

Isometric actions on pseudo-Riemannian nilmanifolds

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Abstract This work deals with the structure of the isometry group of pseudo-Riemannian 2-step nilmanifolds. We study the action by isometries of several groups and we construct examples showing substantial differences with the Riemannian situation; for instance, the action of the nilradical of the isometry group does not need to be transitive. For a nilpotent Lie group endowed with a left-invariant pseudo-Riemannian metric, we study conditions for which the subgroup of isometries fixing the identity element equals the subgroup of isometric automorphisms. This set equality holds for pseudo- H -type Lie groups.

Keywords Pseudo-Riemannian nilmanifolds · Nilpotent Lie groups · Isometry groups · Bi-invariant metrics

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1 Introduction

A pseudo-Riemannian nilmanifold is a pseudo-Riemannian manifold M which admits a transitive action by isometries of a nilpotent Lie group. The interest on these manifolds has been renovated in the last years motivated by their applications not only in mathematics but also in physics (see for instance [8, 10] and references therein). A typical question is the possibility of the extension to the pseudo-Riemannian case of several properties already

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known in the Riemannian situation, a topic which could be quite complicated. Flat pseudo-Riemannian nilmanifolds were investigated in [7]; for non-flat nilmanifolds there are many open problems today.

Let M denote a Riemannian manifold with isometry group $\text{Iso}(M)$. Wolf proved in [21] that if a connected nilpotent Lie group $N \subseteq \text{Iso}(M)$ acts transitively on M then N is unique, it is the nilradical of the isometry group, and the transitive action of N is also simple. Thus, M can be identified with the nilpotent Lie group N equipped with a left-invariant metric. Furthermore, the subgroup H of isometries fixing the identity element coincides with the group H^{aut_N} of isometric automorphisms of N and therefore the isometry group is the semidirect product $\text{Iso}(M) = N \rtimes H^*$ (see [11, 20]).

Later Kaplan [13] studied other isometric actions on a family of 2-step nilmanifolds, namely on H -type Lie groups. It was shown that the group of isometric automorphisms coincides with the group of isometries of N fixing its identity element and the distribution

$$TN = \mathfrak{v}N \oplus \mathfrak{z}N. \quad (1)$$

These subbundles are obtained by left-translation of the splitting at the Lie algebra level

$$\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}, \quad (2)$$

where \mathfrak{n} is the Lie algebra of N , \mathfrak{z} denotes its center and \mathfrak{v} the orthogonal complementary subspace of \mathfrak{z} .

Our goal is to investigate some Lie groups acting by isometries on a fixed pseudo-Riemannian 2-step nilpotent Lie group: the group of isometries preserving the splitting (1), the group of isometric automorphisms and the full isometry group. We get several results concerning this topic, improving the results in [3] and we have examples showing difficulties. As noticed recently by Wolf et al. [1], the question of the structure of the nilradical of the isometry group for a pseudo-Riemannian nilmanifold is subtle.

We exhibit a 2-step nilpotent Lie group N equipped with a left-invariant Lorentzian metric such that:

- the group of isometric automorphisms is smaller than the subgroup of isometries fixing the identity element;
- the Lie group N is not normal into $\text{Iso}(N)$, hence the algebraic structure (*) does not hold;
- the action of the nilradical of $\text{Iso}(N)$ is not transitive on N .

These facts reveal remarkable differences with the Riemannian situation. For Riemannian 2-step nilmanifolds, it is known that every isometry is "compatible" with the splitting (2). Geometrically the subspace \mathfrak{v} corresponds to negative eigenvalues of the Ricci operator, while the subspace \mathfrak{z} is described by the non-negative eigenvalues. We shall see that a similar characterization cannot be achieved for a metric with signature.

Let N denote a 2-step nilpotent Lie group equipped with a pseudo-Riemannian left-invariant metric. If the center is non-degenerate one gets a decomposition of the Lie algebra \mathfrak{n} as in (2). Under these hypothesis,

- (a) the group of isometric automorphisms coincides with the group of isometries fixing the identity element and preserving the splitting (1);
- (b) we get conditions to assert the equality $H = H^{\text{aut}_N}$ so as to obtain the structure (*) for the full isometry group.

In particular, the family of pseudo- H -type Lie groups satisfies the conditions in (b). However, this does not characterize this family.

In the case of degenerate center the situation is more singular. We mainly work with bi-invariant metrics on 2-step nilpotent Lie groups, showing that (a) above does not hold: there are isometric automorphisms not preserving any kind of splitting (1). Even more, we present an example where there is no relationship between those groups.

2 Basic facts and notations

Let $(G, \langle \cdot, \cdot \rangle)$ denote a Lie group equipped with a left-invariant pseudo-Riemannian metric. In this section, we recall some definitions and properties of groups acting by isometries.

Let $\text{Iso}(G)$ denote the (full) isometry group of $(G, \langle \cdot, \cdot \rangle)$. This is a Lie group whenever it is equipped with the compact-open topology. Since left-translations are isometries, it is easy to verify that every $f \in \text{Iso}(G)$ can be written as a product

$$f = L_g \circ \varphi \tag{3}$$

where L_g denotes the translation by the element $g \in G$ and φ is an isometry fixing the identity element. The subgroup of left-translations by elements of G is closed in $\text{Iso}(G)$ and it is isomorphic to G . The subgroup of isometries fixing the identity element denoted by H is also a closed subgroup of $\text{Iso}(G)$ and due to (3) one has

$$\text{Iso}(G) = G \cdot H.$$

Let $\text{Aut}(G)$ denote the group of automorphisms of G and set $\text{Iso}^{\text{aut}}(G) = G \cdot H^{\text{aut}G}$, where $H^{\text{aut}G}$ denotes the group of isometric automorphisms of G , that is $H^{\text{aut}G} = \text{Aut}(G) \cap \text{Iso}(G)$. Since for every automorphism $\phi \in \text{Aut}(G)$ it holds $\phi \circ L_x = L_{\phi(x)} \circ \phi$, it follows that the subgroup of left-translations is a normal subgroup of the group $\text{Iso}^{\text{aut}}(G)$, thus one gets

$$\text{Iso}^{\text{aut}}(G) = G \rtimes H^{\text{aut}G}.$$

A pseudo-Riemannian manifold is called *locally symmetric* if $\nabla R \equiv 0$, where ∇ denotes the covariant derivative with respect to the Levi-Civita connection and R denotes the curvature tensor. The Ambrose–Hicks–Cartan theorem (see for example [17, Thm. 17, Ch. 8]) states that given a complete locally symmetric pseudo-Riemannian manifold M , a linear isomorphism $A : T_p M \rightarrow T_p M$ is the differential of some isometry of M that fixes the point $p \in M$ if and only if it preserves the symmetric bilinear form that the metric induces into the tangent space and if for every $u, v, w \in T_p M$ the following equation holds:

$$R(Au, Av)Aw = AR(u, v)w. \tag{4}$$

Let \mathfrak{g} denote the Lie algebra of G which is identified with the Lie algebra of left-invariant vector fields on G . Then for G connected the following statements are equivalent (see [17, Ch. 11]):

1. $\langle \cdot, \cdot \rangle$ is right-invariant, hence bi-invariant;
2. $\langle \cdot, \cdot \rangle$ is $\text{Ad}(G)$ -invariant;
3. the inversion map $g \rightarrow g^{-1}$ is an isometry of G ;
4. $\langle [u, v], w \rangle + \langle v, [u, w] \rangle = 0$ for all $u, v, w \in \mathfrak{g}$;
5. $\nabla_u w = \frac{1}{2}[u, w]$ for all $u, w \in \mathfrak{g}$, where ∇ denotes the Levi-Civita connection;
6. the geodesics of G starting at the identity element e are the one-parameter subgroups of G .

