

Puzzling initial conditions in the $R_h = ct$ model

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Abstract In recent years, some studies have drawn attention to the lack of large-angle correlations in the observed cosmic microwave background (CMB) temperature anisotropies with respect to that predicted within the standard Λ CDM model. Lately, it has been argued that such a lack of correlations could be explained in the framework of the so-called $R_h = ct$ model without inflation. The aim of this work is to study whether there is a mechanism to generate, through a quantum field theory, the primordial power spectrum presented by these authors. Specifically, we consider two different scenarios: first, we assume a scalar field dominating the early Universe in the $R_h = ct$ cosmological model, and second, we deal with the possibility of adding an early inflationary phase to the mentioned model. During the analysis of the consistency between the predicted and observed amplitudes of the CMB temperature anisotropies in both scenarios, we run into deep issues which indicate that it is not clear how to characterize the primordial quantum perturbations within the $R_h = ct$ model.

1 Introduction

In addition to solve the horizon and flatness problems of the standard Big Bang model, inflation generates a nearly scale invariant power spectrum for density perturbations, which has been exquisitely tested with observations of the cosmic microwave background (CMB) angular spectrum [1–6].

Starting with the Cosmic Background Explorer observations [7], it was noted that the angular two-point correlation function at angular scales larger than 60° is unexpectedly close to zero, contrary to what the standard Λ CDM model predicts. Shortly after, it was rediscovered with the Wilkinson Microwave Anisotropy Probe (WMAP) data [8] and later by the Planck mission [2,9]. This feature at large scales was

studied in detail by several authors, e.g. [10–14]; it has been the source of some controversy (see for instance [15]) and today constitutes one of the persistent large-angle anomalies in the CMB data [16].

Recently, a series of theoretical and observational motivations exposed in [17–19] finished merging into what today is known as the $R_h = ct$ model [20]. This model has received considerable attention over the last few years, since it has been claimed to be favored over the standard Λ CDM by most observational data [21–28]. Even the authors argue that the mentioned horizon problem could be solved in the framework of this model without an inflationary epoch at the beginning of the Universe [29]. Basically, they hold that the Universe can be described by a FLRW cosmology, where the cosmic fluid filling the Universe satisfies, at all times, the overall equation of state $\rho + 3P = 0$, where ρ and P are the total energy density and pressure of the cosmic fluid, respectively. According to the authors, the condition $w = -1/3$ at all times is apparently required by the simultaneous application of the Cosmological Principle and Weyl's postulate [30,31]. We remind the reader that in the standard Λ CDM model, the equation of state $\rho + 3P = 0$ would lead to a Universe with negative curvature.

However, some observational objections were raised, and also the validity of the physical arguments underlying the $R_h = ct$ model have been criticized by a number of authors. Some of them can be found for instance in [32–40]. In particular, the claim made in Ref. [41], regarding that the analysis of the CMB anisotropies in the $R_h = ct$ model is preferred over the Λ CDM, appears to be incorrect; actually the formal computation of the angular power spectrum, i.e. the C_l 's, is absent (in fact, in Sect. 4 we will show explicitly that it is very unlikely that the $R_h = ct$ model can be made consistent with the CMB observational data). Furthermore, the explanation, within this model, concerning how w is kept at $-1/3$ through the transitions from known matter to radiation sounds at least questionable, and the idea that $\rho \propto a^{-2}$ throughout nucle-

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osynthesis, recombination, structure formation and today seems impossible to reconcile with all the observations put together. Some of these criticisms are claimed to have been answered in [42–46]. Nevertheless, after pointing out a number of objections to the $R_h = ct$ model based on recent observational data, in Ref. [47] the authors analyzed the central assumption underlying the original theoretical argument for the model, namely that the comoving Hubble radius should be constant, and showed that it is not required.

In the present manuscript, we will focus on the results presented in [41, 48]. There, the authors analyzed the CMB angular correlation function for a fluctuation spectrum expected from growth in a Universe, whose dynamics is constrained by the equation of state $w = -1/3$. To accomplish this, they mention that since the exact form of the power spectrum emerging from the non-linear growth prior to recombination is unknown, a parameterization for this spectrum can be performed, for example, by assuming a scale-free initial power law spectrum and incorporating in its shape other relevant effects. Then they ensure that it is possible to obtain a better fit than the Λ CDM model to the data corresponding to the angular correlation function, and conclude stating that the absence of power on large scales exhibited by the angular correlation function might be evidence in support of the $R_h = ct$ model simply because it does not require inflation.

In this article, we perform a critical analysis whether there might be a mechanism for generating, through a quantum field theory, the primordial power spectrum presented by those authors in [41]. To do so, we are going to consider two different scenarios: first, we will assume a scalar field dominating the early Universe in the $R_h = ct$ cosmological model, and second, we will deal with the possibility of adding an early inflationary phase to the mentioned model. After that, we will analyze the consistency between the predicted and observed amplitudes of the CMB temperature anisotropies in both scenarios.

The article is organized as follows: in Sect. 2, we review some basics about the $R_h = ct$ model and how to describe classical perturbations in that framework; in Sect. 3, we search for a quantum mechanism to generate the primordial curvature perturbation, and we obtain the primordial power spectra within $R_h = ct$ model with and without an inflationary phase. Later, in Sect. 4, we analyze the amplitudes of the primordial power spectra and the consistency with the amplitude of the CMB temperature anisotropies. In Sect. 5 we make a discussion of our results, and finally in Sect. 6 we summarize our conclusions.

2 Classical perturbations in the $R_h = ct$ model

In this section, we provide a summary of the main characteristics of the $R_h = ct$ Universe. Our main focus is the

cosmological perturbations as presented in Refs. [20, 41, 48]. The first two subsections will be heavily based on the results presented in those references. However, in the last subsection, we will show how to relate the curvature power spectrum with the matter power spectrum proposed in Refs. [41, 48]. We will use units in which $c = \hbar = 1$. We will make use of the reduced Planck mass $M_{\text{P}}^2 = 1/(8\pi G)$ and the “West Coast” signature $(+ - - -)$ for the metric.

2.1 The background

The $R_h = ct$ Universe is characterized by a spatially flat FLRW spacetime, which in comoving coordinates is represented by the line element

$$ds^2 = dt^2 - a^2(t)\delta_{ij}dx^i dx^j.$$

Additionally, the authors of the $R_h = ct$ Universe claim that the total matter components in the Universe combined (dark matter, ordinary matter, radiation and dark energy) behave as a perfect fluid with the overall equation of state¹

$$P = -\frac{\rho}{3}, \quad (1)$$

where ρ and P represent the total energy density and pressure of the Universe, respectively. Therefore, the Friedmann equation $H^2 \equiv (\dot{a}/a)^2 = \rho/(3M_{\text{P}}^2)$ and the continuity equation $\dot{\rho} + 3H(\rho + P) = 0$ (with the dot over functions representing derivative with respect to cosmic time t) lead to a scale factor of the form

$$a(t) = t/t_0, \quad (2)$$

where we have normalized the scale factor to $a(t_0) = 1$ at the present cosmic time t_0 . Consequently, the Hubble radius evolves as $R_h \equiv H^{-1} = t$. This is one of the main features of the R_h model, i.e., the Hubble radius satisfies the relation $H^{-1} = t$ during the whole cosmic evolution and not “just today” as in the standard Λ CDM model. Therefore, the total energy density of the Universe evolves as $\rho \propto 1/a^2$.

