

Aliasing and oblique dual pair designs for consistent sampling

María J. Benac, Pedro G. Massey, and Demetrio Stojanoff *
 Depto. de Matemática, FCE-UNLP, La Plata, Argentina and IAM-CONICET

Abstract

In this paper we study some aspects of oblique duality between finite sequences of vectors \mathcal{F} and \mathcal{G} lying in finite dimensional subspaces \mathcal{W} and \mathcal{V} , respectively. We compute the possible eigenvalue lists of the frame operators of oblique duals to \mathcal{F} lying in \mathcal{V} ; we then compute the spectral and geometrical structure of minimizers of convex potentials among oblique duals for \mathcal{F} under some restrictions. We obtain a complete quantitative analysis of the impact that the relative geometry between the subspaces \mathcal{V} and \mathcal{W} has in oblique duality. We apply this analysis to compute those rigid rotations U for \mathcal{W} such that the canonical oblique dual of $U \cdot \mathcal{F}$ minimize every convex potential; we also introduce a notion of aliasing for oblique dual pairs and compute those rigid rotations U for \mathcal{W} such that the canonical oblique dual pair associated to $U \cdot \mathcal{F}$ minimize the aliasing. We point out that these two last problems are intrinsic to the theory of oblique duality.

AMS subject classification: 42C15, 15A60.

Keywords: frames, oblique duality, majorization, convex potentials, Lidskii's theorem.

Contents

1	Introduction	2
2	Preliminaries	4
2.1	Oblique dual frames and convex potentials	4
2.2	(Log-)majorization and convex functions	6
3	Spectral structure and optimal oblique duals	7
3.1	Spectral structure of oblique duals	7
3.2	Optimal oblique duals with norm restrictions	11
4	Optimal $(\mathcal{V}, \mathcal{W})$-oblique dual pairs with prescribed parameters	13
4.1	Relative geometry between finite dimensional subspaces	13
4.2	Optimal dual pairs by rigid rotations	14
5	Aliasing in oblique duality	19

*Partially supported by CONICET (PIP 0435/10) and Universidad Nacional de La PLata (UNLP 11X681)
 e-mail addresses: mjbenac@gmail.com , massey@mate.unlp.edu.ar , demetrio@mate.unlp.edu.ar

1 Introduction

A finite sequence $\mathcal{F} = \{f_i\}_{i \in \mathbb{I}_n}$ is a frame for a Hilbert space $\mathcal{W} \cong \mathbb{C}^d$ if \mathcal{F} spans \mathcal{W} , where $\mathbb{I}_n = \{1, \dots, n\}$. In this case, a sequence $\mathcal{G} = \{g_i\}_{i \in \mathbb{I}_n}$ in \mathcal{W} is a (classical) dual for \mathcal{F} in \mathcal{W} if the following reconstruction formulas holds:

$$f = \sum_{i \in \mathbb{I}_n} \langle f, f_i \rangle g_i = \sum_{i \in \mathbb{I}_n} \langle f, g_i \rangle f_i, \quad f \in \mathcal{W}. \quad (1)$$

Hence, frames allow for linear encoding-decoding schemes of vectors in \mathcal{W} in terms of linear generators for \mathcal{W} . Moreover, in case $n > d$ then the set of dual frames for \mathcal{F} in \mathcal{W} has a rich structure which plays a key role in applications of finite frame theory to real life situations, such as signal transmission through noisy channels (see [6, 7]). Similarly, applications of finite frame theory have lead to consider the so-called frame design problems, i.e. the existence and construction of frames with prescribed properties, based on the flexibility of finite frames (see [4, 18, 24, 27]).

An important aspect of frames is that of its numerical stability; typically, numerical stability is measured in terms of the spread of the eigenvalues of the so-called frame operator $S_{\mathcal{F}}$ of a frame $\mathcal{F} = \{f_i\}_{i \in \mathbb{I}_n}$, that is given by $S_{\mathcal{F}} = \sum_{i \in \mathbb{I}_n} f_i \otimes f_i$. One of the most important measures of the spread of the spectrum of $S_{\mathcal{F}}$ is given by the frame potential of \mathcal{F} (see [4]) given by $\text{FP}(\mathcal{F}) = \sum_{i, j \in \mathbb{I}_n} |\langle f_i, f_j \rangle|^2 = \text{tr}(S_{\mathcal{F}}^2)$. Indeed, it turns out that minimizers of the frame potential - within appropriate sets of frames - minimize the spread of the spectrum of their frame operators (see [4, 5]). Recently, there has also been interest in the structure of minimizers of the so-called mean squared error of a frame \mathcal{F} given by $\text{MSE}(\mathcal{F}) = \text{tr}(S_{\mathcal{F}}^{-1})$ - within convenient sets of frames (see [16]). This raises the question of whether the minimizers of these two different functionals coincide. It turns out that there is a natural and structural measure of spread of the spectrum of the frame operators, called submajorization, that has proved useful in explaining the spectral and geometrical structure of both frame potential and mean squared error minimizers (see [23, 25, 26]).

In the seminal paper [14] Y. Eldar developed the theory of oblique duality for finite frames, which is an extended setting for linear encoding-decoding schemes in a Hilbert space \mathcal{W} , based on the notion of consistent sampling. As the starting point for this theory, we consider $\mathcal{W}, \mathcal{V} \subset \mathcal{H}$ two subspaces of a finite dimensional Hilbert space \mathcal{H} , such that $\mathcal{W} \oplus \mathcal{V}^{\perp} = \mathcal{H}$ (i.e. $\mathcal{W}^{\perp} + \mathcal{V} = \mathcal{H}$ and $\mathcal{W}^{\perp} \cap \mathcal{V} = \{0\}$). Given a frame $\mathcal{F} = \{f_i\}_{i \in \mathbb{I}_n}$ for \mathcal{W} and a frame $\mathcal{G} = \{g_i\}_{i \in \mathbb{I}_n}$ for \mathcal{V} we say that \mathcal{G} is an oblique dual of \mathcal{F} if the following reconstruction formula holds

$$f = \sum_{i \in \mathbb{I}_n} \langle f, g_i \rangle f_i \quad \text{for every } f \in \mathcal{W}.$$

The theory of oblique duality has been both developed and extended in several ways (see [1, 2, 13, 9, 15]). On the other hand, it has been successfully applied to study duality for finitely generated shift invariant systems on $L^2(\mathbb{R}^d)$ (see [8, 19, 20]).

There are, however, some aspects of oblique duality that remain to be explored, even in the finite dimensional case. In this paper, based on several tools coming from matrix analysis, we consider the following problems in oblique duality. On the one hand, we study the spectral (and geometrical) structure of oblique duals of the frame \mathcal{F} for \mathcal{W} that lie in \mathcal{V} . In this case, we obtain an explicit description of the eigenvalues of the frame operators of oblique duals. With this description at hand, we compute the structure of minimizers of submajorization within the set of oblique duals of \mathcal{F} under some restrictions. These optimal

oblique duals for submajorization turn out to minimize the so-called convex potentials (that include both the frame potential and the mean squared error).

On the other hand, it has been noticed that the relative position of the subspaces \mathcal{V} and \mathcal{W} for which $\mathcal{W} \oplus \mathcal{V}^\perp = \mathcal{H}$ plays a key role when comparing oblique duality to classical duality. This phenomenon has been explored mainly through the angle between the subspaces \mathcal{V} and \mathcal{W} . Yet, the angle between the subspaces \mathcal{V} and \mathcal{W} only provides qualitative measure of the role of the relative geometry of \mathcal{V} and \mathcal{W} in the context of \mathcal{V} -duality. In this paper, we give a detailed description of the role of the relative position of \mathcal{V} and \mathcal{W} in the oblique duality of \mathcal{F} in case the subspaces are finite dimensional. Our analysis relies on a multiplicative Lidskii's inequality and it is based on the complete list of the so-called principal angles between \mathcal{V} and \mathcal{W} . Our results provide sharp quantitative measures of these relations.

We also consider two problems that are intrinsic to oblique duality. We first notice that the so-called canonical oblique dual $(U \cdot \mathcal{F})_{\mathcal{V}}^\#$ of a rigid rotation $U \cdot \mathcal{F} = \{U f_i\}_{i \in \mathbb{I}_n}$ of \mathcal{F} is, in general, not a rigid rotation of the canonical oblique dual $\mathcal{F}_{\mathcal{V}}^\#$ of \mathcal{F} (as opposed to classical duality). Hence, we compute the rigid rotation U_0 of \mathcal{W} such that the oblique dual $(U_0 \cdot \mathcal{F})_{\mathcal{V}}^\#$ is optimal with respect to submajorization, among all such rigid rotations. Again, this implies a family of inequalities in terms of convex potentials that are relevant for numerical analysis purposes. We also compute the exact value of the aliasing norm of the consistent sampling corresponding to subspaces \mathcal{V} and \mathcal{W} and introduce a notion of aliasing for oblique dual pairs. In this context we compute the optimal rigid rotations U_0 that minimize the aliasing of the dual pairs $(U \cdot \mathcal{F}, (U \cdot \mathcal{F})_{\mathcal{V}}^\#)$ for the fixed frame \mathcal{F} .

Throughout the paper we consider finite sequences of vectors \mathcal{F} that are frames for finite dimensional subspaces \mathcal{W} of a possibly infinite dimensional Hilbert space \mathcal{H} , since this is the setting that we shall need for future applications of the results herein; on the other hand, the assumption that \mathcal{H} is finite dimensional does not provide any substantial simplification in the proofs of our results.

The paper is organized as follows. In Section 2 we describe the basic framework of oblique duality between finite sequences of vectors, together with some basic facts about convex potentials. In order to deal with these general convex potentials, we also consider submajorization and log-majorization, which are spectral relations between positive finite rank operators (or positive matrices). In particular, we include a multiplicative analogue of Lidskii's additive inequality that plays a crucial role in this note. In Section 3 we obtain a convenient parametrization of the set of oblique duals of a fixed frame and use it to compute the possible eigenvalues of the frame operators of oblique duals. We then compute the structure of optimal oblique duals for submajorization, under certain restrictions. In Section 4, after recalling some standard notions from functional analysis, we compute the rigid rotations U of \mathcal{W} such that the spread of the eigenvalues of the frame operator of the oblique dual $(U \cdot \mathcal{F})_{\mathcal{V}}^\#$ is minimal, with respect to submajorization. We also consider the combination of the problems of Section 3 and Section 4 i.e., the properties of the optimal oblique dual frame (with norm restrictions) corresponding to the optimal rotation of \mathcal{F} . In Section 5 we compute the exact value of the aliasing norm of the consistent sampling based on the subspaces \mathcal{V} and \mathcal{W} and introduce a notion of aliasing for arbitrary oblique dual pairs $(\mathcal{F}, \mathcal{G})$. In this context, we compute the rigid rotations U of \mathcal{W} that minimize the aliasing for the oblique dual pair $(U \cdot \mathcal{F}, (U \cdot \mathcal{F})_{\mathcal{V}}^\#)$.

2 Preliminaries

In this section we introduce the notations and basic terminology of frame theory and oblique duality. We also describe the convex potentials for finite sequences of vectors (that contain, for example, the Benedetto-Fickus' frame potential) which will serve as a numerical measures of the (relative) spread of the eigenvalues of the frame operators. Finally, we describe some notions from matrix theory that will allow us to deal with these general convex potentials.

2.1 Oblique dual frames and convex potentials

In what follows we consider a fixed complex separable Hilbert space \mathcal{H} . We take

$$\mathbb{M} = \mathbb{I}_p \stackrel{\text{def}}{=} \{1, \dots, p\} \quad \text{for } p \in \mathbb{N} \quad \text{or} \quad \mathbb{M} = \mathbb{N} \quad (2)$$

in such a way that $\dim \mathcal{H} = |\mathbb{M}|$. Let \mathcal{W} be a closed subspace of \mathcal{H} . Recall that a sequence $\mathcal{F} = \{f_i\}_{i \in I}$ in \mathcal{W} is a *frame* for \mathcal{W} if there exist $0 < A < B$ such that

$$A \|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq B \|f\|^2, \quad f \in \mathcal{W}. \quad (3)$$

If only the inequality to the right holds, we say that \mathcal{F} is a Bessel sequence.

In general, given a Bessel sequence $\mathcal{F} = \{f_i\}_{i \in I}$ we consider its *synthesis operator* $T_{\mathcal{F}} \in B(\ell^2(I), \mathcal{H})$ given by $T_{\mathcal{F}}((a_i)_{i \in I}) = \sum_{i \in I} a_i f_i$ which, by hypothesis on \mathcal{F} , is a bounded linear transformation. We also consider $T_{\mathcal{F}}^* \in B(\mathcal{H}, \ell^2(I))$ called the *analysis operator* of \mathcal{F} , given by $T_{\mathcal{F}}^*(f) = (\langle f, f_i \rangle)_{i \in I}$ and the *frame operator* of \mathcal{F} defined by $S_{\mathcal{F}} = T_{\mathcal{F}} T_{\mathcal{F}}^*$. It is straightforward to check that

$$\langle S_{\mathcal{F}} f, f \rangle = \sum_{i \in I} |\langle f, f_i \rangle|^2, \quad f \in \mathcal{H}.$$

Hence, $S_{\mathcal{F}}$ is a positive semidefinite operator; moreover, a Bessel sequence \mathcal{F} in \mathcal{W} is a frame for \mathcal{W} if and only if $S_{\mathcal{F}}$ is an invertible operator when restricted to \mathcal{W} or, equivalently if $T_{\mathcal{F}}$ is a surjective operator onto \mathcal{W} .

In order to describe oblique duality, we fix two closed subspaces $\mathcal{V}, \mathcal{W} \subseteq \mathcal{H}$ such that $\mathcal{W}^\perp \oplus \mathcal{V} = \mathcal{H}$, that is such that $\mathcal{W}^\perp + \mathcal{V} = \mathcal{H}$ and $\mathcal{W}^\perp \cap \mathcal{V} = \{0\}$. Hence, \mathcal{W}^\perp is a common (algebraic) complement of \mathcal{W} and \mathcal{V} . It is well known that in this case $P_{\mathcal{W}}|_{\mathcal{V}} : \mathcal{V} \rightarrow \mathcal{W}$ is a linear bounded isomorphism so, in particular, we see that $\dim \mathcal{V} = \dim \mathcal{W}$ as Hilbert spaces. Moreover, the conditions $\mathcal{W}^\perp \oplus \mathcal{V} = \mathcal{H}$ and $\mathcal{W} \oplus \mathcal{V}^\perp = \mathcal{H}$ are actually equivalent.