By condition 3, the pair $(G, \langle \cdot, \cdot \rangle)$ is a pseudo-Riemannian symmetric space, that is, geodesic symmetries are isometries. Usual computations show that the curvature tensor is

$$R(u, w) = -\frac{1}{4} \operatorname{ad}([u, w]) \quad \text{for } u, w \in \mathfrak{g}. \quad (5)$$

The following lemma is proved by applying the Ambrose–Hicks–Cartan theorem to the Lie group G equipped with a bi-invariant metric and whose curvature formula was given in (5) (see [15]).

Lemma 1 *Let G be a simply connected Lie group with a bi-invariant pseudo-Riemannian metric. Then a linear isomorphism $A : \mathfrak{g} \rightarrow \mathfrak{g}$ is the differential of some isometry in H if and only if for all $u, v, w \in \mathfrak{g}$, the linear map A satisfies the following two conditions:*

1. $\langle Au, Aw \rangle = \langle u, w \rangle$;
2. $A[[u, v], w] = [[Au, Av], Aw]$.

Note: A symmetric bilinear form on a Lie algebra \mathfrak{g} satisfying the condition 4 above is said to be ad-invariant. If it is non-degenerate we just call it a *metric*.

3 A homogeneous Lorentzian manifold of dimension 4

In this section, we study geometrical features of a Lorentzian manifold of dimension 4 and we show that it admits a transitive and simple action by isometries of both a solvable and a nilpotent Lie group.

Set M the pseudo-Riemannian manifold \mathbb{R}^4 with the following Lorentzian metric

$$g = dt(dz + \frac{1}{2}ydx - \frac{1}{2}xdy) + dx^2 + dy^2, \quad (6)$$

where (t, x, y, z) are usual coordinates for \mathbb{R}^4 . Denote $v = (x, y)$ and for each $(t_1, v_1, z_1) \in \mathbb{R}^4$ consider the following differentiable functions of M :

$$L_{(t_1, v_1, z_1)}^N(t_2, v_2, z_2) = \left(t_1 + t_2, v_1 + v_2, z_1 + z_2 + \frac{1}{2}v_1^t J v_2 \right) \quad (7)$$

$$L_{(t_1, v_1, z_1)}^G(t_2, v_2, z_2) = \left(t_1 + t_2, v_1 + R(t_1)v_2, z_1 + z_2 + \frac{1}{2}v_1^t J R(t_1)v_2 \right) \quad (8)$$

where J and $R(t)$ are the linear maps on \mathbb{R}^2 given by

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad R(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \quad t \in \mathbb{R}. \quad (9)$$

Both maps $L_{(t_1, v_1, z_1)}^N$ and $L_{(t_1, v_1, z_1)}^G$ are isometries of (M, g) : in fact, on the basis $\{\partial_t, \partial_x, \partial_y, \partial_z\}$ of $T\mathbb{R}^4$, one has the following differentials:

$$L_{(t_1, x_1, y_1, z_1)*}^N = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{1}{2}y_1 & \frac{1}{2}x_1 & 1 \end{pmatrix}$$

$$L_{(t_1, x_1, y_1, z_1)*}^G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos t_1 & -\sin t_1 & 0 \\ 0 & \sin t_1 & \cos t_1 & 0 \\ 0 & \mu & \nu & 1 \end{pmatrix} \quad \text{with} \quad \begin{aligned} \mu &= \frac{1}{2}(x_1 \sin t_1 - y_1 \cos t_1), \\ \nu &= \frac{1}{2}(x_1 \cos t_1 + y_1 \sin t_1). \end{aligned}$$

Thus, the maps above give isometric left-actions of a solvable Lie group G and a nilpotent Lie group N on (M, g) .

Both the Lie group G and N are modeled on \mathbb{R}^4 with its canonical differentiable structure and with the multiplication map where left-translations are induced by the maps L^G on G and L^N on N .

It is not hard to see that the actions of both groups G and N on M are simple and transitive so that the homogeneous Lorentzian manifold M can be represented as (G, g_G) and (N, g_N) , where g_G and g_N are both given by the same formula (6). Canonical computations show that the metric g_N is left-invariant on N and the metric g_G is bi-invariant on G , that is, left and right-translations by elements of G are isometries for g_G (see the previous section).

Remark 1 The exponential map from \mathfrak{g} to G is not surjective as one verifies with the formula given in Section 4.2 [6], a fact previously proved in [19]. Thus, G is a solvable Lie group which belongs to the class of examples named in Example 3.4 [1] for which there exists a pair of points that cannot be joined by an unbroken geodesic.

The Lie group G known as the oscillator group [19] underlies the Nappi–Witten space [16]. Since the Lorentzian metric g_G is bi-invariant on G , the homogeneous manifold M is symmetric.

3.1 The solvmanifold model

Making use of the model (G, g_G) one obtains the isometry group of (M, g) . Actually as an application of Lemma 1, the group H of isometries of G and hence M fixing the element $e = (0, 0, 0)$ (the identity element of G) have a differential at e with matrix of the form

$$\begin{pmatrix} \varepsilon & 0 & 0 \\ Jv & A & 0 \\ -\frac{1}{2}\|v\|^2 & -(Jv)^t A & \varepsilon \end{pmatrix} \quad \text{with } \varepsilon = \pm 1, A \in \mathbf{O}(2), v \in \mathbb{R}^2$$

where $\mathbf{O}(2)$ denotes the orthogonal group of \mathbb{R}^2 . Thus,

$$H \simeq (\{1, -1\} \times \mathbf{O}(2)) \ltimes \mathbb{R}^2$$

(see [6] for more details). Notice that H has four connected components and the connected component of the identity coincides with the group of inner automorphisms of G

$$H_0 = \{\chi_g : G \longrightarrow G, \chi_g(x) = gxg^{-1} : g \in G\} \simeq \mathbf{SO}(2) \ltimes \mathbb{R}^2.$$

Explicitly for $(t, v, z) \in \mathbb{R}^4$ and $g = (t_0, v_0, z_0)$

$$\begin{aligned} \chi_g(t, v, z) &= (t, v_0 + R(t_0)v - R(t)v_0, & (10) \\ z + \frac{1}{2}v_0^t J R(t_0)v - \frac{1}{2}v_0^t J R(t)v_0 - \frac{1}{2}(R(t_0)v)^t J R(t)v_0). \end{aligned}$$

The following diffeomorphisms

$$\begin{aligned} \psi_1(t, v, z) &= (-t, Sv, -z), \quad \text{where } S(x, y) = (-x, y) \\ \psi_2(t, v, z) &= (-t, R(-t)v, -z), \\ \psi_3(t, v, z) &= \psi_1 \circ \psi_2(t, v, z) = (t, SR(-t)v, z), \end{aligned} \tag{11}$$

constitute isometries of M fixing the element e and they belong to different connected components of H . The other three connected components of H are

$$H_0 \cdot \psi_1, \quad H_0 \cdot \psi_2 \quad \text{and} \quad H_0 \cdot \psi_3.$$

Besides, the group $H^{\text{aut}G}$ of isometric automorphisms of G is not connected since it corresponds to the connected components H_0 and $H_0 \cdot \psi_1$, where ψ_1 is as in (11). Notice that $\chi_{(-\pi,0,0)} \circ \psi_2$ is the group inversion of G .