2.2 Cosmological perturbations

The dynamical evolution of the cosmological perturbations in the $R_h = ct$ Universe follows from Einstein equations $\delta G_{ab} = \delta T_{ab}/M_{\text{P}}^2$. In particular, by using the Newtonian (longitudinal) gauge, the Fourier modes associated to the density contrasts defined as $\delta_{\mathbf{k}}(t) \equiv \delta\rho_{\mathbf{k}}(t)/\bar{\rho}(t)$ (where $\bar{\rho}(t)$ is the background energy density) satisfy

¹ For the specific motivations behind the aforementioned equation of state, we refer the reader to the original work by [20, 31] and a recent rebuttal regarding the consistency of the physical motivations of the $R_h = ct$ Universe [40, 47].

$$\ddot{\delta}_k + (2 - w + 3c_s^2)H\dot{\delta}_k - \frac{3}{2}H^2(1 + 8w - 3w^2 - 6c_s^2)\delta_k = -\frac{k^2c_s^2}{a^2}\delta_k. \tag{3}$$

Therefore, by considering the equation of state associated to the $R_h = ct$ Universe, it is assumed that $w = c_s^2 = -1/3$. Thus, the motion equation for δ_k is given by

$$\ddot{\delta}_k + \frac{3}{t}\dot{\delta}_k - \frac{1}{3}\frac{\Delta_k^2}{t^2}\delta_k = 0 \tag{4}$$

where $\Delta_k \equiv k/(aH)$. Note that $aH = H_0$ is a constant (H_0 denotes the Hubble parameter today). It is not hard to find the solutions of (4); nevertheless, the solutions depend on whether k is greater or less than aH , i.e. $\Delta_k > 1$ or $\Delta_k < 1$. If $\Delta_k > 1$, then the solutions are a growing mode $\delta_k \sim t^\alpha$ (with $\alpha > 0$) and a decaying mode. On the contrary, if $\Delta_k < 1$, then the solutions are a constant and a decaying mode. In the $R_h = ct$ Universe one is primarily interested in the modes such that $k > aH$ since these are the modes that can grow into the large scale structure. As a matter of fact, motivated by the angular correlation of the CMB and the solution corresponding to the growing modes of Eq. (4), the authors of Refs. [41,48] proposed that the initial matter power spectrum is of the form

$$P_\delta(k) \propto k - b \left(\frac{2\pi}{R_e(t_e)} \right)^2 \frac{1}{k} \tag{5}$$

where b is an unknown constant to be adjusted, and R_e is the proper distance to the last scattering surface at time t_e , which corresponds to the cosmic time at the decoupling epoch. The power spectrum (5) can be recast as

$$P_\delta(k) = A\mathcal{H} \left[\frac{k}{\mathcal{H}} - b \left(\frac{\theta_{\max}}{a(t_e)} \right)^2 \frac{\mathcal{H}}{k} \right] \tag{6}$$

where A is the amplitude of the power spectrum, $\mathcal{H} \equiv aH$ and θ_{\max} is the maximum angular size of any fluctuation associated with the CMB emitted at t_e , that is, $\theta_{\max} = [2\pi a(t_e)]/[k_{\max} R_e(t_e)]$; also $k_{\max}/\mathcal{H} = 1$.

2.3 Curvature and matter power spectra in the $R_h = ct$ Universe

Our next step is to relate, also through a classical analysis, the matter power spectrum with the curvature power spectrum in the $R_h = ct$ Universe. Later, we will investigate whether it is possible to find a quantum mechanism for generating the curvature perturbation. If possible, we will relate that spectrum with the matter power spectrum, and then we will compare it with the one proposed in (6).

We start the discussion by switching to conformal time η , i.e. $dt^2 = a^2d\eta^2$. In these coordinates $\mathcal{H} \equiv aH = a'(\eta)/a(\eta)$, where a prime denotes derivative with respect to conformal time. As a matter of fact, using the equation of

state $P = -\rho/3$, the continuity equation $\rho' + 3\mathcal{H}(\rho + P) = 0$ and Friedmann equation $\mathcal{H}^2 = a^2\rho/3M_p^2$, we arrive at the important result

$$\mathcal{H} = H_0. \tag{7}$$

That is, \mathcal{H} is a constant of motion in the $R_h = ct$ Universe and has the value of the Hubble parameter today. For the sake of completeness, we present the explicit form of the scale factor in conformal time coordinates:

$$a(\eta) = e^{H_0(\eta - \eta_0)} \tag{8}$$

where η_0 corresponds to the conformal time today.

The most generic metric associated to a flat FLRW Universe with linear scalar perturbations is

$$ds^2 = a^2(\eta)\{(1 - 2\varphi)d\eta^2 + 2(\partial_i B)dx^i d\eta - [(1 - 2\psi)\delta_{ij} + 2\partial_i \partial_j E]dx^i dx^j\}, \tag{9}$$

where φ, ψ, E, B are scalar functions of the spacetime. In the Newtonian gauge, $\varphi = \Phi, \psi = \Psi$, and $E = B = 0$.

In the absence of anisotropic stress, Einstein equations (EE) $\delta G_{ab} = \delta T_{ab}/M_p^2$ lead to $\Phi = \Psi$. Moreover, considering once again that in the $R_h = ct$ Universe $c_s^2 = w = -1/3$, the equation of motion for the Fourier mode $\Phi_k(\eta)$ that results from combining EE is

$$\Phi_k'' + 2\mathcal{H}\Phi_k' - \frac{k^2}{3}\Phi_k = 0. \tag{10}$$

The general solution of the former equation is a linear combination of $\exp[(q - \mathcal{H})\eta]$ and $\exp[-(q + \mathcal{H})\eta]$, with $q \equiv +\sqrt{k^2/3 + \mathcal{H}^2}$. Furthermore, using Eqs. (7) and (8), we can express the scale factor as $a(\eta) \propto \exp(\mathcal{H}\eta)$. Consequently, if $k \ll \mathcal{H}$ then $q \sim \mathcal{H}$, thus the linearly independent solutions of (10) can be approximated by a constant and a decaying mode $\exp(-2\mathcal{H}\eta) \propto a(\eta)^{-2}$. On the other hand, if $k \gg \mathcal{H}$ then $q \sim k/\sqrt{3}$, and the linearly independent solutions of (10) are approximately given by a growing mode $\Phi_k \sim \exp[(k - \mathcal{H})\eta]$ and a decaying mode $\exp[-(k + \mathcal{H})\eta] \propto \exp(-k\eta)/a(\eta)$ (note that the conformal time η is an increasing variable).

The EE with component $\delta G_{00} = \delta T_{00}/M_p^2$ is useful to relate the density contrasts with the metric perturbation Φ . That is,

$$\delta_k = -\frac{2}{3}\frac{k^2}{\mathcal{H}^2}\Phi_k - 2\Phi_k - \frac{2}{\mathcal{H}}\Phi_k'. \tag{11}$$

As we mentioned in the previous subsection, in the $R_h = ct$ Universe one is interested in the modes such that $k > \mathcal{H}$; that is, the modes whose associated proper wavelength is less than the Hubble radius. These are the modes that evolve as $\Phi_k \sim \exp(k - \mathcal{H})\eta$; consequently $\Phi_k' = (k - \mathcal{H})\Phi$. By using that result, Eq. (11) becomes

$$\delta_k = -\frac{2}{3}\frac{k^2}{\mathcal{H}^2}\Phi_k - 2\frac{k}{\mathcal{H}}\Phi_k. \tag{12}$$

We emphasize that Eq. (12) is valid only for $k > \mathcal{H}$ and $w = c_s^2 = -1/3$.

At this point we have to do a technical digression. The quantum analysis of the field perturbations usually involves the so-called Mukhanov–Sasaki variable, and then one relates that variable with the comoving curvature perturbation \mathcal{R} . We will follow such an analysis in the next section; however, Eq. (12), which will help us to relate the matter power spectrum with the curvature one, was obtained in the Newtonian gauge. Therefore, it will be useful to change from the Newtonian gauge to the comoving gauge. That relation is generically given (for constant w) by [49]

$$\mathcal{R} = -\frac{5+3w}{3+3w}\Phi - \frac{2}{3+3w}\mathcal{H}^{-1}\Phi'. \quad (13)$$

Thus, for $w = -1/3$, Eq. (13) leads to $\mathcal{R} = -2\Phi - \mathcal{H}^{-1}\Phi'$. Moreover, if we focus on the modes such that $k > \mathcal{H}$ and recall that for such modes $\Phi'_k = (k - \mathcal{H})\Phi_k$, then we arrive at

$$-\mathcal{R}_k = \left(1 + \frac{k}{\mathcal{H}}\right)\Phi_k. \quad (14)$$

With Eqs. (12) and (14) at hand, it is straightforward (in the comoving gauge) to relate the corresponding matter and curvature spectra, namely

$$P_\delta(k) \simeq \frac{4}{9}P_{\mathcal{R}}(k) \left(\frac{k^2}{\mathcal{H}^2} + 4\frac{k}{\mathcal{H}} - 2\right), \quad (15)$$

where we have retained only the first three dominant terms in powers of k/\mathcal{H} .

