# On the perturbation of partially hyperbolic endomorphisms

Alejandro Meson

Fernando Vericat*

*Instituto de Física de Líquidos y Sistemas Biológicos (IFLYSIB)*
*CONICET–UNLP and Grupo de Aplicaciones Matemáticas y Estadísticas de la Facultad de Ingeniería (GAMEFI) UNLP*
*La Plata*
*Argentina*

**Abstract**

We analyze when partially hyperbolic endomorphisms can be perturbed in order to be close to one with non-zero Lyapunov exponents and with an unique inverse measure. Problems of this nature were already boarded and solved in the setting of diffeomorphisms. The extension to non-invertible maps presents as one the main difficulties the fact of that multivaluated inverse iterations of the map make that the local unstable manifolds may intersect each other since they depend on the whole prehistory.

## 1. Introduction

The Ergodic Hypothesis, introduced by Boltzmann in Statistical Mechanics, states the agreement of the time-average of the orbit of any point with the spacial-average with respect to a probability. With this inspiration was formulated the Birkhoff ergodic theorem which establishes that for any continuous potential $\psi : X \to \mathcal{R}$ the time-average of the orbit of any point converges to the mean value of $\psi$ with respect to a measure $\mu$ invariant with respect to a measurable map $f : X \to X$, i.e. if $S_n(\psi)(x) := \sum_{i=0}^{n=1} \psi(f^i(x))$ then

$$\lim_{n\to\infty} \frac{1}{n} S_n(\phi)(x) = \int \phi d\mu, \text{ for any } x, \mu - a.e.$$

*E-mail: `vericat@inffysib.unlp.edu.ar`

***Journal of Interdisciplinary Mathematics***
Vol. 16 (2013), No. 6, pp. 343–364

A dynamical system $(M,f)$, with $M$ a compact Riemannian manifold, is *conservative* if it preserves a Lebesgue measure on $M$. In general by *stable ergodicity* is understood that a system remains ergodic under small perturbations. Conservative systems have stable ergodicity if every nearby system remains ergodic with respect to the Lebesgue measure.

Relevant studies about ergodic stability in partially hyperbolic diffeomorphisms [10],[11], revealed that natural measures to analyze ergodic stability in conservative systems are those $\mu$ for which there is a disintegration $\{\mu_x\}$ along the unstable foliation in such a way that any $\mu_x$ is absolutely continuous with respect to $m_x$, where $\{m_x\}$ is the disintegration of the Lebesgue measure. These kind of measures are called *unstable Gibbs states, u–Gibbs states for short* or *SRB (Sinai-Ruelle-Bowen)* –measures. In the dissipative case, the *physical measures* thus called by Eckmann and Ruelle, are adequate candidates to replace the Lebesgue measure. Physical measures for a dynamical system are those for which the set

$$V(\mu) = \left\{x : \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i(x)) = \int \phi d\mu\right\},$$

has positive Lebesgue measure for any continuous map $\phi : M \to \mathbb{R}$. The points in $V(\mu)$ are called *generic points* for $\mu$ or $\mu$*-generic*. Properties and application of these measures in fields like non-equilibrium statistical mechanics and turbulence theory were developed by Eckmann and Ruelle. The *SRB*–states are used to study non-equilibrium statistical mechanics systems close to the equilibrium [6]. Any ergodic *SRB*–measure with a non zero Lyapunov exponent is a physical measure [9] and for partially hyperbolic diffeomorphisms physical measures disintegrate along unstable directions and they are $s-$Gibbs states. Partially hyperbolic systems can have zero Lypaunov exponents, along the central direction, now the existence of ergodic measures is related with the existence of non-zero Lyapunov exponents along the central direction. Therefore the analysis of ergodic stability may be close to the possibility of perturbing partially hyperbolic maps to remove zero central Lyapunov exponents.

An important recent insight is the work of Andersson [2], there the author proved that for partially hyperbolic diffeomorphisms $f$ negative central Lyapunov exponents, with respect to any $u$–Gibbs state the number of physical measures of $f$ is an upper-semi continuous map, i.e. there is a $C^2$ –neighborhood $U$ of $f$ such that if $f$ has $N$ physical measures and $g \in U$ has $M$ physical measures then $M \leq N$. So that if $f$ has an unique physical measure can be obtained $C^2$-closed maps having an unique physical measure. Partially hyperbolic diffeomorphisms such that central Lyapunov exponents with respect to any Gibbs state are negative are called *mostly contracting*. More recently Araujo and Vazquez [1] proved that partially hyperbolic diffeomorphisms with minimal stable foliation can be close to maps with non-zero Lyapunov exponents, this was done for the mostly expanding (Lyapunov exponents with respect to any $u$–Gibbs state are $\geq 0$) and mostly contracting cases.

Among the difficulties found to extend results for diffeomorphisms to non-invertible dynamical systems (endomorphisms) can be mentioned:

(1) Since any point has several preimages by the map $f$ the statistical sums $S_n(\phi)(y) := \sum_{i=0}^{n-1} \phi(f^i(y))$ may have different behaviors at the points $y \in f^{-n}(x)$ and so the Birkhoff ergodic theorem cannot be applied directly in these cases.

(2) The local unstable manifolds at depend on whole sequences $xx_{-1}x_{-2}\ldots$, with $x_{-1+1} \in f^{--1}(x_{-i}), i \geq 1$, each sequence of this kind is called a *prehistory* of $x$, so that they may be intersect each other and by a point may pass infinite unstable manifolds. By this situation for non-inveritible maps $f : M \to M$, is considered the¨natural extensionn̈ $\widehat{M}$ of $M$, which consists of the all preimages of the points in With this notion of hyperbolicity, or partial hyperbolicity, can be done for non-invertible maps, by mean of the decomposition of the tangent of $\widehat{M}$.

For hyperbolic endomorphisms, Mihailescu [8], proved that when the basic set is a connected repellor, i.e. the critical points of the map do not intersect the basic set, can be constructed sequences of measures which weakly converge to a measure $\mu_s$, which is the equilibrium state of the stable potential $\Phi^s = \log|\det(Df|E^s)|$. Then is proved that there exists a measure $\mu^-$ which has absolutely continuous conditionally measures on local stable manifolds, such a measures are called *inverse SRB-measures* or $s$–Gibbs states, indeed is established that $\mu_s = \mu^-$ and $\mu^-$ is the unique inverse measure. By an argument of large deviations can be proved that if is the unique inverse measure then is physical. Uniqueness is ensured since is the only measure which satisfies a Pesin type entropy formula for non-invertible maps, due to Liu, which relates entropy and negative Lyapunov. A feature of connected repellors is that the number of the preimages of any point in the basic set is constant, this property remains under $C^{-1}-$ perturbations [8], i.e. if $f$ has a connected repellor basic set and $g$ is $C^{-1}-$ close to $f$ then the number of images is the same for any point in the basic set of $g$.

In this article we consider a situation like Mihailescu, but in the case of partially hyperbolic endomorphisms, to analyze the problem of perturbing connected repellors to obtain endomorphisms with non-zero Lyapunov exponents, i.e. we board a non-invertible version of the problems solved by Andersson and Araujo-Vazquez. As we mentioned for hyperbolic connected repellors there is an unique $s$-Gibbs state and under perturbation the property of having a basic set which is connected repellor remains, so that there is ergodic stability in the hyperbolic case. We analyze what occurs when there is a central direction, i.e. partially hyperbolic dynamics, we show that there is also an unique $s$-Gibbs state and study the problem of perturbing partially hyperbolic endomorphisms to remove zero Lyapunov exponents.

