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Random multi-player games

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ABSTRACT

The study of evolutionary games with pairwise local interactions has been of interest to many different disciplines. Also, local interactions with multiple opponents had been considered, although always for a fixed amount of players. In many situations, however, interactions between different numbers of players in each round could take place, and this case cannot be reduced to pairwise interactions. In this work, we formalize and generalize the definition of evolutionary stable strategy (ESS) to be able to include a scenario in which the game is played by two players with probability p and by three players with the complementary probability $1 - p$. We show the existence of equilibria in pure and mixed strategies depending on the probability p, on a concrete example of the duel–truel game. We find a range of p values for which the game has a mixed equilibrium and the proportion of players in each strategy depends on the particular value of p. We prove that each of these mixed equilibrium points is ESS. A more realistic way to study this dynamics with high-order interactions is to look at how it evolves in complex networks. We introduce and study an agent-based model on a network with a fixed number of nodes, which evolves as the replicator equation predicts. By studying the dynamics of this model on random networks, we find that the phase transitions between the pure and mixed equilibria depend on probability p and also on the mean degree of the network. We derive mean-field and pair approximation equations that give results in good agreement with simulations on different networks.

Game theory has had remarkable success as a framework to study the behavior of large populations where individuals playing different strategies interact through some game, and they can replicate according to their payoffs. This theory emerged to create a biological context where evolution could be reflected. Recently, however, evolutionary game theory has become of much interest to economists, sociologists, and anthropologists and social scientists in general. It suggested new interpretations and prompted new observational studies. This led us to study the different ways in which populations can interact. Most studies have considered populations where individuals compete in pairwise interactions or even more than two (possibly many) individuals compete simultaneously but always with a fixed amount of players. However, more complicated scenarios were less investigated. The present paper studies a model that incorporates interactions in symmetric games with a random number of players, which allowed us to find games where the coexistence of strategies occurs and is a stable state in time. We give an insight into the conditions to obtain these coexistence states and investigate by many different approaches, what parameters of the game or the environment they depend on.

I. INTRODUCTION

Evolutionary game theory provides a framework to study the behavior of large populations where individuals playing different strategies (or having different biological traits) interact through some game, and they can replicate according to their payoffs. Classically, it is assumed that each individual is equally likely to face any other individual, and a mean-field approximation gives a system of ordinary differential equations for the time evolution of the proportion of individuals in each strategy.

The concept of evolutionary stable strategy (ESS) was introduced by Smith and Price,[1] and it can be characterized in terms of the payoffs among members of the original population (incumbents) and the invaders (or mutants). We can understand it as a refinement of Nash equilibria since it implies a collective no-deviation condition at least up to a critical fraction of players, usually known as the invasion barrier, instead of considering single-player deviations. Hence, an ESS satisfies the additional, stronger condition of stability, which implies that if an ESS is reached, then the proportions of players playing the different strategies do not change over time.[2–5]

Evolutionary games with pairwise interactions were extensively studied in the past.[6–8] More recently, some works have investigated learning and coordination games in two-layer networks[9] as well as multiplayer group interactions.[10,11] Also, local interactions with various opponents had been considered, where in each round, the same fixed number of players is randomly selected from the population to play against each other.[12–14] However, there are many situations in which the number of players can vary over time and even between rounds, as in multiplayer contests, markets, or communication networks access.[15–17] It also happens that the optimal strategies in a two-player game could not be optimal in a three players game, so interactions between multiple players cannot be reduced to pairwise interactions. Therefore, it is interesting to model and study the case in which the game can be played by a different number of players in each round when the strategy must be selected previously, without knowing *a priori* the exact number of players involved. A typical and important example of this situation is auction theory, for instance, in first-price sealed-bid auctions, where the optimal bid depends strongly on the number of bidders, which usually is not known.[18,19]

In this article, we formalize and generalize the definition of evolutionary stable strategy to be able to include this scenario. This question was analyzed previously;[17] however, only two combinations of incumbents and mutants were considered. As we show in Sec. II, all the combinations must be considered, and a hierarchy of payoffs is needed in order to characterize an ESS when the number of players in each interaction is a random variable.

In order to explore these questions, we study here the simplest non-trivial case of the duel–truel game. As usual, in a duel, two players aim to eliminate each other, while in a truel, three players are involved. Truel games were introduced independently by Shubik[20] and Epstein,[21] and their mechanics is as follows. At each time step, an order is drawn between the three players and they shoot cyclically deciding who their target will be (this is usually expressed as the players *shooting* and *killing* each other, although possible applications of this simple game do not need to be so violent). Each player aims at its chosen opponent and with a given probability it achieves the goal of eliminating this player from the game. This process repeats until only one of the three players remains.

The paradox is that the player that has the highest probability of annihilating competitors does not need to be necessarily the winner of this game. This surprising result was already present in the early literature on truels.[20,22–24] The case where this disparity is most clearly seen, in which the one with the least probability of killing turns out to be the one with the highest probability of surviving, is when they are allowed to shoot into the air. Thinking about a round, in the event that two players have perfect aim and one does not, if the player with the worst aim has to shoot first, it will prefer to shoot into the air, instead of shooting at one of the opponents since if they kill one of them, it will be killed by the remaining player. Thus, the perfect ones shoot each other, which results in that the survival probability of the player with the worst aim increases.

For simplicity, we will consider here only two strategies and two classes of players with different probabilities of killing the chosen opponent, *mediocre* players, which have a killing probability equal to $1/2$, and *perfect* players, which have a killing probability equal to 1. The payments of the matrix in the duel game favor the population that uses the strongest strategy, while the payments of the matrix of the truel game favor the population that pays the weakest strategy. In other words, if the local interactions between the players in the population would be pairwise, the ESS would be for the entire population to play strategy *perfect*, whereas if local interactions involve three players, the ESS would be for the entire population to play strategy *mediocre*.

This led us to consider what the ESS would be in a scenario where the number of players is a random variable, the so-called Poisson games.[25] As an example, agents will play a duel with probability $p \in (0, 1)$, and a truel with probability $1 - p$. Let us mention that this game was considered previously,[26] and only the survival probabilities of each strategy were obtained without considering the optimal option of shooting into the air.

Also, we introduce an agent-based model in which players interact in a complex network by copying the strategies of their neighbors, with a dynamics that in mean-field evolves following the replicator equation. Let us observe that the replicator dynamics is usually defined in terms of new individuals entering the population, by selecting a pure strategy with a probability proportional to the payoff given the current mix of agents. Some related works have also explored similar issues[27] but they assume that individuals reproduce instead of copying strategies.

Our approach has an independent interest and has the advantage that can be used in networks with a fixed number of nodes or agents, bypassing the issue of how to add new nodes as agents replicate.

We perform extensive Monte Carlo (MC) simulations of the model in different types of networks and develop an analytical approach based on a pair approximation[28–31] that allows us to obtain approximate equations for the evolution of the fraction of perfect agents in the network. This approach enables us to investigate if the transitions between ESS in pure and mixed strategies found within the Nash equilibrium theory are also observed in complex networks and to identify how the networks' topology affects the existence of mixed equilibria.

This article is organized as follows. In Sec. II, we introduce some notations and definitions from game theory. We characterize the ESS in Sec. III and describe the particular case of the duel–truel game in Sec. IV. In Sec. V, we introduce the agent-based model, develop a mean-field and a pair approximation approach for the system's dynamics, and compare the results with those of MC simulations of the model in various complex networks. Finally, in Sec. VI, we give a summary and conclusions.

II. EVOLUTIONARY GAME THEORY

Let $G = (S, U, \{1, \dots, N\})$ be a finite game, where S is the set of pure strategies $s_1, \dots, s_m$, $\{1, \dots, N\}$ is the set of players, and $U : S^N \to \mathbb{R}^N$ is the payoff of the game when each player selects a strategy.

Let us consider the simplex of mixed strategies, the set of probability vectors,

$$\Delta_S = \left\{x = (x_1, \dots, x_m) : 0 \le x_i, \sum x_i = 1\right\},$$

where a player using the mixed strategy $x \in \Delta_S$ plays the pure strategy s_i with probability x_i.

We extend the payoff U to the set of mixed strategies, in terms of the expected payoffs $E[x^1, \dots, x^N]$, that is, the sum over the possible payoffs multiplied by the probability they occur, where x^j is the mixed strategy of player j for $1 \le j \le N$.

We say that a Nash equilibrium is a vector of mixed strategies, one for each player such that no player can improve its payoff by changing its strategy.[3] Moreover, x is a symmetric Nash equilibrium if

$$E[x, x, \dots, x] \ge E[y, x, \dots, x] \quad \text{for y} \ne \text{x}.$$

On the other hand, let us observe that this definition is restricted only to players involved in a game. The notion of ESS holds for populations of players interacting through some dynamics involving a game, as we describe below.

Given a two-player game with finitely many pure strategies $s_1, \dots, s_m$, let us suppose that the proportion of agents in a population playing strategy s_i is x_i, so the agents define a mixed strategy $x = (x_1, \dots, x_m)$.

Now, the evolution of agents playing s_i is given by the following system of ordinary differential equations:

$$\frac{d}{dt}x_i(t) = x_i\left\{E[e_i, x] - E[x, x]\right\}, \tag{1}$$

where $1 \le i \le m$ and e_i is the ith canonical vector (its ith component is 1 and all other components are 0), which is known as the *replicator equation*. We say that the population distribution x is an *evolutionary stable strategy* if x is a local stable equilibrium of the dynamics.