Fix a frame $\mathcal{F} = \{f_i\}_{i \in I}$ for \mathcal{W} . Following [14, 15] (see also [9]), given a Bessel sequence $\mathcal{G} = \{g_i\}_{i \in I}$ in \mathcal{V} we say that \mathcal{G} is a (oblique) \mathcal{V} -dual of \mathcal{F} if

$$g = \sum_{i \in I} \langle g, f_i \rangle f_i = T_{\mathcal{G}} T_{\mathcal{F}}^* g \quad \text{for every } g \in \mathcal{V}.$$

It turns out (see [14, 15]) that \mathcal{G} is a \mathcal{V} -dual of \mathcal{F} if and only if $T_{\mathcal{G}} T_{\mathcal{F}}^* = P_{\mathcal{V} // \mathcal{W}^\perp}$, where $P_{\mathcal{V} // \mathcal{W}^\perp}$ denotes the oblique projection with range \mathcal{V} and null space \mathcal{W}^\perp . Hence, $T_{\mathcal{G}}$ is surjective onto \mathcal{V} and then \mathcal{G} is a frame for \mathcal{V} ; by taking adjoints in the identity $T_{\mathcal{G}} T_{\mathcal{F}}^* = P_{\mathcal{V} // \mathcal{W}^\perp}$ we also get that $T_{\mathcal{F}} T_{\mathcal{G}}^* = P_{\mathcal{W} // \mathcal{V}^\perp}^* = P_{\mathcal{W} // \mathcal{V}^\perp}$ i.e.

$$f = \sum_{i \in I} \langle f, g_i \rangle g_i \quad \text{for every } f \in \mathcal{W}.$$

We shall consider the set of oblique \mathcal{V} -duals of \mathcal{F} given by

$$\mathcal{D}_{\mathcal{V}}(\mathcal{F}) \stackrel{\text{def}}{=} \{ \mathcal{G} \in \mathcal{V}^I : \mathcal{G} \text{ is a } \mathcal{V}\text{-dual of } \mathcal{F} \} . \quad (4)$$

Remark 2.1. Let $\mathcal{F} = \{f_i\}_{i \in I}$ be a frame for \mathcal{W} . If we set $\mathcal{V} = \mathcal{W}$ then a Bessel sequence \mathcal{G} in \mathcal{W} is a \mathcal{W} -dual of \mathcal{F} if it is a dual frame for \mathcal{F} in the classical sense (see [7]) i.e. $T_{\mathcal{G}} T_{\mathcal{F}}^* = P_{\mathcal{W}}$. Hence

$$\mathcal{D}_{\mathcal{W}}(\mathcal{F}) = \mathcal{D}(\mathcal{F}) \stackrel{\text{def}}{=} \{ \mathcal{G} \in \mathcal{W}^I : \mathcal{G} \text{ is a dual frame for } \mathcal{F} \text{ in } \mathcal{W} \} .$$

Recall that there is a distinguished (classical) dual, called the canonical dual of \mathcal{F} , denoted

$$\mathcal{F}^{\#} = \{f_i^{\#}\}_{i \in I} \quad \text{given by} \quad f_i^{\#} = S_{\mathcal{F}}^{\dagger} f_i \quad \text{for every} \quad i \in I ,$$

where $S_{\mathcal{F}}^{\dagger}$ denotes the Moore-Penrose pseudo-inverse of the (closed range operator) $S_{\mathcal{F}}$. \triangle

In the general setting for oblique duality there also exists a distinguished \mathcal{V} -dual for \mathcal{F} , the so-called *canonical \mathcal{V} -dual* which we denote by

$$\mathcal{F}_{\mathcal{V}}^{\#} = \{f_{\mathcal{V},i}^{\#}\}_{i \in I} \quad \text{given by} \quad f_{\mathcal{V},i}^{\#} = P_{\mathcal{V} // \mathcal{W}^{\perp}} f_i^{\#} = P_{\mathcal{V} // \mathcal{W}^{\perp}} S_{\mathcal{F}}^{\dagger} f_i \quad \text{for every} \quad i \in I , \quad (5)$$

where $\mathcal{F}^{\#} = \{f_i^{\#}\}_{i \in I} \in \mathcal{D}(\mathcal{F})$ denotes the (classical) canonical dual as described in Remark 2.1. It turns out that the encoding-decoding scheme based on the oblique dual pair $(\mathcal{F}, \mathcal{F}_{\mathcal{V}}^{\#})$ has several optimality properties (see [14, 15]).

Convex potentials for finite sequences in \mathcal{H}

In their seminal work [4], Benedetto and Fickus introduced a functional defined on finite sequences of (unit norm) vectors, the so-called *frame potential*, given by

$$\text{FP}(\{f_i\}_{i \in \mathbb{I}_n}) = \sum_{i,j \in \mathbb{I}_n} |\langle f_i, f_j \rangle|^2 . \quad (6)$$

In case $\dim \mathcal{H} = p \in \mathbb{N}$ then one of their major results shows that tight unit norm frames - which form an important class of frames because of their simple reconstruction formulas and robustness properties - can be characterized as (local) minimizers of this functional among unit norm frames. Since then, there has been interest in (local) minimizers of the frame potential within certain classes of frames, since such minimizers can be considered as natural substitutes of tight frames (see for example [5, 23]). Notice that, given $\mathcal{F} = \{f_i\}_{i \in \mathbb{I}_n} \in \mathcal{H}^n$ then $\text{FP}(\mathcal{F}) = \text{tr} S_{\mathcal{F}}^2$. Recently, there has been interest in the structure of frames that minimize other potentials such as the so-called mean squared error (MSE) given by $\text{MSE}(\mathcal{F}) = \text{tr}(S_{\mathcal{F}}^{-1})$ (see [16, 25, 27]). Next, we describe a broad family of potentials introduced in [23], that contain both the frame potential and the MSE.

In what follows we consider the sets

$$\text{Conv}(\mathbb{R}_{\geq 0}) = \{h : [0, \infty) \rightarrow [0, \infty) : h \text{ is a convex function} \}$$

and $\text{Conv}_s(\mathbb{R}_{\geq 0}) = \{h \in \text{Conv}(\mathbb{R}_{\geq 0}) : h \text{ is strictly convex} \}$.

Definition 2.2. Given $h \in \text{Conv}(\mathbb{R}_{\geq 0})$ then the *convex potential* associated to h , denoted by P_h , is defined as follows: for a finite sequence $\mathcal{F} = \{f_i\}_{i \in \mathbb{I}_n} \in \mathcal{H}^n$ with $\mathcal{W} = \text{Span}\{f_i : i \in \mathbb{I}_n\}$ and frame operator $S_{\mathcal{F}} \in B(\mathcal{H})^+$, then

$$P_h(\mathcal{F}) = \sum_{i \in \mathbb{I}_d} h(\lambda_i((S_{\mathcal{F}})_{\mathcal{W}})) ,$$

where $d = \dim \mathcal{W}$ and $(\lambda_i((S_{\mathcal{F}})_{\mathcal{W}}))_{i \in \mathbb{I}_d} \in \mathbb{R}_{>0}^d$ denotes the vector of eigenvalues of the compression $(S_{\mathcal{F}})_{\mathcal{W}} \in B(\mathcal{W})^+$, counting multiplicities and arranged in non-increasing order. \triangle

Remark 2.3. With the notations of Definition 2.2, notice that by construction \mathcal{W} is a reductive subspace for $S_{\mathcal{F}}$, and hence $(S_{\mathcal{F}})_{\mathcal{W}}$ is a well defined positive operator acting on the finite dimensional subspace \mathcal{W} . Moreover,

$$P_h(\mathcal{F}) = \text{tr } h((S_{\mathcal{F}})_{\mathcal{W}})$$

where $h((S_{\mathcal{F}})_{\mathcal{W}}) \in B(\mathcal{W})^+$ is obtained by means of the usual functional calculus from the compression $(S_{\mathcal{F}})_{\mathcal{W}}$ and the trace is taken in the finite dimensional Hilbert space \mathcal{W} . Therefore, in case $h \in \text{Conv}(\mathbb{R}_{\geq 0})$ is such that $h(0) = 0$ we get that

$$P_h(\mathcal{F}) = \text{tr } h((S_{\mathcal{F}})_{\mathcal{W}}) = \text{tr } h(S_{\mathcal{F}}) .$$

In particular, we see that if $h(x) = x^2$ then $P_h(\mathcal{F})$ coincides with the frame potential. \triangle

Fix $h \in \text{Conv}(\mathbb{R}_{\geq 0})$ and consider its associated convex potential P_h . If $\mathcal{F} = \{f_i\}_{i \in \mathbb{I}_n}$ is a finite sequence in \mathcal{H}^n then $P_h(\mathcal{F})$ is a measure of the spread of the eigenvalues of the frame operator of \mathcal{F} . That is, (under suitable normalization hypothesis on \mathcal{F}) the smaller the value $P_h(\mathcal{F})$ is, the more concentrated the non-zero eigenvalues of $S_{\mathcal{F}}$ are (see [23, 25, 26, 27]).

In order to deal with these general convex potentials we consider the notions of submajorization and log-majorization in the next section.

2.2 (Log-)majorization and convex functions

Next we briefly describe sub-majorization, majorization and log-majorization, that are notions from matrix analysis. For a detailed exposition on these relations see [3].

Given $x, y \in \mathbb{R}_{\geq 0}^d$ we say that x is *submajorized* by y , and write $x \prec_w y$, if

$$\sum_{i=1}^k x_i^{\downarrow} \leq \sum_{i=1}^k y_i^{\downarrow} \quad \text{for every } k \in \mathbb{I}_d = \{1, \dots, d\} ,$$

where $z^{\downarrow} \in \mathbb{R}^d$ (respectively $z^{\uparrow} \in \mathbb{R}^d$) denotes the vector obtained by re-arrangement of the entries of $z \in \mathbb{R}^d$ in non-increasing (respectively non-decreasing) order. If $x \prec_w y$ and $\text{tr } x = \sum_{i=1}^d x_i = \sum_{i=1}^d y_i = \text{tr } y$, then x is *majorized* by y , and write $x \prec y$.

Log-majorization between vectors in $\mathbb{R}_{>0}^d$ is a multiplicative analogue of majorization in $\mathbb{R}_{\geq 0}^d$. Indeed, given $x, y \in \mathbb{R}_{>0}^d$ we say that x is *log-majorized* by y , denoted $x \prec_{\log} y$, if

$$\prod_{i=1}^k x_i^{\downarrow} \leq \prod_{i=1}^k y_i^{\downarrow} \quad \text{for every } k \in \mathbb{I}_{d-1} \quad \text{and} \quad \prod_{i=1}^d x_i^{\downarrow} = \prod_{i=1}^d y_i^{\downarrow} .$$

It is known (see [3]) that if $x, y \in \mathbb{R}_{\geq 0}^d$ are such that $x \prec_{\log} y$ then $x \prec_w y$. On the other hand we write $x \leq y$ if $x_i \leq y_i$ for every $i \in \mathbb{I}_d$. It is a standard exercise to show that if $x, y \in \mathbb{R}_{\geq 0}^d$ then $x \leq y \implies x^\downarrow \leq y^\downarrow \implies x \prec_{\log} y \implies x \prec_w y$.

Our interest in majorization is motivated by the relation of this notion with tracial inequalities for convex functions. Indeed, given $x, y \in \mathbb{R}_{\geq 0}^d$ and $h \in \text{Conv}(\mathbb{R}_{\geq 0})$, then (see for example [3]):

1. If one assumes that $x \prec y$, then $\text{tr } h(x) \stackrel{\text{def}}{=} \sum_{i=1}^d h(x_i) \leq \sum_{i=1}^d h(y_i) = \text{tr } h(y)$.
2. If only $x \prec_w y$, but the map h is also non-decreasing, then still $\text{tr } h(x) \leq \text{tr } h(y)$.
3. If $x \prec_w y$, $h \in \text{Conv}_s(\mathbb{R}_{\geq 0})$ is non-decreasing and $\text{tr } h(x) = \text{tr } h(y)$, then there exists a permutation σ of \mathbb{I}_d such that $y_i = x_{\sigma(i)}$ for $i \in \mathbb{I}_d$.

The following result is a multiplicative Lidskii's inequality for matrices, that also contains a detailed description of the case of *equality*. In what follows, given $x = (x_i)_{i \in \mathbb{I}_d}$, $y = (y_i)_{i \in \mathbb{I}_d} \in \mathbb{R}^d$ then $x \circ y = (x_i y_i)_{i \in \mathbb{I}_d} \in \mathbb{R}^d$ denotes the entry-wise product of the vectors. Also, given a selfadjoint matrix $A \in \mathcal{M}_d(\mathbb{C})$ then $\lambda(A) \in \mathbb{R}^d$ denotes the eigenvalues of A , counting multiplicities and arranged in non-increasing order.

Theorem 2.4 ([26]). *Let $S \in \mathcal{G}l(d)^+$ and let $\lambda \in (\mathbb{R}_{> 0}^d)^\downarrow$. Then, for every $V \in \mathcal{M}_d(\mathbb{C})$ such that $\lambda(V^*V) = \lambda$ we have that*

$$\lambda(S) \circ \lambda^\uparrow \prec_{\log} \lambda(VSV^*) \prec_{\log} \lambda(S) \circ \lambda \in (\mathbb{R}_{> 0}^d)^\downarrow. \quad (7)$$

Moreover, if $\lambda(VSV^*) = (\lambda(S) \circ \lambda^\uparrow)^\downarrow$ (resp. $\lambda(VSV^*) = \lambda(S) \circ \lambda$) then there exists an o.n.b. $\{v_i\}_{i \in \mathbb{I}_d}$ of \mathbb{C}^d such that

$$S = \sum_{i \in \mathbb{I}_d} \lambda_i(S) v_i \otimes v_i \quad \text{and} \quad |V| = \sum_{i \in \mathbb{I}_d} \lambda_{d+1-i}^{1/2} v_i \otimes v_i \quad (8)$$

(resp. $S = \sum_{i \in \mathbb{I}_d} \lambda_i(S) v_i \otimes v_i$ and $|V| = \sum_{i \in \mathbb{I}_d} \lambda_i^{1/2} v_i \otimes v_i$). □

3 Spectral structure and optimal oblique duals

In this section we obtain a simple and explicit description of the eigenvalues of the frame operators of oblique duals of a fixed frame. We then apply this result to study the existence and structure of oblique duals that are optimal among oblique duals with some restrictions.

3.1 Spectral structure of oblique duals

Let \mathcal{V} and \mathcal{W} be closed subspaces of the Hilbert space \mathcal{H} such that $\mathcal{W}^\perp \oplus \mathcal{V} = \mathcal{H}$. Let $\mathcal{F} = \{f_i\}_{i \in I}$ be a sequence in \mathcal{W} that is a frame for \mathcal{W} . There are several known characterizations of the elements in $\mathcal{D}_{\mathcal{V}}(\mathcal{F})$ (see for example [9]). In what follows we describe a simple *parametrization* of $\mathcal{D}_{\mathcal{V}}(\mathcal{F})$ in terms of $\mathcal{D}_{\mathcal{W}}(\mathcal{F})$ i.e. the classical dual frames for \mathcal{F} in \mathcal{W} , which is implicit in [9].

Proposition 3.1. *Let \mathcal{V} and \mathcal{W} be closed subspaces of \mathcal{H} such that $\mathcal{W}^\perp \oplus \mathcal{V} = \mathcal{H}$. Let $\mathcal{F} = \{f_i\}_{i \in I}$ be a frame for \mathcal{W} . Then the map*

$$\mathcal{D}(\mathcal{F}) \ni \{g_i\}_{i \in I} \mapsto \{P_{\mathcal{V}/\mathcal{W}^\perp} g_i\}_{i \in I} \in \mathcal{D}_{\mathcal{V}}(\mathcal{F})$$

is a (linear) bijection between $\mathcal{D}(\mathcal{F})$ and $\mathcal{D}_{\mathcal{V}}(\mathcal{F})$ that sends $\mathcal{F}^\#$ to $\mathcal{F}_{\mathcal{V}}^\#$.