Equation (15) is the main result of this subsection. One can immediately observe that if $P_{\mathcal{R}} \propto k^{-1}$, then the resulting matter power spectrum will be of a similar structure as the one shown in Eq. (6), except for a constant term.

In the following section, we will attempt to construct a mechanism for generating the curvature power spectrum $P_{\mathcal{R}}(k)$.

3 Generation of the primordial curvature perturbation

In this section, we will consider two possibilities for generating the primordial curvature perturbations: a scalar field dominating the early $R_h = ct$ Universe, and a preceding inflationary era in the $R_h = ct$ Universe.

Since in the $R_h = ct$ model the combination of different types of matter is such that it mimics a perfect fluid with an overall equation of state $P = -\rho/3$ (which involves a negative pressure), we will make the standard assumption that the early Universe was dominated by a scalar field $\phi(\mathbf{x}, t)$, with some potential $V(\phi)$, such that P and ρ associated to ϕ satisfy $P(\phi) = -\rho(\phi)/3$ at all times. Afterwards, the scalar field should decay into particles of the standard model

and possibly into dark matter particles, and the evolution of the Universe then follow the $R_h = ct$ model. Since we are considering a canonical scalar field, the action is given by

$$S[\phi] = \int d^4x \sqrt{-g} \left(\frac{1}{2} g^{ab} \nabla_a \phi \nabla_b \phi - V(\phi) \right). \quad (16)$$

In contrast with the standard Λ CDM model (plus inflation) in which the end of a different cosmological era is linked to a change in the equation of state, in the $R_h = ct$ Universe the equation of state $P = -\rho/3$ should be satisfied at all times during the evolution of the Universe. As a consequence, we need to provide a condition that marks the end of the early cosmological era dominated by the field ϕ . We propose that the value of the adiabatic speed of sound c_s^2 will help to provide such condition.

For ordinary matter and constant equation of state we know that $w = c_s^2$. However, for a scalar field generically $c_s^2 \neq w$. In particular, for a canonical scalar field (a field with canonical kinetic term), $c_s^2 = 1$ [49]. In fact, in standard slow-roll inflation $c_s^2 = 1$ and $w \simeq -1$. Therefore, in the $R_h = ct$ Universe, we will consider a canonical scalar field that dominates the matter content of the early Universe, and such scalar field will be characterized by $c_s^2 = 1$. Then, at some point during the evolution, the scalar field will decay in such a way that c_s^2 will decrease from $c_s^2 = 1$ to $c_s^2 = -1/3$. Note from Eq. (3) that it is crucial to have $c_s^2 = w = -1/3$ in order to obtain Eq. (4), which results in a solution for the growing modes. It is important to mention that other combinations of w and c_s^2 would lead to a solution of Eq. (3) with a growing mode; in particular, the condition for the $R_h = ct$ model is $w = -1/3$. Hence, other values of c_s^2 , but maintaining $w = -1/3$ could lead to a growing mode in the $R_h = ct$ model. On the other hand, Eq. (4) is the main equation used by the authors of the $R_h = ct$ model to analyze the growth of structure in [41, 48]; and to obtain Eq. (4) from Eq. (3), one must satisfy $c_s^2 = w = -1/3$.

To continue, we split the scalar field into an homogeneous part plus small inhomogeneities, i.e. $\phi(\mathbf{x}, t) = \phi_0(t) + \delta\phi(\mathbf{x}, t)$. The homogeneous part of the field drives the background evolution, that is, the one characterized by the $R_h = ct$ Universe, and the quantum theory of $\delta\phi(\mathbf{x}, t)$ will result in the primordial power spectrum of the perturbations. In the following, we will attempt to construct a quantum theory for $\delta\phi$, but first we will derive some useful quantities to describe the background.

Since the background field, ϕ_0 , drives the evolution of the $R_h = ct$ Universe, we can associate the standard energy-momentum tensor T_β^α to the field ϕ_0 . In particular, from the time component $T_0^0 = \rho(\phi)$, we infer $\rho = \dot{\phi}_0^2/2a^2 + V(\phi)$; additionally, $T_j^i = -P(\phi)\delta_j^i$ implies that $P = \dot{\phi}_0^2/2a^2 - V(\phi)$.

Using the fact that \mathcal{H} is constant [see (7)], and from the continuity equation $\rho' + 3\mathcal{H}(\rho + P) = 0$ applied to the scalar field ϕ_0 , one obtains

$$\phi_0'' = 0. \tag{17}$$

Consequently, from Eqs. (7) and (17), it is clear that in the $R_h = ct$ Universe, \mathcal{H} and ϕ_0' are exactly constants of motion. As a matter of fact, using the Friedmann equations, it can be shown that

$$\frac{\phi_0'}{\mathcal{H}} = \sqrt{2}M_P. \tag{18}$$

Furthermore, using that $\phi_0'' = 0$ and that \mathcal{H} is a constant, we can find a potential that is consistent with $P(\phi) = -\rho(\phi)/3$. This potential turns out to be

$$V(\phi) = \mathcal{H}^4 e^{-2\phi/\mathcal{H}}. \tag{19}$$

Now, let us focus on the linear scalar perturbations. The field perturbations $\delta\phi$ induce metric perturbations $\delta g_{\mu\nu}$ via EE. As we mentioned in Sect. 2.3, Eq. (9) represents the most generic metric associated to a FLRW Universe with scalar perturbations. As is well known, the relativistic perturbation theory has the issue of gauge redundance [50,51]. However, the gravitational part can be characterized by a single, gauge-invariant object known as the Bardeen potential defined as [52]

$$\Phi_B(\mathbf{x}, \eta) = \varphi + \frac{1}{a}[a(B - E)']'. \tag{20}$$

In the same manner, the matter sector can be described by the gauge-invariant field perturbation

$$\delta\phi^{(gi)}(\mathbf{x}, \eta) = \delta\phi + \phi_0'(B - E)'. \tag{21}$$

The Einstein equations relate Φ_B and $\delta\phi^{(gi)}$ through a constraint equation. That implies that the scalar sector can be characterized by a single object; this object is the so-called Mukhanov–Sasaki variable, defined by

$$v(\eta, \mathbf{x}) \equiv a \left[\delta\phi^{(gi)} + \phi_0' \frac{\Phi_B}{\mathcal{H}} \right]. \tag{22}$$

All other relevant quantities can be expressed in terms of $v(\eta, \mathbf{x})$, i.e. it fully characterizes the scalar sector.

Moreover, we can expand the action of our theory, that is, the action of a scalar field minimally coupled to gravity, up to second order in the scalar perturbations, obtaining

$$\delta S^{(2)} = \frac{1}{2} \int d\eta d^3x \left[(v')^2 - \delta^{ij} \partial_i v \partial_j v + \frac{z''}{z} v^2 \right], \tag{23}$$

where $z \equiv a\phi_0'/\mathcal{H}$. From Eq. (18) we obtain in the $R_h = ct$ model

$$z = \sqrt{2}M_P a. \tag{24}$$

From the action (23), the equation of motion is

$$v'' - \nabla^2 v - \frac{z''}{z} v = 0. \tag{25}$$

Notice that Eq. (24) implies that $z''/z = a''/a$. Additionally, the fact that $\mathcal{H}' = 0$ implies that $a''/a = a'^2/a^2 = \mathcal{H}^2$. Thus, the equation of motion can be rewritten as

$$\partial^2 v - \mathcal{H}^2 v = 0, \tag{26}$$

where we have defined the operator $\partial^2 \equiv \partial_\eta^2 - \nabla^2$. Since \mathcal{H}^2 is a positive constant, Eq. (26) is a Klein–Gordon type of equation with the “wrong” mass sign, that is, the motion equation of a free tachyon field. This can also be read directly from action (23), which is $\delta S^{(2)} = \int d\eta d^3x \mathcal{L}$, where

$$\mathcal{L} = \frac{1}{2} \partial^2 v + \frac{1}{2} \mathcal{H}^2 v^2. \tag{27}$$

Thus, quantizing the scalar field $v(\eta, \mathbf{x})$ in the $R_h = ct$ Universe, is equivalent of quantizing a free tachyon with constant mass given by $m^2 = -\mathcal{H}^2 < 0$.