Remark 2 Since $\psi_2 \circ L^G_{(t_1, v_1, z_1)} \circ \psi_2^{-1}$ is not a left-translation, the subgroup of left-translations G is not normal in $\text{Iso}(M)$.

Note that G is normal into $\text{Iso}_0(M)$, the connected component of the identity of $\text{Iso}(M)$, hence

$$\text{Iso}_0(M) = G \rtimes H_0.$$

The Lie algebra of $\text{Iso}(M)$ denoted by \mathfrak{iso} corresponds to the vector space spanned by $\{f_0, f_1, f_2, e_0, e_1, e_2, e_3\}$ obeying the non-trivial Lie bracket relations:

$$\begin{aligned} [f_0, f_1] &= f_2 & [f_0, f_2] &= -f_1 & [f_0, e_1] &= e_2 \\ [f_0, e_2] &= -e_1 & [f_1, e_2] &= e_3 & [f_2, e_1] &= -e_3 \\ [e_0, e_1] &= e_2 & [e_0, e_2] &= -e_1 & [e_1, e_2] &= e_3. \end{aligned}$$

3.2 The nilmanifold model

We study the structure of the isometry group of the nilmanifold M with respect to the nilpotent Lie group N .

The nilpotent Lie group N corresponds to the Lie group $\mathbb{R} \times H_3$, where H_3 denotes the Heisenberg Lie group of dimension 3. Notice that for any element $(t_1, v_1, z_1) \in \mathbb{R}^4$ it holds

$$L^G_{(t_1, v_1, z_1)} = L^N_{(t_1, v_1, z_1)} \circ \chi_{(t_1, 0, 0)} = \chi_{(t_1, 0, 0)} \circ L^N_{(t_1, R(-t_1)v_1, z_1)}, \tag{12}$$

so that the Lie algebra \mathfrak{n} viewed into \mathfrak{iso} is spanned by the vectors $f_0 - e_0, e_1, e_2, e_3$ obeying the non-zero Lie bracket relation

$$[e_1, e_2] = e_3.$$

Since $\chi_{(t_1, 0, 0)} \in H^{\text{aut}N} \subseteq H$ we have

$$\text{Iso}(M) = N \cdot H \quad \text{with } H \simeq (\{1, -1\} \times \text{O}(2)) \rtimes \mathbb{R}^2,$$

but N is not a normal subgroup of $\text{Iso}(M)$.

For the left-invariant metric given in (6) a skew-symmetric derivation D of $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$ must preserve both subspaces \mathfrak{z} and \mathfrak{v} and following canonical computations one gets the non-trivial equalities

$$De_1 = \eta e_2 \quad De_2 = -\eta e_1, \quad \eta \in \mathbb{R}.$$

So the connected component of the identity in the subgroup of isometric automorphisms of N is given by

$$H_0^{\text{aut}N} = \{\chi_{(s, 0, 0)} : s \in \mathbb{R}\}.$$

Recall from (10) that $\chi_{(s, 0, 0)}(t, v, z) = (t, R(s)v, z)$ with $R(s)$ as in (9). Furthermore, $\text{Iso}^{\text{aut}}(N)$ is not connected. In fact, the map ψ_1 defined in (11) is also an isometric automorphism of N .

We already proved that

$$\text{Iso}^{\text{aut}}(N) \subsetneq \text{Iso}(M).$$

Compare this with [3].

The nilradical of \mathfrak{iso} is the ideal of dimension 5 spanned by $\{f_1, f_2, e_1, e_2, e_3\}$ which does not contain the subalgebra \mathfrak{n} . In fact, \mathfrak{n} is not contained in the commutator $[\mathfrak{iso}, \mathfrak{iso}]$. In terms of the isometry functions, the maximal connected normal nilpotent subgroup of $\text{Iso}_0(M)$ corresponds to

$$\tilde{N} = \left\{ L_{(0,w,z)}^G \circ \chi_{(0,v,0)} : v, w, \in \mathbb{R}^2, z \in \mathbb{R} \right\}$$

and usual computations show that the orbit of a point (t_0, v_0, z_0) is given by

$$\mathcal{O}(t_0, v_0, z_0) = \{(t, v, z) \in M : t = t_0\}.$$

This proves that the action of \tilde{N} is not transitive on the nilmanifold M .

We summarize the results for (M, g) .

- N acts transitively on M but it is not a normal subgroup of $\text{Iso}_0(M)$.
- The maximal connected normal nilpotent Lie subgroup of $\text{Iso}(M)$, namely the nilradical, does not contain N and its action on M is not transitive.
- $\text{Iso}^{\text{aut}}(N) \subsetneq \text{Iso}(M)$, where $\text{Iso}^{\text{aut}}(N) = N \rtimes H^{\text{aut}N}$, $\text{Iso}(M) = N \cdot H$.
- $H^{\text{aut}N}$ is not connected.

Compare this with Section 4.2 in [21] and results in [3].

4 Isometric actions for non-degenerate center

This section concerns the study of isometric actions. Let $(N, \langle \cdot, \cdot \rangle)$ denote a simply connected 2-step nilpotent Lie group equipped with a left-invariant pseudo-Riemannian metric and let $\langle \cdot, \cdot \rangle$ denote the metric on its corresponding Lie algebra \mathfrak{n} .

Let \mathfrak{z} be the center of \mathfrak{n} and assume the restriction of $\langle \cdot, \cdot \rangle$ to \mathfrak{z} is non-degenerate. Hence there exists an orthogonal decomposition of \mathfrak{n}

$$\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z} \tag{13}$$

so that \mathfrak{v} is also non-degenerate. This induces on the Lie group N left-invariant orthogonal distributions $\mathfrak{v}N$ and $\mathfrak{z}N$ such that $TN = \mathfrak{v}N \oplus \mathfrak{z}N$.

Denote by $\text{Iso}^{\text{spl}}(N)$ the group of isometries of N that preserves the splitting $TN = \mathfrak{v}N \oplus \mathfrak{z}N$ [3,13]. Notice that left-translations by elements of the group N preserve this splitting. Thus,

$$\text{Iso}^{\text{spl}}(N) = N \cdot H^{\text{spl}}$$

where H^{spl} is the subgroup of isometries which preserve the splitting and fix the identity element of N . When the metric is positive definite, one has [9,13]:

$$\text{Iso}(N) = \text{Iso}^{\text{aut}}(N) = \text{Iso}^{\text{spl}}(N). \tag{14}$$

The purpose here was to analyze the group equalities above in the pseudo-Riemannian case and occasionally to state new relationships between these three groups.

Since the pseudo-Riemannian metric on N is invariant by left-translations, we study the geometry of N as effect from $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$.

