TWO-VALUED STATES ON BAER *-SEMIGROUPS

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In this paper we develop an algebraic framework that allows us to extend families of two-valued states on orthomodular lattices to Baer *-semigroups. We apply this general approach to study the full class of two-valued states and the subclass of Jauch–Piron two-valued states on Baer *-semigroups.

Keywords: Baer *-semigroups, two-valued states, orthomodular lattices.

PACS numbers: 02.10 De.

1. Introduction

Recently, several authors have paid attention to the study of the concept of "state" by extending it to classes of algebras more general than the σ-algebras, as orthomodular posets [6, 26], MV-algebras [7, 15, 16, 22, 27] or effect algebras [9, 29, 30]. In the particular case of quantum mechanics (QM), different families of states are investigated not only because they provide different representations

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of the event structure of quantum systems [21, 32, 33] but also because of their importance in order to understand QM [11, 12, 24, 28].

In [5], a general theoretical framework to study families of two-valued states on orthomodular lattices is given. We shall use these ideas for a general study of two-valued states extended to Baer *-semigroups. Moreover, we investigate varieties of Baer *-semigroups expanded with a unary operation that allows us to capture the notion of two-valued states in an algebraic structure.

The paper is organized as follows: Section 2 contains generalities on universal algebra, orthomodular lattices, and Baer *-semigroups. In Section 3, motivations for a natural extension of the concept of two-valued state from orthomodular lattices to Baer *-semigroups are presented. In Section 4, we introduce the concept of IE_B^*-semigroup. It is presented as a Baer *-semigroup with a unary operation that enlarges the language of the structure. This operation is defined by equations giving rise to a variety denoted by $\mathcal{IE}_B^*$. In this way, $\mathcal{IE}_B^*$ defines a common abstract framework in which several families of two-valued states can be algebraically treated as unary operations on Baer *-semigroups. In Section 5, we give a decidable procedure to extend equational theories of two-valued states on orthomodular lattices to Baer *-semigroups determining sub-varieties of $\mathcal{IE}_B^*$. In Section 6 and Section 7, we apply the results obtained in an abstract way to two important classes of two-valued states, namely the full class of two-valued states and the subclass of Jauch–Piron two-valued states. In Section 8, we study some problems about equational completeness related to subvarieties of $\mathcal{IE}_B^*$. Finally, in Section 9, we introduce subvarieties $\mathcal{IE}_B^*$ whose equational theories are determined by classes of two-valued states on orthomodular lattices.

2. Basic notions

First we recall from [4] some notions of universal algebra that will play an important role in what follows. a *variety* is a class of algebras of the same type defined by a set of equations. If $\mathcal{A}$ is a variety and $\mathcal{B}$ is a subclass of $\mathcal{A}$, we denote by $\mathcal{V}(\mathcal{B})$ the subvariety of $\mathcal{A}$ generated by the class $\mathcal{B}$, i.e. $\mathcal{V}(\mathcal{B})$ is the smallest subvariety of $\mathcal{A}$ containing $\mathcal{B}$. Let $\mathcal{A}$ be a variety of algebras of type τ. We denote by $\text{Term}_{\mathcal{A}}$ the *absolutely free algebra* of type τ built from the set of variables $V = \{x_1, x_2, \ldots\}$. Each element of $\text{Term}_{\mathcal{A}}$ is referred as a *term*. We denote by $\text{Comp}(t)$ the complexity of the term t and by $t = s$ the equations of $\text{Term}_{\mathcal{A}}$.

For $t \in \text{Term}_{\mathcal{A}}$ we often write $t(x_1, \ldots x_n)$ to indicate that the variables occurring in t are among $x_1, \ldots x_n$. Let $A \in \mathcal{A}$. If $t(x_1, \ldots x_n) \in \text{Term}_{\mathcal{A}}$ and $a_1, \ldots, a_n \in A$, by $t^A(a_1, \ldots, a_n)$ we denote the result of application of the term operation t^A to the elements $a_1, \ldots, a_n$. a *valuation* in A is a function $v : V \to A$. Of course, any valuation v in A can be uniquely extended to an $\mathcal{A}$-homomorphism $v : \text{Term}_{\mathcal{A}} \to A$ in the usual way, i.e. if $t_1, \ldots, t_n \in \text{Term}_{\mathcal{A}}$ then $v(t(t_1, \ldots, t_n)) = t^A(v(t_1), \ldots, v(t_n))$. Thus, valuations are identified with $\mathcal{A}$-homomorphisms from the absolutely free algebra. If $t, s \in \text{Term}_{\mathcal{A}}$, $A \models t = s$ means that for each valuation v in A, $v(t) = v(s)$ and $\mathcal{A} \models t = s$ means that for each $A \in \mathcal{A}$, $A \models t = s$.

For each algebra $A \in \mathcal{A}$, we denote by $\mathrm{Con}(A)$ the congruence lattice of A, the diagonal congruence is denoted by Δ and the largest congruence A^2 is denoted by ∇. θ is called *factor congruence* iff there is a congruence θ^* on A such that, $\theta \wedge \theta^* = \Delta$, $\theta \vee \theta^* = \nabla$ and θ permutes with θ^*. If θ and θ^* is a pair of factor congruences on A then $A \cong A/\theta \times A/\theta^*$. A is *directly indecomposable* if A is not isomorphic to a product of two nontrivial algebras or, equivalently, Δ, ∇ are the only factor congruences in A. We say that A is *subdirect product* of a family of $(A_i)_{i \in I}$ of algebras if there exists an embedding $f : a \rightarrow \prod_{i \in I} A_i$ such that $\pi_i f : A \rightarrow A_i$ is a surjective homomorphism for each $i \in I$ where π_i is the projector onto A_i. A is *subdirectly irreducible* iff A is trivial or there is a minimum congruence in $\mathrm{Con}(A) - \Delta$. It is clear that a subdirectly irreducible algebra is directly indecomposable. An important result due to Birkhoff is that every algebra A is a subdirect product of subdirectly irreducible algebras. Thus, the class of subdirectly irreducible algebras rules the valid equations in the variety $\mathcal{A}$.

Now we recall from [14, 20] some notions about orthomodular lattices. a *lattice with involution* [13] is an algebra $\langle L, \vee, \wedge, \neg \rangle$ such that $\langle L, \vee, \wedge \rangle$ is a lattice and $\neg$ is a unary operation on L that fulfills the following conditions: $\neg\neg x = x$ and $\neg(x \vee y) = \neg x \wedge \neg y$. An *orthomodular lattice* is an algebra $\langle L, \wedge, \vee, \neg, 0, 1 \rangle$ of type $\langle 2, 2, 1, 0, 0 \rangle$ that satisfies the following conditions:

1. $\langle L, \wedge, \vee, \neg, 0, 1 \rangle$ is a bounded lattice with involution,
2. $x \wedge \neg x = 0$,
3. $x \vee (\neg x \wedge (x \vee y)) = x \vee y$.

We denote by $\mathcal{OML}$ the variety of orthomodular lattices. Let L be an orthomodular lattice. Two elements a, b in L are *orthogonal* (noted $a \perp b$) iff $a \leq \neg b$. For each $a \in L$ let us consider the interval $[0, a] = \{x \in L : 0 \leq x \leq a\}$ and the unary operation in $[0, a]$ given by $\neg_a x = \neg x \wedge a$. As one can readily realize, the structure $L_a = \langle [0, a], \wedge, \vee, \neg_a, 0, a \rangle$ is an orthomodular lattice.

Boolean algebras are orthomodular lattices satisfying the *distributive law* $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. We denote by **2** the Boolean algebra of two elements. Let L be an orthomodular lattice. An element $c \in L$ is said to be a *complement* of a iff $a \wedge c = 0$ and $a \vee c = 1$. Given a, b, c in L, we write: $(a, b, c)D$ iff $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$; $(a, b, c)D^*$ iff $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ and $(a, b, c)T$ iff $(a, b, c)D$, $(a, b, c)D^*$ hold for all permutations of a, b, c. An element z of L is called *central* iff for all elements $a, b \in L$ we have $(a, b, z)T$. We denote by $Z(L)$ the set of all central elements of L and it is called the *center* of L.

PROPOSITION 2.1. *Let L be an orthomodular lattice. Then we have:*

1. *$Z(L)$ is a Boolean sublattice of L [20, Theorem 4.15],*
2. *$z \in Z(L)$ iff for each $a \in L$, $a = (a \wedge z) \vee (a \wedge \neg z)$ [20, Lemma 29.9].*

Now we recall from [1, 8, 14] some notions about Baer *-semigroups. a *Baer *-semigroup* [8], also called Foulis semigroup [1, 3, 14], is an algebra $\langle S, \cdot, ^*, ', 0 \rangle$

of type $\langle 2, 1, 1, 0 \rangle$ such that, upon defining $1 = 0'$, the following conditions are satisfied:

1. $\langle S, \cdot \rangle$ is a semigroup,
2. $0 \cdot x = x \cdot 0 = 0$,
3. $1 \cdot x = x \cdot 1 = x$,
4. $(x \cdot y)^* = y^* \cdot x^*$,
5. $x^{**} = x$,
6. $x \cdot x' = 0$,
7. $x' \cdot x' = x' = (x')^*$,
8. $x' \cdot y \cdot (x \cdot y)' = y \cdot (x \cdot y)'$.

Let S be a Baer *-semigroup. An element $e \in S$ is a *projector* iff $e = e^* = e \cdot e$. The set of all projectors of S is denoted by $P(S)$. a projector $e \in P(S)$ is said to be closed iff $e'' = e$. We denote by $P_c(S)$ the set of all closed projectors. Moreover we can prove that

$$P_c(S) = \{x' : x \in S\}.$$

We can define a partial order $\langle P(S), \leq \rangle$ as follows,

$$e \leq f \iff e \cdot f = e.$$

In [20, Theorem 37.2] it is proved that, for any $e, f \in P_c(S)$, $e \leq f$ iff $e \cdot S \subseteq f \cdot S$. The facts stated in the next proposition are either proved in [8] or follow immediately from the results in [8].

PROPOSITION 2.2. *Let S be a Baer *-semigroup. Then:*

1. *If $x, y \in P(S)$ and $x \leq y$ then $y' \leq x'$,*
2. $(x \cdot y)'' = (x'' \cdot y)'' \leq y''$,
3. $(x^* \cdot x)'' = x''$,
4. *for each $x \in P_c(S)$, $0 \leq x \leq 1$,*
5. $x \cdot y = 0 \quad$ *iff* $\quad y = x' \cdot y$.

Observe that item 5 was one of the original conditions in the definition of a Baer *-semigroup in [8]. In the presence of conditions 1 . . . 7 of the definition of a Baer *-semigroup, the latter condition is equivalent to condition 8 (see [1, Proposition 2]).