There are various methods proposed for constructing a quantum field theory of a free tachyon in the past [53–58]. Nevertheless, there are some issues that seem to be always present in such theories [59]. Among them, we can mention the non-locality of the tachyonic field, represented in the present paper by the field $v(x)$ [the short-hand notation x refers to a point in spacetime (\mathbf{x}, η)], in the sense that the commutator (as well as the anti-commutator in some methods) $[\hat{v}(x), \hat{v}^\dagger(x')]$ does not vanish for spacelike arguments. Another puzzle is that the energy operator, normally associated to the Hamiltonian, does not have a lower bound on its spectrum, i.e. there are infinitely negative energy states, which requires some reinterpretation principle [60]. But perhaps the most serious difficulty in formulating a theory of tachyons is that the resulting S -matrix is non-unitary. Thus, it is unknown how to describe interactions within the theory of a tachyonic field [59].

In spite of the aforementioned issues, we could proceed in a pragmatic way, and construct a quantum theory of the field $v(x)$ but only considering the modes such that $k > \mathcal{H}$, i.e. modes with a proper wavelength less than the Hubble radius $\lambda_p < H^{-1}$. Also, according to Ref. [41] those modes are the ones that can grow and evolve into large scale structure.² Afterwards, we could compute the quantum two-point correlation function and extract its corresponding power spectrum.

There are various known methods for constructing a quantum theory for a field with the Lagrangian (27) that ignores the “problematic modes”. Among those, we can mention the one proposed by Feinberg [56] and another one developed by Arons and Sudarshan (AS) [57]. We will focus on those

² Note that modes with $k \ll \mathcal{H}$ are always less than \mathcal{H} in the $R_h = ct$ model, therefore they will not be relevant at all observationally.

methods as they illustrate the kind of puzzles one encounters when trying to compute $\langle \hat{v}(\mathbf{x}, \eta) \hat{v}^\dagger(\mathbf{x}', \eta) \rangle$.

Both methods assume that the field $v(x)$ possesses non-vanishing Fourier components only for $k \geq \mathcal{H}$ and is expanded as

$$v(x) = \frac{1}{(2\pi)^{3/2}} \int_{k \geq \mathcal{H}} \frac{d^3k}{\sqrt{2w(k)}} [c_+(\mathbf{k})e^{-iw(k)|\eta|+i\mathbf{k}\cdot\mathbf{x}} + c_-(\mathbf{k})e^{iw(k)|\eta|-i\mathbf{k}\cdot\mathbf{x}}], \quad (28)$$

where $w(k) \equiv +\sqrt{k^2 - \mathcal{H}^2}$. Then one promotes the field v into an operator \hat{v} . The difference between the AS and Feinberg's method is the quantum interpretation of the coefficients $c_+(\mathbf{k})$ and $c_-(\mathbf{k})$.

Feinberg's method follows the traditional approach of promoting $c_+(\mathbf{k}) \rightarrow \hat{c}(\mathbf{k})$ and $c_-(\mathbf{k}) \rightarrow \hat{c}(\mathbf{k})^\dagger$ into annihilation and creation operators respectively. Furthermore, $\hat{c}(\mathbf{k})$ and $\hat{c}(\mathbf{k})^\dagger$ satisfy anti-commutator relations $\{\hat{c}(\mathbf{k}), \hat{c}(\mathbf{k}')^\dagger\} = \delta^3(\mathbf{k} - \mathbf{k}')$. The anti-commutator replaces the commutator since the former is compatible with Lorentz invariance, within the quantization of a free tachyon. However, under a suitable Lorentz transformation, $\hat{c}(\mathbf{k})$ can be converted into $\hat{c}(\mathbf{k})^\dagger$. Thus, the vacuum state defined as $\hat{c}(\mathbf{k})|0\rangle = 0$ is not an invariant vacuum state since in another frame of reference it takes the form $\hat{c}(\mathbf{k})^\dagger|0\rangle = 0$. For this reason, we find Feinberg's method not to be suitable for the problem at hand.

On the other hand, in the AS method *both coefficients are promoted to annihilation operators*. The fact that both operators $\hat{c}_+(\mathbf{k})$ and $\hat{c}_-(\mathbf{k})$ are annihilation operators is needed in this approach in order to preserve the Lorentz invariance symmetry of the vacuum state [53, 59]. Moreover, one also has anti-commutation relations $\{\hat{c}_\pm(\mathbf{k}), \hat{c}_\pm(\mathbf{k}')^\dagger\} = \delta^3(\mathbf{k} - \mathbf{k}')$ and the vacuum state defined as $c_\pm(\mathbf{k})|0\rangle$. Consequently, we can compute $\langle 0|\hat{v}(\mathbf{x}, \eta)\hat{v}^\dagger(\mathbf{x}', \eta)|0\rangle$, which yields

$$\langle 0|\hat{v}(\mathbf{x}, \eta)\hat{v}^\dagger(\mathbf{x}', \eta)|0\rangle = \int_0^\infty \frac{dk}{k} \frac{\sin k|\mathbf{x} - \mathbf{x}'|}{k|\mathbf{x} - \mathbf{x}'|} \frac{k^3}{2\pi^2 w(k)}. \quad (29)$$

In the comoving gauge, the curvature perturbation is given by $\mathcal{R} = v/z$. That is, from (29) we can extract the primordial power spectrum $P_{\mathcal{R}}(k, \eta) = P_v(k)/z(\eta)^2$, which, using Eq. (24), results in

$$P_{\mathcal{R}}(k, \eta) = \frac{1}{2M_{\text{P}}^2 a^2(\eta) w(k)} \simeq \frac{1}{2M_{\text{P}}^2 a^2(\eta) k}. \quad (30)$$

The previous approximated expression is valid only for $k > \mathcal{H}$. As a matter of fact, $P_{\mathcal{R}}(k, \eta) = 0$ for $k < \mathcal{H}$; i.e. there are no "super-Hubble" modes (see footnote 2).

Substituting Eq. (30) into Eq. (15) yields the matter power spectrum,

$$P_\delta(k, \eta) \simeq \frac{2}{9M_{\text{P}}^2 a^2(\eta) \mathcal{H}} \left(\frac{k}{\mathcal{H}} + 4 - 2\frac{\mathcal{H}}{k} \right), \quad (31)$$

which is valid for $k > \mathcal{H}$, while $P_\delta(k) = 0$ if $k < \mathcal{H}$. The quantum theory proposed above resulted in a matter power spectrum (31) of the same structure in k , plus a constant term, as the one in Eq. (6), whose form was proposed by the authors of [41, 48] motivated by observational data. It may be the case that the spectrum (31), including the constant term, could reproduce the results obtained from the one proposed heuristically in Refs. [41, 48], Eq. (6), for some values of the parameters considered in those references. However, the quantum theory of the primordial perturbation in the present section contains at least two fundamental issues: (i) the theory describes a free tachyon field and (ii) the final primordial spectrum, Eq. (30), depends on the scale factor. We will study the implications of the second issue in the next section. Here, let us focus on the first issue.

The fact that the spectrum obtained involved the quantum theory of a free tachyon field could discourage some readers to consider the quantum theory of the field $v(x)$ as a serious mechanism for generating the primordial spectrum in the $R_{\text{h}} = ct$ Universe. The reasons are vast and we entirely subscribe to most of them. However, a possible way to deal with that issue is to abandon the $R_{\text{h}} = ct$ model framework for the early Universe and instead use the standard inflationary paradigm. In other words, we can assume that inflation did occur in the early Universe, but then, after the reheating era, the Universe followed the evolution described by the $R_{\text{h}} = ct$ Universe.