Given $u \in \mathfrak{n}$, denote by ad_u^* the adjoint transformation of ad_u with respect to $\langle \cdot, \cdot \rangle$. One verifies that when $u \in \mathfrak{v}$ and $w \in \mathfrak{z}$ it holds $\text{ad}_u^* w \in \mathfrak{v}$, while $\text{ad}_u^* w = 0$ if $u \in \mathfrak{z}$ or $u, w \in \mathfrak{v}$. Furthermore, each $w \in \mathfrak{z}$ defines a linear transformation $j(w) : \mathfrak{v} \rightarrow \mathfrak{v}$ by

$$j(w)u = \text{ad}_u^* w \quad \text{for all } u \in \mathfrak{v},$$

so that

$$\langle j(w)u, u' \rangle = \langle w, [u, u'] \rangle \quad \text{for all } u, u' \in \mathfrak{v}. \tag{15}$$

Thus, for $w \in \mathfrak{z}$, the map $j(w)$ belongs to $\mathfrak{so}(\mathfrak{v})$, the Lie algebra of skew-symmetric maps of \mathfrak{v} with respect to $\langle \cdot, \cdot \rangle$ and one gets that $j : \mathfrak{z} \rightarrow \mathfrak{so}(\mathfrak{v})$ is a linear homomorphism. As in the Riemannian case, the maps $j(w)$ capture important geometric information of the pseudo-Riemannian space $(N, \langle \cdot, \cdot \rangle)$.

The covariant derivative ∇ relative to the Levi-Civita connection of $(N, \langle \cdot, \cdot \rangle)$ evaluated on left-invariant vector fields is

$$2 \nabla_u w = [u, w] - \text{ad}_u^* w - \text{ad}_w^* u, \quad u, w \in \mathfrak{n},$$

which together with the formula for $j : \mathfrak{z} \rightarrow \mathfrak{so}(\mathfrak{v})$ in (15) gives

$$\begin{cases} \nabla_u w = \frac{1}{2} [u, w] & \text{if } u, w \in \mathfrak{v}, \\ \nabla_u w = \nabla_w u = -\frac{1}{2} j(w)u & \text{if } u \in \mathfrak{v}, w \in \mathfrak{z}, \\ \nabla_u u' = 0 & \text{if } u, u' \in \mathfrak{z}. \end{cases} \tag{16}$$

Since for simply connected nilpotent Lie groups the exponential map $\exp : \mathfrak{n} \rightarrow N$ is a diffeomorphism, it is possible to define smooth maps $b : N \rightarrow \mathfrak{v}$ and $a : N \rightarrow \mathfrak{z}$ such that for a given $n \in N$ one writes

$$n = \exp(b(n) + a(n)). \tag{17}$$

Let $\{b_1, \dots, b_m\}$ be a basis of \mathfrak{v} and $\{a_1, \dots, a_p\}$ be a basis of \mathfrak{z} , then there are defined maps $\{\beta_i, \alpha_j : N \rightarrow \mathbb{R} : i = 1, \dots, m, j = 1, \dots, p\}$ for which

$$b(n) = \sum_{i=1}^m \beta_i(n) b_i, \quad a(n) = \sum_{j=1}^p \alpha_j(n) a_j.$$

Thus, $\varphi = (\beta_1, \dots, \beta_m, \alpha_1, \dots, \alpha_p)$ is a global coordinate system for N where at $n \in N$ it holds

$$\begin{aligned} \frac{\partial}{\partial \beta_i}|_n &= L_{n_*}|_e (b_i + \frac{1}{2} \sum_{k=1}^m [b_i, \beta_k(n) b_k]), \\ \frac{\partial}{\partial \alpha_j}|_n &= L_{n_*}|_e (a_j). \end{aligned} \tag{18}$$

To verify these equalities see formulas for the exponential map in [9].

Let $\gamma : I \rightarrow N$ be a curve on N and write b and a for the vector valued maps $\gamma(t) = \exp(b(t) + a(t))$. Making use of the equalities in (18) one gets

$$\begin{aligned} \frac{d}{dt} \gamma(t) &= \sum_{i=1}^m \dot{\beta}_i \frac{\partial}{\partial \beta_i}|_{\gamma(t)} + \sum_{j=1}^p \dot{\alpha}_j \frac{\partial}{\partial \alpha_j}|_{\gamma(t)} \\ &= L_{\gamma(t)_*}|_e \left(\sum_{i=1}^m \dot{\beta}_i b_i + \sum_{j=1}^p \left(\sum_{k=1}^m \frac{1}{2} c_{ik}^j \dot{\beta}_i \beta_k + \dot{\alpha}_j \right) a_j \right) \\ &= L_{\gamma(t)_*}|_e \left(\dot{b} + \dot{a} + \frac{1}{2} [\dot{b}, b] \right), \end{aligned}$$

where c_{ik}^j denote the structure constants for the Lie algebra. Notice that $L_{\gamma(t)_*}|_e(\dot{b})$ and $L_{\gamma(t)_*}|_e(\dot{a} + \frac{1}{2}[\dot{b}, \dot{b}])$ are the components of $d\gamma/dt$ in the bundles $\mathfrak{v}N$ and $\mathfrak{z}N$, respectively.

So the covariant derivative of $\dot{\gamma}$ along γ is given by

$$\nabla_{d\gamma/dt} d\gamma/dt = L_{\gamma(t)_*}|_e \left(\nabla_{\dot{b} + \dot{a} + \frac{1}{2}[\dot{b}, \dot{b}]} \dot{b} + \left(\dot{a} + \frac{1}{2}[\dot{b}, \dot{b}] \right) \right).$$

Denote by $\sigma = \dot{b} + \dot{a} + \frac{1}{2}[\dot{b}, \dot{b}]$ the curve in \mathfrak{n} . Then

$$\nabla_{\sigma(t)} \dot{b} = \nabla_{\sigma(t)} \left(\sum_{i=1}^m \dot{\beta}_i b_i \right) = \sum_{i=1}^m \ddot{\beta}_i b_i + \sum_{i=1}^m \dot{\beta}_i \nabla_{\sigma(t)} b_i$$

which after (16) equals

$$\nabla_{\sigma(t)} \dot{b} = \ddot{b} + \sum_{i=1}^m \dot{\beta}_i \left(\frac{1}{2}[\dot{b}, b_i] - \frac{1}{2}j \left(\dot{a} + \frac{1}{2}[\dot{b}, \dot{b}] \right) b_i \right) = \ddot{b} - \frac{1}{2}j \left(\dot{a} + \frac{1}{2}[\dot{b}, \dot{b}] \right) \dot{b}. \tag{19}$$

Similar computations give

$$\nabla_{\sigma(t)} \left(\dot{a} + \frac{1}{2}[\dot{b}, \dot{b}] \right) = \ddot{a} + \frac{1}{2}[\ddot{b}, \dot{b}] - \frac{1}{2}j \left(\dot{a} + \frac{1}{2}[\dot{b}, \dot{b}] \right) \dot{b}.$$

Therefore, a curve $\gamma : I \rightarrow N$ is a geodesic if and only if the curve $\sigma : I \rightarrow \mathfrak{n}$ satisfies $\nabla_{\sigma(t)}\sigma(t) = 0$, that is, the following system of equations is satisfied

$$\begin{cases} \ddot{b} - j(\dot{a}_0)\dot{b} = 0 \\ \dot{a} + \frac{1}{2}[\dot{b}, \dot{b}] = \dot{a}_0 \end{cases} \quad \text{where } \dot{a}_0 \text{ denotes } \dot{a}(0). \tag{20}$$