Proof. Let $\mathcal{G} = \{g_i\}_{i \in I} \in \mathcal{D}(\mathcal{F})$ and set $\mathcal{G}' = \{P_{\mathcal{V}/\mathcal{W}^\perp} g_i\}_{i \in I}$. Then $T_{\mathcal{G}'} = P_{\mathcal{V}/\mathcal{W}^\perp} T_{\mathcal{G}}$ and hence

$$T_{\mathcal{G}'} T_{\mathcal{F}}^* = P_{\mathcal{V}/\mathcal{W}^\perp} T_{\mathcal{G}} T_{\mathcal{F}}^* = P_{\mathcal{V}/\mathcal{W}^\perp} P_{\mathcal{W}} = P_{\mathcal{V}/\mathcal{W}^\perp} .$$

Therefore $\mathcal{G}' \in \mathcal{D}_{\mathcal{V}}(\mathcal{F})$ and the map is well defined. To check that the map is injective, let $\mathcal{G} = \{g_i\}_{i \in I}$ and $\mathcal{K} = \{k_i\}_{i \in I} \in \mathcal{D}(\mathcal{F})$ be such that $P_{\mathcal{V}/\mathcal{W}^\perp} g_i = P_{\mathcal{V}/\mathcal{W}^\perp} k_i$ for $i \in I$. Then,

$$P_{\mathcal{V}/\mathcal{W}^\perp} T_{\mathcal{G}} = P_{\mathcal{V}/\mathcal{W}^\perp} T_{\mathcal{K}} \implies P_{\mathcal{V}/\mathcal{W}^\perp} (T_{\mathcal{G}} - T_{\mathcal{K}}) = 0 \implies R(T_{\mathcal{G}} - T_{\mathcal{K}}) \subseteq \mathcal{W}^\perp .$$

But also $R(T_{\mathcal{G}} - T_{\mathcal{K}}) \subseteq \mathcal{W}$, so $R(T_{\mathcal{G}} - T_{\mathcal{K}}) = \{0\}$ and $T_{\mathcal{G}} = T_{\mathcal{K}}$.

Finally we check that the map is surjective. Recall that, since $\mathcal{W}^\perp \oplus \mathcal{V} = \mathcal{H}$, then the map $P_{\mathcal{V}/\mathcal{W}^\perp}|_{\mathcal{W}} : \mathcal{W} \rightarrow \mathcal{V}$ is a linear bounded isomorphism. Thus, given $\mathcal{K} = \{k_i\}_{i \in I} \in \mathcal{D}_{\mathcal{V}}(\mathcal{F})$ there exists a unique Bessel sequence $\mathcal{G} = \{g_i\}_{i \in I}$ in \mathcal{W} such that $P_{\mathcal{V}/\mathcal{W}^\perp} g_i = k_i$ for $i \in I$. Then, $P_{\mathcal{V}/\mathcal{W}^\perp} T_{\mathcal{G}} = T_{\mathcal{K}}$ and therefore

$$P_{\mathcal{V}/\mathcal{W}^\perp} T_{\mathcal{G}} T_{\mathcal{F}}^* = T_{\mathcal{K}} T_{\mathcal{F}}^* = P_{\mathcal{V}/\mathcal{W}^\perp} T_{\mathcal{G}} T_{\mathcal{F}}^* \implies P_{\mathcal{V}/\mathcal{W}^\perp} (T_{\mathcal{G}} T_{\mathcal{F}}^* - P_{\mathcal{W}}) = 0 .$$

Since $R(T_{\mathcal{G}} T_{\mathcal{F}}^* - P_{\mathcal{W}}) \subseteq \mathcal{W}$ then previous equation implies that $T_{\mathcal{G}} T_{\mathcal{F}}^* - P_{\mathcal{W}} = 0$ and hence $\mathcal{G} \in \mathcal{D}(\mathcal{F})$ is such that $\{P_{\mathcal{V}/\mathcal{W}^\perp} g_i\}_{i \in I} = \mathcal{K}$. \square

The previous result allows to obtain several other representations of the \mathcal{V} -duals of \mathcal{F} from the classical theory of dual frames for \mathcal{F} in \mathcal{W} . The following result is an example of this phenomenon

Corollary 3.2. *Let \mathcal{V} and \mathcal{W} be closed subspaces of \mathcal{H} such that $\mathcal{W}^\perp \oplus \mathcal{V} = \mathcal{H}$. Let $\mathcal{F} = \{f_i\}_{i \in I}$ be a frame for \mathcal{W} with canonical \mathcal{V} -dual frame $\mathcal{F}_{\mathcal{V}}^\# = \{f_{\mathcal{V},i}^\#\}_{i \in I}$ defined in Eq. (5). Given any $\mathcal{G} \in \mathcal{V}^I$, then $\mathcal{G} \in \mathcal{D}_{\mathcal{V}}(\mathcal{F}) \iff$ there exists a Bessel sequence $\mathcal{Z} = \{z_i\}_{i \in I} \in \mathcal{V}^I$ such that*

$$T_{\mathcal{Z}} T_{\mathcal{F}}^* f = \sum_{i \in I} \langle f, f_i \rangle z_i = 0 \quad \text{for every } f \in \mathcal{H} \quad \text{and} \quad \mathcal{G} = \{f_{\mathcal{V},i}^\# + z_i\}_{i \in I} .$$

Proof. Let $\mathcal{K} = \{k_i\}_{i \in I} \in \mathcal{D}_{\mathcal{V}}(\mathcal{F})$ be such that $\mathcal{G} = \{P_{\mathcal{V}/\mathcal{W}^\perp} k_i\}_{i \in I}$ as in Proposition 3.1. Since $\mathcal{K} \in \mathcal{D}_{\mathcal{V}}(\mathcal{F})$, it is well known that there exists a Bessel sequence $\mathcal{X} = \{x_i\}_{i \in I}$ in \mathcal{W} such that $\mathcal{K} = \{f_i^\# + x_i\}_{i \in I}$ and such that $T_{\mathcal{X}} T_{\mathcal{F}}^* = 0$, where $T_{\mathcal{X}}$ denotes the synthesis operator of \mathcal{X} (see for example [7]). Hence, $\mathcal{G} = \{P_{\mathcal{V}/\mathcal{W}^\perp} (f_i^\# + x_i)\}_{i \in I}$ which shows that $T_{\mathcal{G}} = T_{\mathcal{F}_{\mathcal{V}}^\#} + P_{\mathcal{V}/\mathcal{W}^\perp} T_{\mathcal{X}}$ with $(P_{\mathcal{V}/\mathcal{W}^\perp} T_{\mathcal{X}}) T_{\mathcal{F}}^* = 0$ and the result holds for $\mathcal{Z} = \{P_{\mathcal{V}/\mathcal{W}^\perp} x_i\}_{i \in I} \in \mathcal{V}$. The converse is straightforward. \square

From now on, we shall restrict our attention to finite sequences of vectors in \mathcal{H} ; accordingly, we shall consider decompositions $\mathcal{W}^\perp \oplus \mathcal{V} = \mathcal{H}$, where \mathcal{V} and \mathcal{W} are finite dimensional subspaces of the Hilbert space \mathcal{H} .

In what follows, we shall be concerned with the spectral properties of frame operators of \mathcal{V} -duals of \mathcal{F} . Thus, we introduce some convenient notations.

Definition 3.3. Let \mathcal{V} and \mathcal{W} be finite dimensional subspaces of the Hilbert space \mathcal{H} such that $\mathcal{W}^\perp \oplus \mathcal{V} = \mathcal{H}$. Let $\mathcal{F} = \{f_i\}_{i \in \mathbb{I}_n}$ be a frame for \mathcal{W} . We consider

$$\mathcal{SD}_{\mathcal{V}}(\mathcal{F}) \stackrel{\text{def}}{=} \{S_{\mathcal{G}} = T_{\mathcal{G}} T_{\mathcal{G}}^* : \mathcal{G} \in \mathcal{D}_{\mathcal{V}}(\mathcal{F})\} \subset B(\mathcal{H})^+ .$$

the set of frame operators of \mathcal{V} -dual frames of \mathcal{F} . \triangle

Proposition 3.4. Let \mathcal{V} and \mathcal{W} be finite dimensional subspaces of the Hilbert space \mathcal{H} such that $\mathcal{W}^\perp \oplus \mathcal{V} = \mathcal{H}$ and let $\dim \mathcal{V} = \dim \mathcal{W} = d$. Let $\mathcal{F} = \{f_i\}_{i \in \mathbb{I}_n}$ be a frame for \mathcal{W} . Then,

$$\mathcal{SD}_{\mathcal{V}}(\mathcal{F}) = \left\{ S_{\mathcal{F}^\#} + B : B \in B(\mathcal{H})^+ , R(B) \subseteq \mathcal{V} \quad \text{and} \quad \text{rk } B \leq n - d \right\} .$$

Proof. Given $\mathcal{G} \in \mathcal{D}_{\mathcal{V}}(\mathcal{F})$, Corollary 3.2 shows that there exists a Bessel sequence $\mathcal{Z} = \{z_i\}_{i \in \mathbb{I}_n}$ in \mathcal{V} such that $T_{\mathcal{G}} = T_{\mathcal{F}^\#} + T_{\mathcal{Z}}$ and $T_{\mathcal{Z}} T_{\mathcal{F}}^* = 0$. Notice that $T_{\mathcal{F}^\#} = P_{\mathcal{V} // \mathcal{W}^\perp} T_{\mathcal{F}^\#} = P_{\mathcal{V} // \mathcal{W}^\perp} S_{\mathcal{F}}^\dagger T_{\mathcal{F}}$ which implies that $T_{\mathcal{Z}} T_{\mathcal{F}^\#}^* = 0$. Hence,

$$S_{\mathcal{G}} = T_{\mathcal{G}} T_{\mathcal{G}}^* = (T_{\mathcal{F}^\#} + T_{\mathcal{Z}})(T_{\mathcal{F}^\#} + T_{\mathcal{Z}})^* = S_{\mathcal{F}^\#} + S_{\mathcal{Z}} ,$$

where $S_{\mathcal{Z}} \in B(\mathcal{H})^+$ is the frame operator of \mathcal{Z} , which is a finite rank operator. Since $T_{\mathcal{Z}} T_{\mathcal{F}}^* = 0$ then $\dim \ker T_{\mathcal{Z}} \geq d$. Therefore, $R(S_{\mathcal{Z}}) = R(T_{\mathcal{Z}})$ so that $R(S_{\mathcal{Z}}) \subset \mathcal{V}$ and $\text{rk } S_{\mathcal{Z}} = \text{rk } T_{\mathcal{Z}} \leq n - d$.

Conversely, let $B \in B(\mathcal{H})^+$ be such that $R(B) \subseteq \mathcal{V}$ and $\text{rk}(B) \leq n - d$. Then, there exists $Z \in B(\mathbb{C}^n, \mathcal{V})$, such that $Z T_{\mathcal{F}}^* = 0$ and $B = ZZ^*$: indeed, since $\dim(R(T_{\mathcal{F}}^*)^\perp) = n - d$ there exists a partial isometry $W \in B(\mathbb{C}^n, \mathcal{V})$ with initial space $\ker W^\perp \subset R(T_{\mathcal{F}}^*)^\perp$ and final space $R(B) = R(B^{1/2})$ so that $Z = B^{1/2} W$ has the desired properties. If we let $\{e_i\}_{i \in \mathbb{I}_n}$ denote the canonical basis of \mathbb{C}^n and $\mathcal{G} = \{(T_{\mathcal{F}^\#} + Z)e_i\}_{i \in \mathbb{I}_n}$ then \mathcal{G} is a finite sequence in \mathcal{V} such that $T_{\mathcal{G}} = T_{\mathcal{F}^\#} + Z$ so that

$$T_{\mathcal{G}} T_{\mathcal{F}}^* = T_{\mathcal{F}^\#} T_{\mathcal{F}}^* = P_{\mathcal{V} // \mathcal{W}^\perp} .$$

Hence $\mathcal{G} \in \mathcal{D}_{\mathcal{V}}(\mathcal{F})$ and $S_{\mathcal{G}} = S_{\mathcal{F}^\#} + ZZ^* = S_{\mathcal{F}^\#} + B$, since $Z T_{\mathcal{F}}^* = 0$. \square

Remark 3.5. Let $A_0 \in \mathcal{M}_d(\mathbb{C})^+$ and consider an integer $m < d$. Define

$$U(A_0, m) \stackrel{\text{def}}{=} \{A_0 + C : C \in \mathcal{M}_d(\mathbb{C})^+ , \text{rk } C \leq d - m\} . \quad (9)$$

We point out that the spectral structure of the set $U(A_0, m)$ is described in [25]. Indeed, given $\mu \in (\mathbb{R}^d)^\downarrow$ then there exists $A = A_0 + C \in U(A_0, m)$ such that $\lambda(A) = \mu$ (i.e. the eigenvalues of A , counting multiplicities and arranged in non-increasing order, coincide with the entries of μ) if and only if

1. $\mu_i \geq \lambda_i(A_0)$ for $i \in \mathbb{I}_d$, in case $m \leq 0$;
2. $\mu_i \geq \lambda_i(A_0)$ for $i \in \mathbb{I}_d$ and $\mu_{d-m+i} \leq \lambda_i(A_0)$ for $i \in \mathbb{I}_m$, in case $m \geq 1$. \triangle

Recall from Eq. (2) that $\mathbb{M} \subseteq \mathbb{N}$ stands for $\mathbb{M} = \mathbb{I}_p$ or $\mathbb{M} = \mathbb{N}$ in such a way that $\dim \mathcal{H} = |\mathbb{M}|$. Henceforth, $\ell_+^1(\mathbb{M})^\downarrow$ denotes the space of sequences $\lambda = (\lambda_i)_{i \in \mathbb{M}}$ with $\lambda_i \geq \lambda_j \geq 0$ for $i, j \in \mathbb{M}$ such that $i \leq j$ and $\text{tr}(\lambda) \stackrel{\text{def}}{=} \sum_{i \in \mathbb{M}} \lambda_i < \infty$.

Let $\mathcal{F} = \{f_i\}_{i \in \mathbb{I}_n} \in \mathcal{H}^n$ be a finite sequence with frame operator $S_{\mathcal{F}} \in B(\mathcal{H})^+$. Hence, $S_{\mathcal{F}}$ is a positive semidefinite finite rank operator, with range $\mathcal{W} = \text{Span}\{f_i : i \in \mathbb{I}_n\} \subset \mathcal{H}$. Let

$d = \dim \mathcal{W}$, and let $(S_{\mathcal{F}})_{\mathcal{W}} \in B(\mathcal{W})^+$ be the compression of $S_{\mathcal{F}}$ to \mathcal{W} (see Remark 2.3); Then, we define

$$\lambda(S_{\mathcal{F}}) = ((\lambda_i((S_{\mathcal{F}})_{\mathcal{W}}))_{i \in \mathbb{I}_d}, 0_{|\mathbb{M}|-d}) \in \ell_+^1(\mathbb{M})^\downarrow,$$

where $(\lambda_i((S_{\mathcal{F}})_{\mathcal{W}}))_{i \in \mathbb{I}_d} \in \mathbb{R}_{>0}^d$ denotes the vector of eigenvalues of the compression $(S_{\mathcal{F}})_{\mathcal{W}} \in B(\mathcal{W})^+$, counting multiplicities and arranged in non-increasing order. It is straightforward to check that $\lambda(S_{\mathcal{F}}) \in \ell_+^1(\mathbb{M})^\downarrow$ coincides with the vector of singular values (or s-numbers) of the compact operator $S_{\mathcal{F}} \in B(\mathcal{H})^+$ (see [28]).