In slow-roll inflation, one has the standard theory of the inflation field, and the end of the inflationary era is achieved when the slow-roll parameters are close to unity. As is well known, the quantum theory of single field slow roll inflation leads to the following expression for the Mukhanov–Sasaki variable:

$$v_k(\eta) \simeq \frac{1}{\sqrt{2k}} \left(1 - \frac{i}{k\eta} \right) e^{-ik\eta}, \quad (32)$$

and $z = \sqrt{2\epsilon} M_{\text{P}} a$, where ϵ is the standard Hubble slow-roll parameter defined as $\epsilon \equiv 1 - \mathcal{H}'/\mathcal{H}$ and during inflation $\epsilon \ll 1$. As a consequence, the primordial spectrum for slow-roll inflation is $P_{\mathcal{R}} \simeq |v_k(\eta)|^2/z^2$, that is,

$$P_{\mathcal{R}}(k) \simeq \frac{1}{4M_{\text{P}}^2 \epsilon a^2 k} \left(1 + \frac{\mathcal{H}^2}{k^2} \right), \quad (33)$$

where we used that $\mathcal{H} \simeq -1/\eta$ during inflation. For the "super-Hubble" modes, i.e. modes that $k \ll \mathcal{H}$ during inflation, one has the familiar result (ignoring the numerical factors)

$$P_{\mathcal{R}}(k) \simeq \frac{H^2}{M_{\text{P}}^2 \epsilon k^3}, \quad (34)$$

that is, the scale invariant primordial spectrum, which remains constant after the "horizon crossing." On the other hand, the "sub-Hubble" modes, which satisfy $k \gg \mathcal{H}$ during inflation, lead to a spectrum of the form

$$P_{\mathcal{R}}(k) \simeq \frac{1}{M_{\text{p}}^2 a^2 \epsilon k}. \tag{35}$$

In the standard Λ CDM model, the sub-Hubble modes are ignored since they decay as $\sim a^{-2}$. However, in the $R_{\text{h}} = ct$ Universe these modes need not to decay so much because in the $R_{\text{h}} = ct$ model there is no ‘‘horizon problem’’ (see Ref. [29]). Consequently, there is no minimum value of e -folds needed for inflation to solve the horizon problem. Thus, the inflationary era could end after a few e -folds and the sub-Hubble modes still contribute to the observable modes in the CMB.

Additionally, in the inflation plus $R_{\text{h}} = ct$ Universe scenario, the dynamical evolution of \mathcal{H} , ignoring the reheating era, is

$$\mathcal{H} = \begin{cases} -\frac{1}{\eta} & \text{if } \eta < \eta_f \text{ (inflation)} \\ H_0 & \text{if } \eta \geq \eta_f \text{ (the } R_{\text{h}} = ct \text{ Universe),} \end{cases} \tag{36}$$

where η_f denotes the conformal time at which inflation ends, thus, $\eta_f = -1/H_0$.

Equation (36) implies that super-Hubble modes $k < \mathcal{H}$ during inflation, stay super-Hubble at all times, namely they do not re-enter the horizon as in the standard Λ CDM model, instead, if a mode satisfies $k < \mathcal{H}$ during inflation, then it also satisfies $k < \mathcal{H}$ during the whole evolution of the $R_{\text{h}} = ct$ Universe. Therefore, taking into account that Φ_k is a constant for super-Hubble modes and that $w = -1/3$, one obtains from Eqs. (11) and (13) the matter power spectrum,

$$P_{\delta}(k) \simeq \frac{H_*^2}{M_{\text{p}}^2 \epsilon_* k^3}, \tag{37}$$

where H_* and ϵ_* are valuated at the time $-k\eta_* = 1$ during inflation.

On the other hand, Eq. (36) implies that if a mode is sub-Hubble during the $R_{\text{h}} = ct$ Universe expansion, $k > \mathcal{H} = H_0$, then it is also sub-Hubble during inflation $-k\eta > 1$ (or equivalently $k > \mathcal{H}_{\text{inf}}$). As a consequence, the primordial spectrum associated to these modes, Eq. (35), should be evaluated at some conformal time $\tilde{\eta}$ after inflation ends, namely when $\epsilon = 1$ and $a(\tilde{\eta}) > a(\eta_f)$. Therefore, after substituting Eq. (35) into Eq. (15), the matter power spectrum associated to the super-Hubble modes, at leading order in k/\mathcal{H} , is

$$P_{\delta}(k) \simeq \frac{k}{M_{\text{p}}^2 a^2(\tilde{\eta}) \mathcal{H}} \tag{38}$$

Thus, adding a standard inflationary era to the $R_{\text{h}} = ct$ model, results in a matter power spectrum of the form

$$P_{\delta}(k) \propto \begin{cases} k & \text{if } k > \mathcal{H} = H_0 \\ k^{-3} & \text{if } k < \mathcal{H} = H_0. \end{cases} \tag{39}$$

The spectrum (39) has a resemblance to the standard prediction of the Λ CDM model but with an important difference. In the traditional model, one divides the parts pro-

portional to k and k^{-3} using the value $k_{\text{eq.}}$, instead of the value H_0 as in (39), where we have defined $k_{\text{eq.}} \equiv 1/\eta_{\text{eq.}}$ and $\eta_{\text{eq.}}$ denotes the conformal time at the epoch of matter-radiation equality. If $k \ll k_{\text{eq.}}$, then the mode enters the horizon (or becomes sub-Hubble) during the matter dominated epoch and $P_{\delta}(k) \propto k$. On the contrary, if $k \gg k_{\text{eq.}}$, then the modes becomes sub-Hubble during the radiation era and $P_{\delta}(k) \propto k^{-3}$.

Also, note that the matter power spectrum shown in (39) is not of the form proposed by the authors of the $R_{\text{h}} = ct$ Universe [see (6)], which we obtained by adding a previous inflationary phase to the $R_{\text{h}} = ct$ model. The only similarity between the two expressions, Eqs. (39) and (6), is in the term that goes as k , the rest of the terms are not equivalent. The spectrum of Eq. (6) was proposed heuristically (not deduced from a physical mechanism) by the original authors of the $R_{\text{h}} = ct$ model in order to provide a solution for the low correlation observed at large angles. Given that Eq. (39) is not the same as (6), we cannot say if the analysis made by the authors of $R_{\text{h}} = ct$ model still is valid for Eq. (39), i.e. we cannot claim that the spectrum (39) solves the low large-angle correlation. In the next section, we will deepen the discussion regarding the viability of the primordial spectra obtained when considering the amplitude of the temperature anisotropies in the CMB.

4 Amplitude of the primordial spectra and the CMB temperature anisotropies

In the previous section, we proposed a mechanism for deriving the primordial spectrum from the quantum fluctuations of a field ϕ that dominated the early Universe, but with the condition that the field must satisfy $P(\phi) = -\rho(\phi)/3$. That procedure resulted, albeit the need of a quantum theory for a free tachyon, in a prediction with a similar shape to the one proposed by the authors of the $R_{\text{h}} = ct$ Universe [41,48], but with the difference that the final primordial spectrum showed in Eq. (30) depends on the scale factor. In the present section, we return to this subject by analyzing the amplitude of the spectrum, which is tightly related to the amplitude of the CMB temperature anisotropies.

We will consider the Sachs–Wolfe effect on the temperature anisotropies. That effect is the dominant source for the anisotropies at large angular scales ($l \leq 20$). It also relates the anisotropies in the temperature observed today on the celestial sphere to the inhomogeneities in the Newtonian potential on the last scattering surface,

$$\frac{\delta T}{T}(\theta, \varphi) \simeq \frac{1}{3} \Phi(\eta_{\text{D}}, \mathbf{x}_{\text{D}}). \tag{40}$$

Here, η_{D} is the conformal time of the decoupling era and $\mathbf{x}_{\text{D}} = R_{\text{D}}(\sin \theta \sin \varphi, \sin \theta \cos \varphi, \cos \theta)$, with R_{D} the comov-

ing radius of the last scattering surface. It is useful to perform a multipolar series expansion $\frac{\delta T}{T} = \sum_{l,m} a_{lm} Y_{lm}(\theta, \varphi)$.

