The effect of finite rank perturbations on Jordan chains of linear operators

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Abstract

A general result on the structure and dimension of the root subspaces of a linear operator under finite rank perturbations is proved: The increase of dimension from the n-th power of the kernel of the perturbed operator to the $(n+1)$-th power differs from the increase of dimension of the corresponding powers of the kernels of the unperturbed operator by at most the rank of the perturbation. This bound is sharp.

1 Introduction

Perturbation theory for linear operators and their spectra is one of the main objectives in operator theory and functional analysis, with numerous applications in mathematics, physics and engineering sciences. In many approaches compact perturbations and perturbations small in size are investigated, e.g. when stability properties of the index, nullity and deficiency of Fredholm and semi-Fredholm operators are analysed. A widely used and well-known fact on the effect of compact perturbations is the following: If S and T are bounded operators in a Banach space, $K = S - T$ is compact and $\lambda \in \mathbb{C}$ is such that $S - \lambda$ is Fredholm, then also $T - \lambda$ is Fredholm and the Fredholm index is preserved. In particular, since $\ker(S - \lambda)$ and $\ker(S - \lambda)^{n+1}/\ker(S - \lambda)^n$ are finite dimensional the same is true for $\ker(T - \lambda)$ and $\ker(T - \lambda)^{n+1}/\ker(T - \lambda)^n$. However, for such an arbitrary compact perturbation K there exists no bound on the dimensions of $\ker(T - \lambda)$ or $\ker(T - \lambda)^{n+1}/\ker(T - \lambda)^n$. The situation is different when the perturbation is not only compact but of finite rank.

In the present note we consider general linear operators S and T in a vector space X such that T is a finite rank perturbation of S. It follows easily that the dimensions of $\ker(S - \lambda)$ and $\ker(T - \lambda)$ differ at most by k if the perturbation $K = S - T$ is an operator with $\operatorname{rank}(K) = k$. Our main objective is to explore

the connections between the kernels of consecutive higher powers of $S - \lambda$ and $T - \lambda$ in more detail, and to prove the following general result on the structure and dimensions of the root subspaces under finite rank perturbations: Given a linear operator S in X, consider the space $\ker(S - \lambda)^{n+1}/\ker(S - \lambda)^n$. Its dimension coincides with the number of linearly independent Jordan chains of S at λ of length at least $n + 1$. It then turns out that the change of the number of these Jordan chains of S at λ under a rank k perturbation is bounded by k,

$$\left| \dim \left(\frac{\ker(S - \lambda)^{n+1}}{\ker(S - \lambda)^n} \right) - \dim \left(\frac{\ker(T - \lambda)^{n+1}}{\ker(T - \lambda)^n} \right) \right| \leq k, \qquad (1.1)$$

and this bound is sharp, see Theorem 2.2 and Example 2.3. Here S and T are defined on subspaces of X and the finite rank perturbation is interpreted in a generalized sense, see Hypothesis 2.1. In particular, our assumptions allow to treat unbounded operators in Banach spaces and finite rank perturbations in resolvent sense. We also emphasize that the dimensions of the root subspaces of the operators S and T may be infinite, and that a finite rank perturbation may turn points from the resolvent set of S into eigenvalues of infinite algebraic multiplicity of T; cf. Example 2.5.

If X is finite dimensional, then S and T are matrices and (1.1) was already proved by S.V. Savchenko in [10, Lemma 2], see also [1, 2, 5, 6, 7, 8, 9] for related results on so-called generic perturbations of matrices. Moreover, there exists a lower bound for the dimension of the root subspace of the perturbed operator T in terms of the dimension of the root subspace of S and the length of the Jordan chains of S at λ; cf. [3, 10]. Such a result was also proved by L. Hörmander and A. Melin in a more general case: the unperturbed operator S is compact and the perturbation $K = T - S$ is of finite rank, see [4, Theorem 3]. In Corollary 2.6 we obtain the same bound for the general setting considered here.

2 Main result

Let X be a vector space over $\mathbb{K}$, where $\mathbb{K}$ stands either for $\mathbb{R}$ or $\mathbb{C}$. Let S and T be linear operators in X defined on some linear subspaces $\operatorname{dom} S$ and $\operatorname{dom} T$ of X, respectively. We consider finite rank perturbations in the following generalized sense:

Hypothesis 2.1. *There exists a linear subspace M contained in $\operatorname{dom} S \cap \operatorname{dom} T$ such that the restrictions $S \restriction M$ and $T \restriction M$ coincide on M and*

$$\max\big\{\dim(\operatorname{dom} S/M), \dim(\operatorname{dom} T/M)\big\} = k < \infty.$$

Three typical situations where the above hypothesis is satisfied are the following:

(i) X is a finite dimensional space, S and T are defined on X and the rank of $S - T$ is k. In this case, for a fixed basis of X, S and T are represented by matrices.

