

Anisotropic effects of background fields on Born–Infeld electromagnetic waves

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Abstract

We show exact solutions of the Born–Infeld theory for electromagnetic plane waves propagating in the presence of static background fields. The non-linear character of the Born–Infeld equations generates an interaction between the background and the wave that changes the speed of propagation and adds a longitudinal component to the wave. As a consequence, in a magnetic background the ray direction differs from the propagation direction—a behavior resembling the one of a wave in an anisotropic medium. This feature could open up a way to experimental tests of the Born–Infeld theory.

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In 1934 Born and Infeld [1,2] proposed a non-linear electrodynamics with the aim of obtaining a finite value for the self-energy of a point-like charge. The Born–Infeld Lagrangian leads to field equations whose spherically symmetric static solution yields a finite value b for the electrostatic field at the origin. The constant b appears in the Born–Infeld Lagrangian as a new universal constant. Following Einstein, Born and Infeld considered the metric tensor $g_{\mu\nu}$ and the electromagnetic field tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ as the symmetric and anti-symmetric parts of a unique field $bg_{\mu\nu} + F_{\mu\nu}$. Then they postulated the Lagrangian density

$$\mathcal{L} = -\frac{1}{4\pi} \left[\sqrt{|\det(bg_{\mu\nu} + F_{\mu\nu})|} - \sqrt{-\det(bg_{\mu\nu})} \right], \quad (1)$$

where the second term is chosen so that the Born–Infeld Lagrangian tends to the Maxwell Lagrangian when $b \rightarrow \infty$. In

four dimensions, this Lagrangian results to be

$$\mathcal{L} = \sqrt{-g} \frac{b^2}{4\pi} \left(1 - \sqrt{1 + \frac{2S}{b^2} - \frac{P^2}{b^4}} \right), \quad (2)$$

where S and P are the scalar and pseudoscalar field invariants

$$S = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} (|\mathbf{B}|^2 - |\mathbf{E}|^2), \quad (3)$$

$$P = \frac{1}{4} {}^*F_{\mu\nu} F^{\mu\nu} = \mathbf{E} \cdot \mathbf{B}, \quad (4)$$

where the dual tensor is ${}^*F_{ij} = 1/2 \epsilon_{ijkl} F^{kl}$.

One of the typical features of non-linear electrodynamics is the appearance of birefringence. However the Born–Infeld Lagrangian is usually mentioned as an exceptional Lagrangian because of the properties of being the unique structural function which [3]: (1) Assures that the theory has a single characteristic surface equation (absence of birefringence); (2) Fulfills the positive energy density and the non-space like energy current character conditions. Due to these conditions, the Lagrangian has time-like or null characteristic surfaces. The Born–Infeld electrodynamic equations can be augmented to a system of hyperbolic conservation laws with interesting properties [4].

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It is a well established fact in non-linear electrodynamics that the presence of background fields modifies the speed of electromagnetic waves. This issue is studied in [3,5–9] by considering the propagation of discontinuities. The result is that the phase velocity is lower than c . Besides, the wave four-vector direction can be described as a null geodesic of an effective geometry that depends on the background field [6]. When there is no background field, the plane wave solution is the same for the Maxwell and the Born–Infeld theory, as was pointed out by Schrödinger [10]. Some conditions for the existence of global smooth spatially periodic planar solutions are studied in [4].

In this work we find out exact solutions for Born–Infeld waves propagating in the presence of a background field. As a still unknown feature, we find that the presence of a magnetic background modifies the ray direction, which does not result to be coincident with the propagation direction (as it would happen in an anisotropic medium).

The Born–Infeld field F satisfies

$$F_{\lambda\mu,\nu} + F_{\nu\lambda,\mu} + F_{\mu\nu,\lambda} = 0, \quad (5)$$

$$(\sqrt{-g}\mathcal{F}^{\mu\nu})_{,\nu} = 0, \quad (6)$$

where $\mathcal{F}_{\mu\nu}$ stand for the components of the 2-form \mathcal{F} (anti-symmetric 2-index covariant tensor) defined as

$$\mathcal{F} = \frac{F - \frac{P}{b^2} *F}{\sqrt{1 + \frac{2S}{b^2} - \frac{P^2}{b^4}}}. \quad (7)$$

Eq. (5) is an identity coming from the definition of $F_{\mu\nu}$, while (6) is the Euler–Lagrange equation that results from varying the Lagrangian (2).