Theorem 3.6 (Spectral structure of \mathcal{V} -duals). *Let \mathcal{V} and \mathcal{W} be finite dimensional subspaces of the Hilbert space \mathcal{H} such that $\mathcal{W}^\perp \oplus \mathcal{V} = \mathcal{H}$ and let $\dim \mathcal{V} = \dim \mathcal{W} = d$. Let $\mathcal{F} = \{f_i\}_{i \in \mathbb{I}_n}$ be a frame for \mathcal{W} and denote $\lambda(S_{\mathcal{F}^\#_{\mathcal{V}}}) = \lambda_{\mathcal{V}}^\# = (\lambda_{\mathcal{V},j}^\#)_{j \in \mathbb{M}}$ and $m = 2d - n$. Given $\mu = (\mu_i)_{i \in \mathbb{M}} \in \ell_+^1(\mathbb{M})^\downarrow$, the following conditions are equivalent:*

1. *There exists $\mathcal{G} \in \mathcal{D}_{\mathcal{V}}(\mathcal{F})$ such that $\lambda(S_{\mathcal{G}}) = \mu$;*

2. *$\mu_i = 0$ for $i \geq d + 1$ and:*

(a) *$\mu_i \geq \lambda_{\mathcal{V},i}^\#$ for $i \in \mathbb{I}_d$, in case $m \leq 0$;*

(b) *$\mu_i \geq \lambda_{\mathcal{V},i}^\#$ for $i \in \mathbb{I}_d$ and $\mu_{d-m+i} \leq \lambda_{\mathcal{V},i}^\#$ for $i \in \mathbb{I}_m$, in case $m \geq 1$.*

Proof. Fix an ONB $\{v_i\}_{i \in \mathbb{I}_d}$ of \mathcal{V} . Let $A_0 \in \mathcal{M}_d(\mathbb{C})^+$ be given by $A_0 = (\langle S_{\mathcal{F}^\#_{\mathcal{V}}} v_j, v_i \rangle)_{i,j \in \mathbb{I}_d}$ and let $m = 2d - n$ (so that $d - m = n - d$). Then $\lambda(S_{\mathcal{F}^\#_{\mathcal{V}}}) = (\lambda(A_0), 0_{|\mathbb{M}|-d}) \in (\ell_+^1(\mathbb{M}))^\downarrow$. Using Proposition 3.4, to each $S_{\mathcal{G}} = S_{\mathcal{F}^\#_{\mathcal{V}}} + B \in \mathcal{SD}_{\mathcal{V}}(\mathcal{F})$ we can associate the element $A_0 + C \in U(A_0, m)$ where $C = (\langle B v_j, v_i \rangle)_{i,j \in \mathbb{I}_d} \in \mathcal{M}_d(\mathbb{C})^+$ in such a way that

$$\lambda(S_{\mathcal{G}}) = (\lambda(A_0 + C), 0_{|\mathbb{M}|-d}) \in \ell_+^1(\mathbb{M})^\downarrow.$$

Conversely, if $A_0 + C \in U(A_0, m)$ then there exists $\mathcal{G} \in \mathcal{D}_{\mathcal{V}}(\mathcal{F})$ such that the matrix corresponding to $S_{\mathcal{G}}$ as above is $A_0 + C$. Thus, the previous remarks show that

$$\{\lambda(S_{\mathcal{G}}) : \mathcal{G} \in \mathcal{D}_{\mathcal{V}}(\mathcal{F})\} = \{(\lambda(A), 0_{|\mathbb{M}|-d}) : A \in U(A_0, m)\}. \quad (10)$$

The proof now follows from Eq. (10) above and Remark 3.5. \square

Remark 3.7. Using Theorem 3.6, in case $\mathcal{V} = \mathcal{W}$ (i.e. classical duality) we recover the structure of classical duals of a frame \mathcal{F} for the Hilbert space \mathcal{W} as described in [25]. \triangle

Corollary 3.8. *There exists $\mathcal{G} \in \mathcal{D}_{\mathcal{V}}(\mathcal{F})$ which is Parseval in \mathcal{V} if and only if:*

(a) *$1 \geq \lambda_{\mathcal{V},i}^\#$ for $i \in \mathbb{I}_d$, in case $m = 2d - n \leq 0$;*

(b) *$1 \geq \lambda_{\mathcal{V},i}^\#$ for $i \in \mathbb{I}_d$ and $1 = \lambda_{\mathcal{V},i}^\#$ for $i \in \mathbb{I}_m$, in case $d - 1 \geq m = 2d - n \geq 1$.*

Proof. Let \mathcal{G} be a frame for \mathcal{V} . Notice that \mathcal{G} is a Parseval in \mathcal{V} , i.e. $S_{\mathcal{G}} = P_{\mathcal{V}}$, if and only if $\lambda_i(S_{\mathcal{G}}) = 1$ for every $i \in \mathbb{I}_d$. Thus, the result now follows from Theorem 3.6. \square

Remark 3.9. With the notations and terminology from Theorem 3.6, notice that Corollary 3.8 can be written as follows: there exists $\mathcal{G} \in \mathcal{D}_{\mathcal{V}}(\mathcal{F})$ which is Parseval in \mathcal{V} if and only if

$$S_{\mathcal{F}^\#_{\mathcal{V}}} \leq P_{\mathcal{V}} \quad \text{and} \quad \dim R(P_{\mathcal{V}} - S_{\mathcal{F}^\#_{\mathcal{V}}}) \leq d - m = n - d = \dim \ker T_{\mathcal{F}}.$$

This last formulation of the existence of Parseval \mathcal{V} -duals formally resembles the characterization in [18, Proposition 2.4] in case of classical duality. \triangle

3.2 Optimal oblique duals with norm restrictions

In applied situations, it is desired to characterize the existence (and find explicit methods of construction) of frames with some prescribed parameters. This kind of problems are referred to as *frame design problems*, and they are at the core of finite frame theory (see for example [4, 5, 16, 23, 25, 26, 27] and the recent book [6]).

Let $\mathcal{W}, \mathcal{V} \subset \mathcal{H}$ be two finite dimensional subspaces such that $\mathcal{W} \oplus \mathcal{V}^\perp = \mathcal{H}$, and let $\dim \mathcal{W} = \dim \mathcal{V} = d$. Given a fixed frame $\mathcal{F} = \{f_i\}_{i \in \mathbb{I}_n} \in \mathcal{W}^n$ for \mathcal{W} we can ask whether there exists $\mathcal{G} \in \mathcal{D}_{\mathcal{V}}(\mathcal{F})$ with some prescribed parameters; and in case such a dual exists we would like to obtain a procedure to construct it. For example, given $\mu \in (\mathbb{R}_{\geq 0}^d)^\downarrow$ we can ask whether there exists $\mathcal{G} \in \mathcal{D}_{\mathcal{V}}(\mathcal{F})$ with $\lambda(S_{\mathcal{G}}) = \mu$. Notice that Theorem 3.6 above completely solves this problem; moreover, the proof of Proposition 3.4 contains a procedure to effectively construct such a dual.

As a consequence of the description of the spectra of elements in $\mathcal{S}(\mathcal{D}_{\mathcal{V}}(\mathcal{F}))$, we see that if $\mathcal{G} \in \mathcal{D}_{\mathcal{V}}(\mathcal{F})$ then $S_{\mathcal{G}} \geq S_{\mathcal{F}^\#}$. This last fact implies that the canonical \mathcal{V} -dual is optimal with respect to several criteria (including convex potentials). Yet, from a numerical point of view the oblique canonical \mathcal{V} -dual might not be the best choice of a \mathcal{V} -dual for \mathcal{F} . For example, the condition number of the frame operator $S_{\mathcal{F}^\#}$ may not be minimal in $\mathcal{D}_{\mathcal{V}}(\mathcal{F})$; indeed, Corollary 3.8 shows that under certain assumptions we can consider a Parseval \mathcal{V} -dual of \mathcal{F} (with minimal condition number).

In order to search for alternate \mathcal{V} -duals that are numerically robust, we proceed as follows: for $t \geq \text{tr}(S_{\mathcal{F}^\#})$ we consider

$$\mathcal{D}_{\mathcal{V},t}(\mathcal{F}) \stackrel{\text{def}}{=} \left\{ \mathcal{G} = \{g_i\}_{i \in \mathbb{I}_n} \in \mathcal{D}_{\mathcal{V}}(\mathcal{F}) : \sum_{i \in \mathbb{I}_n} \|g_i\|^2 \geq t \right\} .$$

Notice that if $t > \text{tr}(S_{\mathcal{F}^\#})$ then the canonical \mathcal{V} -dual is not in $\mathcal{D}_{\mathcal{V},t}(\mathcal{F})$ and therefore, it is natural to ask whether there is an optimal dual in $\mathcal{D}_{\mathcal{V},t}(\mathcal{F})$. Using the well known identity

$$\sum_{i \in \mathbb{I}_n} \|g_i\|^2 = \text{tr}(S_{\mathcal{G}}) = \sum_{i \in \mathbb{I}_d} \mu_i , \quad (11)$$

where $\lambda(S_{\mathcal{G}}) = \mu$, we see that Theorem 3.6 gives a complete solution to a frame design problem in the sense that it allows to get a complete description of the eigenvalue lists of the frame operators of elements in $\mathcal{D}_{\mathcal{V},t}(\mathcal{F})$.

Remark 3.10. Let $A_0 \in \mathcal{M}_d(\mathbb{C})^+$, $t \geq \text{tr}(A_0) \geq 0$ and consider an integer $m < d$. Define

$$U_t(A_0, m) \stackrel{\text{def}}{=} \{A_0 + C : C \in \mathcal{M}_d(\mathbb{C})^+, \text{rk}(C) \leq d - m, \text{tr}(A_0 + C) \geq t\} .$$

The spectral and geometrical structure of the set $U_t(A_0, m)$ is described in [25]; in particular, there it is shown that there exist \prec_w -minimizers within this set. Indeed, using the previous notations, if $\lambda(A_0) = \lambda = (\lambda_i)_{i \in \mathbb{I}_d} \in (\mathbb{R}_{\geq 0}^d)^\downarrow$, we consider $h_m : [\lambda_d, \infty) \rightarrow \mathbb{R}_{\geq 0}$ given by

$$h_m(t) = \sum_{i=\max\{m, 0\}+1}^d (t - \lambda_i)^+ ,$$

where x^+ stands for the positive part of x . Notice that h_m is strictly increasing; hence there exists a unique $c_{\lambda, m}(t) = c \geq \lambda_d$ such that $h_m(c) = t - \text{tr} \lambda$. Then, set

1. $\nu_{\lambda, m}(t) \stackrel{\text{def}}{=}} ((c - \lambda_1)^+ + \lambda_1, \dots, (c - \lambda_d)^+ + \lambda_d) \in (\mathbb{R}_{\geq 0}^d)^\downarrow$, if $m \leq 0$;
2. $\nu_{\lambda, m}(t) \stackrel{\text{def}}{=}} (\lambda_1, \dots, \lambda_m, (c - \lambda_{m+1})^+ + \lambda_{m+1}, \dots, (c - \lambda_d)^+ + \lambda_d) \in \mathbb{R}_{\geq 0}^d$, if $m \in \mathbb{I}_{d-1}$.

Notice that if $t > \text{tr}(A_0)$ then $\nu_{\lambda, m}(t) \in \mathbb{R}_{> 0}^d$. Then, it turns out that (see [25])

1. There exists $A^{\text{op}} \in U_t(A_0, m)$ such that $\lambda(A^{\text{op}}) = \nu_{\lambda, m}(t)^\downarrow$;
2. For every $A \in U_t(A_0, m)$ then $\nu_{\lambda, m}(t) \prec_w \lambda(A)$;
3. If $A = A_0 + B \in U_t(A_0, m)$ then $\lambda(A) = \nu_{\lambda, m}(t)$ if and only if $\nu_{\lambda, m}(t) = \lambda(A_0) + \lambda^\uparrow(B)$ and there exists an ONB $\{z_i\}_{i \in \mathbb{I}_d}$ of \mathbb{C}^d such that

$$A_0 = \sum_{i \in \mathbb{I}_d} \lambda_i z_i \otimes z_i \quad \text{and} \quad B = \sum_{i \in \mathbb{I}_d} \lambda_{d-i+1}(B) z_i \otimes z_i. \quad \triangle$$

The following result shows that there are structural minimizers of arbitrary (strictly) convex potentials in $\mathcal{D}_{\mathcal{V}, t}(\mathcal{F})$, i.e. duals $\mathcal{G} \in \mathcal{D}_{\mathcal{V}, t}(\mathcal{F})$ that simultaneously minimize every convex potential. This is interesting from an applied point of view, since evaluations of convex potentials (e.g. the frame potential as described in Eq. (6)) are typically easier to compute than structural parameters (i.e. computing eigenvalue lists or eigenvectors)

Theorem 3.11 (Optimal duals in $\mathcal{D}_{\mathcal{V}, t}(\mathcal{F})$). *Let \mathcal{V} and \mathcal{W} be finite dimensional subspaces of the Hilbert space \mathcal{H} such that $\mathcal{W}^\perp \oplus \mathcal{V} = \mathcal{H}$ and let $d = \dim \mathcal{V} = \dim \mathcal{W}$. Let $\mathcal{F} = \{f_i\}_{i \in \mathbb{I}_n}$ be a frame for \mathcal{W} and set $\lambda_{\mathcal{V}}^\# \stackrel{\text{def}}{=}} \lambda(S_{\mathcal{F}_{\mathcal{V}}^\#})$. For every $t \geq \text{tr} \lambda_{\mathcal{V}}^\#$ there exists $\nu \in \ell_+^1(\mathbb{M})^\downarrow$ with the following minimality properties:*

1. There exist $\mathcal{G}_{\text{op}} \in \mathcal{D}_{\mathcal{V}, t}(\mathcal{F})$ such that $\lambda(S_{\mathcal{G}_{\text{op}}}) = \nu$;
2. For every non-decreasing function $h \in \text{Conv}(\mathbb{R}_{\geq 0})$ then

$$P_h(\mathcal{G}_{\text{op}}) \leq P_h(\mathcal{G}) \quad , \quad \mathcal{G} = \{g_i\}_{i \in \mathbb{I}_n} \in \mathcal{D}_{\mathcal{V}, t}(\mathcal{F}). \quad (12)$$

Moreover, if we assume further that $h \in \text{Conv}_s(\mathbb{R}_{\geq 0})$ and $\mathcal{G} = \{g_i\}_{i \in \mathbb{I}_n} \in \mathcal{D}_{\mathcal{V}, t}(\mathcal{F})$ attains equality in (12), then $\lambda(S_{\mathcal{G}}) = \nu$ and there exists an ONB $\{x_i\}_{i \in \mathbb{I}_d}$ for \mathcal{V} such that

$$S_{\mathcal{F}_{\mathcal{V}}^\#} = \sum_{i \in \mathbb{I}_d} \lambda_{\mathcal{V}, i}^\# x_i \otimes x_i \quad \text{and} \quad B = S_{\mathcal{G}} - S_{\mathcal{F}_{\mathcal{V}}^\#} = \sum_{i \in \mathbb{I}_d} \lambda_{d-i+1}(B) x_i \otimes x_i.$$

Proof. Fix $\{v_i\}_{i \in \mathbb{I}_d}$ an ONB of \mathcal{V} and let $A_0 \in \mathcal{M}_d(\mathbb{C})^+$ be given by $A_0 = (\langle S_{\mathcal{F}_{\mathcal{V}}^\#} v_j, v_i \rangle)_{i, j \in \mathbb{I}_d}$. Arguing as in the proof of Theorem 3.6, and taking into account the identity in Eq. (11) we see that

$$\{\lambda(S_{\mathcal{G}}) : \mathcal{G} \in \mathcal{D}_{\mathcal{V}, t}(\mathcal{F})\} = \{(\lambda(A), 0_{|\mathbb{M}|-d}) : A \in U_t(A_0, m)\}. \quad (13)$$