## 2. Background

Let $f: M \to M$ be a smooth, for instance $C^2$, endomorphism and $M$ a Riemannian manifold. Let $\Lambda \subset M$ be a compact $f-$ invariant set, i.e. $f(\Lambda) = \Lambda$. A *basic set* for $f$ is a set $\Lambda$ such that neighborhood $V$ of $\Lambda$ with $\Lambda = \bigcap_{n=0}^{\infty} f^n(V)$. The *preimage* of a point $x$ is $f^{-1}(x) = \{y : f(x) = y\}$ and the *n-preimage* of $x$ is $f^{-n}(x) = \{y : f^n(x) = y\}$. A sequence $xx_{-1}\, x_{-2} \ldots . x_{-n+1}$, with $x_{-i+1} \in f^{-1}(x_{-i}), i = 0, 1, \ldots, n = 1$, is called a *n-prehistory* of $x$, a full prehistory of $x$ is a sequence $xx_{-1}\, x_{-2} \ldots$, with $x_{-i+1} \in f^{-1}(x_{-i}), i \geq 1$.

To give a notion of hyperbolicity for endomorphisms is introduced the following concept:

The *natural extension of a* general metric space $X$ is defined as

$$\hat{X} = \{\hat{x} = xx_{-1}x_{-2} \ldots, \textit{with } x_{-i+1} \in f^{-1}(x_{-i}), i \geq 1, x \in X\}. \tag{1}$$

i.e. the set of the full prehistories, with respect to $f$, of points in $X$.

The map $f$ is extended to a map $\hat{f}$ on $\hat{X}$ by the shift $\hat{f}(\hat{x} = xx_{-1}\, x_{-2} \ldots) = (f(x)xx_{-1}x_{-2} \ldots)$. In the natural extension metric space then in the natural extension is put the metric

$$\widehat{d}(\widehat{x},\widehat{y}) = \sum_{n=0}^{\infty} \frac{d(x_{-n},y_{-n})}{2^n}. \tag{2}$$

This makes the natural extension a compact metric space and $\hat{f}$ a homeomorphism. Also can be defined a natural projection by $\pi : \widehat{X} \to X$ by $\pi(\widehat{x} = xx_{-1}\, x_{-2} \ldots) = x$, this map is a bijection.

Dynamical properties of the system $(X,f)$ can be translated to $(\widehat{X}, \widehat{f})$, for instance if $f$ has the property of specification on $X$ and $f$ is positively expansive in $\Lambda$ then $\widehat{f}$ verifies specification and expansiveness on $\widehat{X}$. Any measure $\mu$ in $X$ can be lifted to an unique measure $\hat{\mu}$ in $\hat{X}$ i.e. $\pi_*(\hat{\mu}) = \mu$, where $\pi_*(\hat{\mu})(E) = \hat{\mu}(\pi^{-1}(E))$, for any set $E$ and besides $h_{\hat{\mu}}(\hat{f}) = h_{\mu}(f)$, where $h_{\mu}$ is the measure theoretical entropy.

Let $\Lambda$ be a basic set for $f$ and $\widehat{\Lambda}$ its natural extension, the fibre bundle over $\widehat{\Lambda}$ is defined as

$$T(\hat{\Lambda}) = \{(\hat{x},u) : \hat{x} \in \hat{\Lambda}, v \in T_x(\Lambda)\}$$

, with fibres $T_{\widehat{x}}(\hat{\Lambda})$.

An endomorphism $f : M \to M$ is hyperbolic if the there is a basic set $\Lambda \subset M$ such that for any $x \in \Lambda, \widehat{x} \in \widehat{\Lambda}$ there is a splitting

$$T_{\widehat{x}}(\widehat{\Lambda}) = E^s(x) \oplus E^u(\widehat{x}), \tag{3}$$

such that

$$D(F)(x)(E^s(x)) = E^s(f(x)), D(f)(x)(E^u(\widehat{x})) = E^s(\widehat{f}(\widehat{x}))$$

, and for any $n \geq 1$, and some constant $\lambda > 1$ holds

$$\| D(f^n)(x)(v) \| \leq \lambda^{-n} |v|, \text{ for } v \in E^s(\hat{x})$$

$$\| D(f^n)(x)(v) \| \leq \lambda^{-n} |v|, \text{ for } v \in E^s(x),$$

Now the corresponding definition of partial hyperbolicity is given as: there is an invariant splitting

$$T_{\widehat{x}}(\widehat{\Lambda}) = E^s(x) \otimes E^c(x) \otimes E^u(\widehat{x}),$$

such that there are numbers $0 < \lambda_1 \leq \mu_1 < \lambda_2 \leq \mu_2 < \lambda_3 \leq \mu_3$, $\mu_1 < 1 < \lambda_3$ with

$$C^{-1}\lambda_1^n|v| \leq \|D(f^n)(x)(u)\| \leq C\mu_1^n|v|, \ v \in E^s(x)$$

$$C^{-1}\lambda_2^n|v| \leq \|D(f^n)(x)(v)\| \leq C\mu_2^n|v|, \ v \in E^c(x)$$

$$C^{-1}\lambda_3^n|v| \leq \|D(f^n)(x)(v)\| \leq C\mu_3^n|v|, \ v \in E^u(\widehat{x}) \tag{4}$$

for some constant $C$. The subbundle $E^c$ is called the central direction. These inequalities mean that $E^s$ is uniformly contracting by $f$ and $E^u$ is uniformly expanding, the central direction $E^c$ is dominated by $E^u$ and dominates $E^s$.

Recall that a distribution $E_x \subset T_x(M)$ is *integrable* if there is a foliation $W$ of $M$ with leaves $W(x)$ and a $C^1$ curve $\sigma : \mathbb{R} \to M$ such that $\sigma(t) \in E_{\sigma(t)}$ for any $t$, in particular $T^x(W(x)) = E_x$, for any $x \in M$. In the case of hyperbolic diffeomoprphisms is well known that the stable and unstable sub bundles are integrable, can be defined stable and stable local manifolds $W_\varepsilon^u(x), W_\varepsilon^s(x)$, satisfying the above definition. For endomorphisms occurs that the fibres of the unstable sub bundle at points $\widehat{x} = xx_{-1}x_{-2} \in \widehat{\Lambda}$ depend on the full prehistories of and local unstable manifolds at each $\widehat{x} \in \widehat{\Lambda}$ are defined as:

$$W_\varepsilon^u(\widehat{x}) = \{y \in M : \exists\, \widehat{y}, \pi(\widehat{y}) = \mu : d(x_{-i}, y_{-i}) < \varepsilon, \text{for any } i \geq 0\}.$$

The stable bundle $E^s$ depends only of $x$, local stable manifolds at each $x \in \Lambda$ are defined as:

$$W_\varepsilon^s(x) = \{y \in M : d(f^n(x), f^n(y)) < \varepsilon, \text{for any } n \geq 0\}.$$

$$W_\varepsilon^u(\widehat{x}) = \{y \in M : \exists\, \widehat{y}\ \pi(\widehat{y}) = y : d(x_{-i}, y_{-i}) < \varepsilon, \text{for any } i \geq 0\}.$$

and

$$W_\varepsilon^u(\widehat{x}) = \{y \in M : \exists \widehat{y}, \pi(\widehat{y}) = \mu : d(x_{-i}, y_{-i}) < \varepsilon, \text{for any } i \geq 0\}.$$