(ii) If X is an arbitrary vector space, $\operatorname{dom} S = \operatorname{dom} T$ and

$$\dim(\operatorname{ran}(S - T)) = k.$$

(iii) X is a Banach space, S and T are densely defined closed operators in X, and there exists $\mu \in \mathbb{K}$ in the resolvent set of S and T with

$$\dim\big(\operatorname{ran}\big((S - \mu)^{-1} - (T - \mu)^{-1}\big)\big) = k.$$

Given $\lambda \in \mathbb{K}$, a finite ordered set of non-zero vectors $\{x_0, \ldots, x_{n-1}\}$ in $\operatorname{dom} S$ is a *Jordan chain of length n* at λ if $(S - \lambda)x_0 = 0$ and $(S - \lambda)x_i = x_{i-1}$, $i = 1, \ldots, n - 1$. A Jordan chain of infinite length is defined accordingly. The elements of a Jordan chain are linearly independent. The first $n - 1$ elements of a Jordan chain of length n form a Jordan chain of length $n - 1$. Furthermore, we say that S has k Jordan chains of length n at λ if there exist k linearly independent Jordan chains of length n. The *root subspace $\mathcal{L}_\lambda(S)$ of S at λ* is the collection of all Jordan chains of S at λ,

$$\mathcal{L}_\lambda(S) = \bigcup_{j=1}^{\infty} \ker(S - \lambda)^j.$$

The following theorem is the main result of this article. In the special case that X is finite dimensional it coincides with [10, Lemma 2]. The proof of Theorem 2.2 is given in Section 4.

Theorem 2.2. *Let S and T be linear operators in X satisfying Hypothesis 2.1. Then, the following holds for every $\lambda \in \mathbb{K}$:*

(i) *If $\ker(S - \lambda)^n$ is finite dimensional for some $n \in \mathbb{N}$, then the same holds for $\ker(T - \lambda)^n$ and*

$$\big|\dim \ker(S - \lambda)^n - \dim \ker(T - \lambda)^n\big| \leq k\,n. \tag{2.1}$$

(ii) *If $\ker(S - \lambda)^{n+1}/\ker(S - \lambda)^n$ is finite dimensional for some $n \in \mathbb{N}$, then the same holds for $\ker(T - \lambda)^{n+1}/\ker(T - \lambda)^n$ and*

$$\left|\dim\left(\frac{\ker(S - \lambda)^{n+1}}{\ker(S - \lambda)^n}\right) - \dim\left(\frac{\ker(T - \lambda)^{n+1}}{\ker(T - \lambda)^n}\right)\right| \leq k. \tag{2.2}$$

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The estimates in Theorem 2.2 are sharp in the following sense.

Example 2.3. *In $X = \mathbb{K}^m$ consider a fixed basis $\{e_1, \ldots, e_m\}$ and, with respect to this basis, let the linear operators A_1 and B_1 be given via their $m \times m$ matrix-representation*

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \quad and \quad B_1 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Then A_1 and B_1 satisfy Hypothesis 2.1 with $k = 1$ and $M = \operatorname{span}\{e_2, \ldots, e_m\}$, and we have for $j \leq m$

$$\ker A_1^j = \operatorname{span}\left\{e_1, \ldots, e_j\right\} \quad and \quad \ker B_1^j = \{0\}.$$

Hence the assertions in Theorem 2.2 are sharp for the case $\lambda = 0$ and $k = 1$. In order to obtain sharpness for general $k \in \mathbb{N}$ consider the $(mk \times mk)$-matrices in X^k,

$$A = A_1 \oplus \cdots \oplus A_1 \quad and \quad B = B_1 \oplus \cdots \oplus B_1.$$

In the following corollary the bounds in Theorem 2.2 are considered in the context of the dimensions of the root subspaces.