Let f be an automorphism of N . Its differential at the identity e satisfies $f_*(\mathfrak{z}) \subseteq \mathfrak{z}$ and if moreover f is an isometry its differential also preserves the orthogonal complement $\mathfrak{z}^\perp = \mathfrak{v}$: $f_*(\mathfrak{v}) \subseteq \mathfrak{v}$. In view of $f \circ L_n = L_{f(n)} \circ f$ for every $n \in N$, f preserves the invariant distributions $\mathfrak{z}N$ and $\mathfrak{v}N$ whenever it is an isometric automorphism. Therefore,

$$\text{Iso}^{\text{aut}}(N) \subseteq \text{Iso}^{\text{spl}}(N). \tag{21}$$

Proposition 1 *Let N be a simply-connected 2-step nilpotent Lie group with a left-invariant pseudo-Riemannian metric such that its center is non-degenerate. Then*

$$\text{Iso}^{\text{spl}}(N) = \text{Iso}^{\text{aut}}(N).$$

Proof In view of (21) we should prove that every isometry $f \in H^{\text{spl}}$ is an automorphism of N . The proof follows from the next equality for f , we shall prove:

$$f_*(j(u)w) = j(f_*u)f_*w \quad \text{for all } w \in \mathfrak{v}, u \in \mathfrak{z}.$$

Let $\gamma = \exp(b(t) + a(t))$ denote the geodesic throughout e such that $\dot{\gamma}(0) = w + u$, that is $\dot{b}(0) = w$ and $\dot{a}(0) = u$. Since f_* is an isometry preserving the splitting, the geodesic $\tilde{\gamma} = f \circ \gamma$ can be written as $\tilde{\gamma}(t) = \exp(f_*b(t) + f_*a(t))$. Equations (19) and (20) at $t = 0$ for both geodesics γ and $\tilde{\gamma}$ give

$$\begin{aligned}
 j(f_*u)f_*w &= \left(j(f_* \dot{a}_0)f_* \dot{b} \right)_{t=0} \\
 &= 2 \left(\nabla_{f_*(b+(a+\frac{1}{2}[b,b]))} f_* \dot{b} \right)_{t=0} \\
 &= 2 f_* \left(\nabla_{b+(a+\frac{1}{2}[b,b])} \dot{b} \right)_{t=0} \\
 &= f_*(j(u)w),
 \end{aligned}$$

as intended to prove.

Now consider $w_1, w_2 \in \mathfrak{v}$. For any $u \in \mathfrak{z}$ it holds

$$\begin{aligned}
 \langle f_*[w_1, w_2], u \rangle &= \langle [w_1, w_2], f_*^{-1}u \rangle = \langle w_2, j(f_*^{-1}u)w_1 \rangle = \\
 &= \langle w_2, j(f_*^{-1}u)(f_*^{-1}(f_*w_1)) \rangle = \langle w_2, f_*^{-1}(j(u)f_*w_1) \rangle = \\
 &= \langle f_*w_2, j(u)f_*w_1 \rangle = \langle [f_*w_1, f_*w_2], u \rangle,
 \end{aligned}$$

which together with the fact that \mathfrak{z} is non-degenerate implies that f_* is a Lie algebra automorphism. Consequently $f \in H^{\text{aut}_N}$. □

The proof above extends to the pseudo-Riemannian setting the one performed by Kaplan in [13]. Below we investigate geometrical properties of pseudo-Riemannian 2-step nilpotent Lie groups to get conditions to assert that the group of isometries of N preserving the splitting coincides with the full isometry group of N .

The Ricci tensor of (N, \langle, \rangle) can be seen at the Lie algebra level as the bilinear form on \mathfrak{n} defined throughout the curvature tensor R by

$$\text{Ric}(u, w) = \text{trace}(\xi \longrightarrow R(\xi, u)w) \quad \text{for } u, w \in \mathfrak{n}.$$

Since Ric is a symmetric form on \mathfrak{n} , there exists a linear operator $\text{Rc} : \mathfrak{n} \longrightarrow \mathfrak{n}$ such that

$$\langle \text{Rc } u, w \rangle = \text{Ric}(u, w)$$

which is called the *Ricci operator*. The *scalar curvature* s of N is the trace of the Ricci operator Rc .

In the pseudo-Riemannian case, the formulas for the Ricci operator are slightly different from those in the Riemannian case (see [9]). Recall that a basis $\{w_1, \dots, w_n\}$ of \mathfrak{n} is said to be *orthonormal* if $\langle w_i, w_j \rangle = \pm \delta_{ij}$. The proof of the next proposition follows from usual computations and it can be seen in [18].

Proposition 2 *Let (N, \langle, \rangle) denote a 2-step nilpotent Lie group equipped with a left-invariant pseudo-Riemannian metric such that the center is non-degenerate.*

1. *The Ricci operator leaves \mathfrak{v} and \mathfrak{z} invariant.*
2. *If $\{a_1, \dots, a_p\}$ is an orthonormal basis of \mathfrak{z} then*

$$\text{Rc}|_{\mathfrak{v}} = \frac{1}{2} \sum_{i=1}^p \varepsilon_k j(a_k)^2 \quad \text{with } \varepsilon_k = \langle a_k, a_k \rangle. \tag{22}$$

3. *$\text{Ric}(u, u') = -\frac{1}{4} \text{trace}(j(u)j(u'))$, for all $u, u' \in \mathfrak{z}$.*

We proceed with the study of the eigenvalues of the Ricci operator Rc . Recall that if U is a real vector space its complexification is the vector space

$$U^{\mathbb{C}} = U \otimes_{\mathbb{R}} \mathbb{C}$$

and such that $\dim_{\mathbb{R}} U = \dim_{\mathbb{C}} U^{\mathbb{C}}$. A real linear transformation T of U defines a \mathbb{C} -linear operator on $U^{\mathbb{C}}$ as $T(u \otimes z) = T(u) \otimes z$ for all $u \in U$. In addition, if $U = U_1 \oplus U_2$, then $U^{\mathbb{C}} = U_1^{\mathbb{C}} \oplus U_2^{\mathbb{C}}$ and U_i is invariant under T if and only if $U_i^{\mathbb{C}}$ is invariant under the complex transformation T .

Denote by $\lambda_1, \lambda_2, \dots, \lambda_s$ the different eigenvalues of the (complex) Ricci operator Rc . Since the metric is left-invariant, the eigenvalues of Rc are constant on N . The subspace of $\mathfrak{n}^{\mathbb{C}}$ associated to the eigenvalue λ_i is

$$V_{\lambda_i} = \ker(\text{Rc} - \lambda_i I)^{r_i}, \tag{23}$$

where r_i is the degree of λ_i in the minimal polynomial of Rc . The Jordan decomposition theorem states that

$$\mathfrak{n}^{\mathbb{C}} = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \dots \oplus V_{\lambda_s}. \tag{24}$$

Translating on the left the spaces above, at a generic point $n \in N$ one obtains

$$T_n N^{\mathbb{C}} = L_{n*}|_e V_{\lambda_1} \oplus L_{n*}|_e V_{\lambda_2} \oplus \dots \oplus L_{n*}|_e V_{\lambda_s}. \tag{25}$$

The subspace $L_{n*}|_e V_{\lambda_i}$ of $T_n N^{\mathbb{C}}$ is the one that corresponds to the eigenvalue λ_i of the Ricci tensor at n , Rc_n ; that is,

$$L_{n*}|_e V_{\lambda_i} = \ker(\text{Rc}_n - \lambda_i I)^{r_i}.$$

Given an isometry f of N , it holds

$$f_*|_n \text{Rc}_n = \text{Rc}_{f(n)} f_*|_n \quad \text{for all } n \in N, \tag{26}$$

and this formula is also valid for the corresponding complexified linear transformations. The last two equations yield

$$\begin{aligned} u \in L_{n*}|_e V_{\lambda_i} &\Leftrightarrow (\text{Rc}_n - \lambda_i I)^{r_i} u = 0 \\ &\Leftrightarrow f_*|_n ((\text{Rc}_n - \lambda_i I)^{r_i} u) = 0 \\ &\Leftrightarrow (\text{Rc}_{f(n)} - \lambda_i I)^{r_i} (f_*|_n u) = 0 \\ &\Leftrightarrow f_*|_n(u) \in L_{f(n)*}|_e V_{\lambda_i}. \end{aligned} \tag{27}$$

Therefore, the direct sum of vector spaces in (25) is preserved by isometries.

