Testing Born-Infeld electrodynamics in waveguides

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Waveguides can be employed to test non-linear effects in electrodynamics. We solve Born-Infeld equations for TE waves in a rectangular waveguide. We show that the energy velocity acquires a dependence on the amplitude, and harmonic components appear as a consequence of the non-linear behavior.

Born-Infeld electrodynamics [1, 2, 3, 4] is a non-linear theory that modifies Maxwell electromagnetism in the regime of strong field, which was originally aimed to render finite the self-energy of a point-like charge. The theory was not sufficiently appreciated in its epoch because the attention of the scientific community concentrated mainly on the newborn quantum electrodynamics. However the interest in Born-Infeld theory renewed in the last decades because Born-Infeld-like features emerge in the low energy limit of string theories [5, 6, 7, 8, 9, 10]. Born-Infeld theory introduces a new fundamental constant b, with dimensions of electromagnetic field, which fix the field scale for the transition from Maxwell weak field regime to Born-Infeld strong field regime. Maxwell and Born-Infeld theories have proved to be the sole theories for a massless spin 1 field having causal propagation [11, 12] and absence of birefringence [13, 14] (however birefringence can exist in theories describing interactions between Born-Infeld and scalar fields, as in the case of extended Kaluza-Klein theories [15, 16]). Some Abelian and non-Abelian generalizations of Born-Infeld theory has been mentioned as leading to a non-trivial Galilean limit $c \to \infty$ [17].

Born-Infeld plane waves do not differ from Maxwell plane waves; however, when waves propagate in the presence of background fields or boundaries then Born-Infeld waves depart from Maxwell ones owing to non-linear effects. In fact, if background fields are present then the propagation velocity becomes smaller than c [11, 13, 18] (see Ref.[16] for an equivalent effect in string theory). Moreover, the interaction with the background field causes a typically anisotropic effect since the Poynting vector does not share the wave propagation direction [19]. This kind of effects has not been observed yet, which means that constant b, if it exists, has a very large value [20]. In this work we will study the propagation of Born-Infeld fields in waveguides to show the consequences of Born-Infeld non-linearity in a context susceptible of experimental verification.

Born-Infeld field equations for the antisymmetric field tensor $F_{\mu\nu}$ are

$$\partial_\nu F_{\lambda\mu} + \partial_\mu F_{\nu\lambda} + \partial_\lambda F_{\mu\nu} \;=\; 0\;, \tag{1}$$

$$\partial_\nu \mathcal{F}^{\mu\nu} =\; 0\;, \tag{2}$$

where

$$\mathcal{F}_{\mu\nu} = \frac{F_{\mu\nu} - \frac{P}{b^2}{}^*\!F_{\mu\nu}}{\sqrt{1+\frac{2S}{b^2}-\frac{P^2}{b^4}}} \tag{3}$$

S and P being the scalar and pseudoscalar field invariants

$$S = \frac{1}{4}\;F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}(|\mathbf{B}|^2 - |\mathbf{E}|^2) \tag{4}$$

$$P = \frac{1}{4}\;{}^*F_{\mu\nu}F^{\mu\nu} = \mathbf{E}\cdot\mathbf{B} \tag{5}$$

and ${}^*F_{\mu\nu}$ being the dual field tensor (the tensor resulting from exchanging the roles of $\mathbf{E}$ and $-\mathbf{B}$). When $b \to \infty$ then $\mathcal{F}_{\mu\nu} \to F_{\mu\nu}$; thus Maxwell equations are recovered. Equation (1) means that the field tensor comes from a four-potential $A_\mu:\; F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Equation (2) can be derived from the Born-Infeld Lagrangian

$$L[A_\mu] = -\frac{b^2}{4\,\pi}\;\left(1-\sqrt{1+\frac{2S}{b^2}-\frac{P^2}{b^4}}\right) \tag{6}$$

which goes to Maxwell Lagrangian when $b \to \infty$. The unique solutions of Maxwell equations that also solve Born-Infeld equations (1, 2) are those having vanishing S and P, as happens with plane waves.