We will do an extensive use of geometric language for benefiting from some properties of the exterior derivative d : $d(d \cdots) \equiv 0$, $d\omega \wedge d\omega \equiv 0$ for any 1-form ω (wedge product is the anti-symmetrized tensor product between p -forms). In this language it is $F = dA$, and Eq. (5) read

$$dF = 0, \quad (8)$$

$$d*F = 0. \quad (9)$$

We will use Cartesian coordinates in Minkowski space–time,

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2. \quad (10)$$

We are looking for waves propagating in a uniform background magnetic field. So we propose the solution

$$F = E(\xi) d\xi \wedge dx + B_B dx \wedge dz - B_E dy \wedge dz - B_L dx \wedge dy + \gamma E(\xi) dt \wedge dz, \quad (11)$$

where

$$\xi = z - \beta t \quad (12)$$

is the sole variable in the solution and B_B , B_E , B_L are the components of the uniform background magnetic field $\mathbf{B} = B_E \hat{x} + B_B \hat{y} + B_L \hat{z}$. Eq. (11) means $F_{xz} = -E(\xi) + B_B$, $F_{tx} = -\beta E(\xi)$, etc. So the electric field is $-E(\xi)\hat{x} + \gamma E(\xi)\hat{z}$, and the magnetic field is $-E(\xi)\hat{y} + \mathbf{B}$. Fig. 1 shows the orientation of the wave and the background fields. The terms having

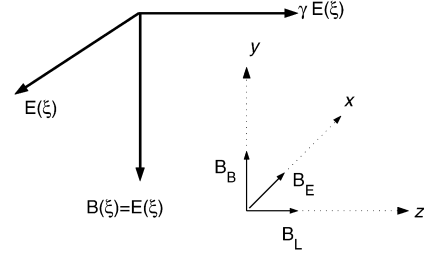


Fig. 1. Schematic picture of the wave and the background fields.

$E(\xi)$ compose the wave; ξ is its phase. $\beta \leq 1$ in (12) takes into account the fact that waves might propagate inside the light cone. The solution (11) fulfills (8) for any function $E(\xi)$ (since $d\xi \wedge d\xi \equiv 0$, etc.).

The invariants S and P for the proposed solution are

$$2S = (1 - \gamma^2 - \beta^2)E(\xi)^2 + B^2 - 2E(\xi)B_B, \quad (13)$$

where $B^2 = B_L^2 + B_E^2 + B_B^2$, and

$$P = (-\beta B_E + \gamma B_L)E(\xi). \quad (14)$$

In order to work out (9), we will compute the numerator in $*\mathcal{F}$. According to (7) it is

$$\begin{aligned} & * \left(F - \frac{P}{b^2} *F \right) \\ &= *F + \frac{P}{b^2} F \\ &= E(\xi) dt(t - \beta dz) \wedge dy - B_B dt \wedge dy - B_E dt \wedge dx \\ &\quad - B_L dt \wedge dz - \gamma E(\xi) dx \wedge dy + b^{-2}(-\beta B_E + \gamma B_L) \\ &\quad \cdot E(\xi)(E(\xi)d\xi \wedge dx + B_B dx \wedge dz - B_E dy \wedge dz \\ &\quad - B_L dx \wedge dy + \gamma E(\xi) dt \wedge dz). \end{aligned} \quad (15)$$

By replacing dz with $d\xi + \beta dt$, the previous result can be rewritten as

$$\begin{aligned} & * \left(F - \frac{P}{b^2} *F \right) \\ &= (-B_E - b^{-2}\beta B_B(-\beta B_E + \gamma B_L)E(\xi)) dt \wedge dx \\ &\quad + ((1 - \beta^2)E(\xi) - B_B \\ &\quad - b^{-2}\beta B_E(\beta B_E - \gamma B_L)E(\xi)) dt \wedge dy \\ &\quad + (-\gamma E(\xi) + b^{-2}B_L(\beta B_E - \gamma B_L)E(\xi)) dx \wedge dy \\ &\quad + d\xi \wedge \cdots \end{aligned} \quad (16)$$

Since $d(d\xi \wedge \cdots) \equiv 0$, because ξ is the only variable in the field F , then the fulfillment of (9) exclusively depends on the behavior of the three first terms in the former result. Taking into account that the differentiation of these terms with respect to ξ will produce three independent components, then (9) can only be satisfied if $*\mathcal{F}$ has components tx , ty and xy equal to constants. Remarkably, the component xy in (16) is linear and homogeneous in $E(\xi)$, but this feature is not shared with the denominator in (7). So, in order to get $*\mathcal{F}_{xy} = \text{const}$, it should be $E(\xi) = 0$ or

$$\gamma = \frac{B_E B_L}{b^2} \beta. \quad (17)$$

This value for γ can be replaced in the components ${}^*\mathcal{F}_{tx}$ and ${}^*\mathcal{F}_{ty}$, which will turn to be constants for β equal to