Usually, the stability is formulated as first and second order conditions in terms of the incumbent population playing x, and any other mutant population distributed in the strategies according to a proportion y,

$$E[x, x] > E[x, y] \tag{2}$$

or

$$\begin{aligned} E[x, x] &= E[x, y], \\ E[x, x] &> E[y, y]. \end{aligned} \tag{3}$$

Since the condition $E[x, x] \ge E[x, y]$ holds because x is a Nash equilibrium, we get that an incumbent strictly outplays a mutant or gets an advantage whenever it plays with another incumbent, while mutants play among them.

Our next task is to extend this definition to symmetric games with a random number of players. For that, we consider a population of agents that play a given game. With certain probability p, two players are chosen at random and play the game against each other, and with the complementary probability $1 - p$, three players are chosen at random and play the game among themselves.

We need to distinguish between the expected payoff $E[e_i, x]$ of an agent playing strategy s_i against a population playing the different strategies with a given proportion x in the two players game, and $E_G[e_i, x]$, the expected payoff in the game with a random number of opponents, which in our case is

$$E_G[e_i, x] = pE[e_i, x] + (1 - p)E[e_i, x, x].$$

Hence, we get the system of ordinary differential equations

$$\frac{d}{dt}x_i(t) = x_i\left\{E_G[e_i, x] - E_G[x, x]\right\}. \tag{4}$$

Let us assume that the strategy x is played by the incumbents, while a small fraction ε of mutants plays y. We say that x is *evolutionary stable* (ES) against y if there exists $\varepsilon_0 > 0$ such that

$$\begin{aligned} & p\left\{\varepsilon E[x, y] + (1 - \varepsilon)E[x, x]\right\} \\ &\quad + (1 - p)\left\{2\varepsilon(1 - \varepsilon)E[x, y, x] + (1 - \varepsilon)^2 E[x, x, x] + \varepsilon^2 E[x, y, y]\right\} \\ &\quad > p\left\{\varepsilon E[y, y] + (1 - \varepsilon)E[y, x]\right\} + (1 - p)\left\{2\varepsilon(1 - \varepsilon)E[y, y, x]\right. \\ &\quad \left. + (1 - \varepsilon)^2 E[y, x, x] + \varepsilon^2 E[y, y, y]\right\} \end{aligned} \tag{5}$$

for any $\varepsilon \le \varepsilon_0$.

We say that x is an *evolutionary stable strategy* (ESS) if there exists $\varepsilon_0 > 0$ such that x is ES against y for any $y \ne x$.

For convenience, let us call $d_{i,j}$ the difference between the expected payoffs of an agent playing x and another agent playing y, when both play against a number of i and j opponents that play x and y, respectively.

Thus,

$$\begin{aligned} d_{1,0} &= E[x, x] - E[y, x], \\ d_{2,0} &= E[x, x, x] - E[y, x, x], \\ d_{0,1} &= E[x, y] - E[y, y], \\ d_{0,2} &= E[x, y, y] - E[y, y, y], \\ d_{1,1} &= E[x, y, x] - E[y, y, x]. \end{aligned}$$

Then, the condition from Eq. (5) is equivalent to the following: a strategy x is ES against a strategy y if either

$$p\, d_{1,0} + (1 - p)d_{2,0} > 0 \tag{6}$$

or

$$p\, (d_{0,1} - d_{1,0}) + (1 - p)2(d_{1,1} - d_{2,0}) > 0, \tag{7}$$

$$p\, d_{1,0} + (1 - p)d_{2,0} = 0, \tag{8}$$

or

$$(1 - p)(-2d_{1,1} + d_{2,0} + d_{0,2}) > 0, \tag{9}$$

$$p\,(d_{0,1}-d_{1,0})+(1-p)2(d_{1,1}-d_{2,0})=0, \tag{10}$$

$$p\,d_{1,0}+(1-p)d_{2,0}=0. \tag{11}$$

We call Eq. (6) label 0 condition, Eq. (7) label 1 condition, and Eq. (9) label 2 condition. The level is given by the power of ε that we are looking at in each case.

It is easy to generalize this idea to games where the number of players can be any integer, whenever the same set of finitely many strategies is used. In the characterization of ESS analyzed previously[17] only two cases of interactions were considered, incumbents against an incumbent vs incumbents against a mutant, or intra-group interactions (all mutants-all incumbent), as in Eqs. (2) and (3).

III. TWO STRATEGY GAMES

For the sake of simplicity, we consider only two strategies, s_1 and s_2, and two symmetric games with these strategies, one for two players and the other for three players. With probability p, a pair of agents are chosen to play a game, and with probability $1-p$ a triplet is chosen to play. We denote by a_{ij} the payoff of an agent (player I) that plays strategy s_i against another agent (player II) that plays strategy s_j, in a two-player game. Similarly, a_{ijk} denotes the payoff of player I who plays strategy s_i against other two agents that play strategies s_j (player II) and s_k (player III), in a three-player game. Here, the payoffs a_{ij} and a_{ijk} correspond to the survival probabilities of player I when it plays a duel and a truel, respectively. We recall that $a_{121}=a_{112}=a_{211}$ and $a_{221}=a_{212}=a_{122}$ due to the symmetry of the game and also $a_{11}=a_{22}=1/2$ and $a_{111}=a_{222}=1/3$.

For the two-player game, if player I plays s_i and player II plays s_j, then I receives the payoff a_{ij} and II receives a_{ji}. The values a_{ij} can be thought as elements of a 2×2 matrix A_2. In the same way, if three players I, II, and III are involved, such that they play strategies s_i, s_j, and s_k, respectively, then I receives the payoff a_{ijk}, II receives a_{jik}, and III receives a_{kij}. In this case, the values a_{ijk} are considered as the elements of two 2×2 matrices A_{3k}. As usual, the matrices A_2, A_{31}, and A_{32} are constant.

The set of mixed strategies is $\Delta_S=\{(\sigma,1-\sigma):\sigma\in[0,1]\}$. The expected payoff of a player using $(\sigma,1-\sigma)$ against an opponent that uses strategy s_j is $\sigma a_{1j}+(1-\sigma)a_{2j}$, and when it plays against two opponents using s_j and s_k is $\sigma a_{1jk}+(1-\sigma)a_{2jk}$. This way of computing expected payoffs also applies when more than one player use mixed strategies.

Let us suppose that a proportion $\sigma(t)$ of the population plays strategy s_1 at time t, and $1-\sigma(t)$ plays s_2. Assuming that the players are selected at random from the population, the time evolution of this proportion is given by the replicator Eq. (4),

$$\frac{d\sigma}{dt}=\sigma\big\{E_G[1,\sigma]-E_G[\sigma,\sigma]\big\}, \tag{12}$$

where $E_G[1,\sigma]$ and $E_G[\sigma,\sigma]$ are short notations for $E_G[(1,0),(\sigma,1-\sigma)]$ and $E_G[(\sigma,1-\sigma),(\sigma,1-\sigma)]$, respectively. Here, we are using the fact that the simplex of strategies is the segment $[0,1]$, and, thus, we can identify a given strategy $(\sigma,1-\sigma)$ with a single number σ, i.e., the probability to play s_1. Finally, Eq. (12) can be expanded to obtain the more symmetric expression

$$\frac{d\sigma}{dt}=\sigma(1-\sigma)\big\{E_G[1,\sigma]-E_G[0,\sigma]\big\}, \tag{13}$$

after using the relation

$$E_G[\sigma,\sigma]=\sigma E_G[1,\sigma]+(1-\sigma)E_G[0,\sigma].$$

A. Pure ESS

Let us consider first the strategy s_1, and let $y=(q,1-q)$ be any mixed strategy. Then, from Eq. (6), we have

$$p\left\{\frac{1}{2}-\left[q\frac{1}{2}+(1-q)a_{21}\right]\right\} + (1-p)\left\{\frac{1}{3}-\left[q\frac{1}{3}+(1-q)a_{211}\right]\right\}>0$$

and

$$(1-q)\left[p\left(\frac{1}{2}-a_{21}\right)+(1-p)\left(\frac{1}{3}-a_{211}\right)\right]>0.$$

It follows that the pure strategy s_1 is an ESS if

$$p\left(\frac{1}{2}-a_{21}\right)+(1-p)\left(\frac{1}{3}-a_{211}\right)>0, \tag{14}$$

whereas the pure strategy s_2 is an ESS if

$$p\left(\frac{1}{2}-a_{12}\right)+(1-p)\left(\frac{1}{3}-a_{122}\right)>0. \tag{15}$$

B. Mixed ESS

Let $x=(\sigma,1-\sigma)$ be a Nash equilibrium. Then, in order to be an ESS, x must satisfy Eq. (7) or Eq. (9) for any $y=(q,1-q)\neq x$.

Condition Eq. (7) is satisfied if

$$(\sigma-q)^2\Bigg(p\,(a_{12}+a_{21}-1)+(1-p)2 \times\Bigg(\left(a_{211}-\frac{1}{3}\right)\sigma+(1-\sigma)\left(a_{122}-\frac{1}{3}\right) + (2\sigma-1)(a_{121}-a_{221})\Bigg)\Bigg)>0, \tag{16}$$

while in case of equality, condition Eq. (9) must holds

$$(\sigma-q)\left(\left(\frac{1}{3}-a_{211}\right)+\left(a_{122}-\frac{1}{3}\right)-2(a_{121}-a_{221})\right)>0. \tag{17}$$

Let us note that this condition will always depend on the sign of the term $(\sigma-q)$. This leads us to conclude that x is an ESS if and only if it satisfies condition equation (16).

