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On the packing chromatic number of hypercubes¹

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Abstract

The packing chromatic number $\chi_\rho(G)$ of a graph G is the smallest integer k needed to properly color the vertices of G in such a way that the distance in G between any two vertices having color i be at least $i + 1$. Goddard et al. [8] found an upper bound for the packing chromatic number of hypercubes Q_n . Moreover, they compute $\chi_\rho(Q_n)$ for $n \leq 5$ leaving as an open problem the remaining cases. In this paper, we obtain a better upper bound for $\chi_\rho(Q_n)$ and we compute the exact value of $\chi_\rho(Q_n)$ for $6 \leq n \leq 8$.

Keywords: packing chromatic number, upper bound, hypercube graphs.

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1 Introduction

We consider finite undirected graphs without loops or multiple edges. The concept of packing coloring comes from the area of frequency planning in wireless networks. This model emphasizes the fact that some frequencies might be used more sparsely than the others. In graph terms, we ask for a partitioning of the vertex set of a graph G into disjoint classes V_1, \dots, V_k (representing frequency usage) according to the following constraint. Each color class V_i is an i -packing, i.e. a set of vertices with the property that any distinct pair $u, v \in V_i$ satisfies $\text{dist}(u, v) \geq i + 1$. Here, $\text{dist}(u, v)$ is the *distance* between u and v , i.e. the length of the shortest path in G from u to v .

Such partitioning into k classes is called a *packing k -coloring*, even though it is allowed that some sets V_i may be empty. The smallest integer k for which exists a packing k -coloring of G is called the *packing chromatic number* of G , and it is denoted by $\chi_\rho(G)$. The notion of packing chromatic number was established by Goddard et al. [8] under the name *broadcast chromatic number*. The term packing chromatic number was introduced by Brešar et al. [4].

Much work has been devoted to the packing chromatic number of graphs ([2], [3], [4], [5], [6], [7], [8]). Fiala and Golovach [5] showed that determining the packing chromatic number is an NP-hard problem for trees. Goddard et al. [8] provided polynomial time algorithms for cographs and split graphs. Recently, Argiroffo et al. [2], [3] gave polynomial time algorithms for special subfamilies of trees, for partner limited graphs and for $(q, q - 4)$ graphs.

In this paper, we are interested in bounding and, whenever possible, finding the packing chromatic number of hypercubes. For any $n \in \mathbb{Z}^+$, the n -dimensional hypercube, denoted Q_n , is the graph in which the vertices are all binary vectors of length n (i.e., the set $\{0, 1\}^n$), and two vertices are adjacent if and only if they differ in exactly one coordinate. Based on coding theory, Goddard et al. [8] gave an asymptotic result for the packing chromatic number of hypercubes. They proved that $\chi_\rho(Q_n) \sim (\frac{1}{2} - O(\frac{1}{n}))2^n$. More precisely, Goddard et al. [8] obtained that $\chi_\rho(Q_n) \leq 2 + (\frac{1}{2} - \frac{1}{4n})2^n$. In the same paper, they also computed $\chi_\rho(Q_n)$ for $1 \leq n \leq 5$, leaving as an open problem the remaining cases.

The *diameter*, $\text{diam}(G)$, of a connected graph G is the maximum distance between two vertices of G . The *Cartesian product* $G \square H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$, vertices (g, h) and (g', h') being adjacent whenever $gg' \in E(G)$ and $h = h'$, or $hh' \in E(H)$ and $g = g'$. Brešar et al. [4] obtained that if G and H are connected graphs on at least two vertices, $\chi_\rho(G \square H) \geq (\chi_\rho(G) + 1)|H| - \text{diam}(G \square H)(|H| - 1) - 1$. Moreover, if

$H = K_n$, the complete graph on $n \geq 2$ vertices, then $\chi_\rho(G \square K_n) \geq n\chi_\rho(G) - (n-1)\text{diam}(G)$. It is well known that the binary hypercube Q_n is isomorphic to the graph $Q_{n-1} \square K_2$. Then, we obtain directly the following lower bound for the packing chromatic number of Q_n .

Corollary 1.1 *Let $n \geq 2$. Then, $\chi_\rho(Q_n) \geq 2\chi_\rho(Q_{n-1}) - (n-1)$.*

In this paper, we improve the upper bound found by Goddard et al. [8]. For this, we use elementary algebraic techniques in order to construct a packing coloring of Q_n , for $n \geq 4$. Furthermore, we obtain the exact values of $\chi_\rho(Q_n)$ for $n = 6, 7$ and 8. Most of our results are given without proof for lack of space.