The stable and unstable manifolds are respectively

$$W^s(x) = (y \in M : d(f^n(x), f^n(y)) \to_n 0\}.$$

$$W^u(\widehat{x}) = \{y \in M : \exists \widehat{y}, \pi(\widehat{y}) = y : d(x_{-i}, y_{-i}) \to_i 0\}.$$

In general the central sub bundle is not integrable, when the sub bundles $E^{cs} := E^c \otimes E^s, E^{cu} := E^c \otimes E^u$ and $E^c$ and $E^c$ are integrable the system is called dynamically coherent, for diffeomorphisms one condition for integrability is: let $\widetilde{W^s}$, $\widetilde{W^u}$ be the lifting to $\widetilde{M}$ of the stable and unstable manifolds respectively, if $\widetilde{W^s}$ and $\widetilde{W^u}$ are quasi-isometric then $E^{cs}, E^{cu}$ and $E^c$ are integrable. Let $W^c$ be the central manifold and by $W^c_\varepsilon(x) = W^c(x) \cap B_\varepsilon(x)$ are denoted the local leaves.

Let $f$ be hyperbolic with basic set $\Lambda$, if $\Lambda$ is a connected repellor, i.e. the critical points of the map do not intersect the basic set, in this case all points have the same number of preimages, for this kind of systems Mihailescu [8] introduced the following measures

$$\mu_n^z = \frac{1}{d^n} \sum_{y \in f^{-n}(z) \cap U} \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(y)}, \tag{5}$$

where $\cup \supset \Lambda, z \in \Lambda$, and $d = card\{f^{-1}(y) \cap U\}$, the sequence $\{\mu_n^z\}$ weakly converges to a measure $\mu_s$, Lebesgue almost all $z \in \Lambda$. The measure $\mu_s$ is precisely the unique Gibbs state of the potential $\Phi^s = \log|Jac(Df|_{E^s})|$.

A partition $\mathcal{A}$ of $M$ is subordinated to the unstable manifold $W^u$, for a given measure $\mu$, if $\mathcal{A}(x) \subset W^u(x),$, for any $\mu - a.e.x$, where $\mathcal{A}(x)$ is the member of $\mathcal{A}$ containing $x$ Let $\{\mu_{x,A}\}$ be the conditional measures of $\mu$ for the partition $\mathcal{A}$ and let $m_{x,\mathcal{A}}$ be Riemannian measure on $\mathcal{A}(x)$, inherited from the Riemannian measure $W^u(x)$ in $W^u(x)$ A $f-$invariant measure $\mu$ has *absolutely continuous conditionally measures* on $W^u$ if has a positive Lyapunov exponents and $\mu_{x,\mathcal{A}}$ is absolutely continuous with respect to $m_{x,\mathcal{A}}$ for any measurable partition $\mathcal{A}$ subordinated to $W^u$ and for any $\mu - a.e.x$. Now measures are those which have absolutely continuous conditionally measures on unstable manifolds. The states are used to study non-equilibrium statistical mechanics systems close to the equilibrium [6].

The measure $\mu_s$ satisfies a Pesin type entropy formula which relates entropy and negative Lyapunov exponents, which was discovered by Liu [7], more specifically: if $\varepsilon$ is the partition by points of then the folding entropy is defined as the conditional entropy $H_\mu(\varepsilon\,|\,f^{-1}(\varepsilon))$ and the entropy production of $(f,\mu)$ is

$$e_\mu(f) := H_\mu(\varepsilon\,|\,f^{-1}(\varepsilon)) + \int \log|D_x(f)|\,d\mu.$$

Let $-\infty \leq \lambda_1(x) \leq \lambda_2(x) \leq \ldots \leq \lambda_{(r)}(x)$ be the Lyapunov spectrum of $(f,\mu)$, and the measure-theoretic entropy of the measure Liu proved that if $m_i(x)$ is the multiplicity of the Lyapunov exponent $\lambda_i(x)$ then

$$h_\mu(f) = H_\mu(\varepsilon\,|\,f^{-1}(\varepsilon)) - \int \textstyle\sum_i m_i(x)\lambda_i^-(x) \tag{6}$$

is valid if and only if has absolutely continuous conditionally measures on stable manifolds, where $a^- = \min(a,0)$ If $\mu$ is a *SRB* $-$measure then $h_\mu(f) = \int \sum_i m_i(x)\lambda_i^+(x)$, with $a^+ = \max(a,0)$ and *so* $e_\mu(f) \geq 0$, to obtain inverse *SRB* $-$measures, recall that inverse is in the sense of having conditional measures on stable manifolds, must be analyzed cases with entropy production $= 0$ or $> 0$. The result of Liu is valid with great generality, not only for hyperbolic or partially hyperbolic endomorphisms, he made assumptions just on the Jacobian of the map.

In [8] was proved if *f* is a $d$ to endomorphism then $H_{\mu_s}(\varepsilon\,|\,f^{-1}(\varepsilon)) = \log d$ and that

$$h_{\mu_s}(f) = \log d - \int \textstyle\sum_i m_i(x)\lambda_i^-(x), \tag{7}$$

therefore the weak limit of $\{\mu_n^z\}$ has absolutely continuous conditionally measures on stable manifolds, now let us call $\mu_f^-$ to such a limit, instead of $\mu_s$.

The *central Lyapunov exponent of f* (recall that we shall assume that dim $E^c = 1$), is the map

$$(f,x) \to \lambda^c(f,x) = \lim_{n\to\infty} \frac{1}{n} \log \| Df^n(x)\,|\,_{E^c} \|,$$

provided the existence of the limit otherwise may be taken *lim* sup or *lim in f*. The integrate central exponent, with respect to the measure $\mu$, is

$$\lambda^c(f,\mu) := \int \lambda^c(f,x)d\mu.$$

For $\mu = \mu_f^-$, we denote $\lambda_-^c(f) = \lambda^c(f, \mu_f^-)$, since $\mu_f^-$ is ergodic, because it is a Gibbs, state results

$$\lambda_-^c(f) = \int \log \| Df|_{E^c} \| d\mu_f^- \tag{8}$$

A foliation $\mathcal{F}$ of is called *minimal* if all of its leaves are dense in $M$. Let $\mathcal{F}^s$ the foliation whose leaves are the stable manifolds $W^s(x)$, we do not need to impose that $\mathcal{F}^s$ be minimal for the uniqueness of the Gibbs state, but shall need to assume that the stable foliation be minimal to ensure that the orbits of Lebesgue almost every point visit any open set $U$ with positive frequency and $\mu(U) > 0$, for any Gibbs state $\mu$.

## 3. Statement of the main result

We present the main result to be proved in this article

**Theorem.** Let $f: M \to M$ be a partially hyperbolic $C^2-$ endomorphism on a compact Riemannian manifold $M$ is assumed that $f$ is a connected repellor, the system is dynamically coherent, the central subbundle is one-dimensional and the stable foliation is minimal. Then $f$ can be perturbed to a $C^2$-close map $g$ which has a $s$-Gibbs state $\mu_g^-$ with $\lambda_-^c(g) \neq 0$.