In the general case when k players are involved in the game with probability p_k, we will have a similar phenomenon. The inequality in the condition for being an ESS for the even labels will depend on the sign of $(\sigma-q)$, then there will be no ESS of those levels.

IV. DUEL OR TRUEL GAME

We now consider a duel–truel random game. The duel and the truel are both matrix games, and we are going to consider only two strategies, s_1 and s_2. In these games, players shoot each other in turns, until only one player remains alive. A player using strategy s_1 kills its opponent with probability 1 when it shoots, which we call *perfect* strategy, while it kills its opponent with probability 1/2 when uses strategy s_2, which we call *mediocre* strategy.

In a round of a duel game, one player is chosen at random (with probability 1/2) to shoot first and then kills its opponent with a probability that depends on the strategy that uses, s_1 or s_2. In case the player misses the target and thus its opponent survives, then the opponent shoots. The shootings and rounds continue until one player finally dies and the other survives, and the game ends. Similarly, in a round of a truel game one player is chosen at random with probability 1/3 to start the shootings. If it kills an opponent, then the remaining opponent shoots, otherwise, one of the two alive opponents is chosen to shoot with probability 1/2. As for the duel, the game ends when only one of the three players remains alive.

Each player's actions are always taken to improve its chances of survival. The element a_{ij} of the duel matrix is the probability that player *I* survives (or the payment it receives), when it uses strategy s_i and plays against player *II* that uses strategy s_j. Analogously, the element a_{ijk} of the truel matrix is the survival probability of player *I*, which uses strategy s_i, when it plays against players *II* and *III* that use strategies s_j and s_k, respectively. The values of these elements are calculated in Appendix A and are given by $a_{11} = a_{22} = 1/2$, $a_{21} = 1/4$, and $a_{12} = 3/4$ for the duel matrix and $a_{111} = a_{222} = 1/3$, $a_{211} = 1/2, a_{121} = a_{112} = 1/4, a_{221} = a_{212} = 17/48$, and $a_{122} = 7/24$ for the truel matrices. The other matrices are obtained by transposing the players' positions. Then, the duel matrix game is given by

	s_1	s_2
s_1	(1/2, 1/2)	(3/4, 1/4)
s_2	(1/4, 3/4)	(1/2, 1/2)

while the truel matrices are given by

	s_1	s_2
s_1	(1/3, 1/3, 1/3)	(1/4, 1/4, 1/2)
s_2	(1/4, 1/2, 1/4)	(7/24, 17/48, 17/48)

when player *III* plays s_1 and by

	s_1	s_2
s_1	(1/2, 1/4, 1/4)	(17/48, 7/24, 17/48)
s_2	(17/48, 17/48, 7/24)	(1/3, 1/3, 1/3)

when player *III* plays s_2. Then, we compute the expected payoffs $E_G[1,\sigma]$ and $E_G[0,\sigma]$ of player *I* when it plays strategies s_1 and s_2, respectively, as

$$E_G[1,\sigma] = p\,[a_{11}\sigma + a_{12}(1-\sigma)] + (1-p) \times \left[a_{122}(1-\sigma)^2 + (a_{121}+a_{112})\sigma(1-\sigma) + a_{111}\sigma^2\right],$$

$$E_G[0,\sigma] = p\,[a_{21}\sigma + a_{22}(1-\sigma)] + (1-p) \times \left[a_{222}(1-\sigma)^2 + (a_{221}+a_{212})\sigma(1-\sigma) + a_{211}\sigma^2\right],$$

and get

$$E_G[1,\sigma] = p\left(\frac{3}{4} - \frac{1}{4}\sigma\right) + (1-p)\left[\frac{7}{24}(1-\sigma)^2 + \frac{1}{2}\sigma(1-\sigma) + \frac{1}{3}\sigma^2\right],$$

$$E_G[0,\sigma] = p\left(\frac{1}{2} - \frac{1}{4}\sigma\right) + (1-p)\left[\frac{1}{3}(1-\sigma)^2 + \frac{17}{24}\sigma(1-\sigma) + \frac{1}{2}\sigma^2\right],$$

after replacing the coefficients a_{ij} and a_{ijk} $(i,j,k = 1,2)$. Then, the replicator equation from Eq. (13) is

$$\frac{d\sigma}{dt} = \sigma(1-\sigma)\left[\frac{p}{4} - (1-p)\left(\frac{1}{24} + \frac{1}{8}\sigma\right)\right]. \tag{18}$$

This equation has three fixed points: $\sigma = 0$, $\sigma = 1$, and

$$\sigma^* = \frac{7p-1}{3(1-p)}. \tag{19}$$

From the Folk theorem,[32] the interior rest point is a Nash equilibrium; however, the boundary points can be Nash equilibria or not.

Since σ^* is the proportion of the population playing strategy s_1, we need $0 < \sigma^* < 1$, which holds for $\frac{1}{7} < p < \frac{2}{5}$. Also, by using condition equations (14) and (15), we get that the Nash equilibrium of the game depends on the value of p that we are using

$$\sigma_{NE} = \begin{cases} 0 & \text{if} \quad p \le \frac{1}{7}, \\ \sigma^* & \text{if} \quad \frac{1}{7} < p < \frac{2}{5}, \\ 1 & \text{if} \quad \frac{2}{5} \le p. \end{cases}$$

The pure Nash equilibria are ESS. In order to analyze the mixed equilibrium, we need to consider inequality equation (7), and this proves that the mixed equilibrium is an ESS for $\frac{1}{7} < p < \frac{2}{5}$.

Let us observe that the killing probabilities 1/2 and 1 for the *mediocre* and *perfect* strategies, respectively, are not critical for the results we obtained. A sensitivity analysis shows that we can vary both probabilities in certain range and the results still hold, changing only the intervals for p.

V. AGENT-BASED DYNAMICS ON COMPLEX NETWORKS

In this section, we introduce and study a model of interacting agents on a complex network, which provides a kinetic approach to the duel–truel problem. The system consists of a population of N agents that play the duel and truel games, and are allowed to update their strategies (perfect vs mediocre) as they interact with other agents. Each agent is located at a node of a complex network of N nodes and degree distribution $\mathscr{P}_k$ (fraction of nodes with k links), and it is allowed to interact with its neighbors in the network. A given agent i $(i = 1,\ldots,N)$ can be in one of two possible states, $\theta_i = 1$ (playing strategy s_1) or $\theta_i = 2$ (playing strategy s_2), corresponding to the strategies of the perfect and mediocre players, respectively. In the initial configuration, each agent takes the state 1 or 2 with equal probability 1/2. We denote by σ_i the fraction of perfect neighbors of agent i, i.e., agent i's neighbors in state $\theta = 1$. The expected payoffs of agent i depend on σ_i, its state θ_i, and the game (duel or truel) that it plays, and they are given by

$$P_1^i = \frac{3}{4} - \frac{1}{4}\sigma_i \quad \text{if}\, \theta_i = 1, \tag{20a}$$

$$P_2^i = \frac{1}{2} - \frac{1}{4}\sigma_i \quad \text{if}\, \theta_i = 2, \tag{20b}$$

when i plays the duel game and by

$$\tilde{P}_1^i = \frac{1}{3}\sigma_i^2 + \frac{1}{2}\sigma_i(1-\sigma_i) + \frac{7}{24}(1-\sigma_i)^2 \quad \text{if}\, \theta_i = 1, \tag{21a}$$

$$\tilde{P}_2^i = \frac{1}{2}\sigma_i^2 + \frac{17}{24}\sigma_i(1-\sigma_i) + \frac{1}{3}(1-\sigma_i)^2 \quad \text{if}\, \theta_i = 2, \tag{21b}$$

when i plays the truel game.

This system of agents is endowed with the following dynamics. In a single time step $\Delta t = 1/N$, an agent i with state θ_i is chosen at random. Then, with probability p agent i plays the duel game: one neighboring agent j of i is randomly chosen, and i adopts the state θ_j of j ($\theta_i \to \theta_i = \theta_j$) with a probability equal to j'payoff P_{θ_j}. With the complementary probability $1-p$, agent i plays the truel game: two random neighbors j and k of i are chosen, and i tries to adopt either the state θ_j of j ($\theta_i \to \theta_i = \theta_j$) with probability $\frac{1}{2}\tilde{P}_{\theta_j}$ or the state θ_k of k ($\theta_i \to \theta_i = \theta_k$) with probability $\frac{1}{2}\tilde{P}_{\theta_k}$. In case that a node with only one neighbor (degree $k=1$) is chosen to play a truel game, nothing happens.