THEOREM 2.3. [20, Theorem 37.8] *Let S be a Baer *-semigroup. For any $e_1, e_2 \in P_c(S)$, we define the following operations:*

$$e_1 \wedge e_2 = e_1 \cdot (e_2' \cdot e_1)',$$
$$e_1 \vee e_2 = (e_1' \wedge e_2')'.$$

Then $\langle P_c(S), \wedge, \vee, ', 0, 1 \rangle$ is an orthomodular lattice with respect to the order $\langle P(S), \leq \rangle$.

We can build a Baer *-semigroup from an orthomodular lattice [8]. In the following we briefly describe this construction.

Let $\langle A, \leq, 0, 1 \rangle$ be a bounded partial ordered set. An order-preserving function $\phi : a \rightarrow A$ is called *residuated function* iff there is another order-preserving function $\phi^+ : a \rightarrow A$, called a *residual function* of ϕ such that $\phi\phi^+(x) \leq x \leq \phi^+\phi(x)$. It can be proved that if ϕ admits a residual function ϕ^+, it is completely determined by ϕ.

REMARK 2.4. We will adopt the notation in [1, §1] in which residuated functions are written on the right. More precisely, if ϕ, ψ are residuated functions, $x\phi$ indicates the value $\phi(x)$ and $\psi\phi$ is interpreted as the function $x\psi\phi = (x\psi)\phi$.

We denote by $S(A)$ the set of residuated functions of A. Let θ be the constant function in A given by $x\theta = 0$. Clearly θ is an order-preserving function and θ^+ is the constant function $x\theta^+ = 1$. Thus $\theta \in S(A)$ and $\langle S(A), \circ, \theta \rangle$, where $\psi \circ \phi = \psi\phi$, is a semigroup.

THEOREM 2.5. [1, Proposition 2] *Let L be an orthomodular lattice. For each $a \in L$ we define*
$$x\phi_a = (x \vee \neg a) \wedge a \quad \text{(Sasaki projection)}.$$

If we define the following unary operations in $S(L)$:

  ϕ^*: *such that* $x\phi^* = \neg((\neg x)\phi^+)$,
  $\phi' := \phi_{\neg 1\phi}$

then

 1. $\langle S(L), \circ, ^*, ', \theta \rangle$ *is a Baer *-semigroup*,
 2. $P_c(S(L)) = \{\phi_a : a \in L\}$,
 3. $f_L : L \rightarrow P_c(S(L))$ *such that* $f_L(a) = \phi_a$ *is an $\mathcal{OML}$-isomorphism.*

If L is an orthomodular lattice, the Baer *-semigroup $\langle S(L), \circ, ^*, ', \theta \rangle$, or $S(L)$ for short, will be referred to as *the Baer *-semigroup of the residuated functions of L.*

Let L be an orthomodular lattice. We say that a Baer *-semigroup S *coordinatizes* L iff L is $\mathcal{OML}$-isomorphic to $P_c(S)$.

3. Two-valued states and Baer *-semigroups

The study of two-valued states becomes relevant in different frameworks. From a physical point of view, two-valued states are distinguished among the set of all classes of states because of their relation to hidden variable theories of quantum mechanics [11]. Another motivation for the analysis of two-valued states is rooted in the study of algebraic and topological representations of the event structures in quantum logic. Examples of them are the characterization of Boolean orthoposets by means of two-valued states [34] and the representation of orthomodular lattices via clopen sets in a compact Hausdorff closure space [33], later extended to orthomodular posets in [17]. We are interested in a theory of two-valued states on Baer *-semigroups as a natural extension of two-valued states on orthomodular lattice.

Formally, a *two-valued state* on an orthomodular L is a function $\sigma : L \rightarrow \{0, 1\}$ satisfying the following:

1. $\sigma(1) = 1$,
2. if $x \perp y$ then $\sigma(x \vee y) = \sigma(x) + \sigma(y)$.

Let L be an orthomodular lattice and $\sigma : L \rightarrow \{0, 1\}$ be a two-valued state. The following properties are derived directly from the definition of two-valued state:

$$\sigma(\neg x) = 1 - \sigma(x) \quad \text{and if} \quad x \leq y \text{ then } \sigma(x) \leq \sigma(y).$$

Based on the above mentioned two properties, in [5], Boolean pre-states are introduced as a general theoretical framework to study families of two-valued states on orthomodular lattices. We shall use these ideas for a general study of two-valued states extended to Baer *-semigroups. Thus, we first give the definition of Boolean pre-state.

DEFINITION 3.1. Let L be an orthomodular lattice. By a *Boolean pre-state* on L we mean a function $\sigma : L \rightarrow \{0, 1\}$ such that:

1. $\sigma(\neg x) = 1 - \sigma(x)$,
2. if $x \leq y$ then $\sigma(x) \leq \sigma(y)$.

EXAMPLE 3.2. Let us consider the orthomodular lattice $MO2 \times \mathbf{2}$ whose Hasse diagram has the following form:
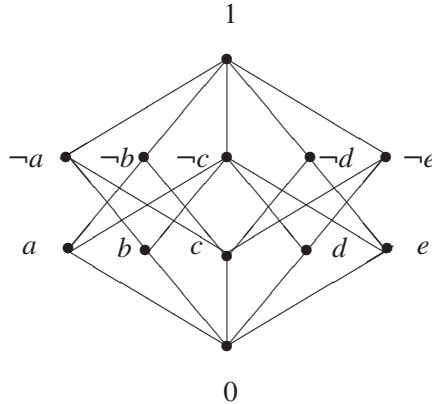
If we define the function $\sigma : MO2 \times \mathbf{2} \rightarrow \{0, 1\}$ such that

$$\sigma(x) = \begin{cases} 1, & \text{if } x \in \{1, \neg a, \neg b, \neg c, \neg d, \neg e\}, \\ 0, & \text{if } x \in \{0, a, b, c, d, e\}, \end{cases}$$

we can see that σ is a Boolean pre-state. This function fails to be a two-valued states since $b \leq \neg c$ but $\sigma(b \vee c) \neq \sigma(b) + \sigma(c)$. In fact, $\sigma(b \vee c) = \sigma(\neg a) = 1$ and $\sigma(b) + \sigma(c) = 0$.

We denote by $\mathcal{E}_B$ the category whose objects are pairs (L, σ) such that L is an orthomodular lattice and σ is a Boolean pre-state on L. Arrows in $\mathcal{E}_B$ are

$(L_1, \sigma_1) \xrightarrow{f} (L_2, \sigma_2)$ such that $f : L_1 \to L_2$ is an OML-homomorphism, and the following diagram is commutative,

$$
\begin{array}{ccc}
 & \sigma_1 & \\
L_1 & \longrightarrow & \{0, 1\} \\
f \downarrow & \equiv \nearrow & \\
 & \sigma_2 & \\
L_2 & &
\end{array}
$$

These arrows are called $\mathcal{E}_B$-homomorphisms.

Let L be an orthomodular lattice and let $\sigma : L \to \{0, 1\}$ be a Boolean pre-state. Since L we can identify with $P_c(S(L))$, we ask whether the Boolean pre-state σ admits a natural extension to the whole of $S(L)$. In other words, whether there exists some kind of function of the form $\sigma^* : S(L) \to \{0, 1\}$ such that the following diagram is commutative,

$$
\begin{array}{ccc}
 & \sigma & \\
L & \longrightarrow & \{0, 1\} \\
f_L \downarrow & \equiv \nearrow & \\
 & \sigma^* & \\
S(L) & &
\end{array}
$$

where f_L is the OML-isomorphism $f_L : L \to P_c(S(L))$ given in Theorem 2.5-3. The simplest way to do this would be to associate with each element $\phi \in S(L)$ an appropriate closed projection $\phi_x \in P_c(S(L))$ and to define $\sigma^*(\phi) = \sigma^*(\phi_x) = \sigma(x)$. An obvious choice for ϕ_x is $\phi'' = \phi_{1\phi}$. In virtue of this suggestion, we introduce the following concept.

DEFINITION 3.3. Let S be a Baer *-semigroup. a *Boolean* pre-state over S is a function $\sigma : S \to \{0, 1\}$ such that

1. $\sigma(x') = 1 - \sigma(x)$,
2. the restriction $\sigma/_{P_c(S)}$ is a Boolean pre-state on $P_c(S)$.

We denote by $\mathcal{E}_B^*$ the category whose objects are pairs (S, σ) such that S is a Baer *-semigroup and σ is a Boolean pre-state on S. The arrows in $\mathcal{E}_B^*$ are $(S_1, \sigma_1) \xrightarrow{f} (S_2, \sigma_2)$ such that $f : S_1 \to S_2$ is a Baer *-semigroup homomorphism, and the following diagram is commutative,

$$
\begin{array}{ccc}
 & \sigma_1 & \\
S_1 & \longrightarrow & \{0, 1\} \\
f \downarrow & \equiv \nearrow & \\
 & \sigma_2 & \\
S_2 & &
\end{array}
$$

These arrows are called $\mathcal{E}_B^*$-homomorphisms. Up to now we have presented a notion that would naturally extend the notion of Boolean pre-state to Baer *-semigroups. However we have not yet proved that this extension may be formally realized. This will be shown in Theorem 3.5. To see this, we first need the following basic results.

PROPOSITION 3.4. *Let S be a Baer *-semigroup and σ be a Boolean* pre-state on S. Then*

1. $\sigma(x'') = \sigma(x)$,
2. *If $x, y \in P(S)$ and $x \leq y$ then, $\sigma(y') \leq \sigma(x')$ and $\sigma(x) \leq \sigma(y)$,*
3. $\sigma(x \cdot y) = \sigma(x'' \cdot y) \leq \sigma(y)$,
4. $\sigma(x^* \cdot x) = \sigma(x)$.