The simplest solutions for a rectangular waveguide can be obtained by proposing the field

$$F \;=\; \frac{\partial u(t,x)}{\partial t}\,dt\wedge dy \;+\; \frac{\partial u(t,x)}{\partial x}\,dx\wedge dy \tag{7}$$

(i.e., $c^{-1}E_y = F_{t\,y} = \partial u/\partial t = -F_{y\,t}$, $B_z = -F_{xy} = -\partial u/\partial x = F_{y\,x}$, and the rest of the components vanish; the symbol $\wedge$ means the antisymmetrized tensor product). This field fulfills Eq.(1) whatever the function $u(t,x)$ is (notice that Eq.(1) is nothing but $dF = 0,$ when written in geometric language). Function $u(t,x)$ will be determined by Eq.(2), together with boundary conditions

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of the type $E_y(t, x_{boundary}) = \partial u/\partial t(t, x_{boundary}) = 0$. We will search for an oscillating solution $u(t, x)$. After obtaining the solution, we will exploit the invariance of Eqs.(1, 2) under the Lorentz group, together with the invariance of the boundary conditions under Lorentz boosts along the z-axis. Thus, by performing a boost $t \to \gamma(t - Vc^{-2}z)$, we will obtain a wave propagating along z. The so built solution has the form

$$\begin{aligned} F &= \frac{\partial u(t,x)}{\partial t}|_{t=\gamma(t-Vc^{-2}z)}\ \gamma\ dt \wedge dy \\ &+ \frac{\partial u(t,x)}{\partial t}|_{t=\gamma(t-Vc^{-2}z)}\ \gamma V c^{-2}\ dy \wedge dz \\ &+ \frac{\partial u(t,x)}{\partial x}|_{t=\gamma(t-Vc^{-2}z)}\ dx \wedge dy \end{aligned} \tag{8}$$

The $F_{y\,x}$ component is a magnetic field B_z along the propagation direction z; $F_{t\,y}$ and $F_{z\,y}$ are transversal electric and magnetic fields $c^{-1}E_y$ and B_x respectively. Therefore, we are building a TE (transverse electric) mode for a rectangular waveguide. Since the solution does not depend on the transversal coordinate y, then it behaves like a $\mathrm{TE}_{n\,0}$ mode in Maxwell theory [20]. Boundary conditions for **E** and **B** do not differ from the usual ones; in fact the continuity of the tangential component of **E** and the normal component of **B** come from Eq.(1) that is shared by Maxwell and Born-Infeld theory. On the other hand, charge and current distributions on the wave guide surfaces guarantee the boundary conditions required by Eq.(2).

When the proposed solution (7) is substituted in Eq.(2) then the following equation for $u(t, x)$ is obtained (notice that $P = 0$):

$$\left[1 + \frac{1}{b^2}\left(\frac{\partial u}{\partial x}\right)^2\right]\ \frac{\partial^2 u}{\partial t^2}\ -\ \frac{2}{b^2}\ \frac{\partial u}{\partial t}\ \frac{\partial u}{\partial x}\ \frac{\partial^2 u}{\partial t\,\partial x} - \left[1 - \frac{1}{b^2}\left(\frac{\partial u}{\partial t}\right)^2\right]\ \frac{\partial^2 u}{\partial x^2}\ =\ 0\,, \tag{9}$$

which is a wave equation for a scalar field $u(t, x)$ derivable from the Lagragian $L[u] = (1 - b^{-2}\ \eta^{\mu\nu}\ \partial_\mu u\ \partial_\nu u)^{1/2}$. The Eq.(9) is called Born-Infeld equation; it is integrable [21, 22] and has a multi-Hamiltonian structure [23]. It is hard to find non-trivial solutions for this equation, especially if one is looking for solutions accomplishing certain boundary conditions. Anyway, solutions written in a parametric way have been found [24, 25]. In Ref.[26] we have developed a complex method for obtaining solutions of Eq.(9) that uses a Maxwellian solution u_M as a seed (notice that Eq.(9) becomes $\Box\ u_M = 0$ when $b \to \infty$); the method leads to that solution behaving like the seed when $b \to \infty$. Since we have to fit periodic boundary conditions for the field in the waveguide, we will use the Maxwellian seed

$$u_M(t, x) = \frac{A}{\kappa}\ \cos\kappa ct\ \sin\kappa x \tag{10}$$

(the value of κ adjusts the boundary conditions to the width of the guide along the x direction). Apart from b, the amplitude A is the only magnitude of the solution having dimensions of field. Therefore, we expect that the corrections to the solution coming from the non-linear features of Eq.(9) will depend on the non-dimensional magnitude A^2b^{-2}. The Born-Infeld solution associated with the seed (10) can been worked out by the method explained in Ref.[26]; at the lowest order in A^2b^{-2} it results

$$\begin{aligned} u(t,x) &= \frac{A}{\kappa}\left(\cos\left[\left(1 - \frac{A^2}{8\,b^2}\right)\ \kappa ct\right] + \frac{A^2}{32\,b^2}\ \cos\left[\left(1 - \frac{A^2}{8\,b^2}\right)\ 3\kappa ct\right]\right) \\ &\left(\sin\left[\left(1 + \frac{A^2}{8\,b^2}\right)\ \kappa x\right] + \frac{A^2}{32\,b^2}\ \sin\left[\left(1 + \frac{A^2}{8\,b^2}\right)\ 3\kappa x\right]\right) + O(A^4b^{-4}) \end{aligned} \tag{11}$$

