on $\mathrm{H}_3(\mathbb{R})$ [23]. Recall that an *algebraic Ricci soliton* on $\mathrm{H}_3(\mathbb{R})$ is a left-invariant pseudo-Riemannian metric such that its Ricci operator Rc satisfies the equality

$$\mathsf{Rc}(g) = c\,\mathsf{Id} + D \qquad \text{where } c \in \mathbb{R} \text{ and } D \text{ is a derivation of } \mathfrak{h}_3,$$

that is $D : \mathfrak{h}_3 \to \mathfrak{h}_3$ is a linear map which satisfies $D[x, y] = [Dx, y] + [x, Dy]$ for all $x, y \in \mathfrak{h}_3$.

A pseudo-Riemannian manifold is called *locally symmetric* if $\nabla R \equiv 0$, where ∇ denotes the covariant derivative with respect to the Levi-Civita connection and R denotes the curvature tensor. The Ambrose–Hicks–Cartan theorem (see for example [21, Thm. 17, Ch. 8]) states that given a complete locally symmetric pseudo-Riemannian manifold M, a linear isomorphism $A : T_pM \to T_pM$ is the differential of some isometry of M that fixes the point $p \in M$ if and only if it preserves the symmetric bilinear form that the metric induces into the tangent space and if for every $u, v, w \in T_pM$ the following equation holds:

$$R(Au, Av)Aw = AR(u, v)w. \tag{12}$$

While in the Riemannian case, the isometry group of a left-invariant metric on a two-step nilpotent Lie group N is the semidirect product of N and the group of isometric automorphism, the question in the pseudo-Riemannian situation is still open in the general case (see [11]). However, for a pseudo-Riemannian left-invariant metric on $\mathrm{H}_3(\mathbb{R})$ with non-degenerate center, the isometry group is the semidirect product $\mathsf{I}(\mathrm{H}_3(\mathbb{R})) = \mathrm{H}_3(\mathbb{R}) \rtimes F(\mathrm{H}_3(\mathbb{R}))$, where $F(\mathrm{H}_3(\mathbb{R}))$ denotes the isotropy subgroup at the identity, which corresponds to the isometric automorphisms, see [11].

Moreover,

- if h_0 is a flat metric on $\mathrm{H}_3(\mathbb{R})$ then $(\mathrm{H}_3(\mathbb{R}), h_0)$ is a space form and hence it is isometric to $\mathbb{R}^3_1$ [21].
- for the non-flat metrics, the action of the isotropy subgroup (of the full isometry group) at the identity element is given by isometric automorphisms [11] so that $\mathsf{I}(\mathrm{H}_3(\mathbb{R}), h_i) = \mathrm{H}_3(\mathbb{R}) \rtimes K_i$, $i = 1, 2$, where K_i denotes the group of (h_i) isometric automorphisms. In [23], this group is described.

Proposition 3.4. *The isometry groups for the Lorentzian left-invariant metrics on* $\mathrm{H}_3(\mathbb{R})$ *are given by*

- $\mathsf{I}(\mathrm{H}_3(\mathbb{R}), h_0) = \mathbb{R}^3 \rtimes \mathsf{O}(2, 1)$,

- $\mathsf{I}(\mathrm{H}_3(\mathbb{R}), h_1) = \mathrm{H}_3(\mathbb{R}) \rtimes \mathsf{O}(2)$,
- $\mathsf{I}(\mathrm{H}_3(\mathbb{R}), h_2) = \mathrm{H}_3(\mathbb{R}) \rtimes \mathsf{O}(1,1)$.

Moreover, both Lorentzian left-invariant non-flat metrics are algebraic Ricci solitons.

Proof. The description of the isometry group for a two-step nilpotent Lie group equipped with a left-invariant metric obtained in [23] and the observations above give the proofs of the isometry groups. Notice that the connected component of the identity are G_0 and G_1 for h_1 and h_2, respectively, (see the description of G_0 and G_1 in the next section).

By computing the Ricci tensor in the case of the naturally reductive metrics h_1 and h_2, one verifies that the corresponding Ricci operators satisfy

$$\mathsf{Rc}(h_1) = \mathsf{Rc}(h_2) = \frac{3}{2}\mathsf{Id} - D \tag{13}$$

where D is the derivation of $\mathfrak{h}_3$ given by

$$D(X_1) = -X_1 \qquad D(X_2) = -X_2 \qquad D(X_3) = -2X_3,$$

showing that both h_1 and h_2 are algebraic Ricci solitons. See also [5]. □

Remark 4. A left-invariant Lorentzian metric on $\mathrm{H}_3(\mathbb{R})$ is flat if and only if the center is degenerate [16]. In [24], a non-flat Lorentzian metric with degenerate center on $\mathbb{R} \times \mathrm{H}_3(\mathbb{R})$ is proved to be naturally reductive and it admits an action by isometries of the free three-step nilpotent Lie group in two generators.