Proof: 1) Is immediate. 2) Suppose that $x, y \in P(S)$ and $x \leq y$. By Proposition 2.2-1, $y' \leq x'$ and taking into account that $x', y' \in P_c(S)$, $\sigma(y') \leq \sigma(x')$. By Proposition 2.2-1 again and since $y' \leq x'$ we have that $x'' \leq y''$. Hence, by item 1, $\sigma(x) = \sigma(x'') \leq \sigma(y'') = \sigma(y)$. 3) By Proposition 2.2-2, $(x \cdot y)'' = (x'' \cdot y)'' \leq y''$. By item 1, $\sigma(x \cdot y) = \sigma((x \cdot y)'') = \sigma((x'' \cdot y)'') = \sigma(x'' \cdot y)$. Since $(x'' \cdot y)''$ and y'' are closed projections, by item 1, we have that $\sigma(x'' \cdot y) = \sigma((x'' \cdot y)'') \leq \sigma(y'') = \sigma(y)$. 4) By Proposition 2.2-3 $(x^* \cdot x)'' = x''$. Then, by item 1, $\sigma(x^* \cdot x) = \sigma((x^* \cdot x)'') = \sigma(x'') = \sigma(x)$. □

THEOREM 3.5. *Let S be a Baer *-semigroup and σ a Boolean pre-state on $P_c(S)$. Then σ_S defined as*

$$\sigma_S(x) = \sigma(x'')$$

is the unique Boolean pre-state on S such that $\sigma_S/_{P_c(S)} = \sigma$.*

Proof: If $x \in S$ then $x'' \in P_c(S)$ and $\sigma(x'')$ is defined. Then σ_S is well defined as a function. Note that if $x \in P_c(S)$ then $\sigma_S(x) = \sigma(x'') = \sigma(x)$ since $'$ is an orthocomplementation on the orthomodular lattice $P_c(S)$. Thus $\sigma_S/_{P_c(S)} = \sigma$. Let $x \in S$. Then $\sigma_S(x') = \sigma(x''') = 1 - \sigma(x'') = 1 - \sigma_S(x)$. Thus σ_S is a Boolean* pre-state on S. Let σ_1 be a Boolean* pre-state on S such that $\sigma_1/_{P_c(S)} = \sigma$. Let $x \in S$. Since $x'' \in P_c(S)$, by Proposition 3.4-2, $\sigma_1(x) = \sigma_1(x'') = \sigma(x'') = \sigma_S(x)$. Hence $\sigma_1 = \sigma_S$ and σ_S is the unique Boolean* pre-state on S such that $\sigma_S/_{P_c(S)} = \sigma$. □

4. An algebraic approach to two-valued states on Baer *-semigroups

In this section we study a variety of Baer *-semigroups enriched with a unary operation that allows us to capture the concept of two-valued states on Baer *-semigroups in an equational theory. We begin this section showing a way to deal with families of Boolean pre-states on orthomodular lattices as varieties in which the concept of Boolean pre-states is captured by adding a unary operation to the orthomodular lattice structure.

Let L be an orthomodular lattice and $\sigma : L \to \{0, 1\}$ be a Boolean pre-state. If we define the function $s : L \to Z(L)$ such that $s(x) = 0^L$ if $\sigma(x) = 0$ and $s(x) = 1^L$ if $\sigma(x) = 1$. Then s has properties s1,...,s5 in the following definition.

DEFINITION 4.1. An *orthomodular lattice with internal Boolean pre-state*, IE_B-lattice for short, is an algebra $\langle L, \wedge, \vee, \neg, s, 0, 1 \rangle$ of type $\langle 2, 2, 1, 1, 0, 0 \rangle$ such that $\langle L, \wedge, \vee, \neg, 0, 1 \rangle$ is an orthomodular lattice and s satisfies the following equations for each $x, y \in L$:

   s1. $s(1) = 1$,
   s2. $s(\neg x) = \neg s(x)$,
   s3. $s(x \vee s(y)) = s(x) \vee s(y)$,
   s4. $y = (y \wedge s(x)) \vee (y \wedge \neg s(x))$,
   s5. $s(x \wedge y) \leq s(x) \wedge s(y)$.

Thus, the class of IE_B-lattices is a variety that we call $\mathcal{IE}_B$. The following proposition provides the main properties of IE_B-lattices.

PROPOSITION 4.2. [5, Proposition 3.5] *Let L be an IE_B-lattice. Then we have:*

1. $\langle s(L), \vee, \wedge, \neg, 0, 1 \rangle$ *is a Boolean sublattice of $Z(L)$,*
2. *If $x \leq y$ then $s(x) \leq s(y)$,*
3. $s(x) \vee s(y) \leq s(x \vee y)$,
4. $s(s(x)) = s(x)$,
5. $x \in s(L)$ *iff $s(x) = x$,*
6. $s(x \wedge s(y)) = s(x) \wedge s(y)$.

A crucial question that must be answered is under which conditions a class of two-valued states over an orthomodular lattice can be characterized by a subvariety of $\mathcal{IE}_B$. To do this, we first need the following two basic results.

PROPOSITION 4.3. [5, Theorem 4.4] *Let L be an IE_B-lattice. Then there exists a Boolean pre-state $\sigma : L \to \{0, 1\}$ such that $\sigma(x) = 1$ iff $\sigma(s(x)) = 1$.*

Observe that the Boolean pre-state in the last proposition is not necessarily unique. When we have an IE_B-lattice and a Boolean pre-state $\sigma : L \to \{0, 1\}$ such that $\sigma(x) = 1$ iff $\sigma(s(x)) = 1$, we say that s, σ are *coherent*. On the other hand, we can build IE_B-lattices from objects in the category $\mathcal{E}_B$ as shown the following proposition.

PROPOSITION 4.4. [5, Theorem 4.10] *Let L be an orthomodular lattice and σ be a Boolean pre-state on L. If we define $\mathcal{I}(L) = \langle L, \wedge, \vee, \neg, s_\sigma, 0, 1 \rangle$, where*

$$s_\sigma(x) = \begin{cases} 1^L, & \text{if } \sigma(x) = 1, \\ 0^L, & \text{if } \sigma(x) = 0, \end{cases}$$

then:

1. $\mathcal{I}(L)$ *is an IE_B-lattice and s_σ is coherent with σ,*
2. *If $(L_1, \sigma_1) \overset{f}{\to} (L_2, \sigma_2)$ is an $\mathcal{E}_B$-homomorphism then $f : \mathcal{I}(L_1) \to \mathcal{I}(L_2)$ is an IE_B-homomorphism.*

Note that $\mathcal{I}$ in the above proposition defines a functor of the form $\mathcal{I} : \mathcal{E}_B \to \mathcal{IE}_B$. Now it is very important to characterize the class $\{\mathcal{I}(L) : L \in \mathcal{IE}_B\}$. To do this, directly indecomposable algebras in $\mathcal{IE}_B$ play an important role and the following proposition provides this result.

PROPOSITION 4.5. [5, Proposition 5.6] *Let* L *be an* IE_B*-lattice, then*

1. L *is directly indecomposable in* $\mathcal{IE}_B$ *iff* $s(L) = \mathbf{2}$,
2. *If* L *is directly indecomposable in* $\mathcal{IE}_B$ *then the function*

$$\sigma_s(x) = \begin{cases} 1, & \text{if } s(x) = 1^L, \\ 0, & \text{if } s(x) = 0^L, \end{cases}$$

is the unique Boolean pre-state coherent with s.

Thus, an immediate consequence of Proposition 4.4 and Proposition 4.5 is the following proposition.

PROPOSITION 4.6. *Let* $\mathcal{D}(\mathcal{IE}_B)$ *be the class of directly indecomposable algebras in* $\mathcal{IE}_B$. *Then*

$$\mathcal{D}(\mathcal{IE}_B) = \{\mathcal{I}(L) : L \in \mathcal{IE}_B\}$$

and $\mathcal{I} : \mathcal{E}_B \to \mathcal{D}(\mathcal{IE}_B)$ *is a categorical equivalence when we consider* $\mathcal{D}(\mathcal{IE}_B)$ *as a category whose arrows are* IE_B*-homomorphisms.*

Since $\mathcal{D}(\mathcal{IE}_B)$ contains the subdirectly irreducible algebras of $\mathcal{IE}_B$, we have that

$$\mathcal{IE}_B \models t = s \quad \text{iff} \quad \mathcal{D}(\mathcal{IE}_B) \models t = s.$$

Hence, the class of orthomodular lattices admitting Boolean pre-states can be identified with the directly indecomposable algebras in $\mathcal{IE}_B$ that determine the variety $\mathcal{IE}_B$. We can use these ideas to give a general criterium to characterize families of two-valued states over orthomodular lattices by a subvariety of $\mathcal{IE}_B$.

Let $\mathcal{A}_I$ be a subvariety of $\mathcal{IE}_B$. We denote by $\mathcal{D}(\mathcal{A}_I)$ the class of directly indecomposable algebras in $\mathcal{A}_I$.

DEFINITION 4.7. Let $\mathcal{A}$ be a subclass of $\mathcal{E}_B$ and let $\mathcal{A}_I$ be a subvariety of $\mathcal{IE}_B$. Then we say that $\mathcal{A}_I$ *equationally characterizes* $\mathcal{A}$ iff the following two conditions are satisfied,

I: for each $(L, \sigma) \in \mathcal{A}$, $\langle \mathcal{I}(L), \wedge, \vee, \neg, s_\sigma, 0, 1 \rangle$ belong to $\mathcal{D}(\mathcal{A}_I)$ where

$$s_\sigma(x) = \begin{cases} 1^L, & \text{if } \sigma(x) = 1, \\ 0^L, & \text{if } \sigma(x) = 0, \end{cases}$$

E: for each $L \in \mathcal{D}(\mathcal{A}_I)$, $(L, \sigma_s) \in \mathcal{A}$ where σ_s, the unique Boolean pre-state coherent with s, is given by $\sigma_s(x) = \begin{cases} 1, & \text{if } s(x) = 1^L, \\ 0, & \text{if } s(x) = 0^L. \end{cases}$

Since $\mathcal{D}(\mathcal{A}_I)$ contains the subdirectly irreducible algebras of $\mathcal{A}_I$, we have that

$$\mathcal{D}(\mathcal{A}_I) \models t = s \quad \text{iff} \quad \mathcal{A}_I \models t = s,$$

where t, s are terms in the language of $\mathcal{A}_I$.

Thus, when we say that a subclass $\mathcal{A}$ of $\mathcal{E}_B$ is equationally characterizable by a subvariety $\mathcal{A}_I$ of $\mathcal{IE}_B$, this means that the objects of $\mathcal{A}$ are identifiable with the directly indecomposable algebras of $\mathcal{A}_I$ according to the items **I** and **E** in Definition 4.7.

Taking into account the concept of IE_B-lattice we introduce a way to study the notion of Boolean* pre-state given in Definition 3.3 via a unary operation added to the Baer *-semigroups structure.

In fact, let S be a Baer *-semigroup. a unary operation s on S that allows us to capture the notion of Boolean* pre-state would have to satisfy the following basic conditions:

a. $s(x') = s(x)'$,
b. The restriction $s/_{P_c(S)}$ defines a unary operation in $P_c(S)$ such that $\langle P_c(S), \vee, \wedge, ', s/_{P_c(S)}, 0, 1 \rangle$ is an IE_B-lattice,
c. s should satisfy a version of Theorem 3.5, i.e. s should be always obtainable as the unique extension of $s/_{P_c(S)}$.

These ideas motivate the following general definition.