The system evolves under this dynamics until it reaches a stationary state, where the fraction of perfect agents remains in a stationary value σ^{stat}. Note from Eqs. (20) and (21) that $P_1^i > P_2^i$ and $\tilde{P}_2^i > \tilde{P}_1^i$ for all $0 \le \sigma_i \le 1$. Therefore, in the long run, we expect a consensus of state-1 agents (all agents in state $\theta = 1, \sigma^{\text{stat}} = 1$) when all agents play the duel game, and a state-2 consensus when all agents play the truel game ($\sigma^{\text{stat}} = 0$). However, a stationary coexistence of both types of agents might be possible when agents are allowed to play duel and truel with probabilities p and $1-p$, respectively. We are interested in exploring the behavior of σ^{stat} with p for networks of different degree distributions, and how σ^{stat} is compared to the value σ^* [Eq. (19)] obtained at the mixed equilibrium of strategies predicted in Sec. IV. In Secs. V A and V B, we study the system on a complete graph (all-to-all interactions) and on random networks with various degree distributions, respectively.

A. Complete graph

In order to gain an insight into the evolution of the system, we investigate in this section the case where each agent can interact with anyone else, which corresponds to the model on a complete graph (CG) or mean-field (MF) ($\mathscr{P}_k = \delta_{k,N-1}$). We define by σ the fraction of agents in state 1 (perfect players); thus, the fraction of state-2 agents (mediocre players) is $1-\sigma$. In the $N \to \infty$ limit, the time evolution of σ is given by the following rate equation:

$$\frac{d\sigma}{dt} = \left[p(P_1 - P_2) + (1-p)(\tilde{P}_1 - \tilde{P}_2)\right]\sigma(1-\sigma), \tag{22}$$

which becomes

$$\frac{d\sigma}{dt} = \left[\frac{1}{4}p - \left(\frac{1}{24} + \frac{s}{8}\right)(1-p)\right]\sigma(1-\sigma) \tag{23}$$

after replacing the expressions for the expected payoffs from Eqs. (20) and (21) and taking $\sigma_i = \sigma$, given that on a CG the fraction of state-1 neighbors of a given agent matches the fraction of state-1 agents of the entire population when $N \gg 1$. The first term in Eq. (22) corresponds to a duel event, which happens with probability p. The gain term $P_1\sigma(1-\sigma)$ represents the transition $21 \to 11$, where two random agents i and j with states $\theta_i = 2$ and $\theta_j = 1$, respectively, are chosen with probability $(1-\sigma)\sigma$, and then i copies j's state with probability P_1, leading to a positive change of $1/N$ in σ. The loss term $P_2\sigma(1-\sigma)$ corresponds to the transition $12 \to 22$, where an agent i with state $\theta_i = 1$ copies the state of another agent j with state $\theta_j = 2$, leading to a negative change of $1/N$ in σ. Analogously, the second term in Eq. (22) corresponds to a truel event (probability $1-p$), where the three possible transitions that make up the gain term $\tilde{P}_1\sigma(1-\sigma)$ are

$$211 \to 111 \quad \text{with prob.}\, (1-\sigma)\sigma^2\,\tilde{P}_1, \tag{24a}$$

$$221 \to 121 \quad \text{with prob.}\, (1-\sigma)^2\sigma\,\frac{1}{2}\tilde{P}_1, \tag{24b}$$

$$212 \to 112 \quad \text{with prob.}\, (1-\sigma)^2\sigma\,\frac{1}{2}\tilde{P}_1, \tag{24c}$$

while the transitions that lead to the loss term $\tilde{P}_2\sigma(1-\sigma)$ are

$$122 \to 222 \quad \text{with prob.}\, \sigma(1-\sigma)^2\,\tilde{P}_2, \tag{25a}$$

$$112 \to 212 \quad \text{with prob.}\, \sigma^2(1-\sigma)\,\frac{1}{2}\tilde{P}_2, \tag{25b}$$

$$121 \to 221 \quad \text{with prob.}\, \sigma^2(1-\sigma)\,\frac{1}{2}\tilde{P}_2. \tag{25c}$$

We note that Eq. (23) agrees with that of the replicator equation (18) derived in Sec. IV, which has three fixed points. The fixed points $\sigma = 0$ and $\sigma = 1$ correspond to the consensus of mediocre and perfect players, respectively, while the third fixed point $\sigma^* = (7p-1)/[3(1-p)]$ corresponds to a mixed state where state-1 and state-2 agents coexist with fractions σ^* and $1-\sigma^*$, respectively. An expansion of Eq. (23) around $\sigma = 0$ to first order in $\epsilon = \sigma \ll 1$ leads to $\frac{d\epsilon}{dt} = \frac{(7p-1)}{24}\epsilon$ and, therefore, $\sigma = 0$ is stable for $p < 1/7$. A similar stability analysis around $\sigma = 1$ leads to $\frac{d\epsilon}{dt} = \frac{(2-5p)}{12}\epsilon$, where $\epsilon = 1-\sigma \ll 1$, and, thus, $\sigma = 1$ is stable for $p > 2/5$. Then, the stable fixed points on a CG are given by

$$\sigma_{\text{CG}} = \begin{cases} 0 & \text{for} \quad p \le \frac{1}{7}, \\ \frac{7p-1}{3(1-p)} & \text{for} \quad \frac{1}{7} < p < \frac{2}{5}, \\ 1 & \text{for} \quad p \ge \frac{2}{5}. \end{cases} \tag{26}$$

In Fig. 1, we plot by a dashed line the stable fixed points σ_{CG} of Eq. (23) as a function of p, which corresponds to the dynamics on

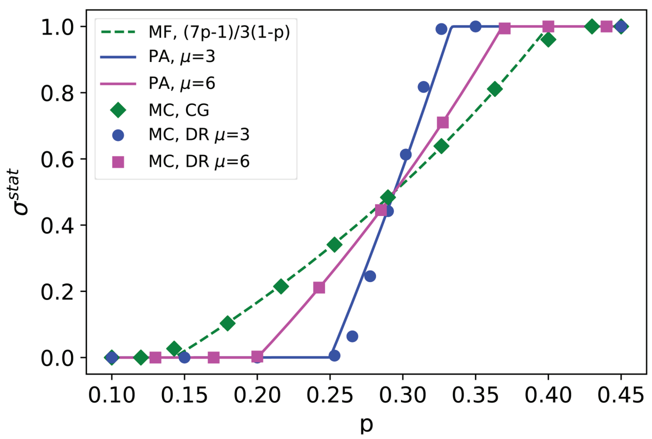

FIG. 1. Stationary fraction of perfect agents σ^{stat} vs duel probability p for the values of the mean degree μ indicated in the legend. The dashed line corresponds to the stable solution σ_{CG} on a CG [Eq. (26)], while solid lines represent the solution from the PA equations (27)–(30). Symbols correspond to the average value of σ at the stationary state obtained from MC simulations on a CG of size $N = 10^3$ (diamonds), and DRRGs of size $N = 10^4$ and degrees $\mu = 6$ (squares) and $\mu = 3$ (circles). Averages were done over different independent realizations of the dynamics, 100 for the CG and 10 for the DRRGs.

a CG. This behavior is the same as that obtained from the theory of Nash equilibrium analyzed in Sec. IV. Therefore, we conclude that the Nash equilibrium corresponds to the stable states of the agent-based model on a CG or MF scenario.

These results show that a stable coexistence of perfect and mediocre players is obtained for values of p in the interval $p_0 < p < p_1$, with $p_0 = 1/7$ and $p_1 = 2/5$, while for $p < p_0$ ($p > p_1$) mediocre (perfects) players dominate. In order to explore how the coexistence phase is affected by the topology of interactions between agents, we study in Sec. V B the duel–truel dynamics on complex networks.

B. Complex networks

In this section, we derive and analyze equations that describe the time evolution of the system in complex networks with a general degree distribution $\mathscr{P}_k$, i.e., the fraction of nodes with k links, subject to the normalization $\sum_{k=1}^{N-1} \mathscr{P}_k = 1$. For that, we use the *homogeneous pair approximation* (PA) approach developed in Refs. 28–31, which is a mean-field approach that takes into account state correlations between first nearest neighbors in the network and neglects correlations to second and higher-order nearest neighbors. This approximation should work well in uncorrelated or random networks like degree-regular random graphs (DRRGs) or Erdös–Renyi (ER) networks, where the neighbors of each node are chosen at random, and thus degree correlations are negligible. In Appendix B, we derive a rate equation for the fraction of perfect agents σ that depends on σ and the fraction of *active links* ρ, i.e., links that connect mediocre and perfect agents (2–1 or 1–2 links) and, therefore, are subject to change. The density ρ allows to capture correlations between the states of neighboring nodes in the network, and together with σ form the following system of two rate equations that describe the dynamics of the model on a complex network (see Appendix B):

$$\frac{d\sigma}{dt} = \frac{\rho}{2}\left[p(P_{1|2} - P_{2|1}) + (1-p)(\tilde{P}_{1|2} - \tilde{P}_{2|1})(1 - \mathscr{P}_1)\right], \quad (27)$$

$$\begin{aligned}\frac{d\rho}{dt} = \frac{\rho}{\mu}\Bigg\{ &\left[pP_{2|1} + (1-p)\tilde{P}_{2|1}\right]\left[(\mu-1)\left(1 - \frac{\rho}{\sigma}\right) - 1\right] \\ &+ \left[pP_{1|2} + (1-p)\tilde{P}_{1|2}\right]\left[(\mu-1)\left(1 - \frac{\rho}{1-\sigma}\right) - 1\right] \\ &+ (1-p)\left(\tilde{P}_{2|1} + \tilde{P}_{1|2}\right)\mathscr{P}_1\Bigg\},\end{aligned} \quad (28)$$

