

Modal-type orthomodular logic

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Abstract

In this paper we enrich the orthomodular structure by adding a modal operator, following a physical motivation. A logical system is developed, obtaining algebraic completeness and completeness with respect to a Kripke-style semantic founded on Baer \star -semigroups as in [20].

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Introduction

In their 1936 seminal paper [1], Birkhoff and von Neumann made the proposal of a non-classical logic for quantum mechanics founded on the basic lattice-order properties of all closed subspaces of a Hilbert space. This lattice-order properties are captured in the orthomodular lattice structure. The orthomodular structure is characterized by a weak form of distributivity called *orthomodular law*. This “weak distributivity”, which is the essential difference with the Boolean structure, makes it extremely intractable in certain aspects. In fact, a general representation theorem for a class of algebras, which has as particular instances the representation theorems as algebras of

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sets for Boolean algebras and distributive lattices, allows in many cases and in a uniform way the choice of a Kripke-style model and to establish a direct relationship with the algebraic model [21]. In this procedure the distributive law plays a very important role. In absence of distributivity this general technique is not applicable, consequently to obtain Kripke-style semantics may be complicated. Such is the case for the orthomodular logic. Indeed, in [12], Goldblatt gives a Kripke-style semantic for the orthomodular logic based on an imposed restriction on the Kripke-style semantic for the orthologic. This restriction is not first order expressible. Thus the obtained semantic is not very attractive. In [20], Miyazaki introduced another approach to the Kripke-style semantic for the orthomodular logic based on the representation theorem by Baer semigroups given by Foulis in [11] for orthomodular lattices. In this way a Kripke-style model is obtained whose universe is given by semigroups with additional operations.

Several authors added modal enrichments to the orthomodular structure based on generalizations of classic modal systems [7, 13, 14], or generalization of quantifiers in the sense of Halmos [15]. In [9] and [10], we have introduced an orthomodular structure enriched with a modal operator called *Boolean saturated orthomodular lattice*. This structure has a rigorous physical motivation and allows to establish algebraic-type versions of the Born rule and the well known Kochen-Specker (KS) theorem [18].

The aim of this paper is to study this structure from a logic-algebraic perspective. The paper is structured as follows. Section 1 contains generalities on universal algebra, orthomodular lattices and Baer \ast -semigroups. In section 2, the physical motivation for the modal enrichment of the orthomodular structure is presented. In section 3 we introduce the class of Boolean saturated orthomodular lattices \mathcal{OML}^\square and we prove that this class conforms a discriminator variety. In section 4, a Hilbert-style calculus is introduced obtaining a strong completeness theorem for the variety \mathcal{OML}^\square . Finally, in section 5, we give a representation theorem by means of a sub-class of Baer \ast -semigroups for \mathcal{OML}^\square . This allows to develop a Kripke-style semantic for the calculus of the precedent section following the approach given in [20]. A strong completeness theorem for these Kripke-style models is also obtained.

1 Basic notions

We freely use all basic notions of universal algebra that can be found in [3]. If K is a class of algebras of the same type then we denote by $\mathcal{V}(K)$ the variety generated by K . Let \mathcal{A} be a variety of algebras of type σ . We denote by $Term_{\mathcal{A}}$ the *absolutely free algebra* of type σ built from the set of variables $V = \{x_1, x_2, \dots\}$. Each element of $Term_{\mathcal{A}}$ is referred to as a *term*. We denote by $Comp(t)$ the complexity of the term t . Let $A \in \mathcal{A}$. If $t \in Term_{\mathcal{A}}$ and $a_1, \dots, a_n \in A$, by $t^A(a_1, \dots, a_n)$ we denote the result of the application of the term operation t^A to the elements a_1, \dots, a_n . A *valuation* in A is a map $v : V \rightarrow A$. Of course, any valuation v in A can be uniquely extended to an \mathcal{A} -homomorphism $v : Term_{\mathcal{A}} \rightarrow A$ in the usual way, i.e., if $t_1, \dots, t_n \in Term_{\mathcal{A}}$ then $v(t(t_1, \dots, t_n)) = t^A(v(t_1), \dots, v(t_n))$. Thus, valuations are identified with \mathcal{A} -homomorphisms from the absolutely free algebra. If $t, s \in Term_{\mathcal{A}}$, $\models_A t = s$ means that for each valuation v in A , $v(t) = v(s)$ and $\models_{\mathcal{A}} t = s$ means that for each $A \in \mathcal{A}$, $\models_A t = s$. We denote by $Con(A)$ the lattice of congruences of A . A *discriminator term* for A is a term $t(x, y, z)$ such that

$$t^A(x, y, z) = \begin{cases} x, & \text{if } x \neq y \\ z, & \text{if } x = y \end{cases}$$

The variety \mathcal{A} is a *discriminator variety* iff there exists a class of algebras K with a common discriminator term $t(x, y, z)$ such that $\mathcal{A} = \mathcal{V}(K)$.

Now we recall from [17] and [19] some notions about orthomodular lattices. A *lattice with involution* [16] is an algebra $\langle L, \vee, \wedge, \neg \rangle$ such that $\langle L, \vee, \wedge \rangle$ is a lattice and \neg is a unary operation on L that fulfills the following conditions: $\neg\neg x = x$ and $\neg(x \vee y) = \neg x \wedge \neg y$. Let $L = \langle L, \vee, \wedge, 0, 1 \rangle$ be a bounded lattice. Given a, b, c in L , we write: $(a, b, c)D$ iff $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$; $(a, b, c)D^*$ iff $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ and $(a, b, c)T$ iff $(a, b, c)D$, $(a, b, c)D^*$ hold for all permutations of a, b, c . An element z of a lattice L is called *central* iff for all elements $a, b \in L$ we have $(a, b, z)T$. We denote by $Z(L)$ the set of all central elements of L and it is called the *center* of L . $Z(L)$ is a Boolean sublattice of L [19, Theorem 4.15]. An *orthomodular lattice* is an algebra $\langle L, \wedge, \vee, \neg, 0, 1 \rangle$ of type $\langle 2, 2, 1, 0, 0 \rangle$ that satisfies the following conditions

1. $\langle L, \wedge, \vee, \neg, 0, 1 \rangle$ is a bounded lattice with involution,
2. $x \wedge \neg x = 0$.

$$3. x \vee (\neg x \wedge (x \vee y)) = x \vee y$$

We denote by \mathcal{OML} the variety of orthomodular lattices. An important characterization of the equations in \mathcal{OML} is given by:

$$\models_{\mathcal{MOML}} t = s \quad \text{iff} \quad \models_{\mathcal{MOML}} (t \wedge s) \vee (\neg t \wedge \neg s) = 1 \quad (1)$$

Therefore we can safely assume that all \mathcal{OML} -equations are of the form $t = 1$, where $t \in \text{Term}_{\mathcal{OML}}$.