Using the Fourier expansion of $\Phi(\eta_D, \mathbf{x}_D)$ and the expression for the Sachs–Wolfe effect, the coefficients a_{lm} can be expressed as

$$a_{lm} \simeq \frac{4\pi i^l}{3} \int \frac{d^3k}{(2\pi)^{3/2}} j_l(kR_D) Y_{lm}^*(\hat{k}) \Phi_k(\eta_D), \tag{41}$$

with j_l the spherical Bessel functions of order l . The observed data is presented in terms of the angular power spectrum defined as $C_l \equiv 1/(2l + 1) \sum_m |a_{lm}|^2$, that is, Eq. (41) yields

$$C_l \simeq \frac{2}{9\pi} \int_0^\infty dk k^2 j_l(kR_D)^2 |\Phi_k(\eta_D)|^2. \tag{42}$$

It is straightforward to check that if $|\Phi_k(\eta_D)|^2 = A/k^3$, with A some constant, then $C_l = A/[l(l + 1)]$. In other words, for $l \leq 20$, the quantity $l(l + 1)C_l$ is a constant A and is equal to the amplitude of the squared temperature anisotropies $\sim 10^{-9}$ [5]. Thus, it is a necessary condition that the squared amplitude of the Newtonian potential at the time of decoupling should scale as $\sim k^{-3}$ if C_l is to be consistent with the temperature anisotropies of the CMB.

In the standard Λ CDM model, the value of $|\Phi_k(\eta_D)|^2$ is determined by the modes that became super-Hubble during inflation. Those modes behave as $|\Phi_k^{\text{inf}}|^2 = A/k^3$ and remained constant during the whole cosmological evolution up until they became sub-Hubble at some point. If the modes became sub-Hubble during the matter dominated epoch, then they remain constant even for $k > \mathcal{H}$. On the other hand, the modes that became sub-Hubble during the radiation dominated epoch decayed as $\sim a^2$. Thus, in the traditional scenario once $|\Phi_k^{\text{inf}}|^2$ is generated during inflation, it remains fixed at that value and then one simply relates $|\Phi_k(\eta_D)|^2 \propto |\Phi_k^{\text{inf}}|^2$. In other words, the amplitude $A = k^3 |\Phi_k^{\text{inf}}|^2$ is fixed during inflation and is the same for all modes up to the decoupling epoch.

Now, let us focus on the value of $|\Phi_k(\eta_D)|^2$ in the $R_h = ct$ Universe. As mentioned previously, the potential Φ_k corresponds to the general solution of Eq. (10), which is a linear combination of $\exp[(q - \mathcal{H})\eta]$ and $\exp[-(q + \mathcal{H})\eta]$, with $q \equiv +\sqrt{k^2/3 + \mathcal{H}^2}$. Moreover, using $a(\eta) \propto (\exp \mathcal{H}\eta)$, the two linearly independent solutions can be rewritten as follows: the first solution is $\exp[(q - \mathcal{H})\eta] = \exp[(q/\mathcal{H} - 1)\mathcal{H}\eta] = \exp[(\alpha - 1)\mathcal{H}\eta] \propto a^{\alpha-1}$. We have defined $\alpha \equiv q/\mathcal{H}$; similarly, the second solution is given by $a^{-\alpha-1}$. Since $\alpha > 0$ the second solution corresponds to a decaying mode; on the other hand, the first solution is explicitly

$$\Phi_k(\eta) = C_k a(\eta)^{\alpha-1}. \tag{43}$$

If $k < \mathcal{H}$ then $\alpha \sim 1$; on the contrary, if $k > \mathcal{H}$ then $\alpha > 1$. Therefore, depending on whether $k < \mathcal{H}$ or $k > \mathcal{H}$, the first linearly independent solution of (10) can be approximated by a constant or a growing mode [which is consistent with

the discussion after Eq. (10)]. In the $R_h = ct$ model one is interested in the growing mode, hence $k > \mathcal{H}$ and $\alpha > 1$.

The primordial spectrum $P_{\mathcal{R}}(k)$ obtained in Eq. (30) can be related to the amplitude of the Newtonian potential $|\Phi_k|^2$ through Eq. (14), which results in

$$|\Phi_k(\eta_p)|^2 = \frac{\mathcal{H}^2}{2M_{\text{p}}^2 a_{\text{p}}^2 k^3}. \tag{44}$$

Note that we have evaluated the scale factor, and consequently the power spectrum, at some conformal time η_p , i.e. $a(\eta_p) = a_p$.

At this point, we will make the assumption that the value of $|\Phi_k|^2$ obtained during the period dominated by the scalar field ϕ [Eq. (44)], when $c_s^2 = 1$, is the same as the one given by (43), when $c_s^2 = -1/3$ at the time η_p . Note, however, that in both cases $w = -1/3$, hence the $R_h = ct$ Universe expansion remains unchanged. In particular, we are assuming that the following condition is satisfied:

$$|\Phi_k(\eta_p)|_{c_s^2=1}^2 = |\Phi_k(\eta_p)|_{c_s^2=-1/3}^2 \tag{45}$$

but $w = -1/3$ in both situations. In other words, we are assuming that the “reheating” period in the $R_h = ct$ Universe is practically instantaneous.

Furthermore, with the condition (45) and expression (44), we can find the explicit value of the integration constant C_k in (43):

$$C_k^2 = \frac{\mathcal{H}^2}{2M_{\text{p}}^2 k^3 a_{\text{p}}^{2\alpha}}. \tag{46}$$

Consequently, the expression for $|\Phi_k(\eta)|^2$ in the $R_h = ct$ model is given by

$$|\Phi_k(\eta)|^2 = \frac{\mathcal{H}^2}{2M_{\text{p}}^2 k^3 a_{\text{p}}^{2\alpha}} a(\eta)^{2\alpha-2}. \tag{47}$$

The Newtonian potential obtained, Eq. (47), has a scale dependence k^{-3} , which could guarantee the same amplitude for all the modes. But, unfortunately, it also carries an additional k -dependence through α ; specifically, for modes $k > \mathcal{H}$, we can approximate $\alpha \simeq k/\mathcal{H}$. Therefore, different modes, grow at a different rate, and for that reason, the amplitude of each mode at the time of decoupling would be different for each mode. Nevertheless, we could make use of the fact that up to this point a_p has remained unspecified. Then, to avoid the mentioned issue, we must have $a_{\text{p}}^{2\alpha} = N^2 a_{\text{D}}^{2\alpha-2}$, with N^2 some normalization constant and a_{D} the scale factor at the time of decoupling. In other words, we are adjusting the value of a_p for each mode in order to achieve that all modes arrive with the same amplitude at the time of decoupling, and thus we obtain a nearly scale invariant spectrum as observed in the CMB.

By using the expression $a_p^{2\alpha} = N^2 a_D^{2\alpha-2}$ and Eq. (47), we obtain the value of $|\Phi_k|^2$ evaluated at the time of decoupling,

$$|\Phi_k(\eta_D)|^2 = \frac{\mathcal{H}^2}{2M_p^2 k^3 N^2}. \tag{48}$$

Afterwards, we could simply adjust N^2 so that

$$\mathcal{H}^2 / (M_p^2 N^2) \simeq 10^{-9}.$$

However, the condition $a_p^{2\alpha} = N^2 a_D^{2\alpha-2}$ can be rewritten as

$$\log a_p = \frac{1}{\alpha} \log N + \left(1 - \frac{1}{\alpha}\right) \log a_D, \tag{49}$$

which implies that if $\alpha \gg 1$ (or equivalently $k \gg \mathcal{H}$) then $a_p \simeq a_D$. That is, for these modes, the epoch dominated by the scalar field ϕ should last up until the decoupling epoch in order to the primordial spectrum obtained can have a consistent amplitude with the corresponding observed in the temperature anisotropies of the CMB.

Another way to show that the spectrum (47) presents some issues with the CMB is to calculate the angular power spectrum $l(l+1)C_l$ from Eq. (42) using precisely Eq. (47). It is known that, for low l , say $l < 20$, the shape of the angular power spectrum must be essentially a constant, independent of l , which results from a nearly scale invariant primordial power spectrum. That is, the region of the angular spectrum where the Sachs–Wolfe effect is dominant must not depend on l in order to be consistent with the CMB. Thus, our next task will be to compute the angular power spectrum using the spectrum (47) for $l < 20$.