$$\beta = \sqrt{\frac{1 + \frac{B_L^2}{b^2}}{1 + \frac{B^2}{b^2}}}. \quad (18)$$

In fact, the values of the examined components result to be independent of the function $E(\xi)$, and (9) is accomplished:

$$\begin{aligned} {}^*\mathcal{F}_{tx} &= -\frac{B_E}{\sqrt{1 + \frac{B^2}{b^2}}}, & {}^*\mathcal{F}_{ty} &= -\frac{B_B}{\sqrt{1 + \frac{B^2}{b^2}}}, \\ {}^*\mathcal{F}_{xy} &= 0. \end{aligned} \quad (19)$$

The obtained values for γ and β imply that

$$\sqrt{1 + \frac{2S}{b^2} - \frac{P^2}{b^4}} = \sqrt{1 + \frac{B^2}{b^2} - \frac{B_BE(\xi)}{b^2\sqrt{1 + \frac{B^2}{b^2}}}}. \quad (20)$$

The value $\beta < 1$ in (18) is the speed of propagation of a Born–Infeld electromagnetic wave in the presence of a uniform magnetic background. The constant $\gamma \neq 0$ in (17) implies the existence of a non-zero electric longitudinal component of the wave due to its interaction with the background. Of course, these differences with the Maxwellian behavior disappear in the limit $b \rightarrow \infty$.

Let us now consider the energy flux for a Born–Infeld wave in the presence of a background field. The energy–momentum tensor in the Born–Infeld theory is

$$\begin{aligned} T^{\mu\nu} &= \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}}{\delta g_{\mu\nu}} \\ &= -\frac{1}{4\pi} \left[F^\mu{}_\rho F^{\nu\rho} + b^2 g^{\mu\nu} \left(1 - \sqrt{1 + \frac{2S}{b^2} - \frac{P^2}{b^4}} \right) \right]. \end{aligned} \quad (21)$$

In particular the energy flux components are quite simple; for instance

$$\begin{aligned} T^{tz} &= -\frac{1}{4\pi} F^t{}_\rho F^{z\rho} = -\frac{1}{4\pi} (F^t{}_x F^{zx} + F^t{}_y F^{zy}) \\ &= \frac{F_{tx} F_{xz} + F_{ty} F_{yz}}{\sqrt{1 + \frac{2S}{b^2} - \frac{P^2}{b^4}}}. \end{aligned} \quad (22)$$

Remarkably, the terms proportional to P in the numerator cancel out. This is a foreseeable feature because P is a pseudoscalar while the energy flux \mathcal{S} is a polar vector. Thus the Poynting vector is

$$\mathcal{S} = \frac{1}{4\pi} \frac{\mathbf{E} \times \mathbf{B}}{\sqrt{1 + \frac{2S}{b^2} - \frac{P^2}{b^4}}}. \quad (23)$$

Then the energy flux vector associated with the solutions obtained above are:

$$\mathcal{S}_x = \frac{1}{4\pi} \frac{\gamma E(\xi)(E(\xi) - B_B)}{\sqrt{1 + \frac{2S}{b^2} - \frac{P^2}{b^4}}}, \quad (24)$$

$$\mathcal{S}_y = \frac{1}{4\pi} \frac{\beta E(\xi) B_L + \gamma E(\xi) B_E}{\sqrt{1 + \frac{2S}{b^2} - \frac{P^2}{b^4}}}, \quad (25)$$

$$\mathcal{S}_z = \frac{1}{4\pi} \frac{\beta E(\xi)(E(\xi) - B_B)}{\sqrt{1 + \frac{2S}{b^2} - \frac{P^2}{b^4}}}. \quad (26)$$

As it is usual when $E(\xi)$ is a periodic function, we will consider just the temporal averaging of the energy flux. Differing from what happens in the Maxwell theory, the non-linear features of the Born–Infeld theory will lead to non-null transversal components of $\langle \mathcal{S} \rangle$. This characteristic can be easily perceived in $\langle \mathcal{S}_x \rangle$. In fact \mathcal{S}_x is an energy flux along the wave polarization direction due to the existence of a longitudinal electric field, which is a consequence of the interaction between the wave and the background field. For a monochromatic wave, $E(\xi) = E_0 \cos(\beta^{-1} \omega \xi)$, the averaged flux at the lower order in b^{-2} is

$$\langle \mathcal{S}_x \rangle = \frac{E_0^2 B_L B_E}{8\pi b^2} + \mathcal{O}(b^{-4}), \quad (27)$$

$$\langle \mathcal{S}_y \rangle = \frac{E_0^2 B_L B_B}{8\pi b^2} + \mathcal{O}(b^{-4}), \quad (28)$$

$$\langle \mathcal{S}_z \rangle = \frac{E_0^2}{8\pi} \left[1 - \frac{4B_B^2 + 2B_E^2 + B_L^2}{2b^2} \right] + \mathcal{O}(b^{-4}). \quad (29)$$