The steps to follow for proving the theorem are outlined next:

- Firstly is established that $f$ has a $s$-Gibbs state $\mu_f^-$ this measure is obtained as weak limit of the sequence like in the hyperbolic case, here are pointed out the necessary modifications to extend the result of [8] to partially hyperbolic systems. $\{\mu_n^z\}$ Like for hyperbolic systems can be proved that if $g$ is a $C^2$-close to a connected repellor then it is also a connected repellor, thus $q$ has an unique $s$-Gibbs state $\mu_g^-$, now the system is stably ergodic. Then is also analyze cases for which $\mu_f^-$ is physical, this fact is indeed no essential in the proof of the theorem, but we add it for a more complete analysis.

- The map $f \to \lambda_-^c(f)$ is lower-semi continuous, from which can be obtained that if $\lambda_-^c(f) > 0$ then $\lambda_-^c(g) > 0$.

- When $\lambda_-^c(f) \leq 0$, are followed the technics of Araujo and Vazquez to define a local perturbation $g$, such that for any point $x$ belonging to a set $W$. with $\mu_g^-(W) > 0$ holds $E_f^c(x) \neq E_g^c(x)$. The perturbation "inclines" the central direction towards the stable subbundle $E_f^s(x)$ in such a way that $\| Df|_{E_f^c} \|$ and $\| Dg|_{E_g^c} \|$ can be compared to establish that $\lambda_-^c(g) < 0$, in this case.

Let $\mu_n^z = \frac{1}{d^n} \sum_{y \in f^{-n}(z) \cap U} \frac{1}{n} \sum_{n=0}^{n-1} \delta_{f^i(y)}$, be the sequence of measures introduced in [8], here $d = card\{f^{-1}(y) \cap U\}, U \supset \Lambda$, recall that in the hyperbolic case was proved that this sequence weakly converges to the Gibbs state $\mu_s$ of the potential $\Phi^s \log|\det(Df|_{E^s})|$, and which is a $s$-Gibbs state. We sketch next how to establish here the validity of this result for partially hyperbolic systems by rewriting the theorem 2 of [8] with the needed modifications.

Any point of $\Lambda$ has the same number of preimages, so there exits a neighborhood $U$ of $\Lambda$ such that any point in $U$ has $d^n$ preimages. In [8] is established that if $\mu \in \mathcal{M}(\Lambda, f), \hat{\mu} \in \mathcal{M}(\hat{\Lambda}, \hat{f})$ with $\pi_*(\hat{\mu}) = \mu$, then for $\hat{E} \subset \hat{\Lambda}$ is valid $\hat{\mu}(\hat{E}) = \lim_{n\to\infty} \mu(\{x_{-n} : \text{there is } a\ \hat{x} = xx_{-1}\cdots x_{-n}\cdots \in \hat{E}\})$.

Let us denote $\sum(n,x) := \frac{1}{n}\sum_{i=0}^{n-1} \delta_{f^i(x)} - \mu_s$, and $\sum(n,x,\phi) = \frac{1}{n}$ $\sum_{i=0}^{n-1} \phi(f^i(x)) - \int \phi d\mu_s$, for a map $\phi : M \to \mathbb{R}$. If $\eta > 0, n \geq 1$ then let set $\hat{E}_n(\eta) = \{\hat{x} \in \hat{\Lambda} : |\sum(n, x_{-n})| \geq \eta\}$, the Birkhoff Ergodic Theorem can be applied in $(\hat{\Lambda}, \hat{f})$ and then for small $\eta$, holds $\hat{\mu}_s(\hat{E}_n(\eta)) \underset{n\to\infty}{\to} 0$. So that for any small $\eta > 0$, and $\gamma > 0, n \geq 1$ there is a $N = N(\eta, \gamma)$ such that

$$\mu_s(\{x_{-m} \in \Lambda \cap f^{-m+n}(x_{-n}) : \sum(n, x_{-n}, \phi) \geq \eta\}) < \gamma,$$

for any $m > N$.

Let us fix a Hölder continuous map $\phi$ it must be proved that if $y, z$ are such that if $d_n(z,y) := \max\{d(f^i(z), f^i(y)) : i = 0, 1, \ldots n-1\} < \varepsilon$ then $\sum(n,y,\phi), \sum(n,z,\phi)$ behave similarly for enough large For this it should be shown that if $\eta > 0$, with $|\sum(n,y,\phi)| \geq \eta$ then there is a $N = N(\eta)$ such that $|\sum(n,z,\phi)| \geq \eta/2$, for any $z$ with $d_n(x,y) < \varepsilon, n \geq N$.

Let $B_{n,\varepsilon}(y)$ be the ball in the metric $d_n$ of radius centered at $y$ if $z \in B_{n,\varepsilon}(y)$ then $z \in W^s_\varepsilon(y)$ or $d(z, W^s_\varepsilon(y)) > 0$. As is pointed out in [8] all the local stable manifolds are contained in the basic set $\Lambda$. In the case of $z \in W^s_\varepsilon(y)$ there is a $\lambda \in (0,1)$ such that $d(f^i(z), f^i(y)) < \lambda^i, i = 0, 1, ... n-1$, so there exists a $N_0 \geq 1$ with

$$|\phi(f^{N_0}(y)) + ... + \phi(f^{n-1}(y)) - \phi(f^{N_0}(z)) - ... - \phi(f^{n-1}(z))| \leq C\lambda^{\alpha N_0},$$

for some constant $C$ and where is the Hölder exponent of $\phi$.

If $z \notin W^s_\varepsilon(y)$, but $z \in B_{n,\varepsilon}(y)$ then the iterations $f^n(z)$ will close exponentially to unstable o central directions, in the hyperbolic case are considered of course the case of unstable manifolds, this is the main difference in the proof for the partially hyperbolic case, but the domination of the central direction allows to manage this situation.

In case of the iterations $f^n(z)$ approach to the unstable direction they are close manifolds at $\widehat{f^n(y)}$ and the distance between their "projections" over this manifolds grows exponentially up to value less than $\varepsilon$. In case of becoming close to central manifolds at $f^n(y)$ the growing of the distance is controlled by the growing in the unstable manifold. Now there exist $N_0, N_1, \lambda \in (0,1)$ such that

$d(f^i(z), f^i(y)) < \lambda^i, i = N_0, ..., N_1 - 1$. The number is taken such that $d(f^i(z), f^j(y)) < \overline{\lambda} d(f^{j+1}(z), f^{j+1}(y)), j = N_1, ..., n-2$, where $\bar{\lambda}$ is some number in $\left(\frac{1}{\inf \Lambda Df|_{E^s}}, 1\right)$.