Remark 1.1 It is clear that this characterization is maintained for each variety \mathcal{A} such that there are terms of the language of \mathcal{A} defining on each $A \in \mathcal{A}$ operations $\vee, \wedge, \neg, 0, 1$ such that $L(A) = \langle A, \vee, \wedge, \neg, 0, 1 \rangle$ is an orthomodular lattice.

Proposition 1.2 [19, Lemma 29.9 and Lemma 29.16] *Let L be an orthomodular lattice then we have that:*

1. $z \in Z(L)$ if and only if $a = (a \wedge z) \vee (a \wedge \neg z)$ for each $a \in L$.
2. If L is complete then, $Z(L)$ is a complete lattice and for each family (z_i) in $Z(L)$ and $a \in L$, $a \wedge \bigvee z_i = \bigvee (a \wedge z_i)$. \square

Now we recall from [11], [19] and [20] some notions about Baer \star -semigroups. A \star -semigroup is an algebra $\langle G, \cdot, \star, 0 \rangle$ of type $\langle 2, 1, 0 \rangle$ that satisfies the following equations:

1. $\langle G, \cdot \rangle$ is a semigroup
2. $0 \cdot x = x \cdot 0 = 0$,
3. $(x \cdot y)^\star = y^\star \cdot x^\star$,
4. $x^{\star\star} = x$.

Let G be a \star -semigroup. An element $e \in G$ is a *projection* iff $e = e^\star = e \cdot e$. The set of all projections of G is denoted by $P(G)$. Let M be a non empty subset of G . If $x \in G$ we define $x \cdot M = \{x \cdot m \in G : m \in M\}$ and $M \cdot x = \{m \cdot x \in G : m \in M\}$. Moreover x is said to be a *left annihilator* of M iff $x \cdot M = \{0\}$ and it is said to be a *right annihilator* of M iff $M \cdot x = \{0\}$. We denote by M^l the set of left annihilators of M and by M^r

the set of right annihilators of M . A \star -semigroup is called *Baer \star -semigroup* iff for each $x \in G$ there exists $e \in G$ such that

$$\{x\}^r = \{y \in G : x \cdot y = 0\} = e \cdot G$$

We do not assume in general that any $e \in P(G)$ can be represented as $\{x\}^r = e \cdot G$ for some $x \in G$. Thus we say that $e \in P(G)$ is a *closed projection* iff there exists $x \in G$ such that $\{x\}^r = e \cdot G$. The set of all closed projections is denoted by $P_c(G)$. Let G be an orthomodular frame. From [19, Lemma 37.2], for each $x \in G$ there exists a unique projection $e_x \in P(G)$ such that $\{x\}^r = e_x \cdot G$. We denote this e_x by x' . Moreover $0'$ is denoted as 1. We can define a partial order $\langle P(G), \leq \rangle$ as follows:

$$e \leq f \iff e \cdot f = e$$

Proposition 1.3 *Let G be a Baer \star -semigroup. For any $e_1, e_2 \in P_c(G)$, we have that:*

1. $e \leq f$ iff $e \cdot G \subseteq f \cdot G$ [19, Theorem 37.2]
2. $x \cdot 1 = 1 \cdot x = x$ [19, Theorem 37.4] □

Theorem 1.4 [19, Theorem 37.8] *Let G be a Baer \star -semigroup. For any $e_1, e_2 \in P_c(G)$, we define the following operation:*

1. $e_1 \wedge e_2 = e_1 \cdot (e_2' \cdot e_1)'$,
2. $e_1 \vee e_2 = (e_1' \wedge e_2')'$.

then $\langle P_c(G), \wedge, \vee, ', 0, 1 \rangle$ is an orthomodular lattice with respect to the order $\langle P(G), \leq \rangle$. □

On the other hand we can build a Baer \star -semigroup from a partial ordered set. Let $\langle A, \leq \rangle$ be a partial ordered set. If $\varphi : A \rightarrow A$ is an order homomorphism then, a *residual map* for φ is an order homomorphism $\varphi^\natural : A \rightarrow A$ such that $(\varphi \circ \varphi^\natural)(x) \leq x \leq (\varphi^\natural \circ \varphi)(x)$ where \circ is the composition of order-homomorphisms. We denote by $G(A)$ the set of order-homomorphisms in A admitting residual maps. If we consider the constant order-homomorphism θ , given by $\theta(x) = 0$, then $\theta \in G(A)$ and $\langle G(A), \circ, \theta \rangle$ is a semigroup.

Theorem 1.5 [11, Theorem 8] *Let A be an orthomodular lattice and we consider the semigroup $\langle G(A), \circ, \theta \rangle$. If for each $\varphi \in G(A)$ we define φ^\star as $\varphi^\star(x) = \neg \varphi^\natural(\neg x)$ then we have that:*

1. $\langle G(A), \circ, \star, \theta \rangle$ is a Baer \star -semigroup,
2. if we define $\mu_a(x) = (x \vee \neg a) \wedge a$ for each $a \in A$ then $P_c(G(A)) = \{\mu_a : a \in A\}$,
3. $f : A \rightarrow P_c(G(A))$ such that $f(a) = \mu_a$ is a \mathcal{OML} -homomorphism. \square

2 Physical motivation of the modally enriched orthomodular structure

In the usual terms of quantum logic, a property of a system is related to a subspace of the Hilbert space \mathcal{H} of its (pure) states or, analogously, to the projector operator onto that subspace. A physical magnitude \mathcal{M} is represented by an operator \mathbf{M} acting over the state space. For bounded self-adjoint operators, conditions for the existence of the spectral decomposition $\mathbf{M} = \sum_i a_i \mathbf{P}_i = \sum_i a_i |a_i\rangle\langle a_i|$ are satisfied. The real numbers a_i are related to the outcomes of measurements of the magnitude \mathcal{M} and projectors $|a_i\rangle\langle a_i|$ to the mentioned properties. Thus, the physical properties of the system are organized in the lattice of closed subspaces $\mathcal{L}(\mathcal{H})$. Moreover, each self-adjoint operator \mathbf{M} has associated a Boolean sublattice W_M of $L(\mathcal{H})$ which we will refer to as the spectral algebra of the operator \mathbf{M} .

Assigning values to a physical quantity \mathcal{M} is equivalent to establishing a Boolean homomorphism $v : W_M \rightarrow \mathbf{2}$, being $\mathbf{2}$ the two elements Boolean algebra. Thus we can say that it makes sense to use the “classical discourse” –this is, the classical logical laws are valid– within the context given by \mathcal{M} .