DEFINITION 4.8. An IE_B^*-*semigroup* is an algebra $\langle S, \cdot, ^*, ', s, 0 \rangle$ of type $\langle 2, 1, 1, 1, 0 \rangle$ such that $\langle S, \cdot, ^*, ', 0 \rangle$ is a Baer *-semigroup and s satisfies the following equations for each $x, y \in S$:

bs1. $s(1) = 1$,
bs2. $s(x') = s(x)'$,
bs3. $s(x)'' = s(x)$,
bs4. $s(x' \vee s(y')) = s(x') \vee s(y')$,
bs5. $y' = (y' \wedge s(x)) \vee (y' \wedge s(x)')$,
bs6. $s(x' \wedge y') \leq s(x') \wedge s(y')$.

Thus, the class of IE_B^*-semigroups is a variety that we call $\mathcal{IE}_B^*$.

PROPOSITION 4.9. *Let S be an IE_B^*-semigroup. Then:*

1. $s(x) \in Z(P_c(S))$,
2. $\langle P_c(S), \vee, \wedge, ', s/_{P_c(S)}, 0, 1 \rangle$ *is an IE_B-lattice and $\langle s(S), \vee, \wedge, ', 0, 1 \rangle$ is a Boolean subalgebra of $Z(P_c(S))$,*
3. $s(x'') = s(x)$,
4. *If $x, y \in P(S)$ and $x \leq y$ then, $s(y') \leq s(x')$ and $s(x) \leq s(y)$,*
5. $s(x \cdot y) = s(x'' \cdot y) \leq s(y)$,
6. $s(x^* \cdot x) = s(x)$.

Proof: 1) and 2). By bs3, for each $x \in S$, $s(x) \in P_c(S)$. Then, by Proposition 2.1-2 and bs5, $s(x) \in Z(P_c(S))$. Since the image of $'$ is $P_c(S)$, from the rest of the axioms, $\langle P_c(S), \vee, \wedge, ', s/_{P_c(S)}, 0, 1 \rangle$ is an IE_B-lattice and $\langle s(S), \vee, \wedge, ', 0, 1 \rangle$ is a Boolean subalgebra of $Z(P_c(S))$. 3, 4, 5, 6) Follow from similar arguments used in the proof of Proposition 3.4. □

THEOREM 4.10. *Let S be a Baer *-semigroup and $\langle P_c(S), \vee, \wedge, ', s, 0, 1 \rangle$ be an IE_B-lattice. Then the operation $s_S : S \to S$ such that*

$$s_S(x) = s(x'')$$

defines the unique IE_B^*-semigroup structure on S such that $s_S/_{P_c(S)} = s$.

Proof: If $x \in S$ then $x'' \in P_c(S)$ and $s(x'')$ is defined. Then s_S is well defined as a function. Since $x \in P_c(S)$ iff $x = x''$, $s_S(x) = s(x'') = s(x)$ for each $x \in P_c(S)$. Thus $s_S/_{P_c(S)} = s$. Now we prove the validity of the axioms bs1, ..., bs6.

bs1) Is immediate. bs2) $s_S(x') = s(x''') = s(x'')' = s_S(x)'$. bs3) $s_S(x)'' = s(x'')'' = s(x'')$ since $s(x) \in P_c(S)$ and $'$ is an orthocomplementation on $P_c(S)$. Hence $s_S(x)'' = s_S(x)$. bs4, bs5 and bs6, follow from the fact that $s_S/_{P_c(S)} = s$ and $\langle P_c(S), \vee, \wedge, ', s, 0, 1 \rangle$ is an IE_B-lattice. Hence s_S defines an IE_B^*-semigroup structure on S such that $s_S/_{P_c(S)} = s$.

Suppose that $\langle S, \cdot, *, ', s_1, 0 \rangle$ is an IE_B^*-semigroup such that $s_1/_{P_c(S)} = s$. Let $x \in S$. Since $x'' \in P_c(S)$, by Proposition 4.9-3, $s_1(x) = s_1(x'') = s(x'') = s_S(x)$. Hence $s_1 = s_S$ and s_S defines the unique IE_B^*-semigroup structure on S such that $s_S/_{P_c(S)} = s$. □

By Proposition 4.9 and Theorem 4.10 we can see that the definition of IE_B^*-semigroup pre-state satisfies the conditions required by the items a, b, c.

COROLLARY 4.11. *Let $\langle L, \vee, \wedge, \neg, s, 0, 1 \rangle$ be an IE_B-lattice and $S(L)$ be the Baer *-semigroup of residuated functions of L. If for each Sasaki projection ϕ_a we define $\bar{s}(\phi_a) = \phi_{s(a)}$, then*

1. *$\langle P_c(S(L)), \vee, \wedge, ', \bar{s}, 0, 1 \rangle$ is an IE_B-lattice and $f : L \to P_c(S(L))$ such that $f(a) = \phi_a$ is an IE_B-isomorphism,*
2. *the operation $\bar{s}_S(\varphi) = \phi_{s(\varphi(1))}$ defines the unique IE_B^*-semigroup structure on $S(L)$ such that L is IE_B-isomorphic to $P_c(S(L))$.*

Proof: 1) By Theorem 2.5, there exists an OML-isomorphism $f : L \to P_c(S(L))$. It is not very hard to see that the composition $\bar{s} = fsf^{-1}$ satisfies bs1,...,bs6 and $f(s(x)) = (fsf^{-1})f(x) = \bar{s}f(x)$, i.e. f preserves $\bar{s}$. Then L is IE_B-isomorphic to $P_c(S(L))$.

2) Let $\varphi \in S(L)$. Then $\bar{s}_S(\varphi) = \phi_{s(\varphi(1))} = \bar{s}(\phi_{\varphi(1)}) = \bar{s}(\phi_{\neg\neg\varphi(1)}) = s(\varphi'')$. Therefore $\bar{s}_S$ is the extension of $\bar{s}$ given in Theorem 4.10. Hence the operation $\bar{s}_S$ defines the unique IE_B^*-semigroup structure on $S(L)$ such that L is IE_B-isomorphic to $P_c(S(L))$. □

COROLLARY 4.12. *Let $\langle S, \cdot, *, ', s, 0 \rangle$ be an IE_B^*-semigroup. Suppose that S_1 is a sub Baer *-semigroup of S and $P_c(S_1)$ is a sub IE_B-lattice of $P_c(S)$. Then the restriction $s/_{S_1}$ defines the unique IE_B^*-semigroup structure on S_1. In this way, S_1 is also a sub IE_B^*-semigroup of S.*

Proof: Let S_1 be a sub Baer *-semigroup of S such that $P_c(S_1)$ is a sub IE_B-lattice of $P_c(S)$. If for each $x \in S_1$ we define $s_{S_1}(x) = s/_{P_c(S_1)}(x'') = s(x'')$

then $s_{S_1} = s/_{S_1}$ and, by Theorem 4.10, it defines the unique IE_B^*-semigroup structure on S_1 that coincides with $s/_{P_c(S_1)}$ in $P_c(S_1)$. In this way S_1 also results as a sub IE_B^*-semigroup of S. $\square$

PROPOSITION 4.13. *Suppose that* $(S_i)_{i \in I}$ *is a family of* IE_B^*-*semigroups. Then,* $\prod_{i \in I} P_c(S_i)$ *is* IE_B-*lattice isomorphic to* $P_c(\prod_{i \in I} S_i)$.

Proof: Since the operations in $\prod_{i \in I} S_i$ are defined pointwise, for each $(x_i)_{i \in I} \in \prod_{i \in I} S_i$, $(x_i)'_{i \in I} = (x'_i)_{i \in I}$. Then it is straightforward to prove that $f((x_i)'_{i \in I}) = (x'_i)_{i \in I}$ defines an OML-isomorphism $f : P_c(\prod_{i \in I} S_i) \to \prod_{i \in I} P_c(S_i)$. We have to prove that this function preserves s. In fact $f(s((x_i)'_{i \in I})) = f((s(x_i))'_{i \in I}) = (s(x_i)')_{i \in I} = (s(x'_i))_{i \in I} = s((x'_i)_{i \in I}) = s(f((x_i)'_{i \in I}))$. Hence f is an IE_B-lattice isomorphism. $\square$

In what follow we study the relation between Boolean* pre-states and IE_B^*-semigroups.

PROPOSITION 4.14. *Let* $\langle S, \cdot, ^*, ', s, 0 \rangle$ *be an* IE_B^*-*semigroup. Then there exists a Boolean* pre-state* $\sigma : S \to \{0, 1\}$ *such that* $s/_{P_c(S)}$ *is coherent with* $\sigma/_{P_c(S)}$.

Proof: By Proposition 4.3 there exists a Boolean pre-state $\sigma_0 : P_c(S) \to \{0, 1\}$ such that $s/_{P_c(S)}$ is coherent with σ_0. By Theorem 3.5, there exists a unique Boolean* pre-state $\sigma : S \to \{0, 1\}$ such that $\sigma_0 = \sigma/_{P_c(S)}$. Hence $s/_{P_c(S)}$ is coherent with $\sigma/_{P_c(S)} = \sigma_0$. $\square$

The following result gives a kind of converse of the last proposition.

PROPOSITION 4.15. *Let* S *be a Baer* *-semigroup and* $\sigma : S \to \{0, 1\}$ *be a Boolean* pre-state. If we define*

$$s_\sigma(x) = \begin{cases} 1^{P_c(S)}, & \text{if } \sigma(x) = 1, \\ 0^{P_c(S)}, & \text{if } \sigma(x) = 0, \end{cases}$$

then $\langle S, \cdot, ^*, ', s_\sigma, 0 \rangle$ *is an* IE_B^*-*semigroup and* $s_\sigma/_{P_c(S)}$ *is coherent with* $\sigma/_{P_c(S)}$.

Proof: By Proposition 4.4, $\langle P_c(S), \wedge, \vee, ', s_\sigma/_{P_c(S)}, 0, 1 \rangle$ is an IE_B-lattice and $s_\sigma/_{P_c(S)}$ is coherent with $\sigma/_{P_c(S)}$. Since $\sigma(x) = \sigma(x'')$ then $s_\sigma(x) = s_\sigma(x'') = s_\sigma/_{P_c(S)}(x'')$. Hence, by Theorem 4.10, s_σ defines the unique IE_B^*-semigroup structure on S_1 that extends $s_\sigma/_{P_c(S)}$. $\square$

5. Varieties of IE_B-lattices determining varieties of IE_B^*-semigroups

When a family of two-valued states over an orthomodular lattice is equationally characterizable by a variety of IE_B-lattices in the sense of Definition 4.7, the problem about the existence of a variety of IE_B^*-semigroups that somehow may be able to equationally characterize the mentioned family of two-valued states may be posed. The following definition provides a "natural candidate" for such a class of IE_B^*-semigroups.