Corollary 2.4. *Let S and T be linear operators in X satisfying Hypothesis 2.1. Assume that the root subspace $\mathcal{L}_\lambda(S)$ of S at $\lambda \in \mathbb{K}$ is finite dimensional. Then, the following holds:*

(i) *If the maximal length of Jordan chains of S at λ is bounded by p then*

$$\left| \dim \mathcal{L}_\lambda(S) - \dim \ker(T - \lambda)^p \right| \leq k\, p.$$

(ii) *If the maximal lengths of Jordan chains of S at λ and Jordan chains of T at λ are bounded by p and q, respectively, then $\mathcal{L}_\lambda(T)$ is finite dimensional and*

$$\left| \dim \mathcal{L}_\lambda(S) - \dim \mathcal{L}_\lambda(T) \right| \leq k\, \max\{p, q\}.$$

Proof. In (i) we have $\mathcal{L}_\lambda(S) = \ker(S - \lambda)^p$. In (ii) we have, in addition, $\mathcal{L}_\lambda(T) = \ker(T - \lambda)^q$. Then (i) and (ii) follow from (2.1). $\square$

We emphasize that in (i) of Corollary 2.4 (where it is assumed that $\mathcal{L}_\lambda(S)$ is finite dimensional) the root subspace $\mathcal{L}_\lambda(T)$ may be infinite dimensional. This will be illustrated by the following example, where a rank one perturbation of a bijective operator generates an infinitely long Jordan chain.

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Example 2.5. *Let $X = \ell^2(\mathbb{N}) \times \ell^2(\mathbb{N})$ and consider the following operators S and T in X:*

$$S\left(\begin{matrix}(x_n)_{n\in\mathbb{N}}\\(y_n)_{n\in\mathbb{N}}\end{matrix}\right) := \left(\begin{matrix}(y_1, x_1, x_2, \dots)\\(y_2, y_3, y_4, \dots)\end{matrix}\right),$$

$$T\left(\begin{matrix}(x_n)_{n\in\mathbb{N}}\\(y_n)_{n\in\mathbb{N}}\end{matrix}\right) := \left(\begin{matrix}(0, x_1, x_2, \dots)\\(y_2, y_3, y_4, \dots)\end{matrix}\right).$$

It is clear that the operator $S - T$ is of rank one, and

$$\ker S = \{0\}.$$

On the other hand T has a Jordan chain at 0 of infinite length, which is given by $\{\left(\begin{smallmatrix}0\\e_n\end{smallmatrix}\right) : n \geq 1\}$ with $\{e_n : n \geq 1\}$ denoting the standard basis in ℓ^2. Hence,

$$\dim \ker T^p = p \quad \text{and} \quad \dim \mathcal{L}_0(T) = \infty.$$

The bound in Corollary 2.4 (ii) can be improved if the number k from Hypothesis 2.1 is small compared to the number of linearly independent Jordan chains of S. The following corollary was obtained in [3, 10] for matrices and in [4, Theorem 3] for compact operators. The proof of Corollary 2.6 below is omitted since it follows the same arguments as the proof of [10, Corollary 1].

Corollary 2.6. *Let S and T be linear operators in X satisfying Hypothesis 2.1. Assume that the root subspace $\mathcal{L}_\lambda(S)$ of S at $\lambda \in \mathbb{K}$ is finite dimensional and let $n_1 \geq n_2 \geq \cdots \geq n_l$ be the lengths of the linearly independent Jordan chains of S at λ. Then, for $k \leq l$ the following holds:*

$$\dim \mathcal{L}_\lambda(S) - n_1 - n_2 - \cdots - n_k \leq \dim \mathcal{L}_\lambda(T). \tag{2.3}$$

We mention that in the situation of Corollary 2.6 the root subspace $\mathcal{L}_\lambda(T)$ may be infinite dimensional (see Example 2.5), and, in this case, the right hand side of (2.3) is ∞.

3 Preparatory statements

In this section we prove Theorem 2.2 for the special case $k = 1$. Notice that it suffices to prove the result for $\lambda = 0$; otherwise replace S and T by $S - \lambda$ and $T - \lambda$. Theorem 2.2 in this situation is formulated below in Proposition 3.3. As a preparation we state two simple lemmas. The first is an immediate consequence of the fact that S and T coincide on the subspace M; cf. Hypothesis 2.1.

Lemma 3.1. *Let S and T be linear operators in X satisfying Hypothesis 2.1. If $\{x_0, \dots, x_n\}$ is a Jordan chain of S at λ such that $x_k \in M$ for every $k = 0, \dots, n$, then $\{x_0, \dots, x_n\}$ is also a Jordan chain of T at λ.*

The next lemma follows from the fact that for a linear operator A in X the mapping $x + \ker A \mapsto Ax$, is an isomorphism between $X/\ker A$ and $\operatorname{ran} A$.