Lemma 2 *Let $(N, \langle \cdot, \cdot \rangle)$ be a 2-step nilpotent Lie group such that $\langle \cdot, \cdot \rangle$ is a pseudo-Riemannian left-invariant metric for which the center is non-degenerate. Assume*

$$\begin{aligned} \mathfrak{v}^{\mathbb{C}} &= V_{\lambda_1} \oplus \dots \oplus V_{\lambda_j}, \\ \mathfrak{z}^{\mathbb{C}} &= V_{\lambda_{j+1}} \oplus \dots \oplus V_{\lambda_s}, \end{aligned} \tag{28}$$

for the different eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_s$ of the Ricci operator Rc with V_{λ_i} the eigenspace corresponding to λ_i . Then every isometry of N preserves the splitting $TN = \mathfrak{v}N \oplus \mathfrak{z}N$, that is,

$$\text{Iso}(N) = \text{Iso}^{\text{spl}}(N).$$

Proof The hypothesis in (28) implies that for any $n \in N$,

$$\begin{aligned} \mathfrak{v}N_n^{\mathbb{C}} &= L_{n*}|_e V_{\lambda_1} \oplus \dots \oplus L_{n*}|_e V_{\lambda_j}, \\ \mathfrak{z}N_n^{\mathbb{C}} &= L_{n*}|_e V_{\lambda_{j+1}} \oplus \dots \oplus L_{n*}|_e V_{\lambda_s}. \end{aligned}$$

Let f be an isometry of N , then $f_*|_n L_n *|_e V_{\lambda_i} = L_{f(n)} *|_e V_{\lambda_i}$ as a consequence of the conditions in (27). Therefore,

$$f_*|_n(\mathfrak{v}N_n^{\mathbb{C}}) = \mathfrak{v}N_{f(n)}^{\mathbb{C}} \quad \text{and} \quad f_*|_n(\mathfrak{z}N_n^{\mathbb{C}}) = \mathfrak{z}N_{f(n)}^{\mathbb{C}}$$

from which we conclude $f_*|_n(\mathfrak{v}N_n) = \mathfrak{v}N_{f(n)}$, $f_*|_n(\mathfrak{z}N_n) = \mathfrak{z}N_{f(n)}$ and so $f \in \text{Iso}^{\text{spl}}(N)$. \square

A large family of nilpotent Lie algebras satisfying the hypothesis of the lemma above is the family of pseudo- H -type Lie algebras.¹

A nilpotent Lie algebra \mathfrak{n} (or its corresponding simply connected Lie group) equipped with a metric $\langle \cdot, \cdot \rangle$ for which the center is non-degenerate is said to be of pseudo- H -type whenever it satisfies

$$j(u)^2 = -\langle u, u \rangle I \quad \text{for all } u \in \mathfrak{z}. \tag{29}$$

Lie algebras with positive definite metric (29) are already known as H -type Lie algebras, introduced by Kaplan in [13]. H -type Lie groups are 2-step nilpotent and they are natural generalizations of the Iwasawa N -groups associated to semisimple Lie groups of real rank one.

Notice that pseudo- H -type Lie algebras are not necessarily non-singular since vectors of zero norm could satisfy (29). For any nilpotent Lie group of pseudo- H -type the three groups in (14) coincide.

Theorem 1 *Let $(N, \langle \cdot, \cdot \rangle)$ denote a pseudo- H -type Lie group. Then*

1. $\text{Iso}^{\text{aut}}(N) = \text{Iso}^{\text{spl}}(N) = \text{Iso}(N)$.
2. *the scalar curvature of $(N, \langle \cdot, \cdot \rangle)$ is negative.*

Proof The first equality in 1 holds after Proposition 1. We use the previous lemma to prove the second one. Indeed, we show that for pseudo- H -type Lie algebras, the Ricci operator is diagonalizable and negative (resp. positive) definite on \mathfrak{v} (resp. on \mathfrak{z}).

Let $\{a_k\}_{k=1}^p$ be an orthonormal basis of \mathfrak{z} . The fact of N being pseudo- H -type implies $j(a_k)^2 = -\langle a_k, a_k \rangle I_m = -\varepsilon_k I_m$ with $\varepsilon_k = \pm 1$ for $k = 1, \dots, p$ and $m = \dim \mathfrak{v}$. Then, according to (22) the Ricci operator satisfies

$$\text{Rc}|_{\mathfrak{v}} = \frac{1}{2} \sum_{k=1}^p \varepsilon_k j(a_k)^2 = -\frac{1}{2} \sum_{k=1}^p \varepsilon_k^2 I_m = -\frac{p}{2} I_m \tag{30}$$

so Rc is negative definite on \mathfrak{v} .

On the other hand for $u \in \mathfrak{z}$

$$\text{Ric}(u, u) = -\frac{1}{4} \text{trace}(j(u)^2) = \frac{1}{4} \text{trace}(\langle u, u \rangle I_m) = \frac{\langle u, u \rangle}{4} m.$$

Hence $\langle \text{Rc} u, u \rangle = \text{Ric}(u, u) = \langle \frac{m}{4} u, u \rangle$ for all $u \in \mathfrak{z}$. Polarizing this identity one gets $\langle \text{Rc} u, u' \rangle = \langle \frac{m}{4} u, u' \rangle$ for any $u, u' \in \mathfrak{z}$ and therefore $\text{Rc} = \frac{m}{4} I_p$ on \mathfrak{z} . In particular, Rc is positive definite on \mathfrak{z} .

Clearly, the eigenvalues of Rc are $\lambda_1 = -p/2$ and $\lambda_2 = m/4$ and the subspace V_{λ_i} in (23) is the eigenspace corresponding to λ_i , for each $i = 1, 2$. Moreover $\mathfrak{v} = V_{\lambda_1}$ and $\mathfrak{z} = V_{\lambda_2}$, so requirements (28) are satisfied and the first assertion of the theorem follows.

From the proof above, the trace of the Ricci operator is $s = -pm/4$, where $p = \dim \mathfrak{z}$ and $m = \dim \mathfrak{v}$, hence negative and 2 follows. \square

¹ We find this name for the first time in [2], see also [12].

A natural question at this point is the validity of the converse of the previous result. The next example gives a negative answer, that is pseudo- H -type Lie groups are not the only ones for which $\text{Iso}^{\text{aut}}(N) = \text{Iso}(N)$.

Recall that solvable Lie groups endowed with a Riemannian left-invariant metric have non-positive scalar curvature (see Theorem 6 [14]). As shown in next example, this assertion does not hold in the indefinite case.