One may define a *global valuation* of the physical magnitudes over $\mathcal{L}(\mathcal{H})$ as a family of Boolean homomorphisms $(v_i : W_i \rightarrow \mathbf{2})_{i \in I}$ such that $v_i | W_i \cap W_j = v_j | W_i \cap W_j$ for each $i, j \in I$, being $(W_i)_{i \in I}$ the family of Boolean sublattices of $\mathcal{L}(\mathcal{H})$. This global valuation would give the values of all magnitudes at the same time maintaining a *compatibility condition* in the sense that whenever two magnitudes shear one or more projectors, the values assigned to those projectors are the same from every context. As we have proved in [8], the KS theorem in the algebraic terms of the previous definition rules out this possibility:

Theorem 2.1 *If \mathcal{H} is a Hilbert space such that $\dim(\mathcal{H}) > 2$, then a global valuation over $\mathcal{L}(\mathcal{H})$ is not possible. \square*

This impossibility to assign values to the properties at the same time satisfying compatibility conditions is a weighty obstacle for the interpretation

of the formalism. B. van Fraassen was the first one to formally include the reasoning of modal logic to circumvent these difficulties presenting a modal interpretation of quantum logic in terms of its semantical analysis [22]. In our case, the modal component was introduced with different purposes: to provide a rigorous framework for the Born rule and mainly, to discuss the restrictions posed by the KS theorem to modalities [9].

To do so we enriched the orthomodular structure with a modal operator taking into account the following considerations:

1) Propositions about the properties of the physical system are interpreted in the orthomodular lattice of closed subspaces of the Hilbert space of the (pure) states of the system. Thus we will retain this structure in our extension.

2) Given a proposition about the system, it is possible to define a context from which one can predicate with certainty about it together with a set of propositions that are compatible with it and, at the same time, predicate probabilities about the other ones (Born rule). In other words, one may predicate truth or falsity of all possibilities at the same time, i.e. possibilities allow an interpretation in a Boolean algebra. In rigorous terms, for each proposition P , if we refer with $\diamond P$ to the possibility of P , then $\diamond P$ will be a central element of a orthomodular structure.

3) If P is a proposition about the system and P occurs, then it is trivially possible that P occurs. This is expressed as $P \leq \diamond P$.

4) Assuming an actual property and a complete set of properties that are compatible with it determines a context in which the classical discourse holds. Classical consequences that are compatible with it, for example probability assignments to the actuality of other propositions, shear the classical frame. These consequences are the same ones as those which would be obtained by considering the original actual property as a possible one. This is interpreted in the following way: if P is a property of the system, $\diamond P$ is the smallest central element greater than P .

From consideration 1) it follows that the original orthomodular structure is maintained. The other considerations are satisfied if we consider a modal operator \diamond over an orthomodular lattice L defined as $\diamond a = \text{Min}\{z \in Z(L) : a \leq z\}$ with $Z(L)$ the center of L under the assumption that this minimum exists for every $a \in L$. In the following section we explicitly show our construction. For technical reasons this algebraic study will be performed using the necessity operator \square instead of the possibility operator \diamond . As usual, it will be then possible to define the possibility operator from the necessity operator.

3 Orthomodular structures and modality

Definition 3.1 Let A be an orthomodular lattice. We say that A is *Boolean saturated* if and only if for all $a \in A$ the set $\{z \in Z(A) : z \leq a\}$ has a maximum. In this case such maximum is denoted by $\Box(a)$.

Example 3.2 In view of Proposition 1.2, orthomodular complete lattices considering $e(a) = \bigvee\{z \in Z(L) : z \leq a\}$ as operator, are examples of boolean saturated orthomodular lattices.

Proposition 3.3 *Let A be an orthomodular lattice. Then A is boolean saturated iff there exists an unary operator \Box satisfying*

$$S1 \quad \Box x \leq x$$

$$S2 \quad \Box 1 = 1$$

$$S3 \quad \Box \Box x = \Box x$$

$$S4 \quad \Box(x \wedge y) = \Box(x) \wedge \Box(y)$$

$$S5 \quad y = (y \wedge \Box x) \vee (y \wedge \neg \Box x)$$

$$S6 \quad \Box(x \vee \Box y) = \Box x \vee \Box y$$

$$S7 \quad \Box(\neg x \vee (y \wedge x)) \leq \neg \Box x \vee \Box y$$

Proof: Suppose that A is Boolean saturated. S1), S2) and S3) are trivial. S4) Since $x \wedge y \leq x$ and $x \wedge y \leq y$ then $\Box(x \wedge y) \leq \Box(x) \wedge \Box(y)$. For the converse, $\Box(x) \leq x$ and $\Box(y) \leq y$, thus $\Box(x) \wedge \Box(y) \leq \Box(x \wedge y)$. S5) Follows from Proposition 1.2 since $\Box(x) \in Z(A)$. S6) For simplicity, let $z = \Box y$. From the precedent item and taking into account that $z \in Z(L)$ we have that $\Box(z \vee x) \wedge \Box(\neg z \vee x) = \Box((z \vee x) \wedge (\neg z \vee x)) = \Box(x)$. Since $\neg z \leq \Box(\neg z \vee x)$ then we have that $1 = z \vee \neg z \leq z \vee \Box(\neg z \vee x)$. Also we have $z \leq \Box(z \vee x)$. Finally $z \vee \Box(x) = (z \vee \Box(z \vee x)) \wedge (z \vee \Box(\neg z \vee x)) = (z \vee \Box(z \vee x)) \wedge 1 = \Box(z \vee x)$ i.e. $\Box(x \vee \Box y) = \Box x \vee \Box y$. S7) Since $\Box(x) \leq x$ then $\neg x \leq \neg \Box x$, we have that $\neg x \vee (y \wedge x) \leq \neg \Box x \vee y$. Using the precedent item $\Box(\neg x \vee (y \wedge x)) \leq \Box(\neg \Box x \vee y) = \neg \Box x \vee \Box y$ since $\neg \Box x \in Z(A)$.

For the converse, let $a \in A$ and $\{z \in Z(A) : z \leq a\}$. By S1 and S5 it is clear that $\Box a \in \{z \in Z(A) : z \leq a\}$. We see that $\Box a$ is the upper bound of the set. Let $z \in Z(A)$ such that $z \leq a$ then $1 = \neg z \vee (a \wedge z)$. Using S2

and $S7$ we have $1 = \Box 1 = \Box(\neg z \vee (a \wedge z)) \leq \neg \Box z \vee a = \neg z \vee a$. Therefore $z = z \wedge (\neg z \vee \Box a)$ and since z is central $z = z \wedge \Box a$ resulting $z \leq \Box a$. Finally $\Box a = \text{Max}\{z \in Z(A) : z \leq a\}$. \square

Note that the operator \Box is an example of quantifier in the sense of Janowitz [15].