If $f^{N_1}(z)$ approaches to $W^u_\varepsilon\left(\widehat{f^{N_1}(y)}\right)$ or $W^c_\varepsilon(f^{N_1}(y))$ then $d(f^i(z), f^i(y))$ will grow exponentially in $W^u_\varepsilon\left(\widehat{f^{N_1}(y)}\right)$ or in some cases in $W^c_\varepsilon(f^{N_1}(y))$ or will have a stable component like in the first case. Further like in [8] follows that for a number $N_2 < n - N_1$

$$|\Sigma(n, y, \psi) - \Sigma(n, z, \psi)| \leq \frac{1}{n}\left[2N_0 \|\psi\| + C\left(\lambda^{\alpha N_0} + \lambda^{-\alpha N_2} + 2N_2 \|\psi\|\right)\right],$$

let $N = N(\eta)$ such that

$\frac{1}{n}\left[2N_0 \|\psi\| + C\left(\lambda^{\alpha N_0} + \lambda^{-\alpha N_2} + 2N_2 \|\psi\|\right)\right] < \eta/2$ for $n > N$, therefore

$$|\Sigma(n,z,\psi)|\geq \eta/2, \textit{for}\,\Lambda \textit{any}\, z\in B_{n,\varepsilon}(y), \tag{9}$$

and

$$|\sum(n,z.\psi)-\sum(n,y.\psi)|<\eta/2,\ z\in B_{n,\varepsilon}(y). \tag{10}$$

Let $F_{n,\varepsilon}$ be $(n,\varepsilon)$-separated set in $\Lambda$ with maximal cardinality and $U$ a neighborhood of $\Lambda$ which points have $d^n n$-preimages. Let $\Lambda\subset V\subset U$ such that all the points of $V$ have $d^n$ $n$-preimages in $U$ If $y\in V$ then $y\in f^n(B_{n,3\varepsilon}(y_i)), y_i\in F_{n,\varepsilon}, i=1,...,d^n$. Let $I(n,x,\psi)=\frac{1}{d^n}$ $\sum_{y\in f^{-n}(x)\cap U}|\sum(n,y,\psi)|$, this expression can be decomposed in pieces $V(y_1,...,y_n)$, for different $y_1,...,y_n\in F_{n,\varepsilon}$ in such away that $\sum(n,y,\phi)$ can be replaced by $\sum(n,\zeta,\phi)$ with $\zeta\in F_{n,\varepsilon} and\ y\in B_{n,3\varepsilon}(\zeta)$. Now $|\sum(n,y,\phi)|\leq |\sum(n,y,\phi)|+\eta/2$, for $n\geq N(\eta)$, and then $\int_v I(n,x,\phi)dm(x)\leq\frac{1}{d^n}$ $\sum_{z_1,...,z_{d^n}\in F_{n,\varepsilon}}\int_{V(y_1,...,y_n)}\sum_{i=1}^{n}|\sum(n,z_i,\phi)|\,dm+\eta/2m(V)\leq\frac{1}{d^n}\sum_{z\in F_{n,\varepsilon}}|\Sigma(n,z,\phi)|$ $\sum_{z\in\{z_1,z_{dn}\}}m(V(y_1,...,y_n))+\frac{\eta}{2}m(V)$, where $m$ is the Lebesgue measure in $\Lambda$.

Thus in a similar way than in [8] obtains

$$\int_V I(n,x,\phi)dm(x)\leq C\sum_{y\in F_{n,\varepsilon}}|\sum(n,y,\phi)|\cdot m\big(f^n(B_{n,3\varepsilon}(y))\big)\frac{1}{d^n}+\frac{\eta}{2}m(V) \tag{11}$$

and

$$\int_V I(n,x,\phi)dm(x)\leq C_1\sum_{y\in F_{n,\varepsilon}}|\sum(n,y,\psi)|\cdot\mu_s((B_{n,\varepsilon/2}(y))+\eta), \tag{12}$$

for some constant $C_1$.

Then is partitioned in disjoint sets $G_{n,\varepsilon}=\{y\in F_{n,\varepsilon}:|\sum(n,y,\psi)|<\eta\}$ and $H_{n,\varepsilon}=\{y\in F_{n,\in}:|\sum(n,y,\phi)|\geq\eta\}$ $\sum_{y\in F_{n,\varepsilon}}|\sum(n,y,\phi)|\cdot\mu_s((B_{n,\varepsilon/2}(y)))=$ $\sum_{y\in G_{n,\varepsilon}}|\sum(n,y,\phi)|\cdot\mu_s((B_{n,\varepsilon/2}(y)))+\sum_{y\in H_{n,\varepsilon}}|\sum(n,y,\phi)|\cdot\mu_s(B_{n,\varepsilon/2}(y))$ if $y\in H_{n,\varepsilon}, z\in B_{n,\varepsilon/2}(y)$ then $|\sum(n,z,\phi)|\geq\eta/2$ and

$B_{n,\varepsilon/2}(y) \cap \Lambda \subset \{z \in \Lambda : |\sum(n,z,\phi)| \geq \eta/2\}$ Therefore

$$\sum_{y \in F_{n,\varepsilon}} |\sum(n,y,\phi)| \cdot \mu_s((B_{n,\varepsilon/2}(y))) \leq$$

$$\eta \sum_{y \in G_{n,\varepsilon}} \mu_s(B_{n,\varepsilon/2}(y)) + 2 \| \phi \| \times \mu_s(z \in \Lambda : |\sum(n,z,\phi)| \geq \eta/2) C_\varepsilon,$$

for a constant $C_\varepsilon$.

Since the balls $B_{n,\varepsilon/2}(y)$ are disjoint $\sum_{y \in G_{n,\varepsilon}} \mu_s(B_{n,\varepsilon/2}(y)) \leq 1$, and also $\mu_s(z \in \Lambda : |\sum(n,z,\phi)| \geq \mu/2) \leq \gamma$ for $n \geq N = N(\eta/2, \gamma)$. The proof continues like in [8] to obtain

$$\int_V I(n,x,\phi) dm(x) \leq C_1(2\eta + C_\varepsilon \| \phi \| \gamma),$$

for arbitrary $\gamma, \eta$.

To see that $\mu_{\bar{f}}$ is physical it must be proved that if $m$ is the Lebesgue measure on $M$, then for any continuous potential $\phi: X \to R$ holds $\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i(x)) = \int \phi d_{\bar{f}}$ for a -a.e. To establish this fact can be analyzed the set of points for which $\frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i)$ apart from the integral, this can be done by mean of elements from the large deviations theory developed by L.S. Young [12]. If $\varphi \in C(X)$ has exponential decay, i.e. there are $C > 0, \varepsilon > 0$, such that for any $x \in \Lambda, n \geq 0$.

$m(B_{n,\varepsilon}(x)) \leq C \exp(-S_n(\varphi)(x))$ then the following large deviation principle is satisfied.

$$\limsup_{n\to\infty} \frac{1}{n} \log m\left\{x : \left| \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i(x)) - \int \phi d\mu \right| > \delta \right\}$$

$$\leq \sup\left\{h_v(f) - \int \varphi dv : \left| \int \phi d\mu - \int \phi dv \right| \geq \delta \right\}.$$

The stable potential $\Phi^s$ is Hölder continuous since the stable subbundles depend Hölder continuously on the base point and verifies a bounded distortion condition [5], from this and [8] can be justified that the large deviation principle holds for $P(\Phi^s)+\Phi^s$ ($P$ is the topological pressure) So that if, $\mu=\mu_{\bar{f}}$, and $f$ is $d-\text{to}-1$, the large deviation principle is expressed as;

$$\limsup_{n\to\infty}\frac{1}{n}\log m\left\{x:\left|\frac{1}{2}\sum_{i=0}^{n-1}\phi(f^i(x))-\int\phi d\mu_{\bar{f}}\right|>\delta\right\}$$
$$\leq \sup\left\{h_v(f)-\lambda_v:\left|\int\phi d\mu_{\bar{f}}-\int\phi dv\right|\geq\delta\right\},$$

where $\lambda_v:=\log d-\int\sum_i m_i(x)\lambda_i^-(x)dv(x)$.