where $\mu \equiv \langle k \rangle = \sum_{k=1}^{N-1} k\,\mathscr{P}_k$ is the mean degree of the network. The terms that involve $\mathscr{P}_1$ correspond to the case where an agent of degree $k = 1$ fails to play a truel game. Here, we denote by $P_{2|1}$ and $\tilde{P}_{2|1}$ the expected payoffs of a state-2 agent that has at least one neighbor in state 1, when it plays a duel and a truel, respectively, and analogously for $P_{1|2}$ and $\tilde{P}_{1|2}$. The expected payoffs are given by Eqs. (20) and (21),

$$P_{2|1} = \frac{1}{2} - \frac{1}{4}\sigma_{2|1}, \quad (29a)$$

$$P_{1|2} = \frac{3}{4} - \frac{1}{4}\sigma_{1|2}, \quad (29b)$$

$$\tilde{P}_{2|1} = \frac{1}{2}\sigma_{2|1}^2 + \frac{17}{24}\sigma_{2|1}(1-\sigma_{2|1}) + \frac{1}{3}(1-\sigma_{2|1})^2, \quad (29c)$$

$$\tilde{P}_{1|2} = \frac{1}{3}\sigma_{1|2}^2 + \frac{1}{2}\sigma_{1|2}(1-\sigma_{1|2}) + \frac{7}{24}(1-\sigma_{1|2})^2, \quad (29d)$$

where $\sigma_{2|1}$ ($\sigma_{1|2}$) is the fraction of state-1 neighbors of a state-2 (state-1) node that has at least one neighbor in the opposite state 1 (2), which can be estimated as (see Sec. 1 of Appendix B)

$$\sigma_{2|1} \simeq \frac{1}{\mu}\left[1 + \frac{(\mu-1)\rho}{2(1-\sigma)}\right] \quad (30a)$$

and

$$\sigma_{1|2} \simeq \frac{(\mu-1)}{\mu}\left(1 - \frac{\rho}{2\sigma}\right). \quad (30b)$$

Now, we can use these equations to study how the mean degree μ of the network and $\mathscr{P}_1$ affect the behavior of σ with p at the stationary state. For that, we integrated numerically the set of Eqs. (27) and (28) together with Eqs. (29) and (30) and plot the stationary value σ^{stat} vs p for different values of μ and $\mathscr{P}_1$, as it is shown by lines in Fig. 1. In this figure, we see the results for a degree distribution $\mathscr{P}_k = \delta_{k,\mu}$, with $\mathscr{P}_1 = 0$ and $\mu = 6$ and 3 (solid lines), which corresponds to DRRGs where each node is connected to other μ random nodes. We have also checked that the solution for the case $\mathscr{P}_k = \delta_{k,N-1}$ matches that of the CG case obtained from Eq. (23) and depicted by a dashed line. We observe that, as μ decreases, the transition values p_0 and p_1 that define the coexistence phase become

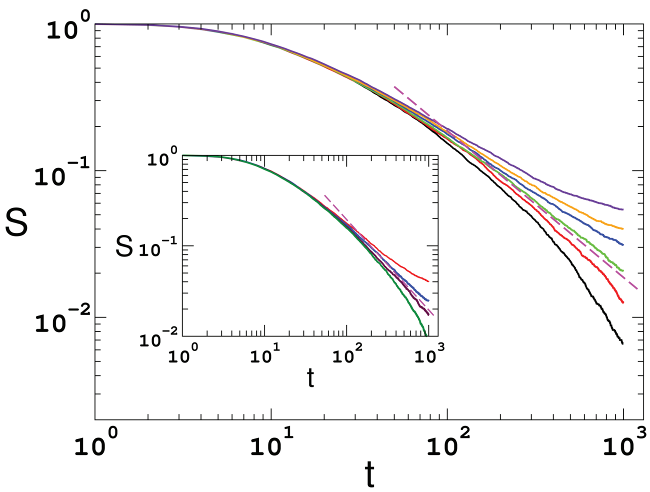

FIG. 2. Survival probability S on a DRRG of $N = 10^4$ nodes, mean degree $\mu = 3$, and various values of p. Main panel: evolution of S after a small perturbation of the two-consensus absorbing state. Curves correspond to $p = 0.265, 0.260, 0.255, 0.250, 0.245$, and 0.240 (from top to bottom). Inset: evolution of S after a small perturbation of the one-consensus absorbing state. Curves correspond to $p = 0.325, 0.330, 0.335$, and 0.340 (from top to bottom). At the transition points $p_0 \simeq 0.250$ and $p_1 \simeq 0.335$, S decays as $S \sim t^{-1}$ (dashed lines).

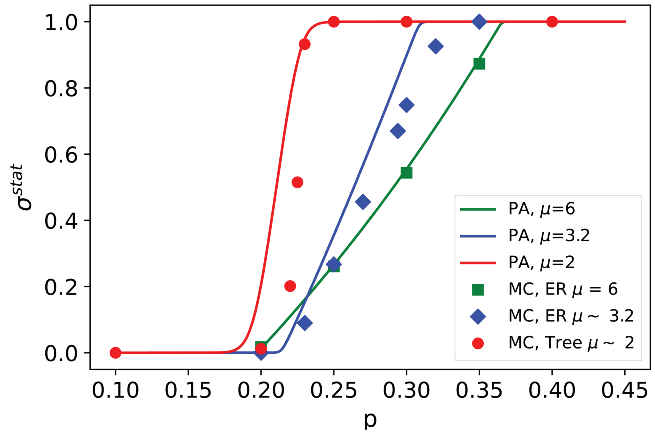

FIG. 3. σ^{stat} vs p for the values of μ indicated in the legend. Solid lines are the solutions from the PA, while symbols correspond to the average value of σ over ten independent realizations at the stationary state obtained from MC simulations on ER networks with $N = 10^4$ nodes and mean degrees $\mu = 6$ (squares) and $\mu \simeq 3.2$ (diamonds) and on a tree network with $N = 10^4$ nodes and $\mu \simeq 2$ (circles).

closer and, therefore, the interval of p for which there is a coexistence of perfect and mediocre players is reduced. This suggests that the coexistence phase tends to shrink and might eventually vanish as the network becomes more sparse by reducing μ.

Although these results were obtained from the PA equations and for DRRGs, we shall see in Sec. V C that a similar behavior is observed for networks with a broader degree distribution.

C. Monte Carlo simulations

We have performed extensive numerical simulations on various networks and checked the analytical results obtained in Secs. V A and V B. We started by running the Monte Carlo (MC) dynamics described in Sec. V on a CG of $N = 10^3$ nodes, and DRRGs of $N = 10^4$ nodes each and mean degrees $\mu = 3$ and $\mu = 6$. We ran the dynamics until the fraction of perfect agents σ reached a stationary value and calculated the average value of σ at the stationary state over 100 independent realizations for the CG and over 10 independent realizations for the DRRGs. Results are shown by symbols in Fig. 1.

We observe a good agreement with the analytical results provided by the MF approximation σ_{CG} on a CG from Eq. (26) (dashed line) and by the PA from Eqs. (27) and (28) (solid lines). However, we see that close to the transition points p_0 and p_1 the simulation points slightly deviate from the analytical prediction. To check if this is due to finite-size effects, we have performed spreading experiments that allowed us to calculate the transition points with high accuracy. These experiments consisted of making a small perturbation of the two absorbing states corresponding to 2-consensus and 1-consensus and see how this perturbation evolves. For instance, for the $\sigma = 0$ absorbing state, we assigned initially the state $\sigma = 0$ to all agents except for a small seed of four random neighboring agents in the network that were assigned the state $\sigma = 1$, and let the system evolve. Then, for values of p below the transition point p_0, the seed quickly vanishes and, thus, all realizations quickly reach the absorbing state, while for $p > p_0$, there are some realizations that survive, where the seed spreads over a fraction of the population. An analogous behavior is obtained around p_1 by perturbing the $\sigma = 1$ absorbing state. To quantify this, we ran the dynamics with these initial conditions and measured the survival probability $S(t)$ of a single run, calculated as the fraction of realizations that did not reach the absorbing state up to time t. Then, the transition point is estimated as the value of p for which $S(t)$ decays as a power law in time. In Fig. 2, we plot the time evolution of $S(t)$ for a DRRG of size $N = 10^4$ and $\mu = 3$. We see that $S(t)$ decays as a power law with exponent -1 at the transition points $p_0 \simeq 0.250$ (main panel) and $p_1 \simeq 0.335$ (inset). These values are very similar to those obtained from the PA, $p_0 \simeq 0.248$ and $p_1 \simeq 0.338$, which confirms that the discrepancies with the numerical simulations in Fig. 1 are due to finite-size effects.