Theorem 3.4 *The class of Boolean saturated orthomodular lattices constitutes a variety which is axiomatized by*

1. *Axioms of \mathcal{OML} ,*
2. *$S1, \dots, S7$.*

Proof: Obvious by Proposition 3.3 \square

Boolean saturated orthomodular lattices are algebras $\langle A, \wedge, \vee, \neg, \Box, 0, 1 \rangle$ of type $\langle 2, 2, 1, 1, 0, 0 \rangle$. The variety of this algebras is noted as \mathcal{OML}^\Box . By simplicity, the set $\text{Term}_{\mathcal{OML}^\Box}$ will be denoted by Term^\Box . Since \mathcal{OML} is a reduct of \mathcal{OML}^\Box we can also assume that all \mathcal{OML}^\Box -equations are of the form $t = 1$. It is well known that \mathcal{OML} is congruence distributive and congruence permutable. Therefore if $A \in \mathcal{OML}^\Box$ and we consider the OML-reduct of A it is clear that $\text{Con}_{\mathcal{OML}^\Box}(A) \subseteq \text{Con}_{\mathcal{OML}}(A)$ resulting A congruence distributive and congruence permutable in the sense of \mathcal{OML}^\Box -congruences. Hence the variety \mathcal{OML}^\Box is congruence distributive and congruence permutable. The following lemma gives basic properties that will be used later:

Lemma 3.5 *Let $A \in \mathcal{OML}^\Box$ and $a, b \in A$ and $z_1, z_2 \in Z(A)$. Then we have:*

1. $\neg \Box a \vee a = 1$,
2. $\neg(a \vee \neg b) \vee (a \vee \neg \Box b) = 1$,
3. $\neg(\neg z_1 \vee z_2) \vee ((\neg(z_1 \vee a) \vee (z_2 \vee a)) = 1$
4. $\Box a \vee \Box b \leq \Box(a \vee b)$,
5. $(\neg \Box a \wedge \neg \Box b) \vee \Box(a \vee b) = 1$,
6. *if $x \leq y$ then $\Box x \leq \Box y$.*

Proof: 1) Since $\Box a \leq a$ then $\neg a \leq \neg \Box a$ and $1 = a \vee \neg a \leq a \vee \neg \Box a$.
2) Since $\neg \Box b \in Z(A)$ and by item 1 we have that $\neg(a \vee \neg b) \vee (a \vee \neg \Box b) = (\neg a \wedge b) \vee (a \vee \neg \Box b) = ((\neg a \vee \neg \Box b) \wedge (b \vee \neg \Box b)) \vee a = ((\neg a \vee \neg \Box b) \wedge 1) \vee a = 1$.
3) $\neg(\neg z_1 \vee z_2) \vee ((\neg(z_1 \vee a) \vee (z_2 \vee a)) \vee (\neg(z_1 \vee z_2) \wedge (z_1 \vee a))) \vee (z_2 \vee a) = \neg((\neg z_1 \wedge a) \vee (z_1 \wedge z_2) \vee (z_2 \wedge a)) \vee (z_2 \vee a) = ((z_1 \vee \neg a) \wedge (\neg z_1 \vee \neg z_2) \wedge (\neg z_2 \vee \neg a)) \vee (z_2 \vee a) = ((z_1 \vee \neg a \vee z_2) \wedge (\neg z_2 \vee \neg x \vee z_2) \wedge (\neg z_2 \vee \neg a \vee z_2)) \vee a = z_1 \vee \neg a \vee z_2 \vee a = 1$. 4) $\Box a \leq a$ and $\Box b \leq b$, $\Box a \vee \Box b \leq a \vee b$. Since $\Box a \vee \Box b \in Z(A)$ it is clear that $\Box a \vee \Box b \leq \Box(a \vee b)$. 5) Immediately from item 4. 6) Suppose that $x \leq y$ then $x = x \wedge y$. By Axiom S4 we have that $\Box x = \Box(x \wedge y) = \Box x \wedge \Box y$, hence $\Box x \leq \Box y$. \square

Lemma 3.6 *Let $A \in \mathcal{OML}^\Box$ and $z \in Z(A)$. Then the binary relation Θ_z on A defined by $a\Theta_z b$ iff $a \wedge z = b \wedge z$ is a congruence on A , such that $A \cong A/\Theta_z \times A/\Theta_{\neg z}$.*

Proof: It is well known that Θ_z is a \mathcal{OML} -congruence and A is \mathcal{OML} -isomorphic to $A/\Theta_z \times A/\Theta_{\neg z}$. Therefore we only need to see the \Box -compatibility. In fact: suppose that $a\Theta_z b$ then $a \wedge z = b \wedge z$. Therefore $\Box(a) \wedge z = \Box(a) \wedge \Box(z) = \Box(a \wedge z) = \Box(b \wedge z) = \Box(b) \wedge \Box(z) = \Box(b) \wedge z$. Hence $\Box(a)\Theta_z \Box(b)$. \square

Proposition 3.7 *Let $A \in \mathcal{OML}^\Box$ then we have that:*

1. *The map $z \rightarrow \Theta_z$ is a lattice isomorphism between $Z(L)$ and the Boolean subalgebra of $\text{Con}(L)$ of factor congruences.*
2. *A is directly indecomposable iff $Z(A) = \{0, 1\}$.*

Proof: 1) Follows from Lemma 3.6 using the same arguments that prove the analog result for orthomodular lattices [2, Proposition 5.2]. 2) Follows from item 1. \square

Proposition 3.8 *Let A be a directly indecomposable \mathcal{OML}^\Box -algebra. Then*

$$t(x, y, z) = (x \wedge \neg \Box((x \wedge y) \vee (\neg x \wedge \neg y))) \vee (z \wedge \Box((x \wedge y) \vee (\neg x \wedge \neg y)))$$

is a discriminator term for A .

Proof: By Proposition 3.7, $Z(A) = \{0, 1\}$. Therefore for each $a \in A - \{1\}$, $\Box(a) = 0$. Let $a, b, c \in A$. Suppose that $a \neq b$. By the characterization of the equations in \mathcal{OML}^\Box we have that $(a \wedge b) \vee (\neg a \wedge \neg b) \neq 1$ and then $t(a, b, c) = a$. If we suppose that $a = b$ then it is clear that $t(a, b, c) = c$. Hence $t(x, y, z)$ is a discriminator term for A . \square

Proposition 3.9 *If \mathcal{A} is a subvariety of \mathcal{OML}^\square then \mathcal{A} is a discriminator variety.*

Proof: Let $SI_{\mathcal{A}}$ be the class of subdirectly irreducible algebras of \mathcal{A} . Each algebra of $SI_{\mathcal{A}}$ is directly indecomposable. Therefore by Proposition 3.8 $SI_{\mathcal{A}}$ admit a common discriminator term. Since $\mathcal{A} = \mathcal{V}(SI_{\mathcal{A}})$ we have that \mathcal{A} is a discriminator variety. \square

4 Hilbert-style calculus for \mathcal{OML}^\square

In this section we build a Hilbert-style calculus $\langle Term^\square, \vdash \rangle$ for \mathcal{OML}^\square . We first introduce some notation. $\alpha \in Term^\square$ is a *tautology* iff $\models_{\mathcal{OML}^\square} \alpha = 1$. Each subset T of $Term^\square$ is referred as *theory*. If v is a valuation, $v(T) = 1$ means that $v(\gamma) = 1$ for each $\gamma \in T$. We use $T \models_{\mathcal{OML}^\square} \alpha$ (read α is *semantic consequence* of T) in the case in which when $v(T) = 1$ then $v(\alpha) = 1$ for each valuation v .