Left-invariant pseudo-Riemannian metrics on two-step nilpotent Lie groups are geodesically complete [10,15].

Remark 5. Natural reductiveness of the Lorentzian metrics on $\mathrm{H}_3(\mathbb{R})$ also follows from results in [7,8].

Relative to the algebraic structure of the isometry group of $(\mathrm{H}_3(\mathbb{R}), h_0)$ usual computations show that $\mathfrak{h}_3$ is not an ideal of the Lie algebra of $\mathsf{I}(\mathrm{H}_3(\mathbb{R}), h_0)$, but $\mathsf{I}(\mathrm{H}_3(\mathbb{R}), h_0) = \mathrm{H}_3(\mathbb{R})\mathsf{O}(2,1)$.

The results of [10] are more specific for left-invariant metrics with non-degenerate center; they were improved in [11]. These observations modify the list given in [6] to obtain the present list in Proposition 3.4.

Therefore our study here revisit previous results in [6–9] giving alternative and improved proofs.

4. Simply connected solvable Lie groups with a bi-invariant metric in dimension four

Our aim now is to describe geometrical features of the simply connected solvable Lie groups of dimension four provided with a bi-invariant metric, more precisely those corresponding to the Lie algebras $\mathfrak{g}_0$ and $\mathfrak{g}_1$ described in Proposition 2.5.

Recall that if G is a connected real Lie group, its Lie algebra $\mathfrak{g}$ is identified with the Lie algebra of left-invariant vector fields on G. Assume G

is endowed with a left-invariant pseudo-Riemannian metric $\langle\,,\,\rangle$. Then the following statements are equivalent (see [21, Ch. 11]):

1. $\langle\,,\,\rangle$ is right-invariant, hence bi-invariant;
2. $\langle\,,\,\rangle$ is $\mathrm{Ad}(G)$-invariant;
3. the inversion map $g \to g^{-1}$ is an isometry of G;
4. $\langle [X,Y],Z\rangle + \langle Y,[X,Z]\rangle = 0$ for all $X,Y,Z \in \mathfrak{g}$;
5. $\nabla_X Y = \frac{1}{2}[X,Y]$ for all $X,Y \in \mathfrak{g}$, where ∇ denotes the Levi Civita connection;
6. the geodesics of G starting at the identity element e are the one parameter subgroups of G.

By (3), the pair $(G,\langle\,,\,\rangle)$ is a pseudo-Riemannian symmetric space. Furthermore, by computing the curvature tensor one has

$$R(X,Y) = -\frac{1}{4}\,\mathrm{ad}([X,Y]) \qquad \text{for } X,Y \in \mathfrak{g}. \tag{14}$$

4.1. Structure of the Lie Groups

The action of e_0 on $\mathfrak{h}_3$ on both Lie algebras $\mathfrak{g}_0$ and $\mathfrak{g}_1$, lifts to a Lie group homomorphism $\rho : \mathbb{R} \to \mathrm{Aut}(\mathrm{H}_3(\mathbb{R}))$ which on $(v,z) \in \mathbb{R}^2 \oplus \mathbb{R}$ has a matrix of the form

$$\rho(t) = \begin{pmatrix} R_i(t) & 0 \\ 0 & 1 \end{pmatrix} \qquad i = 0,1 \tag{15}$$

where

$$R_0(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \quad \text{for } \mathfrak{g}_0, \qquad R_1(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \quad \text{for } \mathfrak{g}_1. \tag{16}$$

Let G_0 and G_1 denote the simply connected Lie groups with respective Lie algebras $\mathfrak{g}_0$ and $\mathfrak{g}_1$. Then G_0 and G_1 are modeled on the smooth manifold $\mathbb{R}^4$, where the algebraic structure is the resulting from the semidirect product of $\mathbb{R}$ and $\mathrm{H}_3(\mathbb{R})$, via ρ. Thus, on G_i for $i = 0,1$, the multiplication is given by

$$(t,v,z)\cdot(t',v',z') = (t+t', v + R_i(t)v', z + z' + \frac{1}{2}v^T J R_i(t) v'). \tag{17}$$