DEFINITION 5.1. Let $\mathcal{A}_I$ be a subvariety of $\mathcal{IE}_I$. Then we define the *subclass* $\mathcal{A}_I^*$ *of* $\mathcal{IE}_I^*$ as

$$\mathcal{A}_I^* = \{S \in \mathcal{IE}_B^* : P_c(S) \in \mathcal{A}_I\}.$$

Before proceeding, we have to make sure that $\mathcal{A}_I^*$ is a nonempty subclass of $\mathcal{IE}_I^*$.

PROPOSITION 5.2. *If $\mathcal{A}_I$ is a nonempty subvariety of $\mathcal{IE}_I$ then $\mathcal{A}_I^*$ is a nonempty subclass of $\mathcal{IE}_B^*$.*

Proof: Suppose that $\langle L, \wedge, \vee, \neg, s, 0, 1 \rangle$ belong to $\mathcal{A}_I$. By Theorem 2.5 we can consider the Baer $*$-semigroup $S(L)$ of residuated functions in L in which L is OML-isomorphic to $P_c(S(L))$. Identifying L with $P_c(S(L))$, by Theorem 4.10, there exists an operation $s_{S(L)}$ on $S(L)$ that defines the unique IE_B^*-semigroup structure on $S(L)$ such that $s_{S(L)/P_c(S(L))} = s$. Hence $S(L) \in \mathcal{A}_I^*$ and $\mathcal{A}_I^*$ is a nonempty subclass of $\mathcal{IE}_B^*$. □

In what follows we shall demonstrate not only that $\mathcal{A}_I^*$ is a variety but also we shall give a decidable method to find an equational system that defines $\mathcal{A}_I^*$ from an equational system that defines $\mathcal{A}_I$. In order to study this we first introduce the following concept.

DEFINITION 5.3. We define the $*$-*translation* $\tau : \text{Term}_{\mathcal{IE}_B} \to \text{Term}_{\mathcal{IE}_B^*}$ as follows:

$\tau(0) = 0$ and $\tau(1) = 1$,
$\tau(x) = x'$ for each variable x,
$\tau(\neg t) = \tau(t)'$,
$\tau(t \wedge s) = (\tau(t)' \cdot \tau(s))' \cdot \tau(s)$,
$\tau(t \vee s) = \tau(\neg(\neg t \wedge \neg s))$,
$\tau(s(t)) = s(\tau(t))$.

PROPOSITION 5.4. *Let S be an IE_B^*-semigroup, $v : \text{Term}_{\mathcal{IE}_B^*} \to S$ be a valuation and τ be the $*$-translation. Then*

1. *for each $t \in \text{Term}_{\mathcal{IE}_B}$, $v(\tau(t)) \in P_c(S)$,*
2. *there exists a valuation $v_c : \text{Term}_{\mathcal{IE}_B} \to P_c(S)$ such that for each $t \in \text{Term}_{\mathcal{IE}_B}$, $v_c(t) = v(\tau(t))$.*

Proof: 1) Let $t \in \text{Term}_{\mathcal{IE}_B}$. If t is the form $\neg r$ then $v(\tau(r)) = v(\tau(\neg r)) = v(\tau(r)') = v(\tau(r))' \in P_c(S)$. If t is the form $s(r)$ then $v(\tau(s(r))) = v(s(\tau(r))) = s(v(\tau(r))) \in Z(P_c(S)) \subseteq P_c(S)$. For the other case we use induction on the complexity of terms in $\text{Term}_{\mathcal{IE}_B}$. If $\text{Comp}(t) = 0$ then t is 0, 1, or a variable x. In these cases $v(\tau(1)) = v(1) = 1^S$, $v(\tau(0)) = v(0) = 0^S$ and $v(\tau(x)) = v(x') = v(x)'$. Thus $v(\tau(t)) \in P_c(S)$. Assume that $v(\tau(t)) \in P_c(S)$ whenever $\text{Comp}(t) < n$. Suppose that $\text{Comp}(t) = n$. We have to consider the case in which t is the form $p \wedge r$. Then $v(\tau(t)) = v(\tau(p \wedge r)) = v((\tau(p)' \cdot \tau(r))' \cdot \tau(r)) = (v(\tau(p))' \cdot v(\tau(r)))' \cdot v(\tau(r)) = v(\tau(r)) \wedge v(\tau(p))$ because $v(\tau(r)) \in P_c(S)$ and $v(\tau(p)) \in P_c(S)$. Thus $v(\tau(t)) \in P_c(S)$. This proves that for each $t \in \text{Term}_{\mathcal{IE}_B}$, $v(\tau(t)) \in P_c(S)$.

2) Consider the valuation $v_c : \mathrm{Term}_{\mathcal{IE}_B} \to P_c(S)$ such that for each variable x, $v_c(x) = v(x')$. Now we proceed by induction on the complexity of terms in $\mathrm{Term}_{\mathcal{IE}_B}$. If $\mathrm{Comp}(t) = 0$ then t is 0, 1, or a variable x. Then, $v_c(1) = 1^S = v(1) = v(\tau(1))$, $v_c(0) = 0^S = v(0) = v(\tau(0))$ and $v_c(x) = v(x') = v(\tau(x))$. Assume that $v_c(t) = v(\tau(t))$ whenever $\mathrm{Comp}(t) < n$. Suppose that $\mathrm{Comp}(t) = n$. We have to consider three possible cases:

t is the form $\neg r$. Then $v_c(t) = v_c(\neg r) = v_c(r)' = v(\tau(r))' = v(\tau(r)') = v(\tau(\neg r)) = v(\tau(t))$,

t is the form $p \wedge r$. $v_c(t) = v_c(p \wedge r) = v_c(p) \wedge v_c(r) = v(\tau(p)) \wedge v(\tau(r)) = (v(\tau(p))' \cdot v(\tau(r)))' \cdot v(\tau(r)) = v((\tau(p)' \cdot \tau(r))' \cdot \tau(r)) = v(\tau(p \wedge r)) = v(\tau(t))$,

t is the form $s(r)$. Then $v_c(t) = v_c(s(r)) = s(v_c(r)) = s(v(\tau(r))) = v(s(\tau(r))) = v(\tau(s(r))) = v(\tau(t))$.

This proves that for each $t \in \mathrm{Term}_{\mathcal{IE}_B}$, $v_c(t) = v(\tau(t))$. $\qquad\square$

PROPOSITION 5.5. *Let S be an IE_B^*-semigroup and $v : \mathrm{Term}_{\mathcal{IE}_B} \to P_c(S)$ be a valuation. Then there exists a valuation $v^* : \mathrm{Term}_{\mathcal{IE}_B^*} \to S$ such that $t \in \mathrm{Term}_{\mathcal{IE}_B}$, $v^*(\tau(t)) = v(t)$.*

Proof: Consider the valuation $v^* : \mathrm{Term}_{\mathcal{IE}_B^*} \to S$ such that for each variable $v^*(x) = v(\neg x)$. Let $t \in \mathrm{Term}_{\mathcal{IE}_B}$. We use induction on the complexity of terms in $\mathrm{Term}_{\mathcal{IE}_B}$. If $\mathrm{Comp}(t) = 0$ then t is 0, 1, or a variable x. Then, $v^*(\tau(1)) = v^*(1) = 1^S = v(1)$, $v^*(\tau(0)) = v^*(0) = 0^S = v(0)$ and $v^*(\tau(x)) = v^*(x') = v^*(x)' = v(\neg x)' = v(x)'' = v(x)$ since $v(x) \in P_c(S)$. Assume that $v^*(\tau(t)) = v(t)$ whenever $\mathrm{Comp}(t) < n$. Suppose that $\mathrm{Comp}(t) = n$. We have to consider three possible cases:

t is the form $\neg r$. Then $v^*(\tau(t)) = v^*(\tau(\neg r)) = v^*(\tau(r)') = v^*(\tau(r))' = v(r)' = v(\neg r) = v(t)$,

t is the form $p \wedge r$. Then $v^*(\tau(t)) = v^*(\tau(p \wedge r)) = v^*((\tau(p)' \cdot \tau(r))' \cdot \tau(r)) = (v^*(\tau(p))' \cdot v^*(\tau(r)))' \cdot v^*(\tau(r)) = (v(p)' \cdot v(r))' \cdot v(r) = v(p) \wedge v(r) = v(p \wedge r) = v(t)$,

t is the form $s(r)$. Then $v^*(\tau(t)) = v^*(\tau(s(r))) = v^*(s(\tau(r))) = s(v^*(\tau(r))) = s(v(r)) = v(s(r)) = v(t)$.

This proves that for each $t \in \mathrm{Term}_{\mathcal{IE}_B}$, $v^*(\tau(t)) = v(t)$. $\qquad\square$

THEOREM 5.6. *Let $\mathcal{A}_I$ be a subvariety of $\mathcal{IE}_I$ and assume that $\{t_i = s_i\}_{i \in I}$ is a set of equations in the language of $\mathcal{IE}_I$ that defines $\mathcal{A}_I$. Then*

$$\mathcal{A}_I^* = \{S \in \mathcal{IE}_B^* : \forall i \in I, S \models \tau(t_i) = \tau(s_i)\}.$$

Proof: On the one hand, assume that $S \in \mathcal{A}_I^*$, i.e. $P_c(S) \models t_i = s_i$ for each $i \in I$. Suppose that there exists $i_0 \in I$ such that $S \not\models \tau(t_{i_0}) = \tau(s_{i_0})$. Then there exists a valuation $v : \mathrm{Term}_{\mathcal{IB}_B} \to S$ such that $v(\tau(t_{i_0})) \neq v(\tau(s_{i_0}))$. By Proposition 5.4 there exists a valuation $v_c : \mathrm{Term}_{\mathcal{IE}_B} \to P_c(S)$ such that for each $t \in \mathrm{Term}_{\mathcal{IE}_B}$, $v_c(t) = v(\tau(t))$. Then $v_c(t_{i_0}) = v(\tau(t_{i_0})) \neq v(\tau(s_{i_0})) = v_c(s_{i_0})$ and $P_c(S) \not\models t_{i_0} = s_{i_0}$ which is a contradiction. Hence $S \models \tau(t_i) = \tau(s_i)$ for each $i \in I$.