Lemma 3.2. *For a linear operator A in X the set $\{x_1 + \ker A, \ldots, x_m + \ker A\}$ is linearly independent in $X/\ker A$ if and only if the set $\{Ax_1, \ldots, Ax_m\}$ is linearly independent in X.*

The next proposition is Theorem 2.2 in the special case $k = 1$ and $\lambda = 0$.

Proposition 3.3. *Let S and T be linear operators in X satisfying Hypothesis 2.1 with $k = 1$. Then the following holds:*

(i) *If $\ker S^n$ is finite dimensional for some $n \in \mathbb{N}$, $n \geq 1$, then the same holds for $\ker T^n$ and*

$$|\dim \ker S^n - \dim \ker T^n| \leq n. \tag{3.1}$$

(ii) *If $\ker S^{n+1}/\ker S^n$ is finite dimensional for some $n \in \mathbb{N}$, $n \geq 1$, then the same holds for $\ker T^{n+1}/\ker T^n$ and*

$$\left|\dim\left(\ker S^{n+1}/\ker S^n\right) - \dim\left(\ker T^{n+1}/\ker T^n\right)\right| \leq 1. \tag{3.2}$$

Proof. First, we show (i) for the case $n = 1$, i.e.

$$|\dim \ker S - \dim \ker T| \leq 1. \tag{3.3}$$

Assume that $\ker S$ is finite dimensional and $\dim \ker T > \dim \ker S + 1$. Then there exist $m := \dim \ker S + 2$ linearly independent vectors $\{x_1, \ldots, x_m\}$ in $\ker T$. If $x_j \in M$ then $Sx_j = Tx_j$. So, if $x_j \in M$ for all $j = 1, \ldots, m$ then $\{x_1, \ldots, x_m\} \subseteq \ker S$, a contradiction.

Hence, there exists $1 \leq k_0 \leq m$ such that $x_{k_0} \in \ker T \setminus M$. After reordering we can assume that $k_0 = m$. As $\dim(\mathrm{dom}\, T/M) \leq 1$ it is easy to see that there exist $\alpha_k \in \mathbb{K}$ such that

$$z_k := x_k - \alpha_k x_m \in M, \qquad k = 1, \ldots, m - 1.$$

Thus $Sz_k = Tz_k = 0$ for $k = 1, \ldots, m - 1$, and we conclude that $\{z_1, \ldots, z_{m-1}\}$ is a linearly independent set in $\ker S$; a contradiction. Therefore, $\dim \ker T \leq \dim \ker S + 1$ and, in particular, $\ker T$ is finite dimensional. By interchanging S and T we also obtain $\dim \ker S - 1 \leq \dim \ker T$ and hence (3.3) follows.

In the following we prove (ii). Let $n \in \mathbb{N}$, $n \geq 1$, such that $\ker S^{n+1}/\ker S^n$ is finite dimensional and set

$$m := \dim(\ker S^{n+1}/\ker S^n) + 2. \tag{3.4}$$

Assume that the set $\{x_{1,n} + \ker T^n, \ldots, x_{m,n} + \ker T^n\}$ is linearly independent in $\ker T^{n+1}/\ker T^n$. For $k = 1, \ldots, m$ construct the following Jordan chains of T at 0:

$$x_{k,0} := T^n x_{k,n}, \quad x_{k,1} := T^{n-1} x_{k,n}, \quad \ldots, \quad x_{k,n-1} := T x_{k,n}.$$

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Then, $x_{k,0} \in \ker T$ for $k = 1, \ldots, m$ and, applying Lemma 3.2 to T^n it follows that

$$\{x_{1,0}, \ldots, x_{m,0}\} \text{ is a linearly independent set in } \ker T. \qquad (3.5)$$

Define the index set $\mathfrak{I}$ by

$$\mathfrak{I} := \{(k, j): \ x_{k,j} \notin M, 1 \leq k \leq m, 0 \leq j \leq n\}.$$

The set $\mathfrak{I}$ is non-empty. Otherwise $\{x_{k,0}, \ldots x_{k,n}\} \subset M$ for every $1 \leq k \leq m$ and, by Lemma 3.1, these m (linearly independent) Jordan chains of T at 0 of length $n+1$ are as well (linearly independent) Jordan chains of S at 0 of length $n+1$, a contradiction to (3.4). Set

$$h := \min\{j: \ (k, j) \in \mathfrak{I} \text{ for some } k \text{ with } 1 \leq k \leq m\}.$$