Assuming that $a_p = a_D/\gamma$, with $\gamma > 1$ a constant and evaluating Eq. (47) at the time of decoupling, we have

$$|\Phi_k(\eta_D)|^2 = \frac{\mathcal{H}^2}{2M_p^2 a_D^2} \frac{\gamma^{2\alpha}}{k^3}. \tag{50}$$

Substituting Eq. (50) into Eq. (42), we find

$$C_l \simeq \frac{1}{9\pi} \frac{\mathcal{H}^2}{M_p^2 a_D^2} \int_{H_0}^{\infty} \frac{dk}{k} j_l(kR_D)^2 \gamma^{2\alpha}. \tag{51}$$

Since the spectrum (47) was obtained for the modes $k > \mathcal{H} = H_0$ note that the lower limit of integration is H_0 . As a matter of fact, the modes with $k < \mathcal{H}$ vanished when we considered the quantum theory of a free tachyon. The value of R_D , which corresponds to the comoving radius of the last scattering surface, was calculated in Refs. [41, 48] in the context of the $R_h = ct$ model resulting $R_D \simeq 10/H_0$. Additionally, for $k > \mathcal{H}$ we can approximate $\alpha \simeq k/\mathcal{H} = k/H_0$. Performing the change of variable in the integral (51) of $x \equiv k/H_0$ yields

$$C_l \simeq \frac{1}{9\pi} \frac{\mathcal{H}^2}{M_p^2 a_D^2} \int_1^{\infty} \frac{dx}{x} j_l(10x)^2 \gamma^{2x}. \tag{52}$$

Noting that the asymptotic form of the functions $j_l(10x)^2$ contribute with a factor of $1/x^2$ as $x \rightarrow \infty$ and that $\gamma > 1$, we can conclude that the integral in Eq. (52) diverges.

We remind the reader that the standard Λ CDM prediction, which is consistent with the CMB data, would have resulted in $l(l+1)C_l = \text{const.}$ for the lowest multipoles (approximately for $l < 20$). On the other hand, the result of Eq. (52) diverges even for the lowest values of l . Thus, in addition to the aforementioned problems, e.g. the need of a quantum theory of a free tachyon and that the epoch dominated by the scalar field ϕ should last up until the decoupling epoch, the resulting angular power spectrum is divergent.

Furthermore, the condition (49) also applies to the situation in which one drops the $R_h = ct$ model in the early Universe in favor of an inflationary era. As we described in Sect. 3, during inflation, the dynamical evolution of the Mukhanov–Sasaki variable v leads to the following expression for the comoving curvature perturbation:

$$|\mathcal{R}_k(\eta)|^2 \simeq \frac{1}{4M_p^2 \epsilon a^2 k} \left(1 + \frac{\mathcal{H}^2}{k^2}\right), \tag{53}$$

which for sub-Hubble modes $k > \mathcal{H}$ is approximated by $|\mathcal{R}_k(\eta)|^2 \simeq (4M_p^2 \epsilon a^2 k)^{-1}$. Next, we evaluate $|\mathcal{R}_k(\eta)|^2$ at some time η_p near the end of inflation, i.e. when $\epsilon = 1$, and we make the assumption that

$$|\mathcal{R}_k(\eta_p)|^2_{w \simeq -1} = |\mathcal{R}_k(\eta_p)|^2_{w = -1/3}, \tag{54}$$

once again neglecting the reheating era.

Using Eq. (14), which allows us to relate $|\mathcal{R}_k(\eta)|^2$ with $|\Phi(\eta)|^2$ when $w = -1/3$, and since we are connecting the inflationary regime with the $R_h = ct$ Universe expansion, we have $\mathcal{H} = H_0$. Consequently, the amplitude of the primordial Newtonian potential is

$$|\Phi(\eta_p)|^2 \simeq \frac{\mathcal{H}^2}{4M_p^2 k^3 a_p^2}, \tag{55}$$

which is essentially the same as the one in (44). Therefore, all the mathematical steps that lead from (44) up to (49) are equivalent, including Eq. (47). And, from condition (49), one is led to conclude that inflation should last until the decoupling epoch in order to the primordial spectrum obtained can have a consistent amplitude with the temperature anisotropies of the CMB. Moreover, the discussion regarding the shape of the angular power spectrum also remains the same since the spectrum obtained in Eq. (47) will be exactly the same in the present scenario of adding an inflationary era to the $R_h = ct$ model.

Given that both approaches, for generating the primordial perturbation, require very unlikely conditions to be compatible with the observed shape and amplitude of the temperature anisotropies, we could do a search for the initial value of $|\Phi_k(\eta_p)|^2$ so that it is consistent with the CMB temperature

anisotropies based solely on the dynamics of the $R_h = ct$ model.

The equation of motion for Φ_k , Eq. (10), implies that

$$\frac{d}{d\eta} \left(\Phi_k(\eta) a(\eta)^{1-\alpha} \right) = 0. \quad (56)$$

That is, $\Phi_k(\eta) a(\eta)^{1-\alpha}$ is a constant of motion in the $R_h = ct$ Universe. Consequently, we have the relation $\Phi_k(\eta_D) = (\Phi_k(\eta_p)/a_p^{\alpha-1}) a_D^{\alpha-1}$, which implies that

$$|\Phi_k(\eta_D)|^2 = \frac{|\Phi_k(\eta_p)|^2}{a_p^{2\alpha-2}} a_D^{2\alpha-2}. \quad (57)$$

We now select the value of the scale factor at the initial time η_p . Hence, we assume that $a_p = C a_D$, with C some normalization constant and $C < 1$, which from Eq. (57) yields

$$|\Phi_k(\eta_D)|^2 = \frac{|\Phi_k(\eta_p)|^2}{C^{2\alpha-2}}. \quad (58)$$

Therefore, if the initial amplitude of the Newtonian potential is of the form

$$|\Phi_k(\eta_p)|^2 = \frac{C^{2\alpha}}{k^3}, \quad (59)$$

then the amplitude of the Newtonian potential at the time of decoupling, Eq. (58), is $|\Phi_k(\eta_D)|^2 = C^2/k^3$, which will be consistent with the amplitude of the temperature anisotropies if $C^2 \simeq 10^{-9}$.

It is evident that Eq. (59) is not equivalent to (44) and/or (55), which corresponds to the primordial amplitude obtained in the two approaches described in the previous section. Thus, any mechanism proposed for generating the primordial curvature perturbation in the $R_h = ct$ Universe must be of the form of Eq. (59) in order to be consistent with the CMB temperature anisotropies. However, neither a scalar field dominating the early Universe satisfying an equation of state $P(\phi) = -\rho(\phi)/3$ nor the inflation yield a primordial spectrum compatible with Eq. (59).

5 Discussion

In this work, we began by proposing that the small inhomogeneities of a scalar field $\delta\phi$ could generate the primordial spectrum in the same fashion as the one in the standard inflationary scenario, but with the important difference that during the period dominated by the scalar field, the equation of state $P(\phi) = -\rho(\phi)/3$ should be satisfied at all times. That is an important condition in the $R_h = ct$ cosmological model.

Under that proposal, the quantum theory of perturbations led to a theory of a free tachyon field with mass $m^2 = -\mathcal{H}^2 = -H_0^2 < 0$. We pushed forward and followed a suitable method for dealing with that kind of theory, which resulted in a matter power spectrum (31) that is similar in

structure in k , plus a constant term, to the one proposed in an empirical manner in Refs. [41,48], Eq. (6). It might be the case that, for some values of the parameters corresponding to the spectrum proposed by the authors of the Refs. [41,48], the spectra (31) and (6) coincide and the analysis of Refs. [41,48] continues to be valid for the matter power spectrum given in Eq. (31). Nevertheless, as we will see in the rest of this discussion there are other important problems associated to the spectrum, Eq. (31).