In this approximation, the transversal part of $\langle \mathcal{S} \rangle$ is parallel to the transversal background field. The angle α between $\langle \mathcal{S} \rangle$ and the direction of propagation is

$$\tan \alpha = \frac{B_L \sqrt{B_E^2 + B_B^2}}{b^2} + \mathcal{O}(b^{-4}). \quad (30)$$

In the same way we can calculate the energy density:

$$\begin{aligned} T^{00} &= \frac{|\mathbf{E}|^2 + b^{-2} P^2}{4\pi \sqrt{1 + \frac{2S}{b^2} - \frac{P^2}{b^4}}} - \frac{b^2}{4\pi} \left(1 - \sqrt{1 + \frac{2S}{b^2} - \frac{P^2}{b^4}} \right) \\ &= \frac{b^2}{4\pi} \left[\frac{1 + b^{-2} |\mathbf{B}|^2}{\sqrt{1 + \frac{2S}{b^2} - \frac{P^2}{b^4}}} - 1 \right]. \end{aligned} \quad (31)$$

Therefore the averaged energy density for the studied solution is

$$\begin{aligned} \langle T^{00} \rangle &= \frac{E_0^2}{8\pi} \left[1 - \frac{3B_B^2 + B_E^2 + B_L^2}{2b^2} \right] + \frac{B^2}{8\pi} \left[1 - \frac{B^2}{4b^2} \right] \\ &\quad + \mathcal{O}(b^{-4}) \end{aligned} \quad (32)$$

which is lower than the corresponding Maxwellian energy density. We remark that, at order b^{-2} , the modulus of the energy velocity $\langle \mathcal{S} \rangle \langle T^{00} \rangle_{\text{wave}}^{-1}$ does not differ from the phase velocity β .

Now we will just display the solutions for the case of a constant electric background field $\mathcal{E} = E_E \hat{x} + E_B \hat{y} + E_L \hat{z}$. Following the same procedure we applied in the previous case, the solution of (8) and (9) is

$$\begin{aligned} F &= E(\xi) d\xi \wedge dx + E_E dt \wedge dx + E_B dt \wedge dy \\ &\quad + E_L dt \wedge dz + \gamma E(\xi) dt \wedge dz \end{aligned} \quad (33)$$

with

$$\beta = \sqrt{1 - \frac{E_B^2 + E_E^2}{b^2}}, \quad \gamma = \frac{E_L E_E}{b^2 \sqrt{1 - \frac{E_B^2 + E_E^2}{b^2}}}. \quad (34)$$

Then

$$\sqrt{1 + \frac{2S}{b^2} - \frac{P^2}{b^4}} = \sqrt{1 - \frac{\mathcal{E}^2}{b^2}} \left(1 + \frac{E_E E(\xi)}{\beta b^2} \right) \quad (35)$$

and the constant components of ${}^*\mathcal{F}$ are

$$\begin{aligned} {}^*\mathcal{F}_{tx} &= -\sqrt{\frac{1 - \frac{E_E^2 + E_B^2}{b^2}}{1 - \frac{\mathcal{E}^2}{b^2}}} E_B, & {}^*\mathcal{F}_{xy} &= -\frac{E_L}{\sqrt{1 - \frac{\mathcal{E}^2}{b^2}}}, \\ {}^*\mathcal{F}_{ty} &= \sqrt{\frac{1 - \frac{E_E^2 + E_B^2}{b^2}}{1 - \frac{\mathcal{E}^2}{b^2}}} E_E, \end{aligned} \quad (36)$$

where $\mathcal{E}^2 = E_B^2 + E_L^2 + E_E^2$. The components of Poynting vector \mathcal{S} are

$$\begin{aligned} \mathcal{S}_x &= \frac{1}{4\pi} \frac{(E_L + \gamma E(\xi))E(\xi)}{\sqrt{1 + \frac{2S}{b^2} - \frac{P^2}{b^4}}}, & \mathcal{S}_y &= 0, \\ \mathcal{S}_z &= -\frac{1}{4\pi} \frac{(E_E - \beta E(\xi))E(\xi)}{\sqrt{1 + \frac{2S}{b^2} - \frac{P^2}{b^4}}}. \end{aligned} \quad (37)$$