Without loss of generality, after a reordering of the indices, assume that $(m, h) \in \mathfrak{I}$, i.e. $x_{m,h} \notin M$. Then,

$$j < h \text{ implies } x_{k,j} \in M \text{ for all } k = 1, \ldots, m. \qquad (3.6)$$

In what follows we construct $m-1$ elements $z_1, \ldots, z_{m-1}$ in $\ker S^{n+1}$ such that $\{z_1 + \ker S^n, \ldots, z_{m-1} + \ker S^n\}$ is linearly independent in $\ker S^{n+1}/\ker S^n$, which is a contradiction to (3.4). We consider three different cases.

Case I: $h = n$. Since $x_{m,n} \notin M$, there exist $\alpha_{k,n} \in \mathbb{K}$ such that

$$z_k := x_{k,n} - \alpha_{k,n} x_{m,n} \in M \cap \ker T^{n+1} \quad \text{for } k = 1, \ldots, m-1.$$

From (3.6) it follows that, for every $k = 1, \ldots, m-1$, the Jordan chain $\{x_{k,0} - \alpha_{k,n} x_{m,0}, \ldots, x_{k,n-1} - \alpha_{k,n} x_{m,n-1}, z_k\}$ of T at 0 is contained in M. Then, by Lemma 3.1 these are also $m-1$ (linearly independent) Jordan chains of S at 0 of length n. In particular, the set $\{z_1 + \ker S^n, \ldots, z_{m-1} + \ker S^n\}$ is linearly independent in $\ker S^{n+1}/\ker S^n$.

Case II: $h = n - 1$. Since $x_{m,n-1} \notin M$, there exist $\alpha_{k,n-1} \in \mathbb{K}$ such that

$$v_{k,n-1} := x_{k,n-1} - \alpha_{k,n-1} x_{m,n-1} \in M \cap \ker T^n \quad \text{for } k = 1, \ldots, m-1.$$

Let $w_{k,n} := x_{k,n} - \alpha_{k,n-1} x_{m,n} \in \ker T^{n+1}$ for $k = 1, \ldots, m-1$ and choose $\alpha_{k,n} \in \mathbb{K}$ such that

$$z_k := w_{k,n} - \alpha_{k,n} x_{m,n-1} \in M \cap \ker T^{n+1} \quad \text{for } k = 1, \ldots, m-1.$$

Since $z_k \in M$ and $v_{k,n-1} \in M$, $k = 1, \ldots, m-1$, we conclude from $T w_{k,n} =$

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$v_{k,n-1}$ together with (3.6) that

$$
\begin{aligned}
S^{n+1} z_k &= S^n S z_k = S^n T z_k \\
&= S^n T(w_{k,n} - \alpha_{k,n} x_{m,n-1}) = S^n(v_{k,n-1} - \alpha_{k,n} x_{m,n-2}) \\
&= S^{n-1} T(v_{k,n-1} - \alpha_{k,n} x_{m,n-2}) \\
&= S^{n-1} T(x_{k,n-1} - \alpha_{k,n-1} x_{m,n-1} - \alpha_{k,n} x_{m,n-2}) \\
&= S^{n-1}(x_{k,n-2} - \alpha_{k,n-1} x_{m,n-2} - \alpha_{k,n} x_{m,n-3}) \\
&\ \ \vdots \\
&= S^2(x_{k,1} - \alpha_{k,n-1} x_{m,1} - \alpha_{k,n} x_{m,0}) \\
&= ST(x_{k,1} - \alpha_{k,n-1} x_{m,1} - \alpha_{k,n} x_{m,0}) \\
&= S(x_{k,0} - \alpha_{k,n-1} x_{m,0}) = T(x_{k,0} - \alpha_{k,n-1} x_{m,0}) = 0,
\end{aligned}
$$

and $S^n z_k = x_{k,0} - \alpha_{k,n-1} x_{m,0} \neq 0$ for all $k = 1, \ldots, m - 1$. By (3.5) the set $\{x_{1,0} - \alpha_{1,n-1} x_{m,0}, \ldots, x_{m-1,0} - \alpha_{m-1,n-1} x_{m,0}\}$ is linearly independent. Then by Lemma 3.2 applied to S^n it follows that the set $\{z_1 + \ker S^n, \ldots, z_{m-1} + \ker S^n\}$ is linearly independent in $\ker S^{n+1}/\ker S^n$.