On the other hand, assume that $S \in \mathcal{IE}_B^*$ and $S \models \tau(t_i) = \tau(s_i)$ for each $i \in I$. Suppose that there exists $i_0 \in I$ such that $P_c(S) \not\models t_{i_0} = s_{i_0}$. Then there

exists a valuation $v : \text{Term}_{\mathcal{I}E_B} \to P_c(S)$ such that $v(t_{i_0}) \neq v(s_{i_0})$. By Proposition 5.5, there exists a valuation $v^* : \text{Term}_{\mathcal{I}E_B^*} \to S$ such that for each $t \in \text{Term}_{\mathcal{I}E_B}$, $v^*(\tau(t)) = v(t)$. Then $v^*(\tau(t_{i_0})) = v(t_{i_0}) \neq v(s_{i_0}) = v^*(\tau(s_{i_0}))$ and $S \not\models \tau(t_{i_0}) = \tau(s_{i_0})$ which is a contradiction. Hence, for each $i \in I$, we have $P_c(S) \models t_i = s_i$. $\square$

6.   Baer *-semigroups and the full class of two-valued states

The category of orthomodular lattices admitting a two-valued state, noted by $\mathcal{T}E_B$, is the full sub-category of $\mathcal{E}_B$ whose objects are E_B-lattices (L, σ) satisfying the following condition,

$$x \perp y \implies \sigma(x \vee y) = \sigma(x) + \sigma(y).$$

In [5, Theorem 6.3] it is proved that the variety

$$\mathcal{I}TE_B = \mathcal{I}E_B + \{s(x \vee (y \wedge \neg x)) = s(x) \vee s(y \wedge \neg x)\}.$$

equationally characterizes $\mathcal{T}E_B$ in the sense of Definition 4.7. Thus, the objects of $\mathcal{T}E_B$ are identifiable to the directly indecomposable algebras of $\mathcal{I}TE_B$. By Theorem 5.6 we can give an equational theory in the frame of Baer *-semigroups that captures the concept of two-valued state. In fact this is done through the variety

$$\mathcal{I}TE_B^* = \mathcal{I}E_B^* + \{s(x' \vee (y' \wedge x'')) = s(x') \vee s(y' \wedge x'')\}$$

7.   Baer *-semigroups and Jauch–Piron two-valued states

The category of orthomodular lattices admiting a Jauch–Piron two-valued state [31], noted by $\mathcal{J}PE_B$, is the full sub-category of $\mathcal{T}E_B$ whose objects are E_B-lattices (L, σ) in $\mathcal{T}E_B$ also satisfying the condition

$$\sigma(x) = \sigma(y) = 1 \implies \sigma(x \wedge y) = 1.$$

In [5, Theorem 7.3] it is proved that the variety

$$\mathcal{I}JPE_B = \mathcal{I}TE_B + \{s(x) \wedge s(\neg x \vee y) = s(x \wedge y)\}$$

equationally characterizes $\mathcal{J}PE_B$ in the sense of Definition 4.7. Thus the objects of $\mathcal{J}PE_B$ are identifiable to the directly indecomposable algebras of $\mathcal{I}JPE_B$. By Theorem 5.6 we can give an equational theory in the frame of Baer *-semigroups that captures the concept of two-valued state. In fact this is done through the variety

$$\mathcal{I}JPE_B^* = \mathcal{I}TE_B^* + \{s(x') \wedge s(x'' \vee y') = s(x' \wedge y')\}.$$

8.   The problem of equational completeness in $\mathcal{A}_I^*$

Let $\mathcal{A}$ be a family of E_B-lattices. Suppose that the subvariety $\mathcal{A}_I$ of $\mathcal{I}E_I$ equationally characterizes $\mathcal{A}$ in the sense of Definition 4.7. Then, through a functor $\mathcal{I}$, $\mathcal{A}$ is identifiable to the directly indecomposable algebras of the variety $\mathcal{A}_I$. In this way, we can state that $\mathcal{A}$ determines the equational theory of $\mathcal{A}_I$. With the natural extension of Boolean pre-states to Baer *-semigroups, encoded in $\mathcal{A}_I^*$, this kind of characterization may be lost. More precisely, the class $\mathcal{A}$ may "not rule"

the equational theory of $\mathcal{A}_I^*$ in the way $\mathcal{A}$ does with $\mathcal{A}_I$. The following example shows such a situation.

EXAMPLE 8.1. Let $\widetilde{\mathcal{B}}$ be the subclass of $\mathcal{E}_B$ formed by the pairs (B, σ) such that B is a Boolean algebra and σ is a Boolean pre-state. $\widetilde{\mathcal{B}}$ is a nonempty class since Boolean homomorphisms of the form $B \to \mathbf{2}$ always exist for each Boolean algebra B and they are examples of Boolean pre-states. It is clear that the class

$$\widetilde{\mathcal{B}}_I = \mathcal{I}E_B + \{x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)\}$$

equationally characterizes the class $\widetilde{\mathcal{B}}$ in the sense of Definition 4.7. Note that $\widetilde{\mathcal{B}}_I$ may be seen as a sub-variety $\widetilde{\mathcal{B}}_I^*$ since, each algebra B in $\widetilde{\mathcal{B}}_I$ in the signature $\langle \wedge, *, \neg, s, 0 \rangle$, where $*$ is the identity, is an IE_B^*-semigroup. Then the equational theory of $\widetilde{\mathcal{B}}_I$, as variety of IE_B^*-semigroups, is determined by the algebras of $\widetilde{\mathcal{B}}$. Note that algebras of $\widetilde{\mathcal{B}}_I$ are commutative Baer *-semigroups and then we have

$$\widetilde{\mathcal{B}}_I \models x \cdot y = y \cdot x.$$

What we want to point out is the following: $\widetilde{\mathcal{B}}_I$ captures (although in some sense a trivial one) the concept of Boolean pre-states over Boolean algebras in a variety. Moreover $\widetilde{\mathcal{B}}$ also determines the equational theory of $\widetilde{\mathcal{B}}_I$ when $\widetilde{\mathcal{B}}_I$ is seen as a variety of IE_B^*-semigroup.

Let us now compare the last result with Definition 5.3 and Theorem 5.6. The variety $\widetilde{\mathcal{B}}_I^*$ given by

$$\begin{aligned}\widetilde{\mathcal{B}}_I^* &= \mathcal{I}E_B^* + \{\tau(x \wedge (y \vee z)) = \tau((x \wedge y) \vee (x \wedge z))\} \\ &= \mathcal{I}E_B^* + \{x' \wedge (y' \vee z') = (x' \wedge y') \vee (x' \wedge z')\}\end{aligned}$$

is the biggest subvariety of $\mathcal{I}E_B^*$ whose algebras have a lattice of closed projections with Boolean structure and then $\widetilde{\mathcal{B}}_I \subseteq \widetilde{\mathcal{B}}_I^*$. We shall prove that $\widetilde{\mathcal{B}}_I \neq \widetilde{\mathcal{B}}_I^*$, i.e. the inclusion, is proper. In fact, let B_4 be the Boolean algebra of four elements $\{0, a, \neg a, 1\}$ endowed with the operation $s(x) = x$, i.e. the identity on B_4. In this case $B_4 \in \widetilde{\mathcal{B}}_I$. According to Theorem 2.5, we consider the Baer *-semigroup $S(B_4)$ of residuated functions of B_4. Since we can identify B_4 with $P_c(S(B_4))$, by Proposition 4.10 we can extend s to $S(B_4)$. Therefore $S(B_4)$ may be seen as an algebra of $\widetilde{\mathcal{B}}_I^*$. Consider the function $\phi : B_4 \to B_4$ such that $0\phi = \phi(0) = 0$, $1\phi = \phi(1) = 1$, $a\phi = \phi(a) = \neg a$ and $(\neg a)\phi = \phi(\neg a) = a$. Note that ϕ is an order preserving function and the composition $\phi\phi = 1_{B_4}$. Hence ϕ is the residual function of itself and then $\phi \in S(B_4)$. Let ϕ_a be the Sasaki projection associated to a. Then $x\phi_a = (x \vee \neg a) \wedge a = x \wedge a$. Note that $a(\phi\phi_a) = (a\phi) \wedge a = \neg a \wedge a = 0$ and $a(\phi_a\phi) = (a\phi_a)\phi = a\phi = \neg a$.

This proves that $\phi\phi_a \neq \phi_a\phi$ and then $S(B_4) \notin \widetilde{\mathcal{B}}_I$ because $\widetilde{\mathcal{B}}_I$ is a variety of commutative Baer *-semigroups. Thus $\widetilde{\mathcal{B}}_I \neq \widetilde{\mathcal{B}}_I^*$ and the inclusion is proper.

Hence, the directly indecomposable algebras of $\widetilde{\mathcal{B}}_I$ considered as IE_B^*-semigroups do not determine the equational theory of $\widetilde{\mathcal{B}}_I^*$. Consequently $\widetilde{\mathcal{B}}$ does not significantly add to the equational theory of $\widetilde{\mathcal{B}}_I^*$.

Taking into account Example 8.1, the following problem may be posed.

> Let $\mathcal{A}$ be a class of E_B-lattices and suppose that the subvariety $\mathcal{A}_I$ of $\mathcal{I}E_I$ equationally characterizes $\mathcal{A}$ in the sense of Definition 4.7. Give a subvariety $\mathcal{G}^*$ of $\mathcal{A}_I^*$ in which we can determine the equational theory of $\mathcal{G}^*$ from the class $\mathcal{A}$.

We conclude this section defining the meaning of the statement that the class $\mathcal{A}$ of E_B-lattices determines the equational theory of a subvariety of $\mathcal{A}_I^*$.

DEFINITION 8.2. Let $\mathcal{A}$ be a class of E_B-lattices. Suppose that the variety $\mathcal{A}_I$ of IE_B-lattices equationally characterizes the class $\mathcal{A}$ and $\mathcal{I} : \mathcal{A} \to \mathcal{D}(\mathcal{A}_I)$ is the functor that provides the categorical equivalence between $\mathcal{A}$ and the category $\mathcal{D}(\mathcal{A}_I)$ of directly indecomposable algebras of $\mathcal{A}_I$. We say that $\mathcal{A}$ *determines the equational theory of a subvariety* $\mathcal{G}^*$ *of* $\mathcal{A}_I^*$ iff there exists a class operator

$$\mathbb{G} : \mathcal{A}_I \to \mathcal{A}_I^*$$

such that:

1. for each $L \in \mathcal{A}$, $\mathbb{G}\mathcal{I}(L)$ is a directly indecomposable algebra in $\mathcal{A}_I^*$,
2. $\mathcal{G}^* = \mathcal{V}(\{\mathbb{G}\mathcal{I}(L) : L \in \mathcal{A}\})$.

In the next section, we will study a class operator, denoted by $\mathbb{S}_0$, that will allow us to define a subvariety of $\mathcal{A}_I^*$ whose equational theory is determinated by $\mathcal{A}$ in the sense of Definition 8.2.