As a remarkable feature, the factor $E_L + \gamma E(\xi)$ in the numerator of \mathcal{S}_x is proportional to the square root in the denominator; thus these factors cancel out and \mathcal{S}_x results to be linear and homogeneous in $E(\xi)$. So there is a main difference according to the Born–Infeld wave propagates in a magnetic or an electric background field: in the second case the direction of the averaged energy flux coincides with the propagation direction:

$$\begin{aligned} \langle \mathcal{S}_x \rangle &= 0, & \langle \mathcal{S}_y \rangle &= 0, \\ \langle \mathcal{S}_z \rangle &= \frac{E_0^2}{8\pi} \left(1 + \frac{2E_E^2 + E_L^2}{2b^2} \right) + \mathcal{O}(b^{-4}), \end{aligned} \quad (38)$$

where E_0 is the wave amplitude.

The speeds of propagation (18) and (34) can be compared with the results obtained in [3,6]. These papers study the propagation of discontinuities in the presence of background fields in a general non-linear theory. It is shown that the equation accomplished by the wave four-vector can be understood as if rays propagate along null geodesics of an effective metric $\bar{g}_{\mu\nu}$ is

$$\bar{g}_{\mu\nu} = \left(b^2 + \frac{1}{2} F_{\rho\sigma} F^{\rho\sigma} \right) g_{\mu\nu} + F_{\mu\lambda} F^{\lambda}_{\nu} \quad (39)$$

being $g_{\mu\nu}$ the space–time metric, and $F_{\mu\nu}$ is the background electromagnetic field where the rays propagate. For a ray propagating along the z direction it is

$$d\bar{s}^2 = \bar{g}_{00} dt^2 + \bar{g}_{zz} dz^2 = 0 \Rightarrow \frac{dz}{dt} = \sqrt{-\frac{\bar{g}_{00}}{\bar{g}_{zz}}}. \quad (40)$$

When the effective metric (39) is evaluated for a magnetic background it results

$$\bar{g}_{00} = \frac{1}{b^2 + \mathcal{B}^2}, \quad \bar{g}_{zz} = \frac{-1}{b^2 + B_L^2}. \quad (41)$$

Instead, for an electric background it is

$$\bar{g}_{00} = \frac{1}{b^2}, \quad \bar{g}_{zz} = \frac{-1}{b^2 - E_B^2 - E_L^2}. \quad (42)$$

Thus the speeds of propagation (18) and (34) are reobtained. This already known consequence of the Born–Infeld theory on the wave propagation in background fields is here added with the knowledge of the exact solutions. These solutions reveal that the background fields not only affect the speed of propagation of the Born–Infeld waves but also they can produce an angle between the ray direction and the propagation direction. This is the case for magnetic backgrounds. Of course, all these effects would be very weak (if they exist), since up to now the Maxwell equations properly describe all of the known classical electromagnetic phenomena. Constant b is the key for passing from the Maxwell theory to the Born–Infeld theory. If the tiny angle α (30) were measured in the laboratory then a way to experimental tests of the Born–Infeld electrodynamics would be opened. The ray deviation is the consequence of the last term in (11), which is a longitudinal electric field that results from the coupling with the background magnetic field. The longitudinal electric field together with the speed of propagation smaller than c are the imprints of the non-linear behavior. At the lowest order in b^{-2} the longitudinal component of the electric field is $b^{-2} B_E B_L E(\xi)$. A similar longitudinal electric field appears in an electric background as well. In this case, however, there is no contribution to the averaged Poynting vector, so the ray does not deviate.

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References

- [1] M. Born, L. Infeld, *Nature* 132 (1933) 1004.
- [2] M. Born, L. Infeld, *Proc. R. Soc. London* 144 (1934) 425.
- [3] J. Plebanski, *Lectures on Non-Linear Electrodynamics*, Nordita Lecture Notes, Copenhagen, 1968.
- [4] Y. Brenier, *Arch. Rational Mech. Anal.* 172 (2004) 65.
- [5] H. Salazar Ibarguen, A. García, J. Plebanski, *J. Math. Phys.* 30 (1989) 11.
- [6] M. Novello, V.A. De Lorenci, J.M. Salim, R. Klippert, *Phys. Rev. D* 61 (2000) 045001.
- [7] M. Novello, J.M. Salim, *Phys. Rev. D* 63 (2001) 083511.
- [8] M. Novello, S.E. Perez Bergliaffa, *AIP Conf. Proc.* 668 (2003) 288.
- [9] G. Boillat, *J. Math. Phys.* 11 (1970) 941.
- [10] E. Schrödinger, *Proc. R. Soc. A* 150 (1935) 465.