The shape factor for conductive heat flow in regular polygonal and circular cross-sections

L.N. Epele,[1,2] H. Fanchiotti,[1,2] and C. A. García Canal[1,2]

[1] *Departamento de Física, Universidad Nacional de La Plata, C.C. 67-1900 La Plata, Argentina*

[2] *IFLP (CONICET), Universidad Nacional de La Plata, C.C. 67-1900 La Plata, Argentina*

Abstract

Different approximate methods have been used for the analysis of the conductive heat flow through cylinders with two or more contours of different forms. In this paper we present a very simple expression, based on conformal mapping and domain variational theory, for the shape factor of a coaxial system consisting of an N-regular polygon and a circle. The results, even for $N = 3$ and with very close boundaries is optimum when compared with the most sophisticated calculations available in the literature.

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I. INTRODUCTION

The study of the characteristic parameters of a coaxial system with a cross-section consisting of an N-regular polygon and a circle, either exterior or interior to each other, is a problem of continuous interest. This is so in very different areas of physics. In particular we should mention electric coaxial lines [1], torsion of bars [2] or conductive heat flow [3]. In fact, several approaches have been proposed with success to solve the Laplace equation in a multiply connected domain of two dimensions, as summarized in the previous references.

In this contribution we combine the standard conformal mapping method with a result of the domain variational theory [1] to obtain iteratively the modified domain functions as Green functions and period matrix. It is precisely this last mathematical object which is directly related with the shape factor, our objective to determine. We are able to compute with any required precision the values of the mentioned shape factor for the conductive heat flow of any coaxial system defined by an N-regular polygon, either external or internal to a circle, for $N = 3$ up. We remark that the computation, using the very simple analytical expression we have obtained from first principles, can be carried on even with the most elementary calculator. Moreover, our expressions do not include free parameters to be adjusted in some way.

II. THEORETICAL BACKGROUND

First we recall the main concepts related to heat conduction. As in many other fields, the thermal resistance in introduced in connection with a flow rate and a driving force. In this case heat, Q, is flowing and the driving force is the temperature difference, ΔT, namely

$$Q = \frac{\Delta T}{R_{th}} \tag{1}$$

On the other hand, the thermal resistance, R_{th}, is directly related to the electric resistance entering the Ohm law,R_{el}, through

$$\frac{R_{th}}{R_{el}} = \frac{\sigma}{\lambda} \tag{2}$$

where σ is the electric conductivity and λ the thermal conductivity. In the electric case, for homogeneous media, one has the following relationship between resistance and capacitance,

C

$$R_{el}\, C = \frac{\epsilon}{\sigma} \tag{3}$$

but C is always factorized as the product of a geometric factor S times the dielectric constant. Consequently, the previous relation reads

$$R_{el} = \frac{1}{\sigma\, S} \tag{4}$$

Now, taking into account (2) one ends with

$$R_{th} = \frac{1}{\lambda\, S} \tag{5}$$

This result, when compared with the definition of the Langmuir shape factor S through $Q = \lambda\, S\, \Delta T$ shows one that the shape factor coincides exactly with the dimensionless geometric factor S of the electric capacitance.

This observation allows us to directly use our previous results on electric coaxial lines [1] based on conformal mapping and domain variational theory.

We summarize in the following the main steps for arriving to an analytical expression for S.

Let us consider, just to fix ideas, the annular domain defined by an external N-regular polygon and an internal circle and perform the conformal mapping [4]

$$w(z) = \frac{N\, b}{B\left(\frac{1}{N}, 1 - \frac{2}{N}\right)\, \cos\left(\frac{\pi}{N}\right)} \int_0^z \frac{dt}{(1 - t^N)^{2/N}} \tag{6}$$

where b is the apothema of the polygon of N sides and $B(x, y)$ is the usual beta function. After the mapping, one ends with a different annular domain now with a circle of radius 1 replacing the polygon and with the inner circle slightly modified. It is precisely to tackle this modification that the domain variational theory is in order. This approach [5], ends with an iterative procedure for obtaining the appropriate functions, Green functions, harmonic measures, period matrices, of the modified domain in terms of the symmetrical ones. The relevant magnitude in the analysis of the modifications is the displacement $\delta_\eta(\theta)$ of the varied boundary in the direction of the normal measured with respect to the original circle [1]. Notice that this approach could be applied directly to the original domain, considering the polygon as a modification of a circle. Nevertheless, the iterative procedure is rapidly convergent for domains bounded by analytic curves. This is the case after the conformal mapping introduced in Eq.(6), because one ends with a circle and an analytic slightly modified circle.