This information is useful to find a basis of the left-invariant vector fields. For G_0 such a basis at every point $(t,x,y,z) \in \mathbb{R}^4$ is given by the following vector fields, each of them evaluated at (t,x,y,z):

$$\begin{aligned}
X_0 &= \partial_t \\
X_1 &= \cos t\,\partial_x + \sin t\,\partial_y + \frac{1}{2}(x\,\sin t - y\,\cos t)\,\partial_z \\
X_2 &= -\sin t\,\partial_x + \cos t\,\partial_y + \frac{1}{2}(x\,\cos t + y\,\sin t)\,\partial_z \\
X_3 &= \partial_z
\end{aligned}$$

and for G_1 it is given by

$$\begin{aligned} X_0 &= \partial_t \\ X_1 &= e^t\,\partial_x - \frac{1}{2} y\, e^t\,\partial_z \\ X_2 &= e^{-t}\,\partial_y + \frac{1}{2} x\, e^{-t}\,\partial_z \\ X_3 &= \partial_z. \end{aligned}$$

These vector fields verify the relations given in (3) and (4), respectively.

For every $i = 0,1$ the bi-invariant metric on G_i induced by the ad-invariant metric on $\mathfrak{g}_i$ described in (9) induces on $\mathbb{R}^4$ the next pseudo-Riemannian metric (in the usual coordinates):

$$\begin{aligned} g_0 &= dz\,dt + dx^2 + dy^2 + \frac{1}{2}(ydx\,dt - xdy\,dt) && \text{for } G_0 \\ g_1 &= dz\,dt + dx\,dy + \frac{1}{2}(ydx\,dt - xdy\,dt) && \text{for } G_1. \end{aligned}$$

4.2. Geodesics

Computing the Christoffel symbols of the Levi-Civita connection for the metrics g_0, g_1 (cf. [21]), a curve $\alpha(s) = (t(s), x(s), y(s), z(s))$ is a geodesic in G_i if its components satisfy the second-order system of differential equations:

- for G_0

$$\begin{cases} t''(s) &= 0, \\ x''(s) &= -t'(s)y'(s), \\ y''(s) &= t'(s)x'(s), \\ z''(s) &= \frac{1}{2}\, t'(s)(x(s)x'(s) + y(s)y'(s)). \end{cases}$$

- for G_1

$$\begin{cases} t''(s) &= 0, \\ x''(s) &= t'(s)x'(s), \\ y''(s) &= -t'(s)y'(s), \\ z''(s) &= -\frac{1}{2}\, t'(s)(x(s)y'(s) + y(s)x'(s)). \end{cases}$$

On the other hand, if $X_e = \sum_{i=0}^{3} a_i X_i(e) \in T_eG$, then the geodesic α through e with initial condition $\alpha'(0) = X_e$ is the integral curve of the left-invariant vector field $X = \sum_{i=0}^{3} a_i X_i$. Suppose $\alpha(s) = (t(s), x(s), y(s), z(s))$ is the curve satisfying $\alpha'(s) = X_{\alpha(s)}$, then its coordinates are as below.

- On G_0, for $a_0 \neq 0$:

$$\begin{aligned} t(s) &= a_0 s, \\ x(s) &= \frac{a_1}{a_0}\sin a_0 s + \frac{a_2}{a_0}\cos a_0 s - \frac{a_2}{a_0}, \\ y(s) &= -\frac{a_1}{a_0}\cos a_0 s + \frac{a_2}{a_0}\sin a_0 s + \frac{a_1}{a_0}, \\ z(s) &= \frac{1}{2}\left[\left(\frac{a_1^2}{a_0} + \frac{a_2^2}{a_0} + 2a_3\right) s - \left(\frac{a_2^2}{a_0^2} + \frac{a_1^2}{a_0^2}\right)\sin a_0 s\right]. \end{aligned}$$

If $a_0 = 0$, it is easy to see that $\alpha(s) = (0, a_1 s, a_2 s, a_3 s)$ is the corresponding geodesic.

- On G_1 for $a_0 \neq 0$:

$$\begin{aligned}
t(s) &= a_0 s,\\
x(s) &= \frac{a_1}{a_0} e^{a_0 s} - \frac{a_1}{a_0},\\
y(s) &= -\frac{a_2}{a_0} e^{-a_0 s} + \frac{a_2}{a_0},\\
z(s) &= \left(\frac{a_1 a_2}{a_0} + a_3\right) s - \frac{a_1 a_2}{a_0^2}\sinh(a_0 s).
\end{aligned}$$

If $a_0 = 0$ again $\alpha(s) = (0, a_1 s, a_2 s, a_3 s)$ is the corresponding geodesic. As a consequence if $X = \sum_{i=0}^{3} a_i X_i(e)$, the exponential map is