We then ran simulations on Erdös–Renyi (ER) networks to explore the impact of the broadness of the degree distribution on the results shown above. We were particularly interested in the case of low mean degrees, where the behavior tends to be quite different from that of MF and where analytical approximations usually fail. As ER networks with low mean degree may get disconnected, we considered the largest connected component of the network as the new system and used the values of its resulting mean degree μ and $\mathscr{P}_1$ in the PA equations. We also ran simulations on networks with a tree-like structure, which have a mean degree close to 2. In Fig. 3, we show by symbols the results of MC simulations for ER networks of $N = 10^4$ nodes each and mean degrees $\mu = 6$ and $\mu = 3.2$, and for tree-like networks of $N = 10^4$ nodes and $\mu \simeq 2$. We also show by

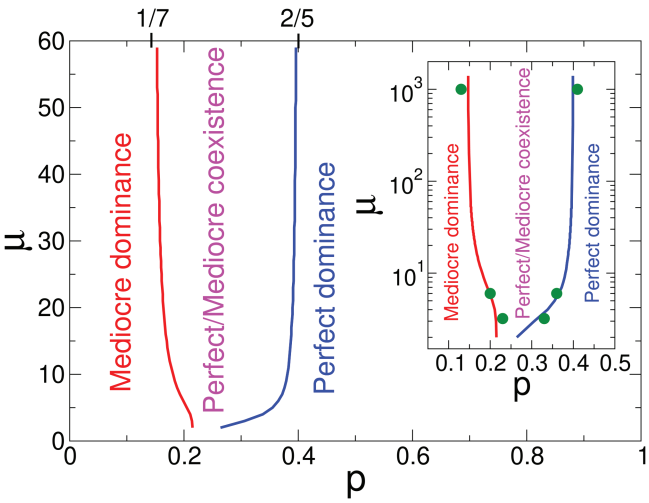

FIG. 4. Phase diagram on the p–μ space showing the transition lines between the coexistence and dominance phases of the model on ER networks, obtained from the PA equations (27) and (28) (solid curves) and from MC simulations on ER networks with mean degrees $\mu = 3.2$ and 6, and a CG with $N = 10^3$ ($\mu = 999$) (solid circles in the inset). The inset shows the plot in linear-log scale for a better visualization of the data points in the range $\mu \in (2, 1000)$.

solid lines the solutions from the PA equations (27) and (28), with the values of μ and $\mathscr{P}_1$ that correspond to each network, as shown in the legend. We see that the agreement with simulations is good for the ER network with $\mu = 6$ but discrepancies arise for the ER with $\mu = 3.2$ and the tree network, which confirms that the PA start to fail for very low values of μ.

We can also see in Fig. 3 that the coexistence phase becomes smaller when μ decreases, as we have seen already for DRRGs (Fig. 1) but it does not seem to vanish completely. For instance, the coexistence phase for the tree network ($\mu \simeq 2$) lays in the small interval $0.20 \lesssim p \lesssim 0.25$. To explore this in more detail, we integrated Eqs. (27) and (28) for several values of $\mu \geq 2$ and $\mathscr{P}_1$ corresponding to an ER network and obtained the transition lines $p_0(\mu)$ and $p_1(\mu)$ that separate the coexistence and dominance phases. This is shown in the phase diagram of Fig. 4, where we see that the coexistence phase increases with μ and approaches the region defined by the interval $[1/7, 2/5]$ that corresponds to the CG. In the inset of Fig. 4, we show the same plot on a linear-log scale for a better visualization, where we added the transition points obtained from MC simulations (circles) for CG and ER networks with $\mu = 3.2$ and 6 (see Figs. 1 and 3).

VI. SUMMARY AND CONCLUSIONS

In this paper, we have extended the definition of evolutionary stable strategy to symmetric games with a random number of players. We derived the conditions under which a given strategy is an ESS, and applied this concept to the study of a population of players that can use one of two possible strategies, perfect and mediocre, and play a duel with probability p or a truel with probability $1 - p$. We showed that the duel–truel game has a mixed equilibrium that is an ESS for values of p in an interval, in which perfect and mediocre players coexist.

We also introduced and studied an agent-based model on complex networks with a microscopic dynamics that leads to an evolution that is well described in mean-field by the replicator equation of the duel–truel game. We showed that the stable solutions of this equation correspond to those of the ESS equilibrium. Moreover, we developed a pair approximation approach to study the dynamics of the model on complex networks with a general degree distribution, which showed that the coexistence phase predicted by the theory is also present when agents interact in complex topologies. We applied this approach to random networks and found that the interval of p for which the stable coexistence of perfect and mediocre players is observed only depends on the mean degree of the network μ but not on higher moments. We found that the coexistence phase shrinks as μ decreases, but it does not seem to vanish completely even for small values of μ. As a consequence, a given unstable mix of the two types of players for some value of p can turn into stable when the mean number of neighbors of a player is increased beyond a threshold. This result implies that the network of interactions affects the stability of the system by inducing a stable coexistence when its connectivity increases.

In order to check these findings, we performed Monte Carlo simulations of the model in degree regular random graphs of low mean degrees, and we could verify that the analytical results were in good agreement with simulations on these networks. The accuracy of the transition points between the coexistence and dominance phases predicted by the pair approximation was also tested by means of spreading experiments. We also carried out simulations on Erdös–Renyi and tree-like networks to investigate the behavior of the model for broad degree distributions and μ close to 2. We found that for low degrees, the analytical approximation gives results that deviate from those of simulations. However, we could verify numerically that, although very small, the coexistence phase is still present in sparse networks.

It would be worthwhile to perform a more in-depth study of the duel–truel game in complex networks with very broad degree distributions, like scale-free networks, and investigate if the observed phenomenology is affected by the width of the distribution. It might also be interesting to extend the theoretical approximations for networks that have degree correlations. Finally, as a future work, we could apply the conditions obtained for ESS to games with interactions higher than three and different strategies and also explore the stability of the coexistence phase for different values of the killing probabilities of perfect and mediocre players.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflict of interest to disclose.

Ethics Approval

Ethics approval is not required.

DATA AVAILABILITY

The data that support the findings of this study are available within the article.

APPENDIX A: CALCULATION OF THE ELEMENTS OF THE DUEL AND TRUEL MATRICES

In this section, we calculate the elements a_{ij} and a_{ijk} of the duel and truel matrices, respectively. The element a_{ij} is the payment that receives player I, who uses strategy s_i, when it plays against player II, which uses strategy s_j, and it corresponds to the probability that player I survives in a duel (player II dies). Analogously, the element a_{ijk} is the payment of player I using strategy s_i when it plays against players II and III that use strategies s_j and s_k, respectively, and corresponds to the survival probability of player I in a truel (players II and III die).

When the game is a duel, one of the two players is chosen randomly (with probability $1/2$) to shoot first. By symmetry, if the two players use the same strategy, then each survives with probability $1/2$, and thus $a_{11} = a_{22} = 1/2$. If player I uses strategy s_2 and player II uses strategy s_1, player I can only survive when it shoots first and kills player II, given that if it misses the target (it does not kill player II), then player II kills it in the next turn. Then, the survival probability of player I is $a_{21} = \frac{1}{2}\frac{1}{2} = \frac{1}{4}$. From this result, it follows that player II survives with the complementary probability $a_{12} = 3/4$.

When the game is a truel, each player survives with the same probability $1/3$ when the three players use the same strategy and, thus, $a_{111} = 1/3$ and $a_{222} = 1/3$.

If player I uses strategy s_1 and players II and III use strategy s_2, we consider that II and III will always shoot I when it is alive, in order to increase their chances of surviving. To obtain the survival probability of player I, we consider the three case scenarios of player I shooting in the first, second, and third turn, where each happens with probability $1/3$. If player I shoots first, it kills II or III. Then, the remaining player shoots I, who survives with probability $1/2$ and then kills the remaining player, ending the game. Therefore, player I survives the game with probability $1/2$. In the case that player I shoots second, the player shooting first shoots I, who survives with probability $1/2$. After that, I kills II or III and then the remaining player shoots I, which survives with probability $1/2$ and then kills this player, ending the game. Following the same line of reasoning, when player I shoots third, it survives only if the other two players fail all together three times. Adding the three cases, the survival probability of player I is $a_{122} = \frac{1}{3}(\frac{1}{2} + \frac{1}{4} + \frac{1}{8}) = \frac{7}{24}$. From this result, players II and III survive with probability $\frac{17}{48}$ each and, therefore, $a_{212} = a_{221} = \frac{17}{48}$.

Finally, we consider the case in which player I uses strategy s_2 and players II and III use s_1. When player I shoots first, we assume that it is allowed to pass its turn by missing the shot on purpose ("shooting into the air"), in order to increase its chances of surviving. Otherwise, if player I kills either II or III, then it will be killed by the remaining player; thus, its survival probability will be zero. Thus, if player I misses the shot, then II will kill III or vice versa and then I will kill the remaining perfect player with probability $1/2$, i.e., surviving with probability $1/2$. When player I shoots second, either I or II is alive, since one killed the other in the first turn to improve its survival chances. Then, I survives with probability $1/2$ to its perfect opponent. Given that player I can never shoot third, it survives the truel with probability $a_{211} = 1/2$. Therefore, perfect players II and III survive with probability $1/4$ each, and thus $a_{112} = a_{121} = 1/4$.

APPENDIX B: DERIVATION OF THE RATE EQUATIONS FOR σ AND ρ

In this section, we derive rate equations for the evolution of the fraction of perfect agents σ and the fraction of active links ρ on random networks with a general degree distribution $\mathscr{P}_k$.