The fact that the quantum theory of the perturbations in the $R_h = ct$ Universe resulted in that of a free tachyon carries deep issues. Among them, perhaps the most important issue in the cosmological context is that the corresponding S -matrix is non-unitary. Therefore, there is no clear way how to describe interactions. The interactions with other fields are important since at some point the scalar field ϕ , dominating the early Universe in the $R_h = ct$ model, should decay into the particles of the Standard Model, and in the absence of a well defined S -matrix, it is a puzzle how to describe such interactions. Another related issue with the non-unitarity of the S -matrix is that the self-interactions of the scalar degree of freedom, characterized here by the Mukhanov–Sasaki variable, result in primordial non-Gaussianities; however, since there is no way to characterize the interactions, one cannot quantify the amount of primordial non-Gaussianities generated by the perturbations in the $R_h = ct$ Universe.

Since in the $R_h = ct$ framework $z''/z = a''/a$, the action given in Eq. (23) will be identical for a hypothetical analysis of tensor modes. Thus, if one wanted to study the tensor case, the quantum field theory used here for scalar perturbations would be equivalent. Therefore, in the light of our results, it is not obvious how tensor modes could be generated within the framework of a quantum theory in the $R_h = ct$ model. In case of adding an inflationary epoch prior to the $R_h = ct$ evolution, tensor modes could be generated but since the modes satisfying $k > \mathcal{H}$ are relevant, their amplitudes will be exponentially suppressed and the suppressing will continue also during the $R_h = ct$ evolution.

Those issues added to the technical and conceptual problems raised by a quantum theory of a free tachyon might suggest that we should abandon the idea of describing the quantum perturbations in the $R_h = ct$ Universe.

The aforementioned problems led us to consider the quantum theory of perturbations during an inflationary era preceding the $R_h = ct$ cosmological expansion. However, given the nature of the $R_h = ct$ Universe, we needed to focus on the primordial spectrum of the sub-Hubble modes. The primordial spectrum that resulted from inflation for the sub-Hubble modes is not consistent with the matter power spectrum proposed by the authors of the $R_h = ct$ Universe.

In fact, the matter power spectrum that we obtained by adding an early inflationary regime in the $R_h = ct$ model, resembles to the traditional one from the Λ CDM model, but

it is not exactly the same. More precisely, the matter power spectrum obtained by adding an early inflationary regime to the $R_h = ct$ model, can be separated into two cases. In one case the spectrum goes as k , and in the second case the spectrum goes as k^{-3} . In the Λ CDM model the matter power spectrum can also be separated into two cases, one that goes as k and a second case where the spectrum goes as k^{-3} . However, as mentioned after Eq. (39), the conditions for the separation into two cases in the $R_h = ct$ model are not the same as the conditions in the standard model. Consequently, the functional form of the Λ CDM matter power spectrum is completely different from the $R_h = ct$ model. Moreover, the matter power spectrum obtained by adding an early era of inflation to the $R_h = ct$ model leads to a matter power spectrum that is different from the one proposed empirically by the authors of the $R_h = ct$ model. The main motivation, as stated by those authors when proposing such a spectrum, was to solve the observed low correlation at large angles in the angular correlation, and since the spectrum that we have obtained, Eq. (39), is not equal to the one heuristically proposed, Eq. (6), we cannot claim that the spectrum in Eq. (39) explains the observed low angular correlation at large angles.

Finally, we investigated the predicted amplitude of the CMB temperature anisotropies following the two approaches described. In the first approach, we considered an early phase dominated by a scalar field ϕ satisfying the equation of state of the $R_h = ct$ model; in the second framework, we assumed an inflationary stage preceding the $R_h = ct$ cosmological evolution. In both approaches, the amplitude of the Newtonian potential at the time of decoupling, which is the main source of the temperature anisotropies at low angular multipoles, depends on the wavenumber k in a non-trivial way. The reason is that in the $R_h = ct$ Universe the evolution of each mode associated to the Newtonian potential evolves as $\sim a^{\alpha-1}$, with $\alpha \simeq k/\mathcal{H}$. As a consequence, we were forced to choose a particular initial condition for the evolution of the modes that translates into adjusting the value of the scale factor for each mode at the initial time η_p . Specifically, we had to choose $a_p^{2\alpha} = N^2 a_D^{2\alpha-2}$, with N some normalization constant and a_D the value of the scale factor at the time of decoupling. Such an election implies that, for modes $k \gg \mathcal{H}$, the initial value of the scale factor is $a_p \simeq a_D$. That is, the era dominated by the scalar field ϕ , in the first approach, or the inflationary era preceding the $R_h = ct$ evolution, should last up to the decoupling epoch in order to the primordial spectrum obtained can have a consistent amplitude with the corresponding observed in the temperature anisotropies of the CMB. Another related problem that emerges from considering the Newtonian potential Eq. (47) is that the predicted angular power spectrum Eq. (52) diverges in the region where the Sachs–Wolfe effect is dominant. The expected behavior for the angular power spectrum, which is consistent with the

CMB data in the Sachs–Wolfe region, is $l(l+1)C_l \simeq$ constant.

We ended our analysis by obtaining the desired form of the initial amplitude of the Newtonian potential based solely on the dynamics of the $R_h = ct$ Universe, and that it could be consistent with the CMB temperature anisotropies. That primordial amplitude should be $|\Phi_k(\eta_p)|^2 = C^{2\alpha}/k^3$, which is not the one obtained from the two approaches considered so far. Thus, we think that any proposed mechanism for generating the primordial spectrum should predict that particular initial amplitude to be consistent with the observed CMB anisotropies. In fact, if some physically motivated mechanism, within the $R_h = ct$ model, can reproduce the primordial spectrum $|\Phi_k(\eta_p)|^2 = C^{2\alpha}/k^3$ then all the concerns raised in our paper would possibly disappear. Nevertheless, neither slow roll inflation based on a single scalar field nor a scalar field dominating the $R_h = ct$ Universe can produce that kind of spectrum.

6 Conclusions

Some studies have drawn attention to the lack of large-angle correlations in the observed CMB temperature anisotropies with respect to that predicted within the standard Λ CDM model. Recently, some authors have suggested that this lack of correlations could be explained in the framework of the so-called $R_h = ct$ model without inflation, by selecting an explicit form for the matter power spectrum and showing that it could achieve a better fit than the Λ CDM model to the data corresponding to the CMB angular correlation function. The aim of this work was to critically investigate whether there may be a mechanism to generate, through a quantum field theory, the primordial power spectrum presented by these authors.

During this search we run into deep issues and, we also studied the possibility of adding an inflationary phase prior to the evolution given by the mentioned model. The resulting power spectrum for the relevant sub-horizon modes within this approach is not consistent with the matter power spectrum displayed by the mentioned authors; thus, it cannot explain the unexpectedly close to zero angular two-point correlation function observed at angular scales larger than 60° .

Also, we analyze the consistency between the predicted and observed amplitudes of the CMB temperature anisotropies with and without the inflationary epoch added prior to the $R_h = ct$ evolution. We found that for modes satisfying $k \gg \mathcal{H}$, the epoch dominated by the scalar field (representing the matter field in the $R_h = ct$ Universe or the inflaton) should last up to the decoupling epoch in order to the primordial spectrum obtained can have a consistent amplitude with the corresponding observed in the temperature anisotropies of the CMB. That is an implausible con-

dition. Additionally, we have performed a brief analysis by focusing on the lowest angular multipoles $l < 20$, where we expect the Sachs–Wolfe effect to be the dominant effect, and we obtained, for $l < 20$, the angular power spectrum $l(l+1)C_l$ is divergent; clearly not consistent with the observations from the CMB.

Finally, we showed the generic form that a primordial curvature power spectrum should exhibit in the $R_h = ct$ framework to be consistent with the CMB temperature anisotropies observed. Neither a scalar field dominating the early Universe satisfying an equation of state $P(\phi) = -\rho(\phi)/3$ nor slow roll inflation based on a single scalar field yield a primordial spectrum compatible with this requirement. Based on the results obtained in this paper, we conclude that (in addition to the criticisms already raised by other authors) it is not clear how to characterize the quantum perturbations within the $R_h = ct$ Universe, rendering this model a very unlikely alternative to the standard Λ CDM model.

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