The image of the original circle of radius a under the transformation (6), is written, at first order in the Fourier expansion, as

$$\rho(\theta) = \rho_0 \; [1 + \alpha_N \; \cos(N\,\theta)] \tag{7}$$

where

$$\rho_0 = \frac{a}{b} \, \frac{\Gamma^2\left(1+\frac{1}{N}\right)}{\Gamma^2\left(1+\frac{2}{N}\right)} \tag{8}$$

with $\Gamma(x)$ is the standard gamma function and

$$\alpha_N = -\frac{2}{N\,(N+1)}\,\rho_0^N \tag{9}$$

The mentioned Fourier coefficients $\delta_\eta(\theta)$, input in the domain variational theory calculation, can be expressed in terms of $\rho(\theta)$. Namely

$$\delta_\eta(\theta) = \rho(\theta) - \rho_0 \tag{10}$$

The period matrix [6], of the domain under study coincides with the geometrical value of the capacitance per unit length of a coaxial line having this domain as the cross-section [7]. For our present case, related to heat flow and as we stated before, this geometrical factor is precisely the shape factor S. Then, the main formula for S with $\rho(\theta)$ in the first approximation (7), results in the very simple expression

$$S^{(1)}(N) = \frac{2\,\pi}{\ln\left(\frac{1}{\rho_0}\right)} \left[1 + \frac{1}{2}\,\frac{1}{\ln\left(\frac{1}{\rho_0}\right)} \left[N\,\frac{1+\rho_0^{2N}}{1-\rho_0^{2N}} - \frac{1}{2}\right]\alpha_N^2\right] \tag{11}$$

One can easily obtain the next approximation to S, that includes the following Fourier coefficient in Eq.(7). Consequently, S now reads

$$\begin{aligned} S^{(2)}(N) = \frac{2\,\pi}{\ln\left(\frac{1}{\rho_1}\right)} \quad & \left[1 + \frac{1}{2}\,\frac{1}{\ln\left(\frac{1}{\rho_1}\right)} \left[\left(N\,\frac{1+\rho_1^{2N}}{1-\rho_1^{2N}} - \frac{1}{2}\right)\alpha_N^2\right.\right. \\ & \left.\left. + \left(2\,N\,\frac{1+\rho_1^{4N}}{1-\rho_1^{4N}} - \frac{1}{2}\right)\alpha_{2N}^2\right]\right] \end{aligned} \tag{12}$$

where

$$\rho_1 = \rho_0 \left[1 + \frac{2\,N+1}{N^2\,(N+1)^2}\,\rho_0^{2N}\right] \tag{13}$$

while

$$\alpha_{2N} = \left[\frac{2\,N+3}{N^2\,(N+1)^2} - \frac{N+2}{N^2\,(2\,N+1)^2}\right]\rho_0^{2N} \tag{14}$$

It is worth noticing that our expression (11) is also valid if the polygon of apothema b is the internal boundary and the circle of radius a is the external one, when ρ_0 is replaced by

$$\rho_0 = \frac{\Gamma^2\left(1+\frac{1}{N}\right)}{\Gamma\left(1+\frac{2}{N}\right)}\,\frac{b}{a}\,\frac{N}{\pi}\,\tan\left(\frac{\pi}{N}\right) \tag{15}$$

and also α_N is replaced by

$$\alpha_n = \frac{2}{N\,(N-1)}\,\rho_0^N \tag{16}$$

III. RESULTS AND FINAL COMMENTS

With all the relevant expression explicit we can now compute any case of interest. We have decided to present en Table 1 our results compared with the ones presented in Ref.[3], just to show the extremely good accuracy of our approach. Notice that in this last reference, the author presented an exhaustive comparison of their results with a series of previous calculations. Consequently one can immediately compare our results with all of them. We have included in the table only the cases $N = 3$ and $N = 4$, because they are, in principle the more critical ones. The other parameter that plays a fundamental role in the goodness of the calculation is the proximity between boundaries, our ratio a/b. For this reason we include in the Table, between brackets, the results including second order corrections for $a/b = 0.7$ and 0.9. This second order correction is at most of 6 per cent for $N = 3$ and $a/b = 0.9$, and negligible small for all the cases not included.

As a final comment we would like to stress that the numerical values speak by themselves showing that we can obtain a very good accuracy added to the simplicity of the analytic expressions used for the calculation.

Certainly, the strong mathematical tool that the domain variational theory is, allows one to consider many other problems related to multiply connected domains either circular or polynomial; either regular or with modification with respect to the symmetrical case; coaxial or not; with two boundaries or many of then. In any case, one ends with quite simple and very accurate analytical expressions without adjustable parameters.

TABLE I: Shape factor S for $N = 3$ and $N = 4$ external polygon coaxial cylinders with inner circle boundary both for the first and second approximations: Eqs.(11) and (12) respectively. Ref.[3] values are indicated for comparison.

a/b	N=3			N=4		
	Eq.(11)	Eq.(12)	Ref.[3]	Eq.(11)	Eq.(12)	Ref [3]
0.1	2.589245	2.589245	2.5892417837	2.641835	2.641835	2.6418293009
0.3	4.731221	4.731282	4.7312803635	4.909771	4.909775	4.90297637284
0.5	7.691014	7.694416	7.6944300913	8.171980	8.172489	8.1724712686
0.7	13.13001	13.20568	13.2052694481	14.54963	14.57357	14.5734159748
0.9	29.28052	31.21268	31.2585633122	36.02949	37.21025	37.1852486539

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