- On G_0, if $a_0 \neq 0$,

$$\exp(X) = \left(a_0, \frac{1}{a_0}(R_0(a_0)J - J)(a_1, a_2)^t, a_3 + \frac{1}{2}\left(\frac{a_1^2}{a_0} + \frac{a_2^2}{a_0}\right)\left(1 - \frac{\sin a_0}{a_0}\right)\right)$$

if $a_0 = 0$,

$$\exp(X) = (0, a_1, a_2, a_3).$$

- On G_1, if $a_0 \neq 0$

$$\exp(X) = \left(a_0, \frac{a_1}{a_0}(e^{a_0} - 1), \frac{a_2}{a_0}(1 - e^{-a_0}), \frac{a_1 a_2}{a_0} + a_3 - \frac{a_1 a_2}{a_0^2}\sinh(a_0)\right)$$

if $a_0 = 0$,

$$\exp(X) = (0, a_1, a_2, a_3).$$

In both cases the geodesic passing through the point $g \in G_i$, $i = 0, 1$ and with derivative the left-invariant vector field X, is the translation on the left of the one-parameter group at e, that is $\gamma(s) = g\exp(sX)$ for $\exp(sX)$ given above.

4.3. Isometries

Let G be a connected Lie group with a bi-invariant metric, and let $\mathsf{I}(G)$ denote the isometry group of G. This is a Lie group when endowed with the compact-open topology. Let φ be an isometry such that $\varphi(e) = x$, for $x \neq e$. Then $L_{x^{-1}} \circ \varphi$ is an isometry which fixes the element $e \in G$. Therefore $\varphi = L_x \circ f$ where f is an isometry such that $f(e) = e$. Let $F(G)$ denote the isotropy subgroup of the identity e of G and let $L(G) := \{L_g : g \in G\}$, where L_g is the translation on the left by $g \in G$. Then $F(G)$ is a closed subgroup of $\mathsf{I}(G)$ and the explanation above says

$$\mathsf{I}(G) = L(G)F(G) = \{L_g \circ f : f \in F(G), g \in G\}. \tag{18}$$

Thus, $\mathsf{I}(G)$ is essentially determined by $F(G)$.

The following lemma is proved by applying Relation (12) in the Ambrose–Hicks–Cartan Theorem to the Lie group G equipped with a bi-invariant metric and whose curvature formula was given in (14). In this way, one gets a geometric proof of the next result (see [17]).

Lemma 4.1. *Let G be a simply connected Lie group with a bi-invariant pseudo-Riemannian metric. Then a linear isomorphism $A : \mathfrak{g} \to \mathfrak{g}$ is the differential of some isometry in $F(G)$ if and only if for all $X, Y, Z \in \mathfrak{g}$, the linear map A satisfies the following two conditions:*

(i) $\langle AX, AY\rangle = \langle X, Y\rangle$;
(ii) $A[[X, Y], Z] = [[AX, AY], AZ]$.

Notice that if G is simply connected, every local isometry of G extends to a unique global one. Therefore the full group of isometries of G fixing the identity is isomorphic to the group of linear isometries of $\mathfrak{g}$ that satisfy condition (ii) of Lemma 4.1. By applying this to our case, one gets the next result.

Theorem 4.2. *Let G be a non-abelian, simply connected solvable Lie group of dimension four endowed with a bi-invariant metric. Then the group of isometries fixing the identity element $F(G)$ is isomorphic to:*

- $(\{1, -1\} \times \mathsf{O}(2)) \ltimes \mathbb{R}^2$ *for* G_0,
- $(\{1, -1\} \times \mathsf{O}(1, 1)) \ltimes \mathbb{R}^2$ *for* G_1.

In particular, the connected component of the identity of $F(G)$ coincides with the group of inner automorphisms $\{I_g : G_i \to G_i,\ I_g(x) = gxg^{-1}\}_{g\in G_i}$, for $i = 0, 1$.

Proof. We proceed with $\mathfrak{g}_0$, the case of $\mathfrak{g}_1$ follows with the same procedure.

Let $A : \mathfrak{g}_0 \to \mathfrak{g}_0$ be a linear isometry that satisfies the conditions of Lemma 4.1.