A large deviation principle for rational maps was established by Comman and Rivera-Letelier [4], in this case is not required exponential decay: let $\hat{\boldsymbol{C}}$ be the Riemann sphere and $f:\hat{\boldsymbol{C}}\to\hat{\boldsymbol{C}}$ be a rational map of degree at least two, satisfying the *topological Collet-Eckmann* condition, i.e there is a constant $C>0$, such that for any $\mu\in\mathcal{M}(\mathcal{J}((f),f)$ holds $\int\log|f'|\,d\mu\geq C$, where $\mathcal{J}((f)$ is the Julia set of $f$. If $\varphi:\mathcal{J}((f)\to\boldsymbol{R}$ is a Hölder continuous map then

$$\limsup_{n\to\infty}\frac{1}{n}\log m\left\{x:\left|\frac{1}{n}\sum_{i=0}^{n-1}\phi(f^i(x))-\int\phi d\mu\right|>\delta\right\}$$
$$\leq \sup\left\{h_v(f)-P(\varphi)-\int\varphi dv:\left|\int\phi d\mu-\int\phi dv\right|\geq\delta\right\}.$$

**Proposition 1.** The measure $\mu_{\bar{f}}$ is physical.

***Proof***. The proof follows the ideas of [12], firstly we claim that $\sup\left\{h_v(f)-\lambda_v:\left|\int\phi d\mu_{\bar{f}}-\int\phi dv\right|\geq\delta\right\}$ if it were $\sup\{h_v(f)-\lambda_v:\left|\int\phi d\mu_{\bar{f}}-\int\phi dv\right|\geq\delta\}=0$, there would be a sequence of measures $(v_n)$ such that $\left|\int\phi d\mu_{\bar{f}}-\int\phi dv_n\right|\geq\delta$ and $hv_n(f)-\lambda_{v_n}\to_n 0$. The map $v\to hv(f)$ is upper semicontinuous, this can be seen for a result by Bowen [3]: for any $\varepsilon>0$, there is a $\delta>0$, such that if $\mathcal{A}$ is a partition with diameter $<\delta$ then $h_v(f)\leq h_v(f,\mathcal{A})-\varepsilon$. This fact joint with $\limsup_{n\to\infty}h_{v_n}(f)\leq h_v(f,A)$

leads that $\nu \mapsto h_\nu(f, A)$ is upper semicontinuous. Now let $\overline{\nu}$ be an accumulation point of $(\nu_n)$ so that since $\log|\det(Df|_{E^s})|$ is continuos and so $\nu \mapsto \lambda_\nu$ does, is obtained $h_{\overline{\nu}}(f) = \lambda_{\overline{\nu}}$. But this is not possible since $\mu_{\bar{f}}$ is the unique measure which verifies this equality. Therefore

$$\limsup_{n\to\infty} \frac{1}{n} m\left\{x : \left| \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i(x)) - \int \phi d\mu_{\bar{f}} \right| > \delta \right\} < 0.$$

Let $Z_{r,n} := \left\{x : \left| \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i(x)) - \int \phi d\mu_{\bar{f}} \right| \geq \frac{1}{r} \right\}$, hence $\limsup_{n\to\infty} \frac{1}{n}$ $\log m(Z_{r,n}) = 0$ and so $\sum_{n=1}^{\infty} m(Z_{r,n})$ converges for any fixed $r$. Finally for $r \to \infty$.

$$\frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i(x)) \to_n \int \phi d\mu_{\bar{f}}, \text{ for } m - a.e.x. \qquad \blacksquare$$

By the comment above the proposition for rational complex maps and potentials $\varphi : \mathcal{J}((f) \to \boldsymbol{R}$ with an unique stable Gibbs state $\mu$ is valid that $\mu$ is physical.

Following the scheme for proving the theorem state:

**Lemma 1.** The map $f \to \lambda_-^c(f) = \int \log \| Df|_{E^c} \| d\mu_{\bar{f}}$ is lower semicontinuous.

***Proof.*** Let $\varepsilon > 0$ and let $N$ be enough large such that $\frac{1}{N} \int \log \| Df^N(x)|_{E_f^c} \| d\mu_{\bar{f}} > \lambda_-^c(f) - \varepsilon$, it can be considered a neighborhood $U$ of $(f, \mu_{\bar{f}})$ such that $\frac{1}{N} \int \log \| Dg^N(x)|_{E_g^c} \| > \lambda_-^c(f) - \varepsilon$, for any $(g, \mu_{\bar{g}}) \in U$ Thus $\lambda_-^c(g) = \lim_{n\to\infty} \frac{1}{n} \int \log \| Dg^n(x)|_{E_g^c} \| > \lambda_-^c(f) - \varepsilon$. $\blacksquare$

From this Lemma and the existence of stable Gibbs states in partially hyperbolic systems is obtained the conclusion of theorem 1 for the case in which the central Lypunov exponent is strictly positive. For the situations in which can be $\lambda_-^c(f) \leq 0$, is used a perturbation map whose construction follows [1].

Let $f: M \to M$ and $x_0 \in M$, let be $U$ a neighborhood of $x_0$ with $f(U) \cap U = \varnothing$. By $B_1^d(0)$ is denoted the unit ball in $\boldsymbol{R}^d$, with $d = \dim E_f^u + \dim E_f^c$. For a manifold $\Sigma$ transversal to $E_f^s$ is considered a $C^2-$ parametrization $\eta: B_1^d(0) \to \sum$, also can be assumed that $T_{x0}(\sum) = E_f^u(x_0) \oplus E_f^c(x_0)$. There is a embedding $i: B_1^s(0) \to M$, where $B_1^s(0)$ is the unit ball in $R^{\dim E_f^s}$ [13]. Now is defined a $C^2$-parametrization.

$\phi: D \to U$, with $D = B_1^d(0) \times B_1^s(0)$ by $\phi(w,z) = y$ where $y$ is the point with of $M$ with $d(y, \eta(w)) = |z|$.

The map $\phi$ can be chosen such that

$$D\phi(w,0)(\boldsymbol{R}^d \times \{0\}) = E_f^u(\phi(w,0)) \oplus E_f^c(\phi(w,0))$$

$$D\phi(w,z)\left(\{0\} \times \boldsymbol{R}^{\dim E_f^s}\right) = E_f^s(\phi(w,z))$$

$$D\phi(w,z)(\{0\} \times \boldsymbol{R} \times \{0\}) = E_f^c(\phi(w,z)).$$

Let $\Phi: \boldsymbol{R} \to [0,1]$ such that

$-\Phi \equiv 0$ en $\boldsymbol{R} - (2,2), \Phi \equiv 1$ en $[-1,1]$ and

$-(s\Phi)' \neq 0$ for any $s \in (2,2)$ except for $\{\pm s_0\}$ for some $s_0 \in (1,2)$. From the map $\Phi$ is defined a function

$$\Phi_\varepsilon: D \to \boldsymbol{R}$$

$$\Phi_\varepsilon((x_1, .., x_u, y), (z_1, ..., z_s)) = \prod_{i=1}^{\dim E_f^u} \Phi\left(\frac{x_i}{\varepsilon}\right)\phi(y) \prod_{i=1}^{\dim E_f^s} \Phi\left(\frac{z_i}{\varepsilon}\right)$$

and then a map

$$h = h_{t,\varepsilon}; D \to D$$

$$h_{t,\varepsilon}(x^u, y, z^s) = (x, y, z^s + ty\Phi_\varepsilon(x^u, y, z^s)e_{u+c+1}) \tag{13}$$