Set $\lambda = \lambda(A_0) = (\lambda_{\mathcal{V}, i}^\#)_{i \in \mathbb{I}_d} \in \mathbb{R}_{> 0}^d$, and notice that $t \geq \text{tr} \lambda$; set $m = 2d - n$ and consider $\nu_{\lambda, m}(t) \in \mathbb{R}_{> 0}^d$ defined as in Remark 3.10. Finally, define

$$\nu = (\nu_{\lambda, m}(t), 0_{|\mathbb{M}|-d}) \in \ell_+^1(\mathbb{M}).$$

If $h \in \text{Conv}(\mathbb{R}_{\geq 0})$ is a non-decreasing function then, by Definition 2.2, we get that

$$P_h(\mathcal{G}) = \sum_{i \in \mathbb{I}_d} h(\lambda_i(S_{\mathcal{G}})) = \text{tr}((S_{\mathcal{G}})_{\mathcal{V}}) \quad \text{for} \quad \mathcal{G} \in \mathcal{D}_{\mathcal{V}}(\mathcal{F}). \quad (14)$$

Hence, the proof now follows from Eqs. (13) and (14) above, Remark 3.10 and the relation between submajorization and non-decreasing convex functions described in Section 2.2. \square

4 Optimal $(\mathcal{V}, \mathcal{W})$ -oblique dual pairs with prescribed parameters

It has long been recognized that for a fixed frame \mathcal{F} for \mathcal{W} , oblique \mathcal{V} -duality offers a much more flexible theory than classical duality, which comes from the fact that we can choose \mathcal{V} from a large class of subspaces (see [9] for example). Moreover, it has also been noticed that the relative position of the subspaces \mathcal{V} and \mathcal{W} for which $\mathcal{W} \oplus \mathcal{V}^\perp = \mathcal{H}$ plays a key role when comparing oblique duality to classical duality. This phenomenon has been studied mainly through the angle between the subspaces \mathcal{V} and \mathcal{W} (see [8, 9, 14] and the definitions below). Yet, the angle between the subspaces \mathcal{V} and \mathcal{W} only provides qualitative measure of the role of the relative geometry of \mathcal{V} and \mathcal{W} in the context of \mathcal{V} -duality. In what follows, we give a detailed description of the role of the relative position of \mathcal{V} and \mathcal{W} in the \mathcal{V} -duality of \mathcal{F} in case the subspaces are finite dimensional. Our analysis relies on a multiplicative Lidskii's inequality and it is based on the complete list of the so-called principal angles between \mathcal{V} and \mathcal{W} . Our results provide sharp quantitative measures of these relations.

4.1 Relative geometry between finite dimensional subspaces

We begin by describing the principal angles and vectors between finite dimensional subspaces. Let $\mathcal{V}, \mathcal{W} \subseteq \mathcal{H}$ be finite dimensional subspaces with $\dim \mathcal{V} = \dim \mathcal{W} = d$. Let $P_{\mathcal{V}}$ and $P_{\mathcal{W}}$ denote the orthogonal projections onto \mathcal{V} and \mathcal{W} respectively. The principal angles

$$0 \leq \theta_1 \leq \dots \leq \theta_d \leq \frac{\pi}{2}$$

are defined (see [17, 22]) in such a way that the positive finite rank operator $|P_{\mathcal{W}} P_{\mathcal{V}}| \in B(\mathcal{H})$ satisfies that

$$\lambda(|P_{\mathcal{W}} P_{\mathcal{V}}|) = (\cos(\theta_1), \dots, \cos(\theta_d), 0_{|\mathbb{M}|-d}) \in \ell_+^1(\mathbb{M})^\downarrow.$$

We say that $w_1, \dots, w_d \in \mathcal{W}$ (respectively $v_1, \dots, v_d \in \mathcal{V}$) are principal vectors (or principal directions) between \mathcal{V} and \mathcal{W} if they are an o.n. basis of \mathcal{W} (respectively if they are an o.n. basis of \mathcal{V}) such that

$$|P_{\mathcal{V}} P_{\mathcal{W}}| w_i = \cos(\theta_i) w_i \quad (\text{ respectively } |P_{\mathcal{W}} P_{\mathcal{V}}| v_i = \cos(\theta_i) v_i) \quad \text{for every } i \in \mathbb{I}_d. \quad (15)$$

An alternative characterization of principal angles and vectors is as follows: given $k \in \mathbb{I}_d$, then define inductively

$$\langle v_k, w_k \rangle = \cos(\theta_k) = \max_{v \in \mathcal{V}} \max_{w \in \mathcal{W}} \langle v, w \rangle$$

subject to the restrictions

$$\|v\| = \|w\| = 1, \quad \langle v, v_i \rangle = 0 \quad \text{and} \quad \langle w, w_i \rangle = 0 \quad \text{for } 0 \leq i \leq k-1,$$

where we set $v_0 = w_0 = 0$. Notice that the principal angles between \mathcal{V} and \mathcal{W} are a qualitative measure of the relative position between these subspaces.

Assume further that $\mathcal{W}^\perp \oplus \mathcal{V} = \mathcal{H}$. Consider the oblique projection $P_{\mathcal{V}/\mathcal{W}^\perp}$. In this case, there exists a connection between the principal angles and vectors between \mathcal{V} and \mathcal{W} and the geometrical and spectral structure of $P_{\mathcal{V}/\mathcal{W}^\perp}$. Indeed, it is known (see [11] and the references therein) that the Moore-Penrose pseudo-inverse of $P_{\mathcal{V}/\mathcal{W}^\perp}$ is given by

$$(P_{\mathcal{V}/\mathcal{W}^\perp})^\dagger = P_{\mathcal{W}} P_{\mathcal{V}} \implies |P_{\mathcal{V}/\mathcal{W}^\perp}|^\dagger = |P_{\mathcal{V}} P_{\mathcal{W}}| \quad \text{and} \quad |(P_{\mathcal{W}/\mathcal{V}^\perp})^*|^\dagger = |P_{\mathcal{W}} P_{\mathcal{V}}|. \quad (16)$$

In this case, since $P_{\mathcal{V}/\mathcal{W}^\perp}$ has rank d , then $|P_{\mathcal{V}} P_{\mathcal{W}}|$ also has rank d and therefore $\theta_d < \pi/2$. Moreover, by Eq. (15), the principal vectors between \mathcal{V} and \mathcal{W} satisfy that

$$|P_{\mathcal{V}/\mathcal{W}^\perp}| w_i = \frac{1}{\cos(\theta_i)} w_i \quad \text{and} \quad |P_{\mathcal{W}/\mathcal{V}^\perp}| v_i = \frac{1}{\cos(\theta_i)} v_i \quad \text{for every } i \in \mathbb{I}_d. \quad (17)$$

Take the polar decomposition $P_{\mathcal{V}} P_{\mathcal{W}} = U |P_{\mathcal{V}} P_{\mathcal{W}}|$ with the unique partial isometry $U \in B(\mathcal{H})$ with initial space \mathcal{W} and final space \mathcal{V} . Hence we have that $|P_{\mathcal{W}} P_{\mathcal{V}}| = U |P_{\mathcal{V}} P_{\mathcal{W}}| U^*$. Therefore, given principal vectors $\{w_i\}_{i \in \mathbb{I}_d} \in \mathcal{W}^d$ between \mathcal{V} and \mathcal{W} , in what follows we shall assume that the corresponding principal vectors $\{v_i\}_{i \in \mathbb{I}_d} \in \mathcal{V}^d$ between \mathcal{V} and \mathcal{W} are given by $v_i = U w_i$ for every $i \in \mathbb{I}_d$. In particular, it holds that

$$P_{\mathcal{W}} v_i = \cos(\theta_i) w_i \quad \text{and} \quad P_{\mathcal{V}} w_i = \cos(\theta_i) v_i \quad \text{for every } i \in \mathbb{I}_d, \quad (18)$$

because, for example, $P_{\mathcal{V}} w_i = P_{\mathcal{V}} P_{\mathcal{W}} w_i = U |P_{\mathcal{V}} P_{\mathcal{W}}| w_i = \cos(\theta_i) U w_i = \cos(\theta_i) v_i$.

Remark 4.1 (On two notions of angle between subspaces). There are two different notions of angle between subspaces in the literature. Next we include their definitions, we compare them and we also relate them to the principal angles defined above. Hence, consider two finite dimensional subspaces $\mathcal{V}, \mathcal{W} \subseteq \mathcal{H}$ with $\dim \mathcal{V} = \dim \mathcal{W} = d$. Let $(\theta_j)_{j \in \mathbb{I}_d}$ denote the principal angles between \mathcal{V} and \mathcal{W} .

1. In [29] the authors introduce the angle $\theta_{\mathcal{V}, \mathcal{W}} \in [0, \pi/2]$ between \mathcal{V} and \mathcal{W} defined by

$$\cos(\theta_{\mathcal{V}, \mathcal{W}}) = \inf_{f \in \mathcal{W}, \|f\|=1} \|P_{\mathcal{V}} f\|.$$

Therefore,

$$\cos(\theta_{\mathcal{V}, \mathcal{W}})^2 = \inf_{f \in \mathcal{W}, \|f\|=1} \langle |P_{\mathcal{V}} P_{\mathcal{W}}|^2 f, f \rangle = \cos(\theta_d)^2. \quad (19)$$

That is, we have the identity $\theta_{\mathcal{V}, \mathcal{W}} = \theta_d$. If we assume further that $\mathcal{W}^\perp \oplus \mathcal{V} = \mathcal{H}$ then Eqs. (17) and (19) provide a simple proof of the identity $\|P_{\mathcal{V}/\mathcal{W}^\perp}\| = \cos(\theta_{\mathcal{V}, \mathcal{W}})^{-1}$.

2. There is yet another notion of angle between subspaces, the so-called Dixmier angle, denoted by $\theta^{\mathcal{V}, \mathcal{W}} \in [0, \pi/2]$ and given by

$$\cos(\theta^{\mathcal{V}, \mathcal{W}}) = \sup_{v \in \mathcal{V}, w \in \mathcal{W}, \|v\|=\|w\|=1} |\langle v, w \rangle| = \|P_{\mathcal{V}} P_{\mathcal{W}}\| = \cos(\theta_1).$$

That is, we have the identity $\theta^{\mathcal{V}, \mathcal{W}} = \theta_1$. If we assume further that $\mathcal{W}^\perp \oplus \mathcal{V} = \mathcal{H}$ then it is well known (see [12]) that $\|P_{\mathcal{V}/\mathcal{W}^\perp}\| = \sin(\theta^{\mathcal{V}, \mathcal{W}^\perp})^{-1}$ which implies that $\sin(\theta^{\mathcal{V}, \mathcal{W}^\perp}) = \cos(\theta_{\mathcal{V}, \mathcal{W}})$ and hence we get that

$$\theta_{\mathcal{V}, \mathcal{W}} = \pi/2 - \theta^{\mathcal{V}, \mathcal{W}^\perp}. \quad \triangle$$

4.2 Optimal dual pairs by rigid rotations

We begin by fixing the following notations:

Notations 4.2. Throughout the rest of the paper we shall consider

1. $\mathcal{V}, \mathcal{W} \subset \mathcal{H}$ two finite dimensional subspaces such that $\mathcal{V} \oplus \mathcal{W}^\perp = \mathcal{H}$;

2. $\angle(\mathcal{V}; \mathcal{W}) = (\theta_j)_{j \in \mathbb{I}_d} \in ([0, \pi/2)^d)^\uparrow$ principal angles ($\dim \mathcal{V} = \dim \mathcal{W} = d$);
3. $\{v_j\}_{j \in \mathbb{I}_d} \in \mathcal{V}^d$, $\{w_j\}_{j \in \mathbb{I}_d} \in \mathcal{W}^d$ principal vectors in \mathcal{V} and \mathcal{W} obeying Eq. (18);
4. $\mathcal{F} = \{f_i\}_{i \in \mathbb{I}_n} \in \mathcal{W}^n$ a frame for \mathcal{W} with

$$\lambda(S_{\mathcal{F}}) = \lambda = (\lambda_i)_{i \in \mathbb{M}} \quad \text{and} \quad \lambda(S_{\mathcal{F}^\#}) = \lambda_{\mathcal{V}}^\# = (\lambda_{\mathcal{V},i}^\#)_{i \in \mathbb{M}}. \quad \triangle$$

Consider the Notations 4.2. In order to have an estimate of $\lambda_{\mathcal{V}}^\#$ notice that

$$S_{\mathcal{F}^\#} = P_{\mathcal{V} // \mathcal{W}^\perp} \cdot \mathcal{F}^\# \implies T_{\mathcal{F}^\#} = P_{\mathcal{V} // \mathcal{W}^\perp} S_{\mathcal{F}}^\dagger T_{\mathcal{F}}$$

and hence

$$S_{\mathcal{F}^\#} = P_{\mathcal{V} // \mathcal{W}^\perp} S_{\mathcal{F}}^\dagger (P_{\mathcal{V} // \mathcal{W}^\perp})^* = P_{\mathcal{V} // \mathcal{W}^\perp} S_{\mathcal{F}}^\dagger P_{\mathcal{W} // \mathcal{V}^\perp}. \quad (20)$$

The previous remarks, together with Lidskii's multiplicative inequalities in Theorem 2.4, allow us to obtain the following bounds in terms of the spectral structure of $S_{\mathcal{F}}$ and the principal angles (i.e. the relative geometry) between \mathcal{V} and \mathcal{W} . We point out that the bounds given in next result are a quantitative measure of how the relative geometry of the subspaces \mathcal{V} and \mathcal{W} impact in oblique duality.

Theorem 4.3. *Consider the Notations 4.2. If we let $\mu = (\lambda_{d-j+1}^{-1} \cos^{-2}(\theta_j))_{j \in \mathbb{I}_d}^\downarrow$ then*

$$\prod_{j \in \mathbb{I}_k} \mu_j \leq \prod_{j \in \mathbb{I}_k} \lambda_{\mathcal{V},j}^\# \leq \left(\prod_{j=d-k+1}^d \lambda_j \cos^2(\theta_j) \right)^{-1}, \quad k \in \mathbb{I}_d. \quad (21)$$

Moreover, $\lambda_{\mathcal{V}}^\# = \mu$ (resp. $\lambda_{\mathcal{V}}^\# = (\lambda_{d-j+1}^{-1} \cos^{-2}(\theta_{d-j+1}))_{j \in \mathbb{I}_d}^\downarrow$) \iff there exist an o.n.b. in \mathcal{W} of principal vectors $\{w_i\}_{i \in \mathbb{I}_d}$ between \mathcal{V} and \mathcal{W} such that

$$S_{\mathcal{F}} = \sum_{j \in \mathbb{I}_d} \lambda_{d-j+1}(S_{\mathcal{F}}) w_j \otimes w_j$$

(resp. $S_{\mathcal{F}} = \sum_{j \in \mathbb{I}_d} \lambda_j(S_{\mathcal{F}}) w_j \otimes w_j$).