2 Packing colorings of Q_n : the upper bound

The principal result of this section is the following :

Theorem 2.1 $\chi_\rho(Q_n) \leq 3 + 2^n(\frac{1}{2} - \frac{1}{2^{\lceil \log_2 n \rceil}}) - 2\lfloor \frac{n-4}{2} \rfloor$, for $n \geq 4$.

In order to prove Theorem 2.1, we derive a packing coloring of Q_n with such a number of colors. For this, we use an algebraic approach for the purpose of construct 2- and 3-packings for Q_n as follows.

Let Γ be a group and C a subset of Γ closed under inverses and identity free. The Cayley graph $\text{Cay}(\Gamma, C)$ is the graph with Γ as its vertex set, two vertices u and v being joined by an edge if and only if $u^{-1}v \in C$. The set C is then called the *connector set* of $\text{Cay}(\Gamma, C)$. It is well known that Q_n is the Cayley graph of the Abelian group \mathbb{Z}_2^n (the elements of \mathbb{Z}_2^n are the binary n -vectors of the set $\{0, 1\}^n$ and the group operation is the sum modulo two coordinatewise), where the connector set is the subset of n -vectors having exactly one coordinate equal to 1. So, in order to construct some packings for Q_n we will analyze the structure of a special subgroup of \mathbb{Z}_2^n .

Let $v \in \mathbb{Z}_2^n$. We denote by v^i the i^{th} coordinate of v , for $1 \leq i \leq n$, that is, $v = (v^1, \dots, v^i, \dots, v^n)$, where v^i is either 0 or 1.

Definition 2.2 Let $n \geq 4$. For $m \in \{4, \dots, n\} \setminus \{2^k + 1 : k \in \mathbb{Z}^+\}$, consider the n -vectors of \mathbb{Z}_2^n defined as follows :

(i) Type I: If m is even,

$$v_m^i = \begin{cases} 1, & \text{if } i \in \{1, 2, m-1, m\} \\ 0, & \text{o.w.} \end{cases}$$

(ii) Type *II*: If m is odd and $m - 1$ is not a power of 2,

$$v_m^i = \begin{cases} 1, & \text{if } i \in \{1, m - 2^{\lfloor \log_2 m \rfloor}, 1 + 2^{\lfloor \log_2 m \rfloor}, m\} \\ 0, & \text{o.w.} \end{cases}$$

We define the subset $W_n \subset \mathbb{Z}_2^n$ as the set containing the vectors v_m defined previously.

Notice that every vector in W_n has exactly four coordinates equal to 1. Moreover, by construction, we have that :

Remark 2.3 Let $n \geq 4$. Then, $|W_n| + 1 = |W_{n+1}|$ if and only if n is not a power of two, otherwise $|W_n| = |W_{n+1}|$. Moreover, if n is even (resp. odd), the new vector in W_{n+1} is of Type *II* (resp. *I*).

Remark 2.4 Let $n \geq 4$. Notice that the distance between any pair of n -vectors in W_n is even.

Besides, by induction on n , we can prove the following lemma.

Lemma 2.5 Let $n \geq 4$. Then, $|W_n| = n - 1 - \lceil \log_2 n \rceil$.

Now, by computing the join of all the 2-subgroups generated by each one of the elements in W_n , we can deduce the following result.

Lemma 2.6 Let $n \geq 4$, and let \mathcal{G}_n be the subgroup of the Abelian group \mathbb{Z}_2^n generated by W_n . Then, the order of \mathcal{G}_n is equal to $2^{|W_n|}$.

The subgroup \mathcal{G}_n plays an important role in the search of i -packings. In particular, we prove that any pair of vertices in \mathcal{G}_n are at distance greater than 3 in Q_n , i.e. the set \mathcal{G}_n is a 3-packing of Q_n . To obtain this, we use the structure in the construction of the sets W_n and we proceed by induction on n .

Lemma 2.7 Let $n \geq 4$. For any $u, v \in Q_n$, $u \neq v$, such that $u, v \in \mathcal{G}_n$, we have that the distance between u and v in Q_n verifies : $\text{dist}(u, v) \geq 4$. Moreover, each element of \mathcal{G}_n has an even number of coordinates (0 included) equal to 1.

Definition 2.8 Let $n \geq 4$. For each j , with $1 \leq j \leq \lfloor \frac{n-4}{2} \rfloor$, let A_j and B_j be 2-subsets of \mathbb{Z}_2^n constructed as follows : A_j is formed by two n -vectors a_{j_1} and a_{j_2} where a_{j_1} has only one 1 in coordinate $j + 2$ and 0 otherwise, and a_{j_2} has 1 in coordinates $t \in [2j + 4] \setminus \{j + 2\}$, and 0 otherwise. The 2-set B_j is formed by two n -vectors b_{j_1} and b_{j_2} , where b_{j_1} (resp. b_{j_2}) is equal to a_{j_1} (resp. a_{j_2}) but with the two first coordinates complemented.