,here $x^u = (x_1, ..., x_u), z^s = (x_1, .., z_s), \{e_i\}$ is the canonical basis in $R^d$ and $e_{u+c+1}$ is the first coordinate along the stable direction. Finally is defined the perturbed map

$$g(x) = \begin{cases} f(x) \text{ if } x \in M - U \\ (f \circ H)(x) \text{ if } x \in U \end{cases} \tag{14}$$

where $H = \phi \circ h \circ \phi^{-1}$

In the construction of can be taken $t = t(\varepsilon)$, with $\varepsilon \to 0$ and with this the $C^2$-distance between $f$ and $g$ can be bounded in the $C^2$-topology, and in this way the perturbation $g$ is $C^2$ – close to $f$ If the stable foliation $\mathcal{F}^s$ is of class $C^2$ then the function $H$ becomes of class $C^2$ Besides if $x \in V = V_\varepsilon := \phi(B_{2\varepsilon}(0))$, then $DH(x)$ preserves the stable and central-stable foliations. From the construction of $g$ can be noticed that the stable directions of $g$ and $f$ agree, so that the same occurs for the stable foliations of $g$ and $f$ Another property of is $Dg(x)(E_f^c(x)) = E_f^c(g(x))$.

The next task is to show that the central directions of $f$ and $g$ apart in a positive angle. For a vector $v = v^c + v^s \in E^{cs}$ its central slope, with respect to the stable direction, is defined as $S_c(v) = S_{c,s}(v) = \frac{\| v^s \|}{\| v^c \|}$.. Following [1] are considered field cones around the central directions of $f$ and $g$

$$C_{\alpha,f}(x) = \{v = v^c + v^s \in {}^{cs}_f(x) : S_c(u) \leq \alpha\}$$

$$C_{\alpha,g}(x) = \{u = v^c + v^s \in {}^{cs}_g(x) : S_c(v) \leq \alpha\},$$

if it is proved that $C_{\alpha,f}(x)$ is not $Dg(x)-$ invariant or $C_{\alpha,g}(x)$ is not $Df(x)$ invariant then $E_f^c(x)$ and $E_g^c$ can be written as graphs of linear maps $\mathcal{L}: E^c_{f,g}(x) \to E^{cs}_{f,g}(x)$.

By a direct calculation if $v = v^c + v^s \in E_f^{cs}(x)$, and $x \in V = \phi(B_{2\varepsilon}(0))$ then follows

$$S_c(Dg(x)v) = \frac{\| (Df(H(x))u)^s \|}{\| (Df(H(x))u)^c \|} =$$

$$= \frac{\| Df(H(x))(DH(x)v)^s \|}{\| (DH(x)v)^s \|} \frac{\| (Df(H(x)))v^c \|}{\| Df(H(x))(DH(x)v)^c \|} S_c(DH(x)v). \tag{15}$$

*If* $\{\omega_t : M \to M\}$ is an Anosov flow $f = w_1$ and then the central direction is the direction of the flow, so $\| Df(x)u^c \| = \| u^c \|$, for any and the above formula reduces, in this case, to

$$S_c(Dg(x)v) = \frac{\| Df(H(x))(DH(x)u)^s \|}{\| (DH(x)u)^s \|} S_c(DH(x)v)$$

When $x \in M - V$

$$S_c(Df(x)v) = S_c(Dg(x)v) = \frac{\| Df(x)v^s \|}{\| v^s \|} \frac{\| v^s \|}{\| v^c \|} \frac{\| v^c \|}{\| Df(x)v^c \|}$$

$$= \frac{\| Df(x)v^s \|}{\| v^s \|} \frac{\| v^c \|}{\| Df(x)v^c \|} S_c(v), \tag{16}$$

which reduces for the case of time Anosov flow to

$$S_c(Df(x)v) = S_c(Dg(x)v) = \frac{\| Df(x)v^s \|}{\| v^s \|} S_c(v).$$

If $x \in V, g^i(x) \notin V,\ i = 1, \ldots, n$ and $g^{n+1}(x) \in V$ then

$$S_c\left(Dg^{n+1}(x)v\right) = \frac{\| Df^{n+1}(H(x))\,((DH(x)v)^s \|}{\| (DH(x)v)^s \|} S_c(DH(x)v).$$

Let be a fixed point of $H$ and $0 < \varepsilon < /4$, for a small $t = t(\varepsilon)$.

$DH(x_0)\left(0, v^c, v^s\right) = \left(0, v^c, tv^c + v^s\right)$. If $v = v^c + v^s \in E_f^{cs}(x_0)$, then $S_c(DH(x_0)v) = \left\| \frac{tv^c + v^s}{v^c} \right\| \geq |t| - S_c(v)$, If $S_c(v) < |t|/4$ then $S_c(DH(x_0)v) > S_c(v)$.

Thus, by the continuous dependence $x \to E_f^{cs}(x)$ there is a $\eta > 0, \alpha > \beta > 0$ such that if $x \in B_\eta(x_0), v \in E_f^{cs}(x)$ and $S_c(v) < \beta$ then

$S_c(DH(x)v) > \alpha > \beta > S_c(v)$ and so $DH(x)v \in C_\beta(x)$. Therefore $DH(x)(C_\beta(x))$ is a can around $DH(x)(E_f^c(x))$.

Let $x \in B_\eta(x_0) \subset V$, such that $g^i(x) \notin V, i = 1,...,n$ and $g^{n+1}(x) \in V$, hence.

$S_c(D_g^{n+1}(x)v) \geq \lambda_1^{n+1} S_c(DH(x)v) > \lambda_1^{n+1}\alpha > S_c(v)(\lambda_1$ from the definition of partial hyperbolicity). Thus if return to after at most $n+1$ iterations then $Dg^{n+1}(C_\beta(x))$ apart from $C_\beta(g^{n+1}(x))$ If $R(x)$ is first-return time map of $x \in B_\eta(x_0)$ to $V$ then $E_f^c(g^{R(x)}(x)) \not\subseteq C_\beta(g^{R(x)}(x))$. Thus $E_f^c(x)$ and $E_g^c(x)$ apart with a positive angle for those $y \in V$ such that $g^R(x) = y$, for some $x \in B_\eta(x_0)$ which does not return to $V$ in a time less than $R$. Let us call $W$ to this set. In particular if $y \in W$ then $E_g^c(y)$ can be written as the graph of a linear map

$$\mathcal{L}: E_f^c(y) \to E_f^s(y).$$

With the next proposition the proof of the theorem will be completed.

**Proposition 2.**

(i) The set $W$ has $\mu_g^-$ positive measure.

(ii) For any $x \in M$ holds.

$\| Dg(x)|_{E_g^c}(x)\| \leq (1 - G(x)) \left\| Df(x)\right|_{E_f^c}(x)\Big\|$, for some measurable function $G: M \to \boldsymbol{R}$, with $G(x) \geq 0$.

***Proof.***

(i) It is assumed that the stable foliation $\mathcal{F}^s$ is minimal. i.e. any leaf $\mathcal{F}^s(x)$ is dense in $M$. The $s$-Gibbs state $\mu_g^-$ is absolutely continuous to the Lebesgue measure along the leaves of the stable foliation $\mathcal{F}_f^s(x) = \mathcal{F}_g^s(x)$, and the frequency of visits to $V$ of any almost $\mu_g^-$ point $x \in V$ if positive. So $\sup p(\mu_g^-) \supset \mathcal{F}_g^s(x)$, for some $x$ and $m(\mathcal{F}_g^s) = 1$. Therefore $\mu_g^-(W) = \mu_g^-(B_\eta(x_0)) > 0$.