Proof. Consider the representation of $S_{\mathcal{F}^\#}$ given in Eq. (20). Denote by $M = |P_{\mathcal{V} // \mathcal{W}^\perp}|$ and let $P_{\mathcal{V} // \mathcal{W}^\perp} = V M$ be the polar decomposition. Notice that $R(M) = \mathcal{W}$; hence, M restricted to (the reducing subspace) \mathcal{W} is invertible. On the other hand, since $R(S_{\mathcal{F}^\#}) = \mathcal{V}$ then the restriction of $S_{\mathcal{F}^\#} = S_{\mathcal{F}}^\dagger$ to (the reducing subspace) \mathcal{W} is also invertible. Hence, Eq. (20) implies that

$$S_{\mathcal{F}^\#} = V (M S_{\mathcal{F}}^\dagger M) V^*. \quad (22)$$

Since V is a partial isometry with initial space \mathcal{W} and final space \mathcal{V} then, Eq. (22) implies that

$$\lambda(S_{\mathcal{F}^\#}) = (\lambda(M_{\mathcal{W}}(S_{\mathcal{F}}^\dagger)_{\mathcal{W}} M_{\mathcal{W}}), 0_{|\mathbb{M}|-d}), \quad (23)$$

where in general $S_{\mathcal{W}} \in B(\mathcal{W})$ denotes the restriction of S to its reducing subspace \mathcal{W} . Since $\lambda((S_{\mathcal{F}}^\dagger)_{\mathcal{W}}) = (\lambda_d^{-1}, \dots, \lambda_1^{-1})$ and $\lambda(M_{\mathcal{W}}) \stackrel{(17)}{=} (\cos(\theta_d)^{-1}, \dots, \cos(\theta_1)^{-1})$, we see that the result is now a consequence of Theorem 2.4 (Lidskii's multiplicative inequalities) and the definition of log-majorization. \square

Consider the Notations 4.2. The previous result suggests that we could take advantage of the relative geometry between the subspaces \mathcal{V} and \mathcal{W} to construct optimal oblique encoding-decoding schemes with prescribed properties. Indeed, let $U \in B(\mathcal{H})$ be a unitary operator such that $U(\mathcal{W}) \subset \mathcal{W}$ i.e. \mathcal{W} is U -invariant. Hence, we could consider the frame $U \cdot \mathcal{F} = \{U f_j\}_{j \in \mathbb{I}_n}$ for \mathcal{W} . Notice that $U \cdot \mathcal{F}$ preserves essentially every property of \mathcal{F} (e.g., linear relations, eigenvalues list of its frame operator, norms of the elements of the frame, etc). In particular, $(U \cdot \mathcal{F})^\# = U \cdot \mathcal{F}^\#$ since $S_{U \cdot \mathcal{F}} = U T_{\mathcal{F}} T_{\mathcal{F}}^* U^* = U S_{\mathcal{F}} U^*$; that is, the (classical) canonical dual frame of $U \cdot \mathcal{F}$ in \mathcal{W} is the rotation by U of the canonical frame for \mathcal{F} in \mathcal{W} . In particular, we get that $S_{U \cdot \mathcal{F}}^\dagger = U S_{\mathcal{F}}^\dagger U^*$. Nevertheless, \mathcal{F} and $U \cdot \mathcal{F}$ can have quite different properties with respect to \mathcal{V} -duality as shown in the following

Example 4.4. Let $\mathcal{H} = \mathbb{C}^3$ and let $\{e_1, e_2, e_3\}$ denote the canonical basis of \mathcal{H} . Set $\mathcal{V} = \overline{\{e_2, \frac{e_1 + e_3}{\sqrt{2}}\}}$ and $\mathcal{W} = \overline{\{e_1, e_2\}}$. Notice that in this case we have that $\mathbb{C}^3 = \mathcal{V} \oplus \mathcal{W}^\perp$. Set

$$\begin{aligned} \mathcal{F}_1 &= \{e_1; (\cos(\pi/3), \sin(\pi/3), 0)\} \subseteq \mathcal{W}, \\ \mathcal{F}_2 &= \{e_2; (\cos(\pi/2 + \pi/3), \sin(\pi/2 + \pi/3), 0)\} \subseteq \mathcal{W}. \end{aligned}$$

Notice that $\mathcal{F}_2 = U \cdot \mathcal{F}_1$ where U is the rotation by (the angle) $\pi/2$ in the plane \mathcal{W} and such that $U e_3 = e_3$. Straightforward computations show that $\lambda(S_{(\mathcal{F}_1)_{\mathcal{V}}^\#}) = (8/3; 1; 0)$ and $\lambda(S_{(\mathcal{F}_2)_{\mathcal{V}}^\#}) = (3.59; 0.74; 0)$. This last fact shows that there is no unitary operator U' such that $U'(\mathcal{V}) = \mathcal{V}$ and such that $(\mathcal{F}_2)_{\mathcal{V}}^\# = U'(\mathcal{F}_1)_{\mathcal{V}}^\#$. \triangle

The previous example shows that the spectral properties of the \mathcal{V} -canonical dual of a frame $U \cdot \mathcal{F}$ indeed depend on U and motivates the construction of unitary operators $U \in B(\mathcal{H})$ with $U(\mathcal{W}) = \mathcal{W}$, such that the dual pair $(U \cdot \mathcal{F}, (U \cdot \mathcal{F})_{\mathcal{V}}^\#)$ induces an optimal encoding-decoding scheme. As a measure of optimality we could consider the minimization of the joint convex potential of the pair among all such pairs; but, since the spectral properties of $U \cdot \mathcal{F}$ are independent of U we are left to compute those unitary operators $U_0 \in B(\mathcal{H})$, with $U_0(\mathcal{W}) = \mathcal{W}$, that minimize - for a non-decreasing function $h \in \text{Conv}(\mathbb{R}_{\geq 0})$ - the convex potential $P_h[(U \cdot \mathcal{F})_{\mathcal{V}}^\#]$ among all unitary operators $U \in B(\mathcal{H})$ such that $U(\mathcal{W}) = \mathcal{W}$. As we shall see, there exist structural solutions to this problem.

Theorem 4.5. Consider the Notations 4.2. Let $\{x_i\}_{i \in \mathbb{I}_d} \in \mathcal{W}^d$ be an ONB of \mathcal{W} such that $S_{\mathcal{F}} x_j = \lambda_j x_j$ for every $j \in \mathbb{I}_d$.

1. Let $U_0 \in B(\mathcal{H})$ be a unitary operator such that $U_0 x_j = w_{d-j+1}$ for every $j \in \mathbb{I}_d$. Then

$$\lambda(S_{(U_0 \cdot \mathcal{F})_{\mathcal{V}}^\#}) = \left((\cos^{-2}(\theta_j) \lambda_{d-j+1}^{-1})_{j \in \mathbb{I}_d}^\downarrow, 0_{|\mathbb{M}|-d} \right). \quad (24)$$

2. If $h \in \text{Conv}(\mathbb{R}_{\geq 0})$ is non-decreasing then

$$P_h((U_0 \cdot \mathcal{F})_{\mathcal{V}}^\#) = \min\{P_h((U \cdot \mathcal{F})_{\mathcal{V}}^\#) : U \in B(\mathcal{H}) \text{ is unitary and } U(\mathcal{W}) = \mathcal{W}\}. \quad (25)$$

Moreover, if we assume further that $h \in \text{Conv}_s(\mathbb{R}_{\geq 0})$ and $U \cdot \mathcal{F}$ attains the minimum of Eq. (25) then there exist principal vectors $\{w'_j\}_{j \in \mathbb{I}_d}$ and a ONB $\{x'_i\}_{i \in \mathbb{I}_d}$ for \mathcal{W} such that $S_{\mathcal{F}} x'_j = \lambda_j x'_j$ and $U x'_j = w'_{d-j+1}$, for $j \in \mathbb{I}_d$.

Proof. Let $U \in B(\mathcal{H})$ be any unitary operator such that $U(\mathcal{W}) = \mathcal{W}$ and let $U \cdot \mathcal{F} = \{U f_i\}_{i \in \mathbb{I}_n}$. Then, notice that

$$(U \cdot \mathcal{F})_{\mathcal{V}}^{\#} = P_{\mathcal{V} // \mathcal{W}^{\perp}} (U \cdot \mathcal{F})^{\#} = P_{\mathcal{V} // \mathcal{W}^{\perp}} U \cdot \mathcal{F}^{\#}$$

and hence, in general we get that

$$S_{(U \cdot \mathcal{F})_{\mathcal{V}}^{\#}} = P_{\mathcal{V} // \mathcal{W}^{\perp}} U S_{\mathcal{F}^{\#}} U^* P_{\mathcal{V} // \mathcal{W}^{\perp}}^* . \quad (26)$$

We can now argue as in the proof of Theorem 4.3 considering $M = |P_{\mathcal{V} // \mathcal{W}^{\perp}}|$ (and the polar decomposition $P_{\mathcal{V} // \mathcal{W}^{\perp}} = V M$) and conclude that

$$\lambda(S_{(U \cdot \mathcal{F})_{\mathcal{V}}^{\#}}) = \left(\lambda \left(M_{\mathcal{W}} (U S_{\mathcal{F}}^{\dagger} U^*)_{\mathcal{W}} M_{\mathcal{W}} \right) , 0_{|\mathbb{M}|-d} \right) , \quad (27)$$

where in general $S_{\mathcal{W}} \in B(\mathcal{W})$ denotes the restriction of S to its reducing subspace \mathcal{W} . Using Eq. (17), if U_0 is as in item 1 then

$$M_{\mathcal{W}} (U_0 S_{\mathcal{F}}^{\dagger} U_0^*)_{\mathcal{W}} M_{\mathcal{W}} w_j = \cos^{-2}(\theta_j) \lambda_{d-j+1}^{-1} w_j \quad \text{for } j \in \mathbb{I}_d .$$

Thus, the previous facts together with Eq. (27) show item 1.

In case $U \in B(\mathcal{H})$ is a unitary operator such that $U(\mathcal{W}) = \mathcal{W}$ then Theorem 4.3 implies that

$$\prod_{j \in \mathbb{I}_k} \lambda_j(S_{(U_0 \cdot \mathcal{F})_{\mathcal{V}}^{\#}}) \leq \prod_{j \in \mathbb{I}_k} \lambda_j(S_{(U \cdot \mathcal{F})_{\mathcal{V}}^{\#}}) \quad , \quad k \in \mathbb{I}_d . \quad (28)$$

As explained in Section 2.2, Eq. (28) implies that

$$\sum_{j \in \mathbb{I}_k} \lambda_j(S_{(U_0 \cdot \mathcal{F})_{\mathcal{V}}^{\#}}) \leq \sum_{j \in \mathbb{I}_k} \lambda_j(S_{(U \cdot \mathcal{F})_{\mathcal{V}}^{\#}}) \quad , \quad k \in \mathbb{I}_d . \quad (29)$$

If $h \in \text{Conv}(\mathbb{R}_{\geq 0})$ is non-decreasing then, by the submajorization relation in Eq. (29), we conclude that

$$P_h((U_0 \cdot \mathcal{F})_{\mathcal{V}}^{\#}) = \sum_{j \in \mathbb{I}_d} h(\lambda_j(S_{(U_0 \cdot \mathcal{F})_{\mathcal{V}}^{\#}})) \leq \sum_{j \in \mathbb{I}_d} h(\lambda_j(S_{(U \cdot \mathcal{F})_{\mathcal{V}}^{\#}})) = P_h((U \cdot \mathcal{F})_{\mathcal{V}}^{\#}) ,$$

which proves Eq. (25). Similarly, the final claim follows from Eq. (27) and Lidskii's multiplicative inequality, as stated in Theorem 2.4 and the properties of log-majorization described in Section 2.2. \square

Consider the Notations 4.2. Then, the previous theorem describes the rigid rotations U_0 that leave invariant \mathcal{W} and such that spectral structure $\lambda((U_0 \cdot \mathcal{F})_{\mathcal{V}}^{\#})$ of the oblique canonical \mathcal{V} -dual of $U_0 \cdot \mathcal{F}$ is optimal with respect to log-majorization.

On the other hand, for a fixed rigid rotation U that leaves invariant \mathcal{W} and for a fixed $t \geq \text{tr}(S_{(U \cdot \mathcal{F})_{\mathcal{V}}^{\#}})$ Theorem 3.11 describes the spectral structure $\lambda_{\mathcal{V}, t}^{\#}(U \cdot \mathcal{F})$ of those oblique \mathcal{V} -duals $\mathcal{G}_t^{\text{op}}(U) \in \mathcal{D}_t(U \cdot \mathcal{F})$ that simultaneously minimize every convex potential within the set $\mathcal{D}_t(U \cdot \mathcal{F})$. It is then natural to wonder whether the spectral structure $\lambda_{\mathcal{V}, t}^{\#}(U_0 \cdot \mathcal{F})$ of $\mathcal{G}_t^{\text{op}}(U_0)$ (optimal dual with trace restriction based on an optimal rigid rotation of \mathcal{F}) has some optimality property. In order to tackle this problem we consider the following result.

Lemma 4.6. *Let $\lambda = (\lambda_i)_{i \in \mathbb{I}_d}$, $\mu = (\mu_i)_{i \in \mathbb{I}_d} \in \mathbb{R}_{\geq 0}^d$, let $m \leq 0$ be an integer and assume that $\lambda \prec_w \mu$. If we let $\nu_{\lambda, m}(t)$ and $\nu_{\mu, m}(t)$ be as in Remark 3.10 (based on λ and μ respectively) for some $t \geq \operatorname{tr} \mu$ ($\geq \operatorname{tr} \lambda$) then we have that $\nu_{\lambda, m}(t) \prec \nu_{\mu, m}(t)$.*

Proof. Recall that by construction $\operatorname{tr}(\nu_{\lambda, m}(t)) = \operatorname{tr}(\nu_{\mu, m}(t)) = t$. Hence, in case $\nu_{\lambda, m}(t) = \frac{t}{d} \cdot \mathbf{1}$ the result follows from the well known relation $\frac{t}{d} \cdot \mathbf{1} \prec \rho$ for every $\rho \in \mathbb{R}^d$ such that $\operatorname{tr} \rho = t$. Otherwise (see Remark 3.10), there exists $1 \leq r \leq d-1$ such that

$$\nu_{\lambda, m}(t) = (\lambda_1, \dots, \lambda_r, c \cdot \mathbf{1}_{d-r}) \quad \text{with} \quad c \leq \lambda_r .$$

On the other hand we can write $\nu_{\mu, m}(t) = (\alpha, \beta) \in (\mathbb{R}_{\geq 0}^d)^\downarrow$ where

$$\alpha = (\mu_i + (c' - \mu_i)^+)_{i=1}^r \in (\mathbb{R}_{\geq 0}^r)^\downarrow \quad \text{and} \quad \beta = (\mu_i + (c' - \mu_i)^+)_{i=r+1}^d \in (\mathbb{R}_{\geq 0}^{d-r})^\downarrow .$$

Therefore, for every $k \in \mathbb{I}_r$ we have that

$$\sum_{i \in \mathbb{I}_k} \lambda_i \leq \sum_{i \in \mathbb{I}_k} \mu_i \leq \sum_{i \in \mathbb{I}_k} (\mu_i + (c' - \mu_i)^+) \implies (\lambda_i)_{i \in \mathbb{I}_r} \prec_w \alpha ,$$

where we have used that $\lambda \prec_w \mu$ in the first inequality above. By [24, Lemma 5.6] we conclude that $\nu_{\lambda, m}(t) \prec_w \nu_{\mu, m}(t)$. The result now follows from the equality $\operatorname{tr}(\nu_{\lambda, m}(t)) = \operatorname{tr}(\nu_{\mu, m}(t))$. \square

Theorem 4.7. *Consider the Notations 4.2 and assume that $n \geq 2d$ (i.e. $2d - n \leq 0$). Let $\{x_i\}_{i \in \mathbb{I}_d} \in \mathcal{W}^d$ be an ONB of eigenvectors for $S_{\mathcal{F}}$ on \mathcal{W} i.e., such that $S_{\mathcal{F}} x_j = \lambda_j x_j$ for every $j \in \mathbb{I}_d$. Let $U_0 \in B(\mathcal{H})$ be a unitary operator such that $U_0 x_j = w_{d-j+1} x_j$ for every $j \in \mathbb{I}_d$. Then,*