The following result can be easily deduced.

Lemma 2.9 *Let $n \geq 6$. Let $g_2 \in \mathbb{Z}_2^n$ (resp. $g_3 \in \mathbb{Z}_2^n$) be the element having only the first coordinate (resp. the second coordinate) equal to 1 and the remaining coordinates equal to 0. For each j , with $1 \leq j \leq \lfloor \frac{n-4}{2} \rfloor$, the 2-sets A_j and B_j in Definition 2.8 verify the following properties :*

- (i) $A_j \cap (g_2 + \mathcal{G}_n) = A_j \cap (g_3 + \mathcal{G}_n) = B_j \cap (g_2 + \mathcal{G}_n) = B_j \cap (g_3 + \mathcal{G}_n) = \emptyset$.
- (ii) A_j (resp. B_j) is a $(2j+2)$ -packing (resp. $(2j+3)$ -packing) of Q_n .

From the previous lemmas, we are able to prove Theorem 2.1.

Proof of Theorem 2.1

Let $n \geq 4$. Let \mathcal{G}_n be the subgroup of the Abelian group \mathbb{Z}_2^n generated by the set W_n (see Definition 2.2). Clearly, the elements of \mathbb{Z}_2^n correspond to the vertices of the binary n -dimensional hypercube Q_n . Let I_1^n , I_2^n and I_3^n be subsets of vertices of Q_n constructed as follows : I_1^n is the set of all vertices in Q_n having an even number (0 included) of coordinates equal to 1. The sets I_2^n and I_3^n are the cosets $g_2 + \mathcal{G}_n$ and $g_3 + \mathcal{G}_n$, resp., where g_2 and g_3 are defined as in Lemma 2.9. By Lemma 2.7, it is clear that both I_2^n and I_3^n are disjoint of I_1^n , because all the elements in I_2^n and in I_3^n have an odd number of coordinates equal to 1. Moreover, I_2^n and I_3^n are disjoint by the Lagrange's Theorem. Therefore, I_i^n is an i -packing of Q_n , for $i = 1, 2, 3$. Now, observe that the family of sets in Definition 2.8 are pairwise disjoint. Furthermore, by Lemma 2.9, for $1 \leq j \leq \lfloor \frac{n-4}{2} \rfloor$, the sets A_j and B_j are $(2j+2)$ -packing and $(2j+3)$ -packing of Q_n respectively. By given a different color greater than $2\lfloor \frac{n-4}{2} \rfloor + 3$ to the remaining vertices in Q_n , we obtain the desired packing coloring of Q_n with $3 + 2^n(\frac{1}{2} - \frac{1}{2^{\lceil \log_2 n \rceil}}) - 2\lfloor \frac{n-4}{2} \rfloor$ colors. \square

3 Lower bounds for $\chi_\rho(Q_n)$: the cases $n = 6, 7$ and 8

As mentioned in the introduction, Goddard et al. [8] computed the packing chromatic numbers of the first five hypercubes : $\chi_\rho(Q_1) = 2$, $\chi_\rho(Q_2) = 3$, $\chi_\rho(Q_3) = 5$, $\chi_\rho(Q_4) = 7$ and $\chi_\rho(Q_5) = 15$. Moreover, by Corollary 1.1 and Theorem 2.1, we know that $25 \leq \chi_\rho(Q_6) \leq 25$, $44 \leq \chi_\rho(Q_7) \leq 49$ and $89 \leq \chi_\rho(Q_8) \leq 95$. Thus, we have the following direct result.

Corollary 3.1 $\chi_\rho(Q_6) = 25$

The main result in this section is the computation of a tight lower bound for the packing chromatic number of hypercubes of dimension 7 and 8. In order

to obtain these lower bounds, we combine the results obtained by Agrell et al. [1] concerning the size of the maximum number of vertices that can be covered by subsets of i -packings in Q_7 (resp. Q_8), for $1 \leq i < 7$ (resp. $1 \leq i < 8$), with the ones concerning the maximum size of balanced independent (and dominating) sets on hypercubes obtained by Ramras [9]. Thus, we obtain that,

Theorem 3.2 $\chi_\rho(Q_7) \geq 49$ and $\chi_\rho(Q_8) \geq 95$.

Therefore, by Theorems 2.1 and 3.2, we conclude that $\chi_\rho(Q_7) = 49$ and $\chi_\rho(Q_8) = 95$.

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