Case III: $0 \leq h \leq n - 2$. In this case we construct, as in Case II, two sets of vectors

$$
\{v_{k,j} \in M \cap \ker T^{j+1} : k = 1, \ldots, m - 1, j = h, \ldots, n - 1\}, \tag{3.7}
$$

and

$$
\{w_{k,j+1} \in \ker T^{j+2} : k = 1, \ldots, m - 1, j = h, \ldots, n - 1\}. \tag{3.8}
$$

By assumption, $x_{m,h} \notin M$. We start the construction with $j = h$, that is, with the definition of the vectors $v_{k,h}$ and $w_{k,h+1}$ for $k = 1, \ldots, m - 1$: There exist $\alpha_{k,h} \in \mathbb{K}$ such that

$$
v_{k,h} := x_{k,h} - \alpha_{k,h} x_{m,h} \in M \cap \ker T^{h+1} \quad \text{for } k = 1, \ldots, m - 1.
$$

Using the same coefficients $\alpha_{k,h} \in \mathbb{K}$, let

$$
w_{k,h+1} := x_{k,h+1} - \alpha_{k,h} x_{m,h+1} \in \ker T^{h+2} \quad \text{for } k = 1, \ldots, m - 1.
$$

Notice that $T w_{k,h+1} = v_{k,h}$ for $k = 1, \ldots, m - 1$. The vectors $v_{k,j}$ and $w_{k,j+1}$ for $k = 1, \ldots, m - 1$ are defined inductively for $j = h + 1, \ldots, n - 1$, in the following way: Fix $j = h + 1, \ldots, n - 1$ and assume that we have constructed $v_{k,j-1} \in M \cap \ker T^j$ and $w_{k,j} \in \ker T^{j+1}$ for $k = 1, \ldots, m - 1$. Then there exist $\alpha_{k,j} \in \mathbb{K}$ such that

$$
v_{k,j} := w_{k,j} - \alpha_{k,j} x_{m,h} \in M \cap \ker T^{j+1} \quad \text{for } k = 1, \ldots, m - 1.
$$

Also, define

$$
w_{k,j+1} := x_{k,j+1} - \sum_{i=0}^{j-h} \alpha_{k,h+i} x_{m,j-i+1} \in \ker T^{j+2} \quad \text{for } k = 1, \ldots, m - 1.
$$

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A straightforward computation shows $Tw_{k,j+1} = v_{k,j}$ for $k = 1, \ldots, m-1$. So, we have constructed the sets in (3.7) and (3.8).

Finally, observe that there also exist $\alpha_{k,n} \in \mathbb{K}$ such that

$$z_k := w_{k,n} - \alpha_{k,n} x_{m,h} \in M \cap \ker T^{n+1} \quad \text{for} \quad k = 1, \ldots, m-1.$$

Hence,

$$\begin{aligned}
Sz_k = Tz_k &= T(w_{k,n} - \alpha_{k,n} x_{m,h}) = v_{k,n-1} - \alpha_{k,n} x_{m,h-1}, \\
S^2 z_k &= S(v_{k,n-1} - \alpha_{k,n} x_{m,h-1}) \\
&= T(v_{k,n-1} - \alpha_{k,n} x_{m,h-1}) \\
&= T(w_{k,n-1} - \alpha_{k,n-1} x_{m,h} - \alpha_{k,n} x_{m,h-1}) \\
&= v_{k,n-2} - \alpha_{k,n-1} x_{m,h-1} - \alpha_{k,n} x_{m,h-2},
\end{aligned}$$