1. *If $U \in B(\mathcal{H})$ is a unitary operator such that $U(\mathcal{W}) = \mathcal{W}$ then $\operatorname{tr}(S_{(U_0 \cdot \mathcal{F})_{\mathbb{V}}^\#}) \leq \operatorname{tr}(S_{(U \cdot \mathcal{F})_{\mathbb{V}}^\#})$.*
2. *If $t \geq \operatorname{tr}(S_{(U \cdot \mathcal{F})_{\mathbb{V}}^\#})$ and we let $\mathcal{G}_t^{\operatorname{op}}(U) \in \mathcal{D}_{\mathbb{V}, t}(U \cdot \mathcal{F})$ (resp. $\mathcal{G}_t^{\operatorname{op}}(U_0) \in \mathcal{D}_{\mathbb{V}, t}(U_0 \cdot \mathcal{F})$) be the optimal dual as in Theorem 3.11 then for every $h \in \operatorname{Conv}(\mathbb{R}_{\geq 0})$*

$$P_h(\mathcal{G}_t^{\operatorname{op}}(U_0)) \leq P_h(\mathcal{G}_t^{\operatorname{op}}(U)) . \quad (30)$$

Proof. As explained in the proof of Theorem 4.5 if U and U_0 are as above then Eq. (29) holds. In this case

$$\operatorname{tr}(S_{(U_0 \cdot \mathcal{F})_{\mathbb{V}}^\#}) = \sum_{j \in \mathbb{I}_d} \lambda_j (S_{(U_0 \cdot \mathcal{F})_{\mathbb{V}}^\#}) \leq \sum_{j \in \mathbb{I}_d} \lambda_j (S_{(U \cdot \mathcal{F})_{\mathbb{V}}^\#}) = \operatorname{tr}(S_{(U \cdot \mathcal{F})_{\mathbb{V}}^\#}) ,$$

which shows item 1. On the other hand, if $t \geq \operatorname{tr}(S_{(U \cdot \mathcal{F})_{\mathbb{V}}^\#}) \geq \operatorname{tr}(S_{(U_0 \cdot \mathcal{F})_{\mathbb{V}}^\#})$ then Eq. (29) together with Lemma 4.6 and Theorem 3.11 (notice that in this case $m = 2d - n \leq 0$) imply that

$$\sum_{j \in \mathbb{I}_k} \lambda_j (S_{\mathcal{G}_t^{\operatorname{op}}(U_0)}) \leq \sum_{j \in \mathbb{I}_k} \lambda_j (S_{\mathcal{G}_t^{\operatorname{op}}(U)}) \quad , \quad k \in \mathbb{I}_d .$$

Hence, Eq. (30) follow from the properties of majorization described in Section 2.2 and Definition 2.2. \square

We conjecture that Theorem 4.7 is also true in case $2d - n \in \mathbb{I}_{d-1}$. We shall consider this problem elsewhere.

5 Aliasing in oblique duality

Let $\mathcal{W}, \mathcal{V} \subset \mathcal{H}$ be closed subspaces such that $\mathcal{V} \oplus \mathcal{W}^\perp = \mathcal{H}$ (or equivalently $\mathcal{W} \oplus \mathcal{V}^\perp = \mathcal{H}$). Recall that in this context the aliasing norm associated to the consistent sampling

$$f \mapsto \tilde{f} = P_{\mathcal{W}/\mathcal{V}^\perp} f \quad , \quad f \in \mathcal{H} \quad (31)$$

(see [14, 21]) is given by

$$A(\mathcal{W}, \mathcal{V}) = \sup_{e \in \mathcal{W}^\perp \setminus \{0\}} \frac{\|P_{\mathcal{W}/\mathcal{V}^\perp} e\|}{\|e\|} = \|P_{\mathcal{W}/\mathcal{V}^\perp} P_{\mathcal{W}^\perp}\| . \quad (32)$$

Notice that the aliasing norm measures the incidence of the orthogonal complement of \mathcal{W} in the overall (oblique) encoding-decoding scheme in Eq. (31) based on these two subspaces. We can interpret $A(\mathcal{W}, \mathcal{V})$ as a measure of the amount of noise that we would get in the oblique encoding-decoding scheme when sampling a perturbed signal $f + e$ that has a component $e \in \mathcal{W}^\perp$. This phenomenon is of interest only when $\mathcal{V} \neq \mathcal{W}$ (as $A(\mathcal{W}, \mathcal{W}) = 0$).

Lemma 5.1. *Consider the Notations 4.2. Then*

1. $|P_{\mathcal{W}^\perp} P_{\mathcal{V}}| v_i = \sin(\theta_i) v_i$ for every $i \in \mathbb{I}_d$.
2. $|P_{\mathcal{W}^\perp} P_{\mathcal{V}/\mathcal{W}^\perp}|^2 w_i = \tan^2(\theta_i) w_i$ for every $i \in \mathbb{I}_d$.

Proof. Recall that with the Notations 4.2, $\{v_i\}_{i \in \mathbb{I}_d}$ is an ONB of \mathcal{V} such that $|P_{\mathcal{W}} P_{\mathcal{V}}| v_i = \cos(\theta_i) v_i$, for $i \in \mathbb{I}_d$. In this case,

$$|P_{\mathcal{W}^\perp} P_{\mathcal{V}}|^2 = P_{\mathcal{V}} - |P_{\mathcal{W}} P_{\mathcal{V}}|^2 \implies |P_{\mathcal{W}^\perp} P_{\mathcal{V}}| v_i = \sin(\theta_i) v_i \quad \text{for every } i \in \mathbb{I}_d . \quad (33)$$

To prove item 2, let us fix $i \in \mathbb{I}_d$. By Eq. (18) we know that $P_{\mathcal{W}} P_{\mathcal{V}} v_i = P_{\mathcal{W}} v_i = \cos(\theta_i) w_i$. On the other hand, recall from Eq. (16) that

$$(P_{\mathcal{V}/\mathcal{W}^\perp})^\dagger = P_{\mathcal{W}} P_{\mathcal{V}} \implies P_{\mathcal{V}/\mathcal{W}^\perp} w_i = \cos(\theta_i)^{-1} v_i , \quad (34)$$

since $v_i \in \mathcal{V} = (\ker P_{\mathcal{W}} P_{\mathcal{V}})^\perp$ and $w_i \in \mathcal{W} = R(P_{\mathcal{W}} P_{\mathcal{V}})$. Similarly, we get that

$$P_{\mathcal{W}/\mathcal{V}^\perp} v_i \stackrel{(34)}{=} P_{\mathcal{V}/\mathcal{W}^\perp}^* (\cos(\theta_i) P_{\mathcal{V}/\mathcal{W}^\perp} w_i) = \cos(\theta_i) |P_{\mathcal{V}/\mathcal{W}^\perp}|^2 w_i \stackrel{(17)}{=} \cos(\theta_i)^{-1} w_i . \quad (35)$$

On the other hand,

$$|P_{\mathcal{W}^\perp} P_{\mathcal{V}/\mathcal{W}^\perp}|^2 = |P_{\mathcal{W}^\perp} P_{\mathcal{V}} P_{\mathcal{V}/\mathcal{W}^\perp}|^2 = P_{\mathcal{W}/\mathcal{V}^\perp} |P_{\mathcal{W}^\perp} P_{\mathcal{V}}|^2 P_{\mathcal{V}/\mathcal{W}^\perp} .$$

Hence, using the previous remarks and Eq. (33) we get that, for every $i \in \mathbb{I}_d$,

$$\begin{aligned} |P_{\mathcal{W}^\perp} P_{\mathcal{V}/\mathcal{W}^\perp}|^2 w_i &\stackrel{(34)}{=} P_{\mathcal{W}/\mathcal{V}^\perp} |P_{\mathcal{W}^\perp} P_{\mathcal{V}}|^2 (\cos(\theta_i)^{-1} v_i) \\ &\stackrel{(33)}{=} \frac{\sin(\theta_i)^2}{\cos(\theta_i)} P_{\mathcal{W}/\mathcal{V}^\perp} v_i \stackrel{(35)}{=} \tan^2(\theta_i) w_i . \end{aligned}$$

This completes the proof. \square

Consider now the Notations 4.2, so in particular \mathcal{V}, \mathcal{W} are finite dimensional. Then, using that $\|P_{\mathcal{W}/\mathcal{V}^\perp}\| = \cos(\theta_d)^{-1}$ we get $A(\mathcal{W}, \mathcal{V}) \leq \cos(\theta_d)^{-1}$ (i.e. with the notations of Remark 4.1 we get $A(\mathcal{V}, \mathcal{W}) \leq \cos(\theta_{\mathcal{W}, \mathcal{V}})^{-1}$, see [14]). Nevertheless, the previous bound for $A(\mathcal{W}, \mathcal{V})$ is not sharp: in case $\mathcal{V} = \mathcal{W}$ then $A(\mathcal{W}, \mathcal{W}) = 0$, yet $\cos(\theta_d)^{-1} = 1$. Next we compute the exact value of the aliasing norm:

Corollary 5.2. *Consider the Notations 4.2. Then,*

$$A(\mathcal{W}, \mathcal{V}) = \tan(\theta_d) .$$

Proof. By the definition of the aliasing (32) and Lemma 5.1

$$A(\mathcal{W}, \mathcal{V}) = \|P_{\mathcal{W}/\mathcal{V}^\perp} P_{\mathcal{W}^\perp}\| = \| |P_{\mathcal{W}^\perp} P_{\mathcal{V}/\mathcal{W}^\perp} | \| = \max_{i \in \mathbb{I}_d} \tan(\theta_i) = \tan(\theta_d) . \quad \square$$

Consider the Notations 4.2. Let $\mathcal{G} = \{g_i\}_{i \in \mathbb{I}_d} \in \mathcal{D}_{\mathcal{V}}(\mathcal{F})$ i.e., such that $T_{\mathcal{F}} T_{\mathcal{G}}^* = P_{\mathcal{W}/\mathcal{V}^\perp}$. Then, when applying the encoding-decoding scheme induced by the pair $(\mathcal{F}, \mathcal{G})$ the orthogonal complement \mathcal{W}^\perp may also have an incidence in the sampling process; in case $\mathcal{V} \neq \mathcal{W}$, if we sample the perturbation $f + e$ for $f \in \mathcal{W}$ and $e \in \mathcal{W}^\perp$ then, there is a corresponding perturbation of the coefficients $T_{\mathcal{G}}^* f$ given by $T_{\mathcal{G}}^* e$: in this case, the squared norm (energy) of the perturbation is $\|T_{\mathcal{G}}^* e\|^2 = \langle S_{\mathcal{G}} e, e \rangle$. Thus, we introduce the following

Definition 5.3. *Let $\mathcal{W}, \mathcal{V} \subset \mathcal{H}$ be closed subspaces such that $\mathcal{W} \oplus \mathcal{V}^\perp = \mathcal{H}$. Let $\mathcal{F} = \{f_i\}_{i \in I}$ and $\mathcal{G} = \{g_i\}_{i \in I}$ be frames for \mathcal{W} and \mathcal{V} respectively such that $\mathcal{G} \in \mathcal{D}_{\mathcal{V}}(\mathcal{F})$. Then, we define the aliasing relative to the oblique dual pair $(\mathcal{F}, \mathcal{G})$, denoted $A(\mathcal{F}, \mathcal{G})$ given by*

$$A(\mathcal{F}, \mathcal{G}) = \sup_{e \in \mathcal{W}^\perp \setminus \{0\}} \frac{\|T_{\mathcal{G}}^* e\|}{\|e\|} = \sup_{e \in \mathcal{W}^\perp \setminus \{0\}} \frac{\langle S_{\mathcal{G}} e, e \rangle^{1/2}}{\|e\|} . \quad \triangle$$

With the notations of Definition 5.3, notice that $A(\mathcal{F}, \mathcal{G})$ is a normalized measure of the relative incidence of \mathcal{W}^\perp in the analysis of perturbed signals in terms of \mathcal{G} , in the sense that

$$\|T_{\mathcal{G}}^* e\| \leq A(\mathcal{F}, \mathcal{G}) \|e\| \quad , \quad e \in \mathcal{W}^\perp .$$

There is an alternative interpretation of the aliasing $A(\mathcal{F}, \mathcal{G})$ that is as follows: with the previous notations, let $e \in \mathcal{W}^\perp$: then

$$\|T_{\mathcal{G}}^* e\|^2 = \sum_{i \in I} |\langle e, g_i \rangle|^2 = \sum_{i \in I} |\langle e, P_{\mathcal{W}^\perp} g_i \rangle|^2 = \|T_{\widehat{\mathcal{G}}}^* e\|^2 ,$$

where $\widehat{\mathcal{G}} = \{P_{\mathcal{W}^\perp} g_i\}_{i \in \mathbb{I}_n}$ is considered as a finite sequence in \mathcal{W}^\perp . Therefore,

$$A(\mathcal{F}, \mathcal{G}) = \sup_{e \in \mathcal{W}^\perp \setminus \{0\}} \left(\sum_{i \in I} \frac{|\langle e, P_{\mathcal{W}^\perp} g_i \rangle|^2}{\|e\|^2} \right)^{1/2} = \sup_{e \in \mathcal{W}^\perp \setminus \{0\}} \frac{\|T_{\widehat{\mathcal{G}}}^* e\|}{\|e\|}$$

can be considered as a measure of the (normalized) residual sampling power of $\widehat{\mathcal{G}} = P_{\mathcal{W}^\perp} \cdot \mathcal{G}$ in \mathcal{W}^\perp .

Assume further that $\mathcal{F} = \{f_i\}_{i \in I}$ is a Parseval frame for \mathcal{W} i.e. $S_{\mathcal{F}} = P_{\mathcal{W}}$. Then, $\mathcal{F}_{\mathcal{V}}^\# \in \mathcal{D}_{\mathcal{V}}(\mathcal{F})$ is such that $S_{\mathcal{F}_{\mathcal{V}}^\#} = |P_{\mathcal{W}/\mathcal{V}^\perp}|^2$; hence in this case

$$A(\mathcal{F}, \mathcal{F}_{\mathcal{V}}^\#) = \sup_{e \in \mathcal{W}^\perp \setminus \{0\}} \frac{\langle S_{\mathcal{F}_{\mathcal{V}}^\#} e, e \rangle^{1/2}}{\|e\|} = \sup_{e \in \mathcal{W}^\perp \setminus \{0\}} \frac{\|P_{\mathcal{W}/\mathcal{V}^\perp} e\|}{\|e\|} = \mathcal{A}(\mathcal{W}, \mathcal{V}) .$$

In opposition to $A(\mathcal{W}, \mathcal{V})$, the aliasing $A(\mathcal{F}, \mathcal{G})$ depends on the particular choice of oblique dual frames $(\mathcal{F}, \mathcal{G})$ considered, and not only on the subspaces \mathcal{V} and \mathcal{W} . Then, it is natural to consider the problem of designing frames \mathcal{F} and \mathcal{G} for \mathcal{W} and \mathcal{V} respectively, such that $(\mathcal{F}, \mathcal{G})$ is an oblique dual pair and such that $A(\mathcal{F}, \mathcal{G})$ is minimum.