1. Equation for σ

In Fig. 5(a) we show an schematic illustration of the transition $21 \to 11$ in a single update event, which leads to a change of $1/N$ in σ. This transition involves the following processes and associated probabilities. A node i with state $\theta_i = 2$ and degree k is chosen at random with probability $(1-\sigma)\mathscr{P}_k$. Then, with probability $n^1_{2,k}/k$, a neighbor j with state $\theta_j = 1$ is chosen at random, where $n^1_{2,k}$ $(0 \le n^1_{2,k} \le k)$ is the number of neighbors of i in the opposite state $\theta = 1$, i.e., the number of *active links* connected to node i. We recall that an active link is a link that connects nodes with different states (2–1 or 1–2 links). Finally, i copies j's state with a probability equal to the mean payoff of j, which we denote by $P_{1|2}$ or $\tilde{P}_{1|2}$, when i plays

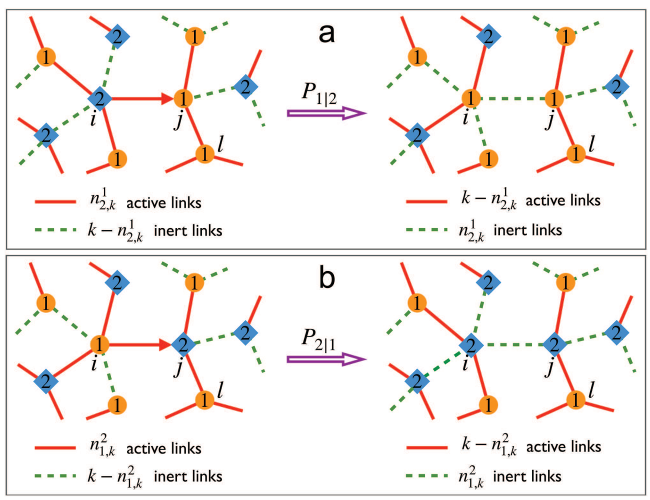

FIG. 5. (a) Update event in which a node i with state $\theta_i = 2$ (diamond) adopts the state $\theta_j = 1$ of its neighboring node j (circle) with probability $P_{1|2}$ (j's payoff). The change in σ and ρ is $1/N$ and $\frac{2\left(k-2n^1_{2,k}\right)}{\mu N}$, respectively. (b) Update event in which a node i with state $\theta_i = 1$ (circle) adopts the state $\theta_j = 2$ of its neighboring node j (diamond) with probability $P_{2|1}$ (j's payoff). The change in σ and ρ is $-1/N$ and $\frac{2\left(k-2n^2_{1,k}\right)}{\mu N}$, respectively.

a duel or truel, respectively. Here, we use the subindex 1|2 to denote that j has state $\theta_j = 1$ and at least one neighbor (node i) with state $\theta_i = 2$ [see Fig. 5(a)].

Analogously, with probability $\sigma\, \mathscr{P}_k \frac{n^2_{1,k}}{k} P_{2|1}$, a randomly chosen node i with state $\theta_i = 1$ and degree k copies a random neighbor j with state $\theta_j = 2$ when it plays a duel [see Fig. 5(b)], where $n^2_{1,k}$ is the number of state-2 neighbors of i and $P_{2|1}$ is the expected payoff of j, which has at least a neighbor with state 1 (node i). This corresponds to a 12 → 22 transition, leading to a change of $-1/N$ in σ. The expression for the probability of a truel game is the same but replacing $P_{2|1}$ by $\tilde{P}_{2|1}$. Adding these four terms corresponding to the two transitions and the two playing strategies, we can write the mean change of σ in a time step $\Delta t = 1/N$ as

$$\frac{d\sigma}{dt} = \frac{(1-\sigma)}{1/N}\left\{p\sum_{k=1}^{N-1}\mathscr{P}_k\sum_{n^1_{2,k}=0}^{k}B\left(n^1_{2,k}\right)\frac{n^1_{2,k}}{k}P_{1|2} + (1-p)\sum_{k=2}^{N-1}\mathscr{P}_k\sum_{n^1_{2,k}=0}^{k}B\left(n^1_{2,k}\right)\frac{n^1_{2,k}}{k}\tilde{P}_{1|2}\right\}\frac{1}{N}$$
$$+\frac{\sigma}{1/N}\left\{p\sum_{k=1}^{N-1}\mathscr{P}_k\sum_{n^2_{1,k}=0}^{k}B\left(n^2_{1,k}\right)\frac{n^2_{1,k}}{k}P_{2|1} + (1-p)\sum_{k=2}^{N-1}\mathscr{P}_k\sum_{n^2_{1,k}=0}^{k}B\left(n^2_{1,k}\right)\frac{n^2_{1,k}}{k}\tilde{P}_{2|1}\right\}\left(-\frac{1}{N}\right), \quad \text{(B1)}$$

where $B\left(n^1_{2,k}\right)$ is the probability that $n^1_{2,k}$ active links are connected to a node of state 2 and degree k and analogously for $B\left(n^2_{1,k}\right)$. The first and second summations in each term run over the degrees k and the number of active links around a node of degree k, respectively. We note that the summations over k in the second and fourth terms, which corresponds to a truel game, start from $k = 2$ because a node with only one neighbor (degree $k = 1$) cannot play the truel game. The mean payoffs of a neighbor j of i are given by Eqs. (20) and (21),

$$P_{2|1} = \frac{1}{2} - \frac{1}{4}\sigma_{2|1}, \quad \text{(B2a)}$$

$$\tilde{P}_{2|1} = \frac{1}{2}\sigma^2_{2|1} + \frac{17}{24}\sigma_{2|1}(1-\sigma_{2|1}) + \frac{1}{3}(1-\sigma_{2|1})^2 \quad \text{(B2b)}$$

if $\theta_j = 2$ and $\theta_i = 1$, and

$$P_{1|2} = \frac{3}{4} - \frac{1}{4}\sigma_{1|2}, \quad \text{(B3a)}$$

$$\tilde{P}_{1|2} = \frac{1}{3}\sigma^2_{1|2} + \frac{1}{2}\sigma_{1|2}(1-\sigma_{1|2}) + \frac{7}{24}(1-\sigma_{1|2})^2 \quad \text{(B3b)}$$

if $\theta_j = 1$ and $\theta_i = 2$, where we denote by $\sigma_{2|1}$ and $\sigma_{1|2}$ the fraction of state-1 neighbors of j when it has state $\theta_j = 2$ and $\theta_j = 1$, respectively. These fractions can be estimated as

$$\sigma_{2|1} \simeq \sum_{k'=1}^{N-1}\frac{k'\mathscr{P}_{k'}}{\mu}\frac{\langle n^1_{2,k'}\rangle}{k'} \quad \text{and} \quad \sigma_{1|2} \simeq \sum_{k'=1}^{N-1}\frac{k'\mathscr{P}_{k'}}{\mu}\frac{\langle n^1_{1,k'}\rangle}{k'}, \quad \text{(B4)}$$

where $\langle n^1_{2,k'}\rangle$ ($\langle n^1_{1,k'}\rangle$) is the mean number of neighbors of j that are in state 1 when $\theta_j = 2$ ($\theta_j = 1$). We have also used that j has degree k' with probability $k'\mathscr{P}_{k'}/\mu$, given that in uncorrelated networks the probability that a random node (node i) is connected to a node of degree k' (node j) is proportional to its degree. Then, the mean numbers $\langle n^1_{2,k'}\rangle$ and $\langle n^1_{1,k'}\rangle$ can be approximated as

$$\langle n^1_{2,k'}\rangle \simeq 1 + (k'-1)P(1|21) \quad \text{(B5a)}$$

and

$$\langle n^1_{1,k'}\rangle \simeq (k'-1)P(1|12), \quad \text{(B5b)}$$

where $P(1|21)$ ($P(1|12)$) is the conditional probability that a neighbor $l \neq i$ of j has state $\theta_l = 1$ given that $\theta_j = 2$ ($\theta_j = 1$) and $\theta_i = 1$ ($\theta_i = 2$) (see Fig. 5). Equations (B5a) and (B5b) take into account that j has at least one neighbor (node i) with state 1 and 2, respectively. Within a PA, $P(1|21)$ and $P(1|12)$ can be approximated as $P(1|2)$ and $P(1|1)$, respectively, if we neglect correlations between second nearest neighbors, i.e., between θ_l and θ_i. Then, if we denote by ρ the fraction of active links, $P(1|2)$ and $P(1|1)$ can be calculated from Bayes' equality

$$\rho = P(2)P(1|2) + P(1)P(2|1), \quad \text{(B6)}$$

where $P(2) = (1-\sigma)$ and $P(1) = \sigma$ are the probabilities that a randomly chosen node is in state 2 and 1, respectively. As the two terms in right hand side of Eq. (B6) are equal, we have that

$$P(1|2) = \frac{\rho}{2(1-\sigma)}, \quad \text{(B7a)}$$

$$P(2|1) = \frac{\rho}{2\sigma}, \quad \text{(B7b)}$$

and, thus,

$$P(1|1) = 1 - P(2|1) = 1 - \frac{\rho}{2\sigma}. \quad \text{(B8)}$$

Using these expressions for $P(1|2)$ and $P(1|1)$, we obtain from Eq. (B5) that

$$\langle n^1_{2,k'}\rangle \simeq 1 + (k'-1)\frac{\rho}{2(1-\sigma)} \quad \text{(B9a)}$$

and

$$\langle n^1_{1,k'}\rangle \simeq (k'-1)\left(1 - \frac{\rho}{2\sigma}\right). \quad \text{(B9b)}$$

Plugging these expressions into Eq. (B4) and performing the summations, we finally obtain

$$\sigma_{2|1} \simeq \frac{1}{\mu}\left[1+\frac{(\mu-1)\rho}{2(1-\sigma)}\right]$$

and

$$\sigma_{1|2} \simeq \frac{(\mu-1)}{\mu}\left(1-\frac{\rho}{2\sigma}\right), \tag{B10}$$

which are quoted in Eq. (30) of the main text.