Example 1 Let H_3 be the Heisenberg Lie group and denote with \mathfrak{h}_3 its Lie algebra which is spanned by vectors e_1, e_2, e_3 , satisfying the non-zero Lie bracket relation $[e_1, e_2] = e_3$. Consider the left-invariant metric on H_3 induced by the metric on \mathfrak{h}_3 given by

$$-\langle e_1, e_1 \rangle = \langle e_2, e_2 \rangle = \langle e_3, e_3 \rangle = 1;$$

in particular the center of \mathfrak{h}_3 is non-degenerate. After the computation of $j(e_3)$ it holds $j(e_3)^2 = I$, implying that \mathfrak{h}_3 is not a pseudo- H -type Lie algebra.

By Proposition 2, the Ricci operator satisfies $\text{Rc}|_{\mathfrak{v}} = \frac{1}{2}I$ and $\text{Rc}(e_3) = -\frac{1}{2}e_3$. So Rc is diagonalizable and the scalar curvature of $(H_3, \langle \cdot, \cdot \rangle)$ is $s = 1/2$. Also $\mathfrak{v} = V_{1/2}$ and $\mathfrak{z} = V_{-1/2}$, thus Lemma 2 leads us to

$$\text{Iso}(H_3) = \text{Iso}^{\text{aut}}(H_3).$$

In [6], it is shown that this non-flat pseudo-Riemannian nilpotent Lie group satisfies $\text{Iso}^{\text{aut}}(H_3) = H_3 \rtimes \text{O}(1, 1)$.

Remark 3 The Lorentzian nilpotent Lie group (N, g_N) in the previous section shows that in the pseudo-Riemannian case the isometries do not need to preserve the splitting $\mathfrak{v}N \oplus \mathfrak{z}N$, even when the center is non-degenerate (compare with [3]). Using Proposition 2, one gets that the Ricci tensor of this manifold is non-zero but nilpotent. Then the Lie algebra \mathfrak{n} of N is the subspace defined in (23) corresponding to the zero eigenvalue, that is $V_0 = \mathfrak{n}$. Since V_0 has non-trivial intersections with \mathfrak{v} and \mathfrak{z} , a decomposition as in (28) is not possible.

The fact that the Ricci operator is nilpotent implies that the scalar curvature of the Lie groups (G, g_G) and (N, g_N) are zero but they are not Ricci flat. Note that G is 3-step solvable (see Theorem 7 in [14]).

Once one knows that for $(N, \langle \cdot, \cdot \rangle)$ it holds $H^{\text{aut}N} = H$, the isometry group can be computed as follows. Recall that since N is simply connected, we do not distinguish between the group of automorphisms of N and of \mathfrak{n} . The isotropy group H is given by

$$H = \{(\phi, T) \in \text{O}(\mathfrak{z}, \langle \cdot, \cdot \rangle_{\mathfrak{z}}) \times \text{O}(\mathfrak{v}, \langle \cdot, \cdot \rangle_{\mathfrak{v}}) : Tj(w)T^{-1} = j(\phi w), \quad w \in \mathfrak{z}\}, \quad (31)$$

where $\text{O}(V, \langle \cdot, \cdot \rangle)$ denotes the group of isometric linear maps of $(V, \langle \cdot, \cdot \rangle)$. The Lie algebra of H is given by

$$\mathfrak{h} = \{(A, B) \in \mathfrak{so}(\mathfrak{z}, \langle \cdot, \cdot \rangle_{\mathfrak{z}}) \times \mathfrak{so}(\mathfrak{v}, \langle \cdot, \cdot \rangle_{\mathfrak{v}}) : [B, j(w)] = j(Aw), \quad w \in \mathfrak{z}\}, \quad (32)$$

where $\mathfrak{so}(V, \langle \cdot, \cdot \rangle)$ denotes the set of skew-symmetric linear maps of $(V, \langle \cdot, \cdot \rangle)$. In fact, let ψ denote an orthogonal automorphism of $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$. Thus, $\psi(\mathfrak{z}) \subseteq \mathfrak{z}$ and since $\mathfrak{v} = \mathfrak{z}^\perp$ then $\psi(\mathfrak{v}) \subseteq \mathfrak{v}$. Set $\phi := \psi|_{\mathfrak{z}}$ and $T := \psi|_{\mathfrak{v}}$, thus $(\phi, T) \in \text{O}(\mathfrak{z}, \langle \cdot, \cdot \rangle_{\mathfrak{z}}) \times \text{O}(\mathfrak{v}, \langle \cdot, \cdot \rangle_{\mathfrak{v}})$ such that

$$\begin{aligned} \langle \phi^{-1}[u, v], x \rangle &= \langle [Tu, Tv], j(x) \rangle \quad \text{if and only if} \\ \langle j(\phi x)u, v \rangle &= \langle j(x)Tu, Tv \rangle \end{aligned}$$

which implies (31). By deriving (31) one gets (32). The following proposition is the correct version of that in [18].

Proposition 3 *Let N denote a simply connected 2-step nilpotent Lie group endowed with a left-invariant pseudo-Riemannian metric, with respect to which the center is non-degenerate and such that $H = H^{\text{aut}N}$. Then the group of isometries is*

$$\text{Iso}(N) = N \rtimes H^{\text{aut}N},$$

where N acts as the group of left-translations by elements of N and the isotropy subgroup H is given by (31) with Lie algebra as in (32).

5 Isometric actions for degenerate center

Let N denote a simply connected nilpotent Lie group with Lie algebra \mathfrak{n} and set a direct sum of vector spaces

$$\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}.$$

Note that \mathfrak{v} does not necessarily coincide with \mathfrak{z}^\perp . By translations on the left, set $TN = \mathfrak{v}N \oplus \mathfrak{z}N$, a decomposition of left-invariant distributions. Define $\text{Iso}^{\text{spl}}(N)$ as the group of isometries preserving this splitting.

Assume $(N, \langle \cdot, \cdot \rangle)$ is endowed with a bi-invariant metric (in this case $\mathfrak{z}^\perp \subseteq \mathfrak{z}$). Here we show that the three groups, $\text{Iso}(N)$, $\text{Iso}^{\text{aut}}(N)$ and $\text{Iso}^{\text{spl}}(N)$, could be very different and moreover there could be no relationship between $\text{Iso}^{\text{aut}}(N)$ and $\text{Iso}^{\text{spl}}(N)$.

Lemma 3 *Let $(N, \langle \cdot, \cdot \rangle)$ be a 2-step nilpotent Lie group equipped with a bi-invariant metric, let \mathfrak{n} denote its Lie algebra. Then for any direct sum decomposition $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$, the automorphism $\text{Ad}(n)$ does not preserve \mathfrak{v} , whenever n is a non-central element.*

Proof Since the Lie group is 2-step nilpotent, for any $n \in N$ there exists $w \in \mathfrak{n}$ such that $n = \exp(w)$ and $\text{Ad}(n) = I + \frac{1}{2} \text{ad}_w$. Consider a direct sum as vector spaces of the form $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$. Suppose n is non-central then $n = \exp(w)$ with $w \notin \mathfrak{z}$ and let $u \in \mathfrak{n}$ be such that $[w, u] \neq 0$. Write $u = b + a$ with $a \in \mathfrak{z}$ and $b \in \mathfrak{v} - \{0\}$. Then $\text{Ad}(n)(b) = b + \frac{1}{2}[w, b] = b + \frac{1}{2}[w, u]$ having non-zero component on \mathfrak{z} . \square

Remark 4 Let N denote a pseudo-Riemannian nilpotent Lie group admitting a lattice Γ such that $\Gamma \backslash N$ is compact. Let $\text{Iso}_0^{\text{spl}}(\Gamma \backslash N)$ denote the connected component of the subgroup of isometries preserving a fixed splitting of $T(\Gamma \backslash N)$. Cordero and Parker [4] conclude that $\text{Iso}_0^{\text{spl}}(\Gamma \backslash N) \simeq T^m$ is an m -dimensional torus. Its proof should make use of the fact that there are no non-trivial inner automorphisms in this group (Lemma 3).