Lemma 4.1 *Let γ and $\alpha \in Term^\square$. Then we have*

1. *If v is a valuation then $v(\alpha) = 1$ iff $v(\Box\alpha) = 1$.*
2. *$\gamma \models_{\mathcal{OML}^\square} \alpha$ iff $\gamma \models_{\mathcal{OML}^\square} \Box\alpha$ iff $\Box\gamma \models_{\mathcal{OML}^\square} \alpha$ iff $\Box\gamma \models_{\mathcal{OML}^\square} \Box\alpha$.*

Proof: 1) If $v(\alpha) = 1$ then $1 = \Box 1 = \Box(v(\alpha)) = v(\Box\alpha)$. The converse follows from the fact $1 = v(\Box\alpha) = \Box(v(\alpha)) \leq v(\alpha)$. 2) Immediate from the item 1. \square

Definition 4.2 Consider by definition the following binary connective

$$\alpha R \beta \text{ for } (\alpha \wedge \beta) \vee (\neg\alpha \wedge \neg\beta)$$

The calculus $\langle Term^\square, \vdash \rangle$ is given by the following axioms:

- A0 $1R(\alpha \vee \neg\alpha)$ and $0R(\alpha \wedge \neg\alpha)$,
- A1 $\alpha R\alpha$,
- A2 $\neg(\alpha R\beta) \vee (\neg(\beta R\gamma) \vee (\alpha R\gamma))$,
- A3 $\neg(\alpha R\beta) \vee (\neg\alpha R\neg\beta)$,
- A4 $\neg(\alpha R\beta) \vee ((\alpha \wedge \gamma)R(\beta \wedge \gamma))$,

- A5 $(\alpha \wedge \beta)R(\beta \wedge \alpha)$,
- A6 $(\alpha \wedge (\beta \wedge \gamma))R((\alpha \wedge \beta) \wedge \gamma)$,
- A7 $(\alpha \wedge (\alpha \vee \beta))R\alpha$,
- A8 $(\neg\alpha \wedge \alpha)R((\neg\alpha \wedge \alpha) \wedge \beta)$,
- A9 $\alpha R\neg\neg\alpha$,
- A10 $\neg(\alpha \vee \beta)R(\neg\alpha \wedge \neg\beta)$,
- A11 $(\alpha \vee (\neg\alpha \wedge (\alpha \vee \beta)))R(\alpha \vee \beta)$,
- A12 $(\alpha R\beta)R(\beta R\alpha)$,
- A13 $\neg(\alpha R\beta) \vee (\neg\alpha \vee \beta)$,
- A14 $(\Box\alpha \vee \alpha)R\alpha$,
- A15 $\Box(\alpha \vee \neg\alpha)R(\alpha \vee \neg\alpha)$,
- A16 $\Box\Box\alpha R\Box\alpha$,
- A17 $\Box(\alpha \wedge \beta)R(\Box\alpha \wedge \Box\beta)$,
- A18 $((\alpha \wedge \Box\beta) \vee (\alpha \wedge \neg\Box\beta))R\alpha$,
- A19 $\Box(\alpha \vee \neg\Box\beta)R(\Box\alpha \vee \neg\Box\beta)$,
- A20 $\Box(\alpha \vee \Box\beta)R(\Box\alpha \vee \Box\beta)$,
- A21 $(\Box(\neg\alpha \vee (\beta \wedge \alpha)) \vee (\neg\Box\alpha \vee \Box\beta))R(\neg\Box\alpha \vee \Box\beta)$,
- A22 $\neg(\alpha \vee \neg\beta) \vee (\alpha \vee \neg\Box\beta)$,
- A23 $\neg(\gamma \vee \neg\beta) \vee (\neg(\beta \vee \alpha) \vee (\gamma \vee \alpha))$,
- A24 $\Box(\alpha \vee \beta) \vee (\neg\Box\alpha \wedge \neg\Box\beta)$.

and the following inference rules:

$$\frac{\alpha, \neg\alpha \vee \beta}{\beta} \quad \text{disjunctive syllogism (DS)}$$

$$\frac{\alpha}{\Box\alpha}, \quad \text{necessitation (N)}$$

Let T be a theory. A *proof* from T is a sequence $\alpha_1, \dots, \alpha_n$ in $Term^\square$ such that each member is either an axiom or a member of T or follows from some preceding member of the sequence using *DS* or *N*. $T \vdash \alpha$ means that α is provable in T , that is, α is the last element of a proof from T . If $T = \emptyset$, we use the notation $\vdash \alpha$ and in this case we will say that α is a theorem of $\langle Term^\square, \vdash \rangle$. T is *inconsistent* if and only if $T \vdash \alpha$ for each $\alpha \in Term^\square$; otherwise it is *consistent*.

Proposition 4.3 *Let T be a theory and $\alpha, \beta, \gamma \in Term^\square$. Then we have*

1. $T \vdash \alpha R \beta \implies T \vdash \beta R \alpha$
2. $T \vdash \alpha R \beta$ and $T \vdash \beta R \gamma \implies T \vdash \alpha R \gamma$
3. $T \vdash \alpha R \beta \implies T \vdash \neg \alpha R \neg \beta$
4. $T \vdash \alpha R \beta$ and $T \vdash \alpha \wedge \gamma \implies T \vdash \beta \wedge \gamma$
5. $T \vdash \alpha R \beta$ and $T \vdash \alpha \vee \gamma \implies T \vdash \beta \vee \gamma$
6. $T \vdash \alpha R \beta \implies T \vdash \Box \alpha R \Box \beta$
7. $\vdash \alpha \vee \neg \alpha$
8. $T \vdash \alpha \implies T \vdash \alpha \vee \beta$

Proof:

1)

- (1) $T \vdash \alpha R \beta$
- (2) $T \vdash (\alpha R \beta) R (\beta R \alpha)$ by A12
- (3) $T \vdash \neg((\alpha R \beta) R (\beta R \alpha)) \vee (\neg(\alpha R \beta) \vee (\beta R \alpha))$ by A13
- (4) $T \vdash (\neg(\alpha R \beta) \vee (\beta R \alpha))$ by DS 2,2
- (5) $T \vdash \beta R \alpha$ by DS 1,4

2) Is easily from A2 and two application of the *DS*.

3) Follows from A3.

4)

- (1) $T \vdash \alpha R \beta$

- (2) $T \vdash \alpha \wedge \gamma$
- (3) $T \vdash \neg(\alpha R \beta) \vee ((\alpha \wedge \gamma)R(\beta \wedge \gamma))$ by A4
- (4) $T \vdash (\alpha \wedge \gamma)R(\beta \wedge \gamma)$ by DS 1,2
- (5) $\neg(\alpha \wedge \gamma)R(\beta \wedge \gamma) \vee (\neg(\alpha \wedge \gamma) \vee (\beta \wedge \gamma))$ by A4
- (6) $T \vdash \beta \wedge \gamma$ by DS 5,4,2

5) Follows by item 4, A9 and A10.