As $\sigma_{2|1}$ and $\sigma_{1|2}$ only depend on σ, ρ, and μ, and so the payoffs from Eqs. (B2) and (B3), we can pull $P_{1|2}$, $\tilde{P}_{1|2}$, $P_{2|1}$, and $\tilde{P}_{2|1}$ out of the summations of Eq. (B1). Then, expressing the second and fourth summations over k as $\sum_{k=2}^{N-1}\mathscr{P}_k = \sum_{k=1}^{N-1}\mathscr{P}_k - \mathscr{P}_1$, we can write

$$\begin{aligned}\frac{d\sigma}{dt} &= (1-\sigma)\left\{\left[pP_{1|2}+(1-p)\tilde{P}_{1|2}\right]\sum_{k=1}^{N-1}\mathscr{P}_k\sum_{n^1_{2,k}=0}^{k}B\left(n^1_{2,k}\right)\frac{n^1_{2,k}}{k}-(1-p)\tilde{P}_{1|2}\mathscr{P}_1\sum_{n^1_{2,1}=0}^{1}B\left(n^1_{2,k}\right)n^1_{2,1}\right\}\\ &\quad-\sigma\left\{\left[pP_{2|1}+(1-p)\tilde{P}_{2|1}\right]\sum_{k=1}^{N-1}\mathscr{P}_k\sum_{n^2_{1,k}=0}^{k}B\left(n^2_{1,k}\right)\frac{n^2_{1,k}}{k}-(1-p)\tilde{P}_{2|1}\mathscr{P}_1\sum_{n^2_{1,1}=0}^{1}B\left(n^2_{1,1}\right)n^2_{1,1}\right\}\end{aligned} \tag{B11}$$

$$\begin{aligned}&= (1-\sigma)\left\{\left[pP_{1|2}+(1-p)\tilde{P}_{1|2}\right]\sum_{k=1}^{N-1}\mathscr{P}_k\frac{\langle n^1_{2,k}\rangle}{k}-(1-p)\tilde{P}_{1|2}\mathscr{P}_1\langle n^1_{2,1}\rangle\right\}\\ &\quad-\sigma\left\{\left[pP_{2|1}+(1-p)\tilde{P}_{2|1}\right]\sum_{k=1}^{N-1}\mathscr{P}_k\frac{\langle n^2_{1,k}\rangle}{k}-(1-p)\tilde{P}_{2|1}\mathscr{P}_1\langle n^2_{1,1}\rangle\right\},\end{aligned} \tag{B12}$$

where $\langle n^1_{2,k}\rangle$ and $\langle n^2_{1,k}\rangle$ are the first moments of $B\left(n^1_{2,k}\right)$ and $B\left(n^2_{1,k}\right)$, respectively. These moments can be calculated by means of the conditional probabilities from Eq. (B7) as

$$\langle n^1_{2,k}\rangle = kP(1|2) = \frac{k\rho}{2(1-\sigma)} \tag{B13a}$$

and

$$\langle n^2_{1,k}\rangle = kP(2|1) = \frac{k\rho}{2\sigma}. \tag{B13b}$$

Finally, replacing these expressions for the first moments in Eq. (B12) and rearranging terms we arrive at the following equation for the evolution of the fraction of perfect agents quoted in Eq. (27) of the main text:

$$\frac{d\sigma}{dt} = \frac{\rho}{2}\left[p(P_{1|2}-P_{2|1})+(1-p)(\tilde{P}_{1|2}-\tilde{P}_{2|1})(1-\mathscr{P}_1)\right], \tag{B14}$$

where the payoffs P are given by Eqs. (B2), (B3), and (B10).

2. Equation for ρ

To derive an equation for the fraction of active links ρ, we follow an approach similar to that of Sec. 1 of Appendix B for σ. In Fig. 5(a), we describe the change in ρ in a time step $\Delta t = 1/N$, when a node i with state $\theta_i = 2$ and degree k copies the state $\theta_j = 1$ of a random neighbor j (21 → 11 transition). The $n^1_{2,k}$ active links (2–1) connected to node i become inert (1–1) and vice versa, leading to a local change $\Delta n^1_{2,k} = k - 2n^1_{2,k}$ in the number of active links and a change $\Delta\rho = \frac{2(k-2n^1_{2,k})}{\mu N}$ in ρ, where $\mu N/2$ is the total number of links in the system. Similarly, in a 12 → 22 transition, the change is $\Delta\rho = \frac{2(k-2n^2_{1,k})}{\mu N}$. Considering the two transitions and the two playing strategies, we can write the following equation analogous to Eq. (B1):

$$\begin{aligned}\frac{d\rho}{dt} &= \frac{(1-\sigma)}{1/N}\left\{p\sum_{k=1}^{N-1}\mathscr{P}_k\sum_{n^1_{2,k}=0}^{k}B\left(n^1_{2,k}\right)\frac{n^1_{2,k}}{k}P_{1|2}\frac{2\left(k-2n^1_{2,k}\right)}{\mu N}\right.\\ &\quad\left.+(1-p)\sum_{k=2}^{N-1}\mathscr{P}_k\sum_{n^1_{2,k}=0}^{k}B\left(n^1_{2,k}\right)\frac{n^1_{2,k}}{k}\tilde{P}_{1|2}\frac{2\left(k-2n^1_{2,k}\right)}{\mu N}\right\}\\ &\quad+\frac{\sigma}{1/N}\left\{p\sum_{k=1}^{N-1}\mathscr{P}_k\sum_{n^2_{1,k}=0}^{k}B\left(n^2_{1,k}\right)\frac{n^2_{1,k}}{k}P_{2|1}\frac{2\left(k-2n^2_{1,k}\right)}{\mu N}\right.\\ &\quad\left.+(1-p)\sum_{k=2}^{N-1}\mathscr{P}_k\sum_{n^2_{1,k}=0}^{k}B\left(n^2_{1,k}\right)\frac{n^2_{1,k}}{k}\tilde{P}_{2|1}\frac{2\left(k-2n^2_{1,k}\right)}{\mu N}\right\}.\end{aligned} \tag{B15}$$

As we have done in Sec. 1 of Appendix B, we can pull $P_{1|2}$, $\tilde{P}_{1|2}$, $P_{2|1}$, and $\tilde{P}_{2|1}$ out of the summations, combine the first and second summations over k into one, as well as the third and fourth summations and perform the summations over the number of active links connected to node i, to obtain

$$\frac{d\sigma}{dt} = \frac{2(1-\sigma)}{\mu}\left\{\left[pP_{1|2} + (1-p)\tilde{P}_{1|2}\right]\sum_{k=1}^{N-1}\frac{\mathscr{P}_k}{k}\left[k\langle n^1_{2,k}\rangle - 2\langle (n^1_{2,k})^2\rangle\right] - (1-p)\tilde{P}_{1|2}\mathscr{P}_1\left(k\langle n^1_{2,1}\rangle - 2\langle (n^1_{2,1})^2\rangle\right)\right\}$$
$$+ \frac{2\sigma}{\mu}\left\{\left[pP_{2|1} + (1-p)\tilde{P}_{2|1}\right]\sum_{k=1}^{N-1}\frac{\mathscr{P}_k}{k}\left[k\langle n^2_{1,k}\rangle - 2\langle (n^2_{1,k})^2\rangle\right] - (1-p)\tilde{P}_{2|1}\mathscr{P}_1\left[\langle n^2_{1,1}\rangle - 2\langle (n^2_{1,1})^2\rangle\right]\right\}. \quad \text{(B16)}$$

Here, $\langle (n^1_{2,k})^2\rangle$ and $\langle (n^2_{1,k})^2\rangle$ are the second moments of the Binomial distributions $B\left(n^1_{2,k}\right)$ and $B\left(n^2_{1,k}\right)$, respectively, which are given by

$$\langle (n^1_{2,k})^2\rangle = kP(1|2) + k(k-1)P(1|2)^2 = \frac{k\rho}{2(1-\sigma)} + \frac{k(k-1)\rho^2}{4(1-\sigma)^2} \quad \text{(B17a)}$$

and

$$\langle (n^2_{1,k})^2\rangle = kP(2|1) + k(k-1)P(2|1)^2 = \frac{k\rho}{2\sigma} + \frac{k(k-1)\rho^2}{4\sigma^2}. \quad \text{(B17b)}$$

Finally, plugging these expressions for the second moments and the expressions from Eq. (B13) for the first moments into Eq. (B16) we obtain, after doing some algebra, the following equation for the evolution of the fraction of active links:

$$\frac{d\rho}{dt} = \frac{\rho}{\mu}\left\{\left[pP_{2|1} + (1-p)\tilde{P}_{2|1}\right]\left[(\mu-1)\left(1-\frac{\rho}{\sigma}\right) - 1\right]\right.$$
$$+\left[pP_{1|2} + (1-p)\tilde{P}_{1|2}\right]\left[(\mu-1)\left(1-\frac{\rho}{1-\sigma}\right) - 1\right]$$
$$\left. + (1-p)\left(\tilde{P}_{2|1} + \tilde{P}_{1|2}\right)\mathscr{P}_1\right\},$$

which is quoted in Eq. (28) of the main text.

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