(ii) Let us firstly consider the case $E_f^c(x) \neq E_g^c(x)$, so that $E_g^c(x)$ is the graph of a linear map $\mathcal{L}: E_f^c(x) \to E_f^s(x)$, i.e. any $u$ belonging to the graph of $\mathcal{L}$ can be written as $u = v + \mathcal{L}v, v \in E_f^c(x) - \{0\}$, $\mathcal{L}v \in E_f^s(x)$, also $Dg(x)(\mathcal{L}v) \in E_f^s(x)$.

Let $x \in M - V$ and

$$Dg(x) : E_f^c(x) \oplus E_f^s(x) \to E_f^c(x) \oplus E_f^s(x),$$

$\frac{\| Dg(x)u \|^2}{\| u \|^2} = \frac{\| Dg(x)(v + \mathcal{L}v) \|^2}{\| v + \mathcal{L}v \|^2} = \frac{\| Dg(x)v \|^2 + \| Dg(x)\mathcal{L}v \|^2}{\| v \|^2 + \| \mathcal{L}v \|^2}$, the last equality is because the $Dg$-invariance of the decomposition $E_f^c(x) \oplus E_f^s(x)$.

Thus $$\frac{\| Dg(x)u \|^2}{\| u \|^2} = \frac{\| Dg(x)(v + \mathcal{L}v) \|^2}{\| v + \mathcal{L}v \|^2} \left[ \frac{\frac{\| Dg(x)\mathcal{L}v \|^2}{\| \mathcal{L}v \|^2} + 1}{\frac{\| Dg(x)v \|^2}{\| v \|^2} + 1} \right].$$

If $u = v + w \in E_f^c(x) \oplus E_f^s(x)$ then $S_c(Dg(x)u) = \frac{\| Dg(x)w \|}{\| w \|} \frac{\| w \|}{\| u \|}$

$\frac{\| v \|}{\| Dg(x)v \|}$, since $w \in E_f^s(x) = E_g^s(x)$ and $g$ is a contraction along $E_g^s$ we have that $\frac{\| Dg(x)w \|}{\| w \|} < \lambda < 1$, for some number $\lambda$. By the domination in the central subbundle holds $\frac{\| Dg(x)v \|}{\| v \|} > \lambda$. Consequently

$$S_c(Dg(x)u) < S_c(u).$$

Then

$\frac{\| Dg(x)w \|}{\| Dg(x)v \|} < \frac{\| w \|}{\| v \|}$ and $\frac{\| Dg(x)\mathcal{L}v \|^2}{\| \mathcal{L}v \|^2} < \frac{\| Dg(x)v \|^2}{\| v \|^2}$. Thus

$\| Dg(x)|_{E_g^c}(x) \| \leq (1 - G(x)) \| Df(x)|_{E_f^c}(x) \|$ for some $G(x) > 0$.

Let $x \in V$, in this case is considered

$$Dg(x) = Df(H(x))DH(x) \; : \; DH(x)\left(E_f^c(x)\right) \to DH(x)\left(E_f^s(x)\right),$$

and the graph of the map $DH(x) \circ \mathcal{L}$ So, similarly than the earlier situation

$$\| Dg(x)|_{E_g^c}(x) \| = \| Df(H(x))\, DH(x)\left(E_g^c(x)\right) \| \leq (1 - G(x)) \| Df(x)\left(E_f^c(x)\right) \|.$$

Let us consider now the case $E^c_f(x) = E^c_g(x)$, i.e. $\mathcal{L} \equiv 0$. Let $x \in M - V$, since $Dg(x)\big(E^c_f(x)\big) = E^c_f(g(x))$, we have

$$\| Dg(x)|_{E^c_g}(x) \| = \| Df(x)|_{E^c_f}(x) \| = 1.$$

Let $x \in V$,

$\| Dg(x)|_{E^c_g}(x) \| = \| Df(H(x))DH(x)\big(E^c_g(x)\big) \| = \| Df(H(x))DH(x)\big(E^c_f(x)\big) \|$, if were $DH(x)\big(E^c_f(x)\big) = E^c_f(x)$ then $\| Dg(x)| E^c_f(x) DH(x)\big(E^c_f(x)\big) \|$, If then $DH(x)\big(E^c_f(x)\big)$ would be the graph of linear map

$\mathcal{L}_{H(x)} : E^c_f(H(x)) \to E^s_f(H(x))$ and like in the case $E^c_f(x) \neq E^c_g(x)$ obtains

$$\| Dg(x)|_{E^c_f(H(x))} \| = \| Df(H(x))DH(x)\big(E^c_g(x)\big) \| \leq$$

$$(1 - G(x)) \| Df(H(x))DH(x)\big(E^c_f(H(x))\big) \|.$$

By $i$) the set in points in which $G$ is strictly positive has $\mu^-_g$ positive measure, thus $\lambda^c_-(g) = \int \log \| Dg|_{E^c} \| \, d\mu^-_g \leq (1 - G(x)) \int \log \| Df|_{E^c} \| d\mu^-_f < 0.$

∎

## References


[1] V. Araujo and C. H. Vazquez., Abundance of non-zero Lyapunov exponents, arXiv:1012.2320.

[2] Robust ergodic properties in partially nhyperbolic dynamics, *Trans. Amer. Math. Soc*. Vol. 362 (4), 2010, pp. 1831–1867.

[3] R. E. Bowen., Entropy expansive maps, *Trans. Amer. Math. Soc.* Vol. 164, 1972, pp. 323–331.

[4] H. Comman and J. Rivera-Letelier., Large deviation principles for non-uniformly hyperbolic rational maps, *Ergod. Th & Dynam. Sys,* Vol. 31, 2011, pp. 321–349.

[5] A. Katok-B., Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems,* Cambridge Univ. Pres. 1995.

[6] G. Gallavotti and D. Ruelle., states and non-equilibrium Statistical Mechanics close to equilibrium, *Commun Math. Phys* , Vol. 190, 1997, pp. 279–285

[7] P. D. Liu., Invariant measures satisfying an equality relating entropy, folding entropy and negative Lyapunov exponents, *Commun Math. Phys,* Vol. 284, 2008, pp. 391–406.

[8] E. Mahailescu., Physical measures for inverse iterates near hyperbolic repellors, *J. Stat. Phys.,* Vol. 139, 2010, pp. 800–819.

[9] Y. Pesin., Characteristic Lyapunov exponents and smooth Ergodic Theory, *Russian Math. Surveys,* Vol. 32(4), 1977, pp. 55–114.

[10] C. Pugh and M. Schub., Stable ergodicity, *Bull Amer. Math. Soc.*, Vol. 41(1), 2004, pp. 1–41.

[11] F. Rodriguez Hertz, M. A Rodriguez Hertz, A. Tahzibi and R. Ures., Maximazing measures for partially hyperbolic systems with compact center leaves, arXiv:1010.3372 (2010).

[12] L. S. Young., Some large deviation results for dynamical systems, *Trans. Amer. Math. Soc.* Vol. 318(2), 1900, pp. 525–543.

[13] M. Schub., Global stability of dynamical systems, Springer-Verlag, 1987.

*Received August, 2011*