6)

- (1) $T \vdash \alpha R \beta$
- (2) $T \vdash (\alpha \wedge \beta) \vee (\neg \alpha \wedge \neg \beta)$ equiv 1
- (3) $T \vdash (\alpha \wedge \beta) \vee \neg(\alpha \vee \beta)$ by item 5 and A10
- (4) $\vdash \neg((\alpha \wedge \beta) \vee \neg(\alpha \vee \beta)) \vee ((\alpha \wedge \beta) \vee \neg \Box(\alpha \vee \beta))$ by A22
- (5) $T \vdash (\alpha \wedge \beta) \vee \neg \Box(\alpha \vee \beta)$ by DS 4,3
- (6) $T \vdash \Box((\alpha \wedge \beta) \vee \neg \Box(\alpha \vee \beta))$ by DS 4,3
- (7) $T \vdash \Box(\alpha \wedge \beta) \vee \neg \Box(\alpha \vee \beta)$ by A13, A19
- (8) $T \vdash (\Box \alpha \wedge \Box \beta) \vee \neg \Box(\alpha \vee \beta)$ by item 5 and A17
- (9) $\neg((\Box \alpha \wedge \Box \beta) \vee \neg \Box(\alpha \vee \beta)) \vee (\neg(\Box(\alpha \vee \beta) \vee (\neg \Box \alpha \wedge \neg \Box \beta)) \vee ((\Box \alpha \wedge \Box \beta) \vee (\neg \Box \alpha \wedge \neg \Box \beta)))$ by A23
- (10) $\Box(\alpha \vee \beta) \vee (\neg \Box \alpha \wedge \neg \Box \beta)$ by A24
- (11) $(\Box \alpha \wedge \Box \beta) \vee (\neg \Box \alpha \vee \neg \Box \beta)$ by SD 8,9,10
- (12) $T \vdash \Box \alpha R \Box \beta$ equiv 1

7) Follows from A1 and A13.

8)

- (1) $T \vdash \alpha$
- (2) $T \vdash (\alpha \vee \neg \alpha)R((\alpha \vee \neg \alpha) \vee \beta)$ by A3, A8, A10
- (3) $T \vdash (\alpha \vee \neg \alpha) \vee \beta$ by item 7 and A13

- (4) $\vdash \neg((\alpha \wedge \beta) \vee \neg(\alpha \vee \beta)) \vee ((\alpha \wedge \beta) \vee \neg\Box(\alpha \vee \beta))$ by A22
- (5) $\vdash \neg\alpha \vee (\alpha \vee \beta)$ by 4, A5, A3, A10
- (6) $\vdash \alpha \vee \beta$ by DS 1, 5 \square

Proposition 4.4 *Axioms of the $\langle Term^\square, \vdash \rangle$ are tautologies.*

Proof: For A0... A13 see ([17, Chapter 4.15]). A22...24 follow from Proposition 3.5. \square

Theorem 4.5 *Let T be a theory. If for each $\alpha \in Term^\square$ we consider the set $[\alpha] = \{\beta : T \vdash \alpha R \beta\}$ then $L_T = \{[\alpha] : \alpha \in Term^\square\}$ determines a partition in equivalence classes of $Term^\square$. Defining the following operation in L_T*

$$\begin{aligned} [\alpha] \wedge [\beta] &= [\alpha \wedge \beta] & \neg[\alpha] &= [\neg\alpha] & 0 &= [0] \\ [\alpha] \vee [\beta] &= [\alpha \vee \beta] & \Box[\alpha] &= [\Box\alpha] & 1 &= [1] \end{aligned}$$

then we have

1. $\langle L_T, \vee, \wedge, \neg, \Box, 0, 1 \rangle$ is a Boolean saturated orthomodular lattice.
2. $T \vdash \alpha$ if and only if $[\alpha] = 1$

Proof: 1) By A1 and Proposition 4.3 (item 1 and 2) $L_T = \{[\alpha] : \alpha \in Term^\square\}$ is a partition in equivalence classes of $Term^\square$. By Proposition 4.3 (item 3 and 6) \vee, \wedge, \neg, \Box are well defined in L_T . By A0...A13 and ([17, Proposition 4.15. 1]) L_T is an orthomodular lattice. By A14...A21 and Proposition 4.4, L_T is boolean saturated. 2) Assume that $T \vdash \alpha$, then we have that:

- (1) $T \vdash \alpha$
- (2) $T \vdash \alpha R(\alpha \wedge (\alpha \vee \neg\alpha))$ by A7
- (3) $T \vdash \alpha \wedge (\alpha \vee \neg\alpha)$ by 1 and A13
- (4) $T \vdash (\alpha \wedge (\alpha \vee \neg\alpha)) \vee (\neg\alpha \wedge \neg(\alpha \vee \neg\alpha))$ by 3 and Prop. 4.3 8
- (5) $T \vdash \alpha R(\alpha \vee \neg\alpha)$ equiv in 4

resulting $[\alpha] = 1$. On the other hand, if $[\alpha] = 1$, we have that $T \vdash \alpha R(\alpha \vee \neg\alpha)$. Using Proposition 4.3 7 and Ax13, it results $T \vdash \alpha$. \square

The following theorem establishes the strong completeness for $\langle Term^\square, \vdash \rangle$ with respect to the variety \mathcal{OML}^\square .

Theorem 4.6 *Let $\alpha \in Term^\square$ and T be a theory. Then we have that:*

$$T \vdash \alpha \iff T \models_{\mathcal{OML}^\square} \alpha$$

Proof: If T is inconsistent, this result is trivial. Assume that T is consistent. \implies) Immediate. \impliedby) Suppose that T does not prove α . Then, by Proposition 4.5, $[\alpha] \neq 1$. Then the projection $p : Term^\square \rightarrow L_T$ with $p(\varphi) = [\varphi]$ is a valuation such that $p(\varphi) = 1$ for each $\varphi \in T$ and $p(\alpha) \neq 1$. Finally we have that not $T \models_{\mathcal{OML}^\square} \alpha$. \square

Corollary 4.7 (Compactness) *Let $\alpha \in Term^\square$ and T be a theory. Then we have that, $T \models_{\mathcal{OML}^\square} \alpha$ iff there exists a finite subset $T_0 \subseteq T$ such that $T_0 \models_{\mathcal{OML}^\square} \alpha$.*

Proof: In view of Theorem 4.6, if $T \models_{\mathcal{OML}^\square} \alpha$ then $T \vdash \alpha$. If $\varphi_1, \dots, \varphi_m, \alpha$ is a proof of α from T , we can consider the finite set $T_0 = \{\varphi_i \in T : \varphi_i \in \{\alpha_1, \dots, \alpha_n\}\}$. Using again Theorem 4.6 we have $T_0 \models_{\mathcal{OML}^\square} \alpha$. \square

We can also establish a kind of deduction theorem.