Remark 5.4. Consider the notations 4.2. Given $\mathcal{G} \in \mathcal{D}_{\mathcal{V}}(\mathcal{F})$, it is easy to see from Definition 5.3 that

$$A(\mathcal{F}, \mathcal{G}) = \|T_{\mathcal{G}}^* P_{\mathcal{W}^\perp}\| = \|P_{\mathcal{W}^\perp} S_{\mathcal{G}} P_{\mathcal{W}^\perp}\|^{1/2} \quad (36)$$

On the other hand, as a consequence of Proposition 3.4 we see that $S_{\mathcal{F}_V^\#} \leq S_{\mathcal{G}} \implies P_{\mathcal{W}^\perp} S_{\mathcal{F}_V^\#} P_{\mathcal{W}^\perp} \leq P_{\mathcal{W}^\perp} S_{\mathcal{G}} P_{\mathcal{W}^\perp}$. Therefore

$$A(\mathcal{F}, \mathcal{F}_V^\#) = \|P_{\mathcal{W}^\perp} S_{\mathcal{F}_V^\#} P_{\mathcal{W}^\perp}\|^{1/2} \leq \|P_{\mathcal{W}^\perp} S_{\mathcal{G}} P_{\mathcal{W}^\perp}\|^{1/2} = A(\mathcal{F}, \mathcal{G}).$$

That is, in this case the canonical oblique dual pair $(\mathcal{F}, \mathcal{F}_V^\#)$ minimizes the aliasing among all oblique dual pairs $(\mathcal{F}, \mathcal{G})$ for $\mathcal{G} \in \mathcal{D}_{\mathcal{V}}(\mathcal{F})$. \triangle

Consider the notations 4.2 and let $U \in B(\mathcal{H})$ be a unitary operator such that $U(\mathcal{W}) = \mathcal{W}$. As shown in previous sections, the spectral structure of $S_{(U \cdot \mathcal{F})_V^\#}$ depends on the choice of such a unitary. Therefore it is natural to consider the unitary operators U as before, that minimize the aliasing $A(U \cdot \mathcal{F}, \mathcal{G})$ where $\mathcal{G} \in \mathcal{D}_{\mathcal{V}}(U \cdot \mathcal{F})$. Remark 5.4 shows that in this case we can restrict attention to the oblique dual pairs of the form $(U \cdot \mathcal{F}, (U \cdot \mathcal{F})_V^\#)$ for a unitary operator $U \in B(\mathcal{H})$ such that $U(\mathcal{W}) = \mathcal{W}$. The following result fits into the previous analysis scheme and links optimal solutions of this problem to optimal solutions of the problem considered in Theorem 4.5.

Theorem 5.5. Consider the Notations 4.2. Let $\{x_i\}_{i \in \mathbb{I}_d} \in \mathcal{W}^d$ be an ONB of eigenvectors for $S_{\mathcal{F}}$ on \mathcal{W} i.e., such that $S_{\mathcal{F}} x_j = \lambda_j x_j$ for every $j \in \mathbb{I}_d$.

1. If $U \in B(\mathcal{H})$ is a unitary operator such that $U(\mathcal{W}) = \mathcal{W}$ then

$$A(U \cdot \mathcal{F}, (U \cdot \mathcal{F})_V^\#) \geq \max_{j \in \mathbb{I}_d} \frac{\tan(\theta_j)}{\lambda_{d-j+1}^{1/2}}.$$

2. If $U_0 \in B(\mathcal{H})$ is a unitary operator such that $U_0 x_j = w_{d-j+1}$ for every $j \in \mathbb{I}_d$, then

$$A(U_0 \cdot \mathcal{F}, (U_0 \cdot \mathcal{F})_V^\#) = \max_{j \in \mathbb{I}_d} \frac{\tan(\theta_j)}{\lambda_{d-j+1}^{1/2}}.$$

In particular, the lower bound in item 1. above is sharp.

Proof. Let $U \in B(\mathcal{H})$ be a unitary operator such that $U(\mathcal{W}) = \mathcal{W}$. Then, as in the proof of Theorem 4.5, we have that

$$S_{(U \cdot \mathcal{F})_V^\#} = P_{\mathcal{V} // \mathcal{W}^\perp} U S_{\mathcal{F}^\#} U^* P_{\mathcal{W} // \mathcal{V}^\perp}$$

Therefore, by Eq. (36) we deduce that

$$A((U \cdot \mathcal{F})_V^\#, U \cdot \mathcal{F})^2 = \|P_{\mathcal{W}^\perp} P_{\mathcal{V} // \mathcal{W}^\perp} U S_{\mathcal{F}^\#} U^* P_{\mathcal{W} // \mathcal{V}^\perp} P_{\mathcal{W}^\perp}\|. \quad (37)$$

On the other hand, by Lemma 5.1, we have that

$$|P_{\mathcal{W}^\perp} P_{\mathcal{V} // \mathcal{W}^\perp}|^2 w_i = \tan^2(\theta_i) w_i \quad \text{for every } i \in \mathbb{I}_d. \quad (38)$$

In particular, $\text{rk}(|P_{\mathcal{W}^\perp} P_{\mathcal{V} // \mathcal{W}^\perp}|) = \#\{i \in \mathbb{I}_d : \theta_i > 0\} \stackrel{\text{def}}{=} k$. Let P denote the orthogonal projection onto $\mathcal{N} = R(|P_{\mathcal{W}^\perp} P_{\mathcal{V} // \mathcal{W}^\perp}|)$. Then, by the interlacing inequalities (see [3]) we get that the eigenvalues $\lambda(P(US_{\mathcal{F}^\#}U^*)P) = (\mu_i)_{i \in \mathbb{M}}$ satisfy that:

$$\mu_i \geq \lambda_{(d-k+i)}(S_{\mathcal{F}^\#}) = \lambda_{k-i+1}^{-1} \quad \text{if } 1 \leq i \leq k \quad \text{and} \quad \mu_i = 0 \quad \text{if } i \geq k+1. \quad (39)$$

We apply Lidskii's multiplicative inequality in Theorem 2.4 (to the operators acting on \mathcal{N} , where $|P_{\mathcal{W}^\perp} P_{\mathcal{V}/\mathcal{W}^\perp}|$ acts as an invertible operator) and get

$$\begin{aligned} (\tan^2(\theta_{d-k+i}))_{i \in \mathbb{I}_k} \circ (\mu_i)_{i \in \mathbb{I}_k} &\prec_w (\lambda_i(|P_{\mathcal{W}^\perp} P_{\mathcal{V}/\mathcal{W}^\perp}| (P U S_{\mathcal{F}^\#} U^* P) |P_{\mathcal{W}^\perp} P_{\mathcal{V}/\mathcal{W}^\perp}|))_{i \in \mathbb{I}_k} \\ &= (\lambda_i(P_{\mathcal{W}^\perp} P_{\mathcal{V}/\mathcal{W}^\perp} U S_{\mathcal{F}^\#} U^* P_{\mathcal{V}/\mathcal{W}^\perp} P_{\mathcal{W}^\perp}))_{i \in \mathbb{I}_k} \in \mathbb{R}^k, \end{aligned}$$

where the last equality follows by taking polar decomposition of $P_{\mathcal{W}^\perp} P_{\mathcal{V}/\mathcal{W}^\perp}$. In particular, using the sub-majorization relation, Eq. (37) and the inequalities in (39) we see that

$$A(U \cdot \mathcal{F}, (U \cdot \mathcal{F})_{\mathcal{V}}^\#)^2 \geq \max_{i \in \mathbb{I}_k} \left\{ \frac{\tan^2(\theta_{d-(k-i)})}{\lambda_{k-i+1}} \right\} = \max_{i \in \mathbb{I}_d} \left\{ \frac{\tan^2(\theta_j)}{\lambda_{d-j+1}} \right\},$$

which shows item 1. Fix the unitary $U_0 \in B(\mathcal{H})$ of item 2. Then $(U_0 S_{\mathcal{F}^\#} U_0^*) w_i = \lambda_{d-i+1}^{-1} w_j$ for $i \in \mathbb{I}_d$. Recall from Eq. (38) that $|P_{\mathcal{W}^\perp} P_{\mathcal{V}/\mathcal{W}^\perp}| w_i = \tan(\theta_i) w_i$ for $i \in \mathbb{I}_d$. Then

$$\lambda(|P_{\mathcal{W}^\perp} P_{\mathcal{V}/\mathcal{W}^\perp}| U_0 S_{\mathcal{F}^\#} U_0^* |P_{\mathcal{W}^\perp} P_{\mathcal{V}/\mathcal{W}^\perp}|) = \left(\left(\frac{\tan^2(\theta_j)}{\lambda_{d-j+1}} \right)_{j \in \mathbb{I}_d}^\downarrow, 0_{|\mathbb{M}|-d} \right).$$

This shows that canonical oblique dual pair corresponding to $U_0 \cdot \mathcal{F}$ attains the minimal aliasing. \square

Remark 5.6. Consider the Notations 4.2. If $\mathcal{G} \in \mathcal{D}_{\mathcal{V}}(\mathcal{F})$ then the compression

$$(S_{\mathcal{G}})_{\mathcal{W}^\perp} = P_{\mathcal{W}^\perp} S_{\mathcal{G}} |_{\mathcal{W}^\perp} \in B(\mathcal{W}^\perp)^+$$

of $S_{\mathcal{G}}$ to \mathcal{W}^\perp is a (operator valued) measure of the incidence of \mathcal{W}^\perp in the encoding-decoding scheme based on the oblique dual pair $(\mathcal{F}, \mathcal{G})$. By Eq. (36), it follows that $A(\mathcal{F}, \mathcal{G}) = \|(S_{\mathcal{G}})_{\mathcal{W}^\perp}\|^{1/2}$, where $\|T\|$ stands for the operator norm of $T \in B(\mathcal{W}^\perp)$. We can consider other (natural) scalar valued measures of the form

$$A_h(\mathcal{F}, \mathcal{G}) = \text{tr } h((S_{\mathcal{G}})_{\mathcal{W}^\perp}),$$

for $h \in \text{Conv}(\mathbb{R}_{\geq 0})$ non-decreasing and such that $h(0) = 0$ (which is well defined since $(S_{\mathcal{G}})_{\mathcal{W}^\perp}$ is a finite rank positive operator and $h(0) = 0$). A careful inspection of the proof of Theorem 5.5 shows that, with the notations of that result,

$$A_h(U_0 \cdot \mathcal{F}, (U_0 \cdot \mathcal{F})_{\mathcal{V}}^\#) \leq A_h(U \cdot \mathcal{F}, (U \cdot \mathcal{F})_{\mathcal{V}}^\#)$$

for every unitary operator $U \in B(\mathcal{H})$ such that $U(\mathcal{W}) = \mathcal{W}$, i.e. that the rigid rotation U_0 has several other optimal properties. \triangle

Acknowledgments. We would like to thank professors Gustavo Corach and Jorge Antezana for fruitful conversations related with the content of this work.

References

- [1] E. Andruchow, J. Antezana, G. Corach, Sampling formulae and optimal factorizations of projections. *Sampl. Theory Signal Image Process.* 7 (2008), no. 3, 313-331.
- [2] J. Antezana, G. Corach. Singular value estimates of oblique projections. *Linear Algebra Appl.* 430 (2009), no. 1, 386-395.
- [3] R. Bhatia, *Matrix Analysis*, Berlin-Heidelberg-New York, Springer 1997.

- [4] J.J. Benedetto, M. Fickus, Finite normalized tight frames. *Frames. Adv. Comput. Math.* 18 (2003), no. 2-4, 357-385.
- [5] P.G. Casazza, M. Fickus, J. Kovacevic, M.T. Leon, J.C. Tremain, A physical interpretation of tight frames. *Harmonic analysis and applications*, 51–76, *Appl. Numer. Harmon. Anal.*, Birkhäuser Boston, MA, 2006.
- [6] Eds. P. G. Casazza and G. Kutyniok, *Finite Frames: Theory and Applications*, Applied and Numerical Harmonic Analysis. Birkhäuser/Springer, New York, 2013.
- [7] O. Christensen, *An introduction to frames and Riesz bases*. Applied and Numerical Harmonic Analysis. Birkhäuser Boston, Inc., Boston, MA, 2003.
- [8] O. Christensen, Y.C. Eldar, Oblique dual frames and shift-invariant spaces. *Appl. Comput. Harmon. Anal.* 17 (2004), no. 1, 48-68.
- [9] O. Christensen, Y.C. Eldar, Characterization of oblique dual frame pairs. *EURASIP J. Appl. Signal Process.* (2006), 1-11.
- [10] G. Corach, J. Giribet. Oblique projections and sampling problems. *Integral Equations Operator Theory* 70 (2011), no. 3, 307-322.
- [11] G. Corach, A. Maestriperieri, Polar decomposition of oblique projections. *Linear Algebra Appl.* 433 (2010), no. 3, 511-519.
- [12] F. Deutsch, *The angle between subspaces of a Hilbert space. Approximation theory, wavelets and applications (Maratea, 1994)*, 107-130, *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, 454, Kluwer Acad. Publ., Dordrecht, 1995.
- [13] T.G. Dvorkind, Y.C. Eldar, Robust and Consistent Sampling. *IEEE Signal Process. Lett.* 16(9) (2009), 739-742.
- [14] Y.C. Eldar, Sampling with arbitrary sampling and reconstruction spaces and oblique dual frame vectors. *J. Fourier Anal. Appl.* 9 (2003), no. 1, 77-96.
- [15] Y.C. Eldar, T. Werther, General framework for consistent sampling in Hilbert spaces. *Int. J. Wavelets Multiresolut. Inf. Process.* 3 (2005), no. 4, 497-509.
- [16] M. Fickus, D. G. Mixon and M. J. Poteet, Frame completions for optimally robust reconstruction, *Proceedings of SPIE*, 8138: 81380Q/1-8 (2011).
- [17] G. H. Golub and C. F. Van Loan, *Matrix Computations*, 3rd ed., Johns Hopkins University Press, Baltimore, MD, 1996.
- [18] D. Han, Frame representations and Parseval duals with applications to Gabor frames. *Trans. Amer. Math. Soc.* 360 (2008), no. 6, 3307-3326.
- [19] A.A. Hemmat, J.P. Gabardo. The uniqueness of shift-generated duals for frames in shift-invariant subspaces. *J. Fourier Anal. Appl.* 13 (2007), no. 5, 589-606.
- [20] A.A. Hemmat, J.P. Gabardo, Properties of oblique dual frames in shift-invariant systems. *J. Math. Anal. Appl.* 356 (2009), no. 1, 346-354.
- [21] A.J.E.M. Janssen, The Zak transform and sampling theorems for wavelet subspaces, *IEEE Trans. Signal Processing*, 41 (1993), 3360-3364.
- [22] A.V. Knyazev, M.E. Argentati, Principal angles between subspaces in an A -based scalar product: algorithms and perturbation estimates. *SIAM J. Sci. Comput.* 23 (2002), no. 6, 2008-2040.
- [23] P. Massey, M. Ruiz, Minimization of convex functionals over frame operators. *Adv. Comput. Math.* 32 (2010), no. 2, 131-153.
- [24] P. Massey, M. Ruiz, D. Stojanoff, Duality in reconstruction systems. *Linear Algebra Appl.* 436 (2012), no. 3, 447-464.
- [25] P. Massey, M. Ruiz, D. Stojanoff, Optimal dual frames and frame completions for majorization. *Appl. Comput. Harmon. Anal.* 34 (2013), no. 2, 201-223.
- [26] P. Massey, M. Ruiz, D. Stojanoff, Multiplicative Lidskii's inequalities and optimal perturbations of frames (submitted).
- [27] M. J. Poteet, *Parametrizing finite frames and optimal frame completions*, Doctoral thesis, Graduate School of Engineering and Management, Air Force Institute of Technology, Air University.
- [28] B. Simon, *Trace ideals and their applications*. Second edition. *Mathematical Surveys and Monographs*, 120. American Mathematical Society, Providence, RI, 2005.
- [29] M. Unser, A. Aldroubi, A general sampling theory for nonideal acquisition devices, *IEEE Trans Signal Processing*, vol. 42 (1994) no. 11, 2915-2925.