9.   The class operator $\mathbb{S}_0$

Let $\langle L, \wedge, \vee, \neg, s, 0, 1 \rangle$ be an IE_B-lattice. By Corollary 4.11, we consider the IE_B^*-semigroup $S(L)$ of residuated functions of L. By abuse of notation, we also denote by s the operation $s_{S(L)}$ on $S(L)$ where $s_{S(L)}(x) = s(x'')$. Let $S_0(L)$ be the sub Baer *-semigroup of $S(L)$ generated by the Sasaki projections on L. In the literature, $S_0(L)$ is refereed to as the *small Baer* *-*semigroup of products of Sasaki projections on L* [1, 8]. By Corollary 4.12, $S_0(L)$ with the restriction $s/_{S_0(L)}$ is a sub IE_B^*-semigroup of $S(L)$. This IE_B^*-semigroup will be denoted by $\mathbb{S}_0(L)$.

Since $P_c(\mathbb{S}_0(L))$ is IE_B-isomorphic to L, if $\mathcal{A}_I$ is a subvariety of $\mathcal{I}E_I$ and $L \in \mathcal{A}_I$ then $\mathbb{S}_0(L) \in \mathcal{A}_I^*$. These results allow us to define the following class operator,

$$\mathbb{S}_0 : \mathcal{A}_I \to \mathcal{A}_I^* \quad \text{s.t. } L \mapsto \mathbb{S}_0(L)$$

PROPOSITION 9.1. *Let* L_1, L_2 *be two* IE_B-*lattices and* $f : L_1 \to L_2$ *be an* IE_B-*homomorphism.*

1. *If* $\phi_{a_1} \dots \phi_{a_n}$ *are Sasaki projections in* L_1*, then for each* $x \in L_1$ *we have* $f(x\phi_{a_1} \dots \phi_{a_n}) = f(x)\phi_{f(a_1)} \dots \phi_{f(a_n)}$.
2. *If* f *is a surjective function then there exists a unique* IE_B^*-*homomorphism* $g : \mathbb{S}_0(L_1) \to \mathbb{S}_0(L_2)$ *such that, identifying* L_1 *with* $P_c(\mathbb{S}_0(L_1))$*, * $g/_{L_1} = f$*. Moreover, g is a surjective function.*

3. *If f is bijective, then $\mathbb{S}_0(L_1)$ and $\mathbb{S}_0(L_2)$ are IE_B^*-isomorphic.*

Proof: 1) We use induction on n. Suppose that $n = 2$. Then

$$f(x\phi_{a_1}\phi_{a_2}) = f((((x \vee \neg a_1) \wedge a_1) \vee \neg a_2) \wedge a_2)$$
$$= (((f(x) \vee \neg f(a_1)) \wedge f(a_1)) \vee \neg f(a_2)) \wedge f(a_2)$$
$$= f(x)\phi_{f(a_1)}\phi_{f(a_2)}.$$

Suppose that the result holds for $m < n$. Then

$$f(x\phi_{a_1}\ldots\phi_{a_n}) = f((x\phi_{a_1}\ldots\phi_{a_{n-1}} \vee \neg a_n) \wedge a_n)$$
$$= (f(x)\phi_{f(a_1)}\ldots\phi_{f(a_{n-1})} \vee \neg f(a_n)) \wedge f(a_n)$$
$$= f(x)\phi_{f(a_1)}\ldots\phi_{f(a_n)}.$$

2) Suppose that $f : L_1 \to L_2$ is a surjective IE_B-homomorphism. If $\phi \in \mathbb{S}_0(L_1)$, then $\phi = \phi_{a_1}\ldots\phi_{a_n}$ where ϕ_{a_i} are Sasaki projections on L_1. We define the function $g : \mathbb{S}_0(L_1) \to \mathbb{S}_0(L_2)$ such that

$$g(\phi) = g(\phi_{a_1}\ldots\phi_{a_n}) = \phi_{f(a_1)}\ldots\phi_{f(a_n)}.$$

We first prove that g is well defined. Suppose that $\phi = \phi_{a_1}\ldots\phi_{a_n} = \phi_{c_1}\ldots\phi_{c_m}$. Let $b \in L_2$. Since f is a surjective function then there exists $a \in L_1$ such that $f(a) = b$. Then, by the item 1,

$$b\phi_{f(c_1)}\ldots\phi_{f(c_m)} = f(a)\phi_{f(c_1)}\ldots\phi_{f(c_m)}$$
$$= f(a\phi_{c_1}\ldots\phi_{c_m})$$
$$= f(a\phi_{a_1}\ldots\phi_{a_n})$$
$$= f(a)\phi_{f(a_1)}\ldots\phi_{f(a_n)}$$
$$= b\phi_{f(a_1)}\ldots\phi_{f(a_n)}.$$

Thus $g(\phi_{a_1}\ldots\phi_{a_n}) = g(\phi_{c_1}\ldots\phi_{c_m})$ and g is well defined. Note that for each $a \in L_1$, $g(\phi_a) = \phi_{f(a)}$, and then $g/_{L_1} = f$ identifying L_1 with $P_c(\mathbb{S}_0(L_1))$. The surjectivity of g follows immediately from the surjectivity of f. By definition of g, it is immediate that g is a $\langle \circ, *, 0 \rangle$-homomorphism where $\psi \circ \phi = \psi\phi$. We prove that g preserves the operation $'$. Suppose that $\phi = \phi_{a_1}\ldots\phi_{a_n}$. By Theorem 2.5, $\phi' = \phi_{\neg 1\phi} = \phi_{\neg 1\phi_{a_1}\ldots\phi_{a_n}}$. By the item 1 we have that

$$g(\phi') = g(\phi_{\neg 1\phi_{a_1}\ldots\phi_{a_n}})$$
$$= \phi_{f(\neg 1\phi_{a_1}\ldots\phi_{a_n})}$$
$$= \phi_{\neg f(1)\phi_{f(a_1)}\ldots\phi_{f(a_n)}}$$
$$= \phi_{\neg 1 g(\phi)}$$
$$= g(\phi)'.$$

Thus, g preserves the operation $'$. Now we prove that g preserves s. By Proposition 4.9–3, $s(\phi) = s(\phi'')$. Then there exists $a \in L_1$ such that $\phi'' = \phi_a$. By Corollary

4.11, $g(s(\phi)) = g(s(\phi_a)) = g(\phi_{s(a)}) = \phi_{f(s(a))} = \phi_{s(f(a))} = s(\phi_{f(a)}) = s(g(\phi_a)) = s(g(\phi'')) = s(g(\phi)'') = s(g(\phi))$ and g preserves the operation s. Hence g is a surjective IE_B^*-homomorphism such that, identifying L_1 with $P_c(\mathbb{S}_0(L_1))$, $g/_{L_1} = f$. We have to prove that g is unique. Suppose that there exists an IE_B^*-homomorphism $h : \mathbb{S}_0(L_1) \to \mathbb{S}_0(L_2)$ such that $h/_{L_1} = f$. Let $\phi = \phi_{a_1} \ldots \phi_{a_n} \in \mathbb{S}_0(L_1)$. Then $h(\phi) = h(\phi_{a_1} \ldots \phi_{a_n}) = h(\phi_{a_1}) \ldots (\phi_{a_n}) = f(\phi_{a_1}) \ldots f(\phi_{a_n}) = g(\phi_{a_1} \ldots \phi_{a_n}) = g(\phi)$. Thus, $h = g$ and this proves thet of g is unique.

3) To prove this item, we assume that f is bijective and use the function g of item 2. Then we have to prove that g is injective. Suppose that $g(\phi) = g(\psi)$ where $\varphi, \psi \in \mathbb{S}_0(L_1)$. Suppose that $\phi = \phi_{a_1} \ldots \phi_{a_n}$ and $\psi = \phi_{c_1} \ldots \phi_{c_m}$. By 1, for each $x \in L_1$ we have that

$$
\begin{aligned}
f(x\phi_{a_1} \ldots \phi_{a_n}) &= f(x)\phi_{f(a_1)} \ldots \phi_{f(a_n)} \\
&= f(x)g(\phi_{a_1} \ldots \phi_{a_n}) \\
&= f(x)g(\phi) \\
&= f(x)g(\psi) \\
&= f(x)\phi_{f(c_1)} \ldots \phi_{f(c_m)} \\
&= f(x\phi_{c_1} \ldots \phi_{c_m}).
\end{aligned}
$$

Since f is bijective, $\phi_{a_1} \ldots \phi_{a_n}(x) = \phi_{c_1} \ldots \phi_{c_m}(x)$ and then $\phi = \psi$. Thus g is bijective. $\qquad\square$

PROPOSITION 9.2. *Let A be a sub IE_B-lattice of L. Then there exists a sub IE_B^*-semigroup S_A of $\mathbb{S}_0(L)$ such that A is IE_B-isomorphic to $P_c(S_A)$.*

Proof: Consider the set

$$
S_A = \bigcup_{n \in \mathbb{N}} \{\phi_{a_1} \phi_{a_2} \ldots \phi_{a_n} : a_i \in a \},
$$

where ϕ_{a_i} are Sasaki projections on L. Note that, in general, $S_A \neq \mathbb{S}_0(A)$ since the domain of Sasaki projections ϕ_{a_i} is L (and not A). In [1, Proposition 10] it is proved that S_A is a sub Baer *-semigroup of $\mathbb{S}_0(L)$ in which A is OML-isomorphic to $P_c(S_A)$ and then A is IE_B-isomorphic to $P_c(S_A)$. Thus, by Corollary 4.12, S_A is a sub IE_B^*-semigroup of $\mathbb{S}_0(L)$. $\qquad\square$

PROPOSITION 9.3. *Let S be an IE_B^*-semigroup and for each $a \in S$ we define the function $\psi_a : P_c(S) \to P_c(S)$ such that $\psi_a(x) = (xa)''$. Then*

1. *if $a \in P_c(S)$ then $\psi_a = \phi_a$,*
2. *$f : S \to S(P_c(S))$ such that $f(a) = \psi_a$ is an IE_B^*-homomorphism,*
3. *let S_0 be the sub IE_B^*-semigroup of S generated by $P_c(S)$. If we consider the restriction $f/_{S_0}$ then $\mathrm{Imag}(f/_{S_0}) = \mathbb{S}_0(P_c(S))$.*

Proof: 1) Suppose that $a \in P_c(S)$. By [20, Lemma 37.10], $\psi_a(x) = (xa)'' = (x \vee \neg a) \wedge a = x\phi_a$. Hence $\psi_a = \phi_a$.

2) In [1, Proposition 7] is proved that f preserves the operations $\langle \cdot, {}^*, {}', 0 \rangle$. Then we have to prove that f preserves the operation s. Note that, by item 1, $f/_{P_c(S)}$ is the IE_B-isomorphism $a \mapsto \phi_a$. Then, by Corollary 4.11, $f(s(a)) = s(f(a))$ for each $a \in P_c(S)$. Taking into account that for each $a \in S$, $s(a) = s(a'')$, we have that $f(s(a)) = f(s(a'')) = s(f(a'')) = s(f(a)'') = s(f(a))$. Thus f is an IE_B^*-homomorphism.