Corollary 4.8 *Let $\gamma, \alpha \in Term^\square$ and T be a theory. Then we have that:*

$$T \cup \{\gamma\} \vdash \alpha \text{ iff } T \vdash \neg\Box\gamma \vee \alpha$$

Proof: By Theorem 4.6 we will prove that $T \cup \{\gamma\} \models_{\mathcal{OML}^\square} \alpha$ iff $T \models_{\mathcal{OML}^\square} \neg\Box\gamma \vee \alpha$. By Corollary 4.7 $T \cup \{\gamma\} \models_{\mathcal{OML}^\square} \alpha$ iff there exists $\varphi_1 \dots \varphi_n \in T$ such that $(\varphi_1 \wedge \dots \wedge \varphi_n) \wedge \gamma \models_{\mathcal{OML}^\square} \alpha$. Let $\varphi = \varphi_1 \wedge \dots \wedge \varphi_n$. Then $\varphi \wedge \gamma \models_{\mathcal{OML}^\square} \alpha$ implies that $(\varphi \wedge \gamma) \vee \neg\Box(\gamma) \models_{\mathcal{OML}^\square} \neg\Box\gamma \vee \alpha$ and then $\varphi \vee \neg\Box\gamma \models_{\mathcal{OML}^\square} \neg\Box\gamma \vee \alpha$ since for each valuation v , $v(\Box\gamma)$ is a central element and $v(\gamma \vee \neg\Box\gamma) = 1$. Since $\varphi \models_{\mathcal{OML}^\square} \varphi \vee \neg\Box\gamma$ we have that $\varphi \models_{\mathcal{OML}^\square} \Box\gamma \vee \alpha$ thus $T \models_{\mathcal{OML}^\square} \neg\Box\gamma \vee \alpha$.

On the other hand, if $T \models_{\mathcal{OML}^\square} \neg\Box\gamma \vee \alpha$ we can consider again $\varphi = \varphi_1 \wedge \dots \wedge \varphi_n$ such that $\varphi_1 \dots \varphi_n \in T$ and $\varphi \models_{\mathcal{OML}^\square} \neg\Box\gamma \vee \alpha$. Therefore $\varphi \wedge \Box\gamma \models_{\mathcal{OML}^\square} \Box\gamma \wedge (\neg\Box\gamma \vee \alpha)$ and then $\varphi \wedge \Box\gamma \models_{\mathcal{OML}^\square} \Box\gamma \wedge \alpha$ since for

each valuation v , $v(\Box\gamma \wedge (\neg\Box\gamma \vee \alpha)) = v(\Box\gamma \wedge \alpha)$ taking into account that $v(\Box\alpha)$ is always a central element. Since $\Box\gamma \wedge \alpha \models_{\mathcal{OML}^\square} \alpha$ we have that $\varphi \wedge \Box\gamma \models_{\mathcal{OML}^\square} \alpha$. Applying Lemma 4.1 we have that $\Box(\varphi \wedge \Box\gamma) \models_{\mathcal{OML}^\square} \alpha$ hence $\Box\varphi \wedge \Box\gamma \models_{\mathcal{OML}^\square} \alpha$ and $\varphi \wedge \gamma \models_{\mathcal{OML}^\square} \alpha$ in view of Axiom $S4$ of \mathcal{OML}^\square . Thus $T \cup \{\gamma\} \models_{\mathcal{OML}^\square} \alpha$. \square

5 Modal orthomodular frames and Kripke-style semantics

In order to establish a Kripke-style semantics $\langle Term^\square, \vdash \rangle$ we first introduce the concept of modal Baer semigroups which constitute a sub-class of Baer \star -semigroups.

Definition 5.1 A *modal Baer semigroup* is a Baer \star -semigroup G such that $\langle P_c(G), \wedge, \vee, ', 0, 1 \rangle$ is a Boolean saturated orthomodular lattice. A *modal orthomodular frame* is a pair $\langle G, u \rangle$ such that G is a modal Baer semigroup and u is a valuation $u : Term^\square \rightarrow P_c(G)$

We denote by \mathcal{MOF} the class of all modal orthomodular frames. The following result is a representation theorem by modal Baer semigroups of Boolean saturated orthomodular lattices.

Theorem 5.2 Let $A \in \mathcal{OML}^\square$, then there exists a modal Baer semigroup $G(A)$ such that A is \mathcal{OML}^\square -isomorphic to $P_c(G(A))$.

Proof: Let $A \in \mathcal{OML}^\square$. By Theorem 1.5 there exists a Baer \star -semigroup G such that A is \mathcal{OML} -isomorphic to $P_c(G(A))$. Since \mathcal{OML} -isomorphisms preserve supremum of central elements we have that $P_c(G(A)) \in \mathcal{OML}^\square$ and then, $G(A) \in \mathcal{MBS}$. \square

Note that we can easily prove that $\models_{\mathcal{OML}^\square} t = 1$ iff for all modal Baer semigroups G we have that $\models_{P_c(G)} t = 1$.

Proposition 5.3 Let $\langle G, u \rangle$ be a modal orthomodular frame and $t, s \in Term^\square$. Then we have that:

1. $u(t \wedge s) \cdot G = (u(t) \cdot G) \cap (u(s) \cdot G)$,
2. $u(\neg t) \cdot G = \{x \in G : \forall y \in u(t) \cdot G, y^\star \cdot x = 0\}$

3. $u(\Box t) \cdot G = \bigcup \{x \cdot G : x \in Z(P_c(G)) \text{ and } x \leq v(t)\}$.

Proof: 1) Follows from an analogous argument used in [20, Theorem 3.13].
 2) See the proof of [20, Lemma 3.16-3]. 3) We first note that $u(t) \geq \Box u(t) = u(\Box t) \in Z(P_c(G))$. Thus $u(\Box t) \cdot G \in \{x \cdot G : x \in Z(P_c(G)) \text{ and } x \leq u(t)\}$ and then $u(t) \cdot G \subseteq \bigcup \{x \cdot G : x \in Z(P_c(G)) \text{ and } x \leq u(t)\}$. On the other hand, if $x \in Z(P_c(G))$ and $x \leq u(t)$ then $x \cdot G \subseteq u(\Box t) \cdot G$ since $x \leq u(\Box t)$. Hence $\bigcup \{x \cdot G : x \in Z(P_c(G)) \text{ and } x \leq u(t)\} \subseteq u(t) \cdot G$. \square

Definition 5.4 Let $\langle G, u \rangle$ be a modal orhomodular frame. Then we define inductively the *forcing relation* $\models_{\langle G, u \rangle}^x \subseteq G \times Term^\square$ as follows:

1. $\models_{\langle G, u \rangle}^x p$ iff $x \in u(p) \cdot G$, for each variable $p \in Term^\square$,
2. $\models_{\langle G, u \rangle}^x \alpha \wedge \beta$ iff $\models_{\langle G, u \rangle}^x \alpha$ and $\models_{\langle G, u \rangle}^x \beta$,
3. $\models_{\langle G, u \rangle}^x \neg \alpha$ iff $\forall g \in G, \models_{\langle G, u \rangle}^g \alpha \implies g' \cdot x = 0$,
4. $\models_{\langle G, u \rangle}^x \Box \alpha$ iff $x = z \cdot g$ such that $z \in Z(P_c(G))$ and $\models_{\langle G, u \rangle}^z \alpha$.