and, in the same way, we show that

$$S^{n-h} z_k = v_{k,h} - \sum_{i=1}^{n-h} \alpha_{k,h+i} x_{m,h-i},$$

where $x_{m,l} = 0$ if $l < 0$. Also, observe that

$$\begin{aligned}
S^{n-h+1} z_k &= S\left(v_{k,h} - \sum_{i=1}^{n-h} \alpha_{k,h+i} x_{m,h-i}\right) \\
&= T\left(v_{k,h} - \sum_{i=1}^{n-h} \alpha_{k,h+i} x_{m,h-i}\right) \\
&= T\left(x_{k,h} - \alpha_{k,h} x_{m,h} - \sum_{i=1}^{n-h} \alpha_{k,h+i} x_{m,h-i}\right) \\
&= x_{k,h-1} - \sum_{i=0}^{n-h} \alpha_{k,h+i} x_{m,h-i-1}, \\
S^{n-h+2} z_k &= S\left(x_{k,h-1} - \sum_{i=0}^{n-h} \alpha_{k,h+i} x_{m,h-i-1}\right) \\
&= T\left(x_{k,h-1} - \sum_{i=0}^{n-h} \alpha_{k,h+i} x_{m,h-i-1}\right) \\
&= x_{k,h-2} - \sum_{i=0}^{n-h} \alpha_{k,h+i} x_{m,h-i-2}, \\
&\quad\vdots \\
S^n z_k &= x_{k,0} - \alpha_{k,h} x_{m,0}, \quad \text{and} \\
S^{n+1} z_k &= 0.
\end{aligned}$$

Furthermore, by (3.5) the set $\{x_{1,0} - \alpha_{1,h}x_{m,0}, \ldots, x_{m-1,0} - \alpha_{m-1,h}x_{m,0}\}$ is linearly independent in $\ker S$. Applying Lemma 3.2 to S^n it follows that the set $\{z_1 + \ker S^n, \ldots, z_{m-1} + \ker S^n\}$ is linearly independent in $\ker S^{n+1}/\ker S^n$.

Summing up, we have shown in Cases I-III above that there exists a linearly independent set $\{z_1 + \ker S^n, \ldots, z_{m-1} + \ker S^n\}$ in $\ker S^{n+1}/\ker S^n$, which contradicts (3.4). Therefore,

$$\dim(\ker T^{n+1}/\ker T^n) \leq \dim(\ker S^{n+1}/\ker S^n) + 1,$$

and, in particular, $\ker T^{n+1}/\ker T^n$ is finite dimensional. Then, (3.2) follows by interchanging S and T. Finally, (3.1) is a consequence of (3.3) and repeated applications of (3.2). $\qquad\square$

Before proving Theorem 2.2 in Section 4 we will improve the upper bound in (ii) of Proposition 3.3 for a particular class of rank-one perturbations.

Assume that S is a linear operator in X and M is a linear subspace in $\operatorname{dom} S$ such that $\dim(\operatorname{dom} S/M) = k$. Then, there exist linearly independent vectors $x_1, \ldots, x_k \in (\operatorname{dom} S) \setminus M$ such that

$$\operatorname{dom} S = M \dotplus \operatorname{span}\{x_1, \ldots, x_k\}$$

We define the restrictions

$$S_p := S \restriction \big(M \dotplus \operatorname{span}\{x_1, \ldots, x_p\}\big), \qquad 1 \leq p \leq k.$$

Lemma 3.4. *Given $2 \leq p \leq k$, if $\ker S_p^{n+1}/\ker S_p^n$ is finite dimensional for some $n \in \mathbb{N}$, then the same holds for $\ker S_{p-1}^{n+1}/\ker S_{p-1}^n$ and*

$$\dim\left(\frac{\ker S_p^{n+1}}{\ker S_p^n}\right) - 1 \leq \dim\left(\frac{\ker S_{p-1}^{n+1}}{\ker S_{p-1}^n}\right) \leq \dim\left(\frac{\ker S_p^{n+1}}{\ker S_p^n}\right).$$

Proof. By Proposition 3.3 only the second inequality needs to be proved. Assume that $\dim\big(\ker S_p^{n+1}/\ker S_p^n\big) = i < \infty$ and that the set $\{z_1 + \ker S_{p-1}^n, \ldots, z_{i+1} + \ker S_{p-1}^n\}$ is linearly independent in $\ker S_{p-1}^{n+1}/\ker S_{p-1}^n$. Then, since $\ker S_{p-1}^{n+1} \subset \ker S_p^{n+1}$, there exist $\alpha_1, \ldots, \alpha_{i+1} \in \mathbb{K}$ (not all equal to zero) such that

$$z := \alpha_1 z_1 + \cdots + \alpha_{i+1} z_{i+1} \in \ker S_p^n.$$

Together with $z \in \operatorname{dom} S_{p-1}^{n+1} \subset \operatorname{dom} S_{p-1}^n$ we conclude $z \in \ker S_{p-1}^n$, a contradiction, and Lemma 3.4 is shown. $\qquad\square$