In fact, if the left side of Eq.(9) is computed for the wave (11) then an oscillating function of order A^4b^{-4} will be obtained instead of zero. A Lorentz boost along the z-axis turns the solution (11) into a propagating wave. The phase of the temporal factor changes to $(1 - A^2/(8b^2))\ \ \kappa\gamma(t - Vc^{-2}z)$ but the phase of the transversal sector remains invariant. Thus, the frequency and the wave vector are

$$\omega\ \simeq\ \left(1 - \frac{A^2}{8\,b^2}\right)\ \kappa\gamma \tag{12a}$$

$$k_\parallel\ \simeq\ \left(1 - \frac{A^2}{8\,b^2}\right)\ \kappa\gamma V c^{-2} \tag{12b}$$

$$k_\perp\ \simeq\ \left(1 + \frac{A^2}{8\,b^2}\right)\ \kappa \tag{12c}$$

Since $\gamma = (1 - V^2c^{-2})^{-1/2}$ then we obtain the dispersion relation

$$\omega^2 - c^2k_\parallel^2 \simeq \left(\frac{1 - \frac{A^2}{8\,b^2}}{1 + \frac{A^2}{8\,b^2}}\right)^2\ c^2k_\perp^2 \simeq \left(1 - \frac{A^2}{2\,b^2}\right)c^2k_\perp^2 \tag{13}$$

The energy velocity is the velocity V relative to the frame where the solution is the stationary wave between two boundaries shown in Eq.(11) (in this frame the mean value of the Poynting vector vanishes). By dividing Eqs. (12b) and (12a) it is obtained

$$V = \frac{c^2k_\parallel}{\omega}$$

then, using the dipersion relation, the energy velocity results

$$\frac{V^2}{c^2}\ =\ 1 - \left(1 - \frac{A^2}{2\,b^2}\right)\ \frac{c^2\,k_\perp^2}{\omega^2}\ +\ O(A^4b^{-4}) \tag{14}$$

where ω is the source frequency and $k_\perp$ is determined by the waveguide width. The Born-Infeld field configuration in the waveguide is

$$\begin{aligned}\frac{E_y}{\gamma\, A} &= -\sin(\omega t - k_\| z)\sin k_\perp x \\ &+\frac{A^2}{32\, b^2}\Big[4\sin(\omega t - k_\| z)\sin k_\perp x \\ &-\sin(\omega t - k_\| z)\sin 3k_\perp x \\ &-3\sin 3(\omega t - k_\| z)\sin k_\perp x\Big] + O(A^4 b^{-4})\end{aligned} \tag{15}$$

$$B_x = -V c^{-1} E_y \tag{16}$$

$$\begin{aligned}\frac{B_z}{A} &= -\cos(\omega t - k_\| z)\cos k_\perp x \\ &-\frac{A^2}{32\, b^2}\Big[4\cos(\omega t - k_\| z)\cos k_\perp x \\ &+\cos 3(\omega t - k_\| z)\cos k_\perp x \\ &+3\cos(\omega t - k_\| z)\cos 3k_\perp x\Big] + O(A^4 b^{-4})\end{aligned} \tag{17}$$

Eq.(14) shows that the energy velocity V for Born-Infeld fields propagating in hollow waveguides increases with the amplitude A, for $A \ll b$. Although the effect is expected to be very weak, since the Born-Infeld constant b should be a very large field, the Eq.(14) shows that the influence of A on the energy velocity can be magnified by employing a source wavelength slightly bigger than the transverse dimension of the waveguide, i.e. when the energy velocity is very small and the propagating mode is near the cutoff. In particular, the TE_{10} mode will not propagate if the frequency is lower than $\omega_{cutoff} \simeq (1 - b^{-2}A^2/4)\,\pi c/d$, d being the waveguide width along the x direction; then, the cutoff frequency decreases with the amplitude. These features are susceptible of experimental test. Another effect coming from the non-linearity of the theory is the presence of harmonics into the field (15-17); so, even in the TE_{10} mode the waveguide will not be traveled by a unique wavelength, but a set of harmonics will enter with amplitudes proportional to powers of $A^2 b^{-2}$.

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