The fact that N is endowed with a bi-invariant metric implies that the conjugation map χ_n given by $\chi_n(g) = ngn^{-1}$ is an isometry for all $n \in N$. The lemma above says that none of these automorphisms preserve any possible splitting $TN = \mathfrak{v}N \oplus \mathfrak{z}N$ with the exception of the trivial ones, thus $\text{Iso}^{\text{spl}}(N) \neq \text{Iso}^{\text{aut}}(N)$ and $\text{Iso}^{\text{spl}}(N) \neq \text{Iso}(N)$ since N is non-abelian.

As already mentioned for bi-invariant metrics, the map: $g \rightarrow g^{-1}$ is an isometry which is not an automorphism, unless the group is abelian. Thus $\text{Iso}^{\text{spl}}(N) \neq \text{Iso}^{\text{aut}}(N)$. As stated above, the group $\text{Iso}^{\text{aut}}(N)$ is not contained into the group $\text{Iso}^{\text{spl}}(N)$. The next example shows that $\text{Iso}^{\text{spl}}(N)$ could not be contained in $\text{Iso}^{\text{aut}}(N)$.

Example 2 Consider \mathbb{R}^6 with the canonical differentiable structure and let g denote the following pseudo-Riemannian metric on \mathbb{R}^6 :

$$g = dx_1 dx_6 + dx_3 dx_4 - dx_2 dx_5. \quad (33)$$

This manifold admits an isometric transitive and simple action of the 2-step nilpotent Lie group N which is modeled on \mathbb{R}^6 with multiplication operation such that for $p = (x_1, x_2, x_3, x_4, x_5, x_6)$ and $q = (y_1, y_2, y_3, y_4, y_5, y_6)$, it holds

$$p \cdot q = \left(x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4 + \frac{1}{2}(x_1y_2 - x_2y_1), \right. \\ \left. x_5 + y_5 + \frac{1}{2}(x_1y_3 - x_3y_1), x_6 + y_6 + \frac{1}{2}(x_2y_3 - x_3y_2) \right). \tag{34}$$

The corresponding metric on N induced by (33) is invariant under left- and right-translations, hence g is a bi-invariant (pseudo-Riemannian) metric on N . Its Lie algebra \mathfrak{n} admits a basis $\{e_i\}_{i=1}^6$ obeying the non-zero Lie bracket relations

$$[e_1, e_2] = e_4, \quad [e_1, e_3] = e_5, \quad [e_2, e_3] = e_6.$$

The Lie algebra \mathfrak{n} is the free 2-step nilpotent Lie algebra on three generators and N is its corresponding simply connected Lie group. At the Lie algebra level, the bi-invariant metric (33) induces the ad-invariant metric verifying [5]

$$\langle e_1, e_6 \rangle = \langle e_3, e_4 \rangle = -\langle e_2, e_5 \rangle = 1.$$

The center \mathfrak{z} of \mathfrak{n} is spanned by e_4, e_5, e_6 and it is totally isotropic ($\mathfrak{z}^\perp = \mathfrak{z}$). Moreover,

$$\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z} \tag{35}$$

where $\mathfrak{v} = \text{span}\{e_1, e_2, e_3\}$ is also a totally isotropic subspace. The splitting of TN by left-invariant distributions associated to (35) at each point p of N is given by

$$\mathfrak{v}N_p = \text{span}\{X_1|_p, X_2|_p, X_3|_p\} \quad \text{and} \quad \mathfrak{z}N_p = \text{span}\{X_4|_p, X_5|_p, X_6|_p\}$$

where X_i is the left-invariant vector field of N such that $X_i|_0 = e_i$. Canonical computations show that $\langle \partial_i, \partial_j \rangle = \langle X_i, X_j \rangle$ for all i, j , where $\partial_i = \frac{\partial}{\partial x_i}$.

For each $\tau \in \mathbb{R}$ consider the diffeomorphism F^τ , defined as

$$F^\tau(x_1, x_2, x_3, x_4, x_5, x_6) = (\cosh \tau x_1 + \sinh \tau x_3, x_2, \sinh \tau x_1 + \cosh \tau x_3, \\ \cosh \tau x_4 - \sinh \tau x_6, x_5, -\sinh \tau x_4 + \cosh \tau x_6)$$

which is an isometry of N . Indeed, its differential at p , in the ordered basis $\{\partial_1, \partial_3, \partial_4, \partial_6, \partial_2, \partial_5\}$ of T_pN , is

$$dF_p^\tau = \begin{pmatrix} A & 0 & 0 \\ 0 & A^{-1} & 0 \\ 0 & 0 & I_{2 \times 2} \end{pmatrix}, \quad \text{where} \quad A = \begin{pmatrix} \cosh \tau & \sinh \tau \\ \sinh \tau & \cosh \tau \end{pmatrix} \in \mathcal{O}(1, 1).$$

The map F^τ preserves the metric in (33) for all $\tau \in \mathbb{R}$. Moreover, it preserves the splitting $TN = \mathfrak{v}N \oplus \mathfrak{z}N$. In fact $dF_p^\tau(\mathfrak{z}N_p) = \mathfrak{z}N_{F^\tau(p)}$ and

$$dF_p^\tau X_1|_p = \cosh \tau X_1|_{F^\tau(p)} + \sinh \tau X_3|_{F^\tau(p)} \in \mathfrak{v}N_{F^\tau(p)}, \\ dF_p^\tau X_2|_p = X_2|_{F^\tau(p)} \in \mathfrak{v}N_{F^\tau(p)}, \\ dF_p^\tau X_3|_p = \sinh \tau X_1|_{F^\tau(p)} + \cosh \tau X_3|_{F^\tau(p)} \in \mathfrak{v}N_{F^\tau(p)}.$$

Therefore $dF_p^\tau(\mathfrak{v}N_p) = \mathfrak{v}N_{F^\tau(p)}$ and $F^\tau \in \text{Iso}^{\text{spl}}(N)$

Finally, notice that dF_0^τ is not a Lie algebra isomorphism if $\tau \neq 0$, therefore $F^\tau \notin \text{Iso}^{\text{aut}}(N)$ for $\tau \neq 0$. Furthermore, $\text{Iso}(N) = N \cdot \mathcal{O}(3, 3)$ by Lemma 1.

We have already proved that on the Lie group N of the previous example, there are isometries preserving a fixed splitting which are not automorphisms.

Proposition 4 *Let $(N, \langle \cdot, \cdot \rangle)$ denote a 2-step nilpotent Lie group endowed with a bi-invariant metric. Then the center is degenerate [5] and*

- $\text{Iso}^{\text{spl}}(N) \neq \text{Iso}^{\text{aut}}(N)$;
- $\text{Iso}^{\text{spl}}(N) \subsetneq \text{Iso}(N)$ and $\text{Iso}^{\text{aut}}(N) \subsetneq \text{Iso}(N)$.

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