4 Proof of Theorem 2.2

We start the proof with some preparations. By assumption S and T satisfy Hypothesis 2.1. We discuss the case

$$\dim\big(\operatorname{dom} S/M\big) = k \quad \text{and} \quad \dim\big(\operatorname{dom} T/M\big) = l \leq k.$$

Then there exist linearly independent vectors $x_1, \ldots, x_k \in (\operatorname{dom} S) \setminus M$ and $y_1, \ldots, y_l \in (\operatorname{dom} T) \setminus M$ such that

$$\operatorname{dom} S = M \dotplus \operatorname{span}\{x_1, \ldots, x_k\} \quad \text{and} \quad \operatorname{dom} T = M \dotplus \operatorname{span}\{y_1, \ldots, y_l\}.$$

Also, we can assume that $\operatorname{span}\{x_1, \ldots, x_k\} \cap \operatorname{span}\{y_1, \ldots, y_l\} = \{0\}$ (otherwise M can be enlarged). Next, consider the restrictions

$$S_p := S \restriction \left(M \dotplus \operatorname{span}\{x_1, \ldots, x_p\} \right), \qquad 1 \leq p \leq k,$$

and

$$T_q := T \restriction \left(M \dotplus \operatorname{span}\{y_1, \ldots, y_q\} \right), \qquad 1 \leq q \leq l.$$

Clearly $S = S_k$ and $T = T_l$. As mentioned before, it is sufficient to prove Theorem 2.2 for $\lambda = 0$. Let $\ker S^{n+1}/\ker S^n$ be finite dimensional for some $n \in \mathbb{N}$, $n \geq 1$. Applying repeatedly Lemma 3.4 to $S = S_k, S_{k-1}, \ldots, S_2$, we see that $\ker S_1^{n+1}/\ker S_1^n$ is finite dimensional and

$$\dim\left(\frac{\ker S^{n+1}}{\ker S^n}\right) - (k-1) \leq \dim\left(\frac{\ker S_1^{n+1}}{\ker S_1^n}\right) \leq \dim\left(\frac{\ker S^{n+1}}{\ker S^n}\right). \qquad (4.1)$$

The operators S_1 and T_1 satisfy Hypothesis 2.1 with $k = 1$. Hence, by Proposition 3.3, $\ker T_1^{n+1}/\ker T_1^n$ is finite dimensional and

$$\left|\dim\left(\ker S_1^{n+1}/\ker S_1^n\right) - \dim\left(\ker T_1^{n+1}/\ker T_1^n\right)\right| \leq 1. \qquad (4.2)$$

Similarly, repeated application of Lemma 3.4 to $T_2, T_3, \ldots, T_l = T$ shows that $\ker T^{n+1}/\ker T^n$ is finite dimensional and

$$\dim\left(\frac{\ker T^{n+1}}{\ker T^n}\right) - (l-1) \leq \dim\left(\frac{\ker T_1^{n+1}}{\ker T_1^n}\right) \leq \dim\left(\frac{\ker T^{n+1}}{\ker T^n}\right). \qquad (4.3)$$

Since $l \leq k$, notice that $-(k-1) \leq -(l-1)$. Therefore with (4.1), (4.2) and (4.3)

$$\begin{aligned}
\dim\left(\ker S^{n+1}/\ker S^n\right) &- \dim\left(\ker T^{n+1}/\ker T^n\right) \\
&\geq \dim\left(\ker S_1^{n+1}/\ker S_1^n\right) - \dim\left(\ker T^{n+1}/\ker T^n\right) \\
&\geq \dim\left(\ker T_1^{n+1}/\ker T_1^n\right) - 1 - \dim\left(\ker T^{n+1}/\ker T^n\right) \\
&\geq -(l-1) - 1 \\
&\geq -(k-1) - 1 = -k.
\end{aligned}$$

An analogous calculation for the upper bound shows

$$\dim\left(\ker S^{n+1}/\ker S^n\right) - \dim\left(\ker T^{n+1}/\ker T^n\right) \leq k,$$

which yields

$$\left|\dim\left(\ker S^{n+1}/\ker S^n\right) - \dim\left(\ker T^{n+1}/\ker T^n\right)\right| \leq k,$$

and assertion (ii) in Theorem 2.2 holds. Finally, assertion (i) in Theorem 2.2 follows from

$$\left|\dim \ker S - \dim \ker T\right| \leq k,$$

which is shown in a similar way as in the proof of Proposition 3.3, and a repeated application of (2.2).

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