The relation $\models_{\langle G, u \rangle}^x \alpha$ is read as α is true at the point x in the modal orhomodular frame $\langle G, u \rangle$ and by $\models_{\langle G, u \rangle} \alpha$ we understand that for each $x \in G$, $\models_{\langle G, u \rangle}^x \alpha$. Generalizing, if T is a theory, $\models_{\langle G, u \rangle} T$ means that, for each $\beta \in T$ we have that $\models_{\langle G, u \rangle} \beta$. With these elements we can establish a notion of *consequence* in the Kripke-style sense that will be noted by $T \models_{\mathcal{MOF}} \alpha$.

$$T \models_{\mathcal{MOF}} \alpha \text{ iff } \forall \langle G, u \rangle \in \mathcal{MOF}, \models_{\langle G, u \rangle} T \implies \models_{\langle G, u \rangle} \alpha$$

Let $\alpha \in Term^\square$, T be a theory and $\langle G, u \rangle$ be an orhomodular frame. Then we consider the following sets:

$$\begin{aligned} |\alpha|_{\langle G, u \rangle} &= \{x \in G : \models_{\langle G, u \rangle}^x \alpha\} \\ |T|_{\langle G, u \rangle} &= \bigcap_{\beta \in T} |\beta|_{\langle G, u \rangle} \end{aligned}$$

Proposition 5.5 Let $\alpha \in Term^\square$, T be a theory and $\langle G, u \rangle$ be a modal orhomodular frame. Then we have that:

1. $|\alpha|_{\langle G, u \rangle} = u(\alpha) \cdot G$,

$$2. \models_{\langle G, u \rangle} T \quad \text{iff} \quad | T |_{\langle G, u \rangle} = G$$

Proof: 1) We use induction on the complexity of terms. If α is a variable the proposition results trivial. If α is $\beta \wedge \gamma$ or $\neg \beta$ we refer to [20, Lemma 3.16]. Suppose that α is $\Box \beta$. We prove that $u(\Box \beta) \cdot G \subseteq | \Box \beta |_{\langle G, u \rangle}$. By Proposition 1.3-1 and by inductive hypothesis, we have that $u(\Box \beta) \cdot G = \Box(u(\beta)) \cdot G \subseteq u(\beta) \cdot G = | \beta |_{\langle G, u \rangle}$. Then $\Box(u(\beta)) \cdot 1 = \Box(u(\beta)) \in | \beta |_{\langle G, u \rangle}$ i.e., $\models_{\langle G, u \rangle}^{\Box(u(\beta))} \beta$. Thus if $x \in u(\Box \beta) \cdot G$ then $x = \Box(u(\beta)) \cdot g$ and, taking into account that $\Box(u(\beta)) \in Z(P_c(G))$, it results that $x \in | \Box \beta |_{\langle G, u \rangle}$. On the other hand, if $x \in | \Box \beta |_{\langle G, u \rangle}$, then $x = z \cdot g$ such that $z \in Z(P_c(G))$ and $z \in | \beta |_{\langle G, u \rangle}$. Then by inductive hypothesis we have that $z = u(\beta) \cdot g_0 \leq u(\beta) \cdot 1 = u(\beta)$. By the lattice-order definition of $\Box(u(\beta))$ it is clear that $z \leq \Box(u(\beta)) = u(\Box(\beta))$. Therefore $z \cdot G \subseteq u(\Box(\beta)) \cdot G$. Hence $x \in u(\Box(\beta)) \cdot G$.

2) Using the above item, $\models_{\langle G, u \rangle} T$ iff $\forall \beta \in T, \models_{\langle G, u \rangle} \beta$ iff $\forall \beta \in T, | \beta |_{\langle G, u \rangle} = G$ iff $G = \bigcap_{\beta \in T} | \beta |_{\langle G, u \rangle} = | T |_{\langle G, u \rangle}$ \square

Theorem 5.6 [Kripke style completeness] *Let $\alpha \in Term^\Box$ and T be a theory. Then we have*

$$T \models_{\mathcal{OML}^\Box} \alpha \iff T \models_{\mathcal{MOF}} \alpha$$

Proof: Suppose that $T \models_{\mathcal{OML}^\Box} \alpha$. By Corollary 4.7 there exists $\gamma_1, \dots, \gamma_n \in T$ such that if we consider γ as $\gamma_1 \wedge \dots \wedge \gamma_n$ then $\gamma \models_{\mathcal{OML}^\Box} \alpha$. By Corollary 4.8 we have that $\models_{\mathcal{OML}^\Box} \neg \Box \gamma \vee \alpha$. Let $\langle G, u \rangle$ be a modal orthomodular frame such that $\models_{\langle G, u \rangle} T$. By Proposition 5.5 and Proposition 1.3-1 we have that $| \gamma |_{\langle G, u \rangle} = G$ and then $u(\gamma) = 1$. But $u(\gamma) = 1$ implies $u(\Box \gamma) = 1$ and $\neg u(\Box \gamma) = 0$. Therefore, necessarily $u(\alpha) = 1$, and $| \alpha |_{\langle G, u \rangle} = G$. Hence $\models_{\langle G, u \rangle} \alpha$.

On the other hand we assume that $T \models_{\mathcal{MOF}} \alpha$. Suppose that $T \not\models_{\mathcal{OML}^\Box} \alpha$. Then there exists $A \in \mathcal{OML}^\Box$ and a valuation $v : Term^\Box \rightarrow A$ such that $v(T) = 1$ but $v(\alpha) \neq 1$. By Theorem 5.3 there exists a modal Baer semigroup $G(A)$ such that A is \mathcal{OML}^\Box -isomorphic to $P_c(G(A))$ being $f : A \rightarrow P_c(G(A))$ such isomorphism. Consider the modal orthomodular frame $\langle G(A), fv \rangle$. Then for each $\beta \in T$ we have that $| \beta |_{\langle G(A), fv \rangle} = fv(\beta) \cdot G(A) = 1 \cdot G(A) = G(A)$. Therefore $| T |_{\langle G(A), fv \rangle} = G(A)$ and $\models_{\langle G(A), fv \rangle} T$ in view of Proposition 5.5. By Proposition 1.3-1 $| \alpha |_{\langle G(A), fv \rangle} = fv(\alpha) \cdot G(A) \neq G(A)$ again since $fv(\alpha) < 1$. Then $\not\models_{\langle G(A), fv \rangle} \alpha$ which is a contradiction. Hence $T \models_{\mathcal{OML}^\Box} \alpha$. \square

Conclusions

We have developed a logical system based on the orthomodular structure of propositions about quantum systems enriched with a modal operator. We have obtained algebraic completeness and completeness with respect to a Kripke-style semantic founded on Baer $*$ -semigroups. The importance of this structure from a physical perspective deals with the interpretation of quantum mechanics in terms of modalities.

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