3) Suppose that $\varphi \in \mathbb{S}_0(P_c(S))$. Then $\varphi = \phi_{a_1} \ldots \phi_{a_n}$ for some $a_1, \ldots, a_n$ in $P_c(S)$. If we consider the element $a = a_1 a_2 \ldots a_n$ then $a \in S_0$ and, since f is an IE_B^*-homomorphism, $f(a) = f(a_1 a_2 \ldots a_n) = f(a_1) f(a_2) \ldots f(a_n) = \phi_{a_1} \phi_{a_2} \ldots \phi_{a_n} = \varphi$. $\text{Imag}(f/_{S_0}) = \mathbb{S}_0(P_c(S))$. $\square$

PROPOSITION 9.4. *Let $(L_i)_{i \in I}$ be a family of IE_B-lattices. Then*

1. *if $\vec{a} = (a_i)_{i \in I} \in \prod_{i \in I} L_i$, then the Sasaki projection $\phi_{\vec{a}} : \prod_{i \in I} L_i \to \prod_{i \in I} L_i$ satisfies that for each $\vec{x} = (x_i)_{i \in I}$, $\vec{x}\phi_{\vec{a}} = (x_i \phi_{a_i})_{i \in I}$,*
2. *if $\vec{a} = (a_i)_{i \in I}$ and $\vec{b} = (b_i)_{i \in I}$ are elements in $\prod_{i \in I} L_i$, then for each $\vec{x} = (x_i)_{i \in I}$, $\vec{x}\phi_{\vec{a}}\phi_{\vec{b}} = (x_i \phi_{a_i} \phi_{b_i})_{i \in I}$,*
3. *$\mathbb{S}_0(\prod_{i \in I} L_i)$ is IE_B^*-isomorphic to $\prod_{i \in I} \mathbb{S}_0(L_i)$.*

Proof: 1) Let $\vec{a} = (a_i)_{i \in I} \in \prod_{i \in I} L_i$. Then

$$\vec{x}\phi_{\vec{a}} = ((x_i)_{i \in I} \vee \neg(a_i)_{i \in I}) \wedge (a_i)_{i \in I}$$
$$= ((x_i \vee \neg a_i) \wedge a_i)_{i \in I}$$
$$= (x_i \phi_{a_i})_{i \in I}.$$

2) Let $\vec{a} = (a_i)_{i \in I}$ and $\vec{b} = (b_i)_{i \in I}$ be two elements in $\prod_{i \in I} L_i$ and $\vec{x} = (x_i)_{i \in I}$. Then, by item 1, we have that

$$\vec{x}\phi_{\vec{a}}\phi_{\vec{b}} = (\vec{x}\phi_{\vec{a}})\phi_{\vec{b}}$$
$$= ((x_i \phi_{a_i})_{i \in I})\phi_{\vec{b}}$$
$$= (x_i \phi_{a_i} \phi_{b_i})_{i \in I}.$$

3) Follows from 2. $\square$

PROPOSITION 9.5. *Let L be an IE_B-lattice. Then L is directly indecomposable iff $\mathbb{S}_0(L)$ is directly indecomposable.*

Proof: Suppose that $\mathbb{S}_0(L)$ admits a nontrivial decomposition in direct products of IE_B^*-semigroups, i.e. $\mathbb{S}_0(L) = \prod_{i \in I} S_i$. Then, by Proposition 4.13, we can see that $L \approx_{IE_B} P_c(\mathbb{S}_0(L)) \approx_{IE_B} P_c(\prod_{i \in I} S_i) \approx_{IE_B} \prod_{i \in I} P_c(S_i)$. Thus L admits a nontrivial decomposition in direct products of IE_B-lattices.

Suppose that L admits a nontrivial decomposition in direct products of IE_B-lattices, i.e. $L = \prod_{i \in I} L_i$. Then, by Proposition 9.4-3, $\mathbb{S}_0(L) = \mathbb{S}_0(\prod_{i \in I} L_i) \approx_{IE_B^*} \prod_{i \in I} \mathbb{S}_0(L_i)$. Thus $\mathbb{S}_0(L)$ admits a nontrivial decomposition in direct products of IE_B^*-semigroups. $\square$

Let $\mathcal{A}_I$ be a variety of IE_B-lattices. We denote by $\mathcal{G}^*(\mathcal{A}_I)$ the sub variety of $\mathcal{A}_I^*$ generated by the class $\{\mathbb{S}_0(L) : L \in \mathcal{A}_I\}$. More precisely,

$$\mathcal{G}^*(\mathcal{A}_I) = \mathcal{V}(\{\mathbb{S}_0(L) : L \in \mathcal{A}_I\})$$

We also introduce the following subclass of $\mathcal{G}^*(\mathcal{A}_I)$,

$$\mathcal{G}_D^*(\mathcal{A}_I) = \{\mathbb{S}_0(L) : L \in \mathcal{D}(\mathcal{A}_I)\},$$

where $\mathcal{D}(\mathcal{A}_I)$ is the class of the direct indecomposable algebras of $\mathcal{A}_I$. By Proposition 9.5, we can see that $\mathcal{G}_D^*(\mathcal{A}_I)$ is a subclass of the direct indecomposable algebras of $\mathcal{A}_I^*$.

THEOREM 9.6. *Let $\mathcal{A}_I$ be a variety of IE_B-lattices. Then*

$$\mathcal{G}^*(\mathcal{A}_I) \models t = r \quad iff \quad \mathcal{G}_D^*(\mathcal{A}_I) \models t = r.$$

Proof: As regards to the nontrivial direction assume that

$$\mathcal{G}_D^*(\mathcal{A}_I) \models t(x_1, \ldots, x_n) = r(x_1, \ldots, x_n).$$

Let $\mathbb{S}_0(L) \in \mathcal{G}^*(\mathcal{A}_I)$. By the subdirect representation theorem, there exists an IE_B-lattice embedding $\iota : L \hookrightarrow \prod_{i \in I} L_i$ where $(L_i)_{i \in I}$ is a family of subdirectly irreducible algebras in $\mathcal{A}_I$. Therefore, $L_i \in \mathcal{D}(\mathcal{A}_I)$ and $\mathbb{S}_0(L_i) \in \mathcal{G}_D^*(\mathcal{A}_I)$ for each $i \in I$. By Proposition 9.2, there exists an IE_B^*-semigroup embedding $\iota_F : F \hookrightarrow \mathbb{S}_0(\prod_{i \in I} L_i)$ where L is IE_B-isomorphic to $P_c(F)$. By Proposition 9.4, we can assume that the IE_B^*-semigroup embedding ι_F is of the form $\iota_F : F \hookrightarrow \prod_{i \in I} \mathbb{S}_0(L_i)$. By Proposition 9.3, if we consider the sub IE_B^*-semigroup F_0 of F generated by $P_c(F)$ then there exists a surjective IE_B^*-homomorphisms $f : F_0 \to \mathbb{S}_0(L)$. The following diagram provides some intuition

$$F_0 \hookrightarrow F \overset{\iota_F}{\hookrightarrow} \prod_{i \in I} \mathbb{S}_0(L_i)$$

$$f \downarrow$$

$$\mathbb{S}_0(L)$$

Since F_0 can be embedded into a direct product $\prod_{i \in I} \mathbb{S}_0(L_i)$, where $\mathbb{S}_0(L_i) \in \mathcal{G}_D^*(\mathcal{A}_I)$ for each $i \in I$, by hypothesis, we have that

$$F_0 \models t(x_1, \ldots, x_n) = r(x_1, \ldots, x_n).$$

Let $\vec{a} = (a_1 \ldots a_n)$ be a sequence in $\mathbb{S}_0(L)$. Since f is surjective, there exists a sequence $\vec{m} = (m_1, \ldots, m_n)$ in F_0 such that $f(\vec{m}) = (f(m_1), \ldots, f(m_n)) = \vec{a}$. Since $t^{F_0}(\vec{m}) = r^{F_0}(\vec{m})$, then $t^{\mathbb{S}_0(L)}(\vec{a}) = f(t^{F_0}(\vec{m})) = f(r^{F_0}(\vec{m})) = t^{\mathbb{S}_0(L)}(\vec{a})$. Hence, $\mathbb{S}_0(L) \models t(x_1, \ldots, x_n) = r(x_1, \ldots, x_n)$ and the equation holds in $\mathcal{G}^*(\mathcal{A}_I)$. $\qquad\square$

Even though the study of equations in $\text{Exp}(\mathcal{A}_I)$ is quite treatable from the result obtained in Theorem 9.6, we do not have, in general, a full description of the equational system that defines the variety $\text{Exp}(\mathcal{A}_I)$. The following corollary provides an interesting property of $\text{Exp}(\mathcal{A}_I)$.

COROLLARY 9.7. *Let $\mathcal{A}_I$ be a variety of IE_B-lattices. Then*

$$\mathcal{G}^*(\mathcal{A}_I) \models s(x) \cdot y = y \cdot s(x).$$

Proof: Let S be an algebra in $\mathcal{G}^*_D(\mathcal{A}_I)$. Then for each $x \in S$, $s(x) \in \{0, 1\}$ and $s(x) \cdot y = y \cdot s(x)$. Hence, by Theorem 9.6, $\mathcal{G}^*(\mathcal{A}_I) \models s(x) \cdot y = y \cdot s(x)$. □

Let $\mathcal{A}_I$ be a variety of IE_B-lattices. Note that the assignment $L \mapsto \mathbb{S}_0(L)$ defines a class operator of the form

$$\mathbb{S}_0 : \mathcal{A}_I \to \mathcal{G}^*(\mathcal{A}_I) \subseteq \mathcal{A}_I^*.$$

Taking into account Definition 8.2, by Proposition 9.5 and Theorem 9.6 we can establish the following result.

THEOREM 9.8. *Let $\mathcal{A}$ be a class of E_B-lattices. Suppose that the variety $\mathcal{A}_I$ of IE_B-lattices equationally characterizes the class $\mathcal{A}$. Then the class $\mathcal{A}$ determines the equational theory of $\mathcal{G}^*(\mathcal{A}_I)$.*

This last theorem provides a solution to the problem posed in Section 8.1.

10. Final remarks

We have developed an algebraic framework that allows us to extend families of two-valued states on orthomodular lattices to Baer *-semigroups. To do so, we have explicitly enriched this variety with a unary operation that captures the concept of two-valued states on Baer *-semigroups as an equational theory. Moreover, a decidable method to find the equational system is given. We have also applied this general approach to study the full class of two-valued states and the subclass of Jauch–Piron two-valued states on Baer *-semigroups.

Acknowledgements

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