Since $C^1(\mathfrak{g}_0)$ coincides with $C^2(\mathfrak{g}_0)$ it follows that $AC^1(\mathfrak{g}_0) \subseteq C^1(\mathfrak{g}_0)$. We also have $[C^1(\mathfrak{g}_0), C^1(\mathfrak{g}_0)] = span\{e_3\}$ and from the relation $-Ae_3 = [Ae_1, [Ae_1, Ae_0]]$ one has $Ae_3 = a_{33}e_3$. Thus, we may assume that in the basis $\{e_0, e_1, e_2, e_3\}$ the map A has a matrix of the form

$$\begin{pmatrix} a_{00} & 0 & 0 & 0 \\ a_{10} & a_{11} & a_{12} & 0 \\ a_{20} & a_{21} & a_{22} & 0 \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

From $\langle Ae_0, Ae_3\rangle = 1$ it follows that

$$a_{00}a_{33} = 1. \tag{19}$$

From $\langle Ae_i, Ae_j\rangle = \delta_{ij}$, for $i, j = 1, 2$ one gets that

$$\tilde{A} := \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathsf{O}(2). \tag{20}$$

Now $A[e_0, [e_1, e_0]] = [Ae_0, [Ae_1, Ae_0]] = Ae_0$ implies

$$a_{00}^2 a_{11} = a_{11}, \qquad a_{00}^2 a_{21} = a_{21} \tag{21}$$

and

$$a_{31} = -a_{00}(a_{10}a_{11} + a_{20}a_{21}). \tag{22}$$

Equations (19), (20) and (21) assert

$$a_{00} = a_{33} = \pm 1. \tag{23}$$

Now from $A[e_0,[e_2,e_0]] = [Ae_0,[Ae_2,Ae_0]] = Ae_2$ one has

$$a_{32} = -a_{00}(a_{10}a_{12} + a_{22}a_{20}). \tag{24}$$

Set $w = (a_{10}, a_{20})^T$, from (22) and (24) it follows that $(a_{31}, a_{32}) = \mp w^T \widetilde{A}$.

Finally, the relation $\langle Ae_0, Ae_0\rangle = 0$ implies $a_{30} = \mp\frac{1}{2}||w||^2$. Therefore

$$A = \begin{pmatrix} \pm 1 & 0 & 0 \\ w & \tilde{A} & 0 \\ \mp\frac{1}{2}||w||^2 & \mp w^T\tilde{A} & \pm 1 \end{pmatrix} \tag{25}$$

where $w \in \mathbb{R}^2$ and $\tilde{A} \in \mathsf{O}(2)$. Moreover, any matrix of the form (25) verifies (i) and (ii) of Lemma 4.1. This gives a group isomorphic to $(\{1,-1\}\times\mathsf{O}(2))\ltimes\mathbb{R}^2$ for which the identity component corresponds to those matrices of the form (25) with $a_{00} = a_{33} = 1$ and $\widetilde{A} \in \mathsf{SO}(2) = \{R_0(t) : t \in \mathbb{R}\}$.

On the other hand, the set of isometric automorphisms of $\mathfrak{g}_0$ coincides with the set $\mathrm{Ad}(G_0)$, that is, the matrices of the form

$$\mathrm{Ad}(t,v) = \begin{pmatrix} 1 & 0 & 0 \\ Jv & R_0(t) & 0 \\ -\frac{1}{2}||v||^2 & -(Jv)^T R_0(t) & 1 \end{pmatrix}, \qquad v \in \mathbb{R}^2$$

being $A(t,v) = \mathrm{Ad}(t,v,z)$ for $v = (x,y)$. By dimension and since $Ad(G_0)$ is connected, it must coincide with the identity component.

The procedure for $\mathfrak{g}_1$ is the same. In this case we obtain that in the basis $\{e_0,\cdots,e_3\}$, the matrix of a linear isometry of $\mathfrak{g}_1$ that satisfies the conditions of Lemma 4.1 is of the form

$$A = \begin{pmatrix} \pm 1 & 0 & 0 \\ w & \tilde{A} & 0 \\ \mp\frac{1}{2}||w||^2 & \mp w^T\tilde{J}\tilde{A} & \pm 1 \end{pmatrix} \tag{26}$$

with $w = (x,y)^T \in \mathbb{R}^2$, $||w||^2 = 2xy$, $\tilde{A} \in \mathsf{O}(1,1)$ and $\tilde{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

The matrix $A(t,v)$ of $\mathrm{Ad}(t,v,z)$ with $v = (x,y)$ is of the form (26) with $a_{00} = 1, w = (-x,y)$ and $\tilde{A} = R_1(t)$. □

Remark 6. For G_0 compare with [3]. In [11], one can see that at the connected component of the identity one has $\mathsf{I}_0(G_0) = G_0 \rtimes \mathrm{Inn}(G_0)$ while the semidirect structure is no longer true for the full isometry group $\mathsf{I}(G_0) = G_0F(G_0)$ as in (18).

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