Frame completions with prescribed norms: local minimizers and applications

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Dedicated to the memory of María Amelia Muschietti

Abstract

Let $\mathcal{F}_0 = \{f_i\}_{i\in\mathbb{I}_{n_0}}$ be a finite sequence of vectors in $\mathbb{C}^d$ and let $\mathbf{a} = (a_i)_{i\in\mathbb{I}_k}$ be a finite sequence of positive numbers, where $\mathbb{I}_n = \{1\,,\,\dots\,,\,n\}$ for $n \in \mathbb{N}$. We consider the completions of $\mathcal{F}_0$ of the form $\mathcal{F} = (\mathcal{F}_0, \mathcal{G})$ obtained by appending a sequence $\mathcal{G} = \{g_i\}_{i\in\mathbb{I}_k}$ of vectors in $\mathbb{C}^d$ such that $\|g_i\|^2 = a_i$ for $i \in \mathbb{I}_k$, and endow the set of completions with the metric $d(\mathcal{F}, \tilde{\mathcal{F}}) = \max\{\, \|g_i - \tilde{g}_i\| : \ i \in \mathbb{I}_k\}$ where $\tilde{\mathcal{F}} = (\mathcal{F}_0, \tilde{\mathcal{G}})$. In this context we show that local minimizers on the set of completions of a convex potential P_φ, induced by a strictly convex function φ, are also global minimizers. In case that $\varphi(x) = x^2$ then P_φ is the so-called frame potential introduced by Benedetto and Fickus, and our work generalizes several well known results for this potential. We show that there is an intimate connection between frame completion problems with prescribed norms and frame operator distance (FOD) problems. We use this connection and our results to settle in the affirmative a generalized version of Strawn's conjecture on the FOD.

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Contents

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1 Introduction

A family $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n} \in (\mathbb{C}^d)^n$ is a frame for $\mathbb{C}^d$ if it generates $\mathbb{C}^d$. Equivalently, $\mathcal{F}$ is a frame for $\mathbb{C}^d$ if there exist positive constants $0 < A \leq B$ such that

$$A\ \|f\|^2 \leq \sum_{i\in\mathbb{I}_n} |\langle f, f_i\rangle|^2 \leq B\ \|f\|^2 \quad \text{for} \quad f \in \mathbb{C}^d\,. \tag{1}$$

As a (possibly redundant) set of generators, a frame $\mathcal{F}$ provides linear representations of vectors in $\mathbb{C}^d$. Indeed, it is well known that in this case

$$f = \sum_{i\in\mathbb{I}_n} \langle f, g_i\rangle\ f_i = \sum_{i\in\mathbb{I}_n} \langle f, f_i\rangle\ g_i \tag{2}$$

for certain frames $\mathcal{G} = \{g_i\}_{i\in\mathbb{I}_n}$, that are the so-called dual frames of $\mathcal{F}$. Thus, a vector (signal) $f \in \mathbb{C}^d$ can be encoded in terms of the coefficients $(\langle f, g_i\rangle)_{i\in\mathbb{I}_n}$; these coefficients can be sent (one by one) through a transmission channel and the receiver can then reconstruct f, by decoding the sequence of coefficients using the reconstruction formula in Eq. (2).

Frames are of interest in applied situations, in which their redundancy can be used to deal with real-life problems, such as noise in the transmission channel (leading to what is known in the literature as robust frame designs). The stability of the reconstruction algorithm in Eq. (2) also plays a central role in applications of frame theory. The consideration of these features of frames motivated the introduction of *unit norm tight* frames, which are those frames for which we can choose $A = B$ in Eq. (1) and such that $\|f_i\| = 1$ for $i \in \mathbb{I}_n$. It turns out that unit norm tight frames have several optimality properties related with erasures of the frame coefficients and numerical stability of their reconstruction formula [10, 22].

In the seminal paper [3] Benedetto and Fickus gave another characterization of unit norm tight frames, in terms of a convex functional known as the frame potential. Indeed, given a finite sequence $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n}$ in $\mathbb{C}^d$ then the frame potential of $\mathcal{F}$, denoted $\text{FP}(\mathcal{F})$, is given by

$$\text{FP}(\mathcal{F}) = \sum_{i\,,\,j\in\mathbb{I}_n} |\langle f_i\,,\,f_j\rangle|^2\,. \tag{3}$$

Benedetto and Fickus showed that if we endow the set of unit norm frames with n elements in $\mathbb{C}^d$ with the metric $d(\mathcal{F}, G) = \max\{\|f_i - g_i\| :\ i \in \mathbb{I}_n\}$ then unit norm tight frames are characterized as local minimizers of the frame potential, and that they are actually global minimizers of this functional (among unit norm frames). This was the first indirect proof of the existence of unit norm tight frames (for $n \geq d$). In applications, it is sometimes useful to consider frames $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n}$ with norms prescribed by a sequence $\mathbf{a} = (a_i)_{i\in\mathbb{I}_n}$ i.e. such that $\|f_i\|^2 = a_i$ for $i \in \mathbb{I}_n$. The consideration of these families raised the question of whether there exist tight frames with (arbitrary) prescribed norms, leading to what is known as frame design problems. It turns out that a complete solution to the frame design problem can be obtained in terms of the Schur-Horn theorem, which is a central result in matrix analysis; moreover, this characterization showed that for some sequences $\mathbf{a}$ there is no tight frame with norms given by $\mathbf{a}$ (see [2, 5, 7, 12, 15, 16, 17, 25]).

The absence of tight frames in the class $\mathbb{F}_{\mathbb{C}^d}(\mathbf{a})$ of frames with norms prescribed by a fixed sequence $\mathbf{a}$ led to the consideration of substitutes of tight frames within this class i.e., frames in $\mathbb{F}_{\mathbb{C}^d}(\mathbf{a})$ that had some optimality properties within this class. A complete solution of the optimal design problem with prescribed norms with respect to the frame potential was given in [8] where the global minimizers of FP were computed; moreover, the authors obtained a crucial property of these optimal frame designs: they showed that if we endow $\mathbb{F}_{\mathbb{C}^d}(\mathbf{a})$ with the metric $d(\mathcal{F}, G) = \max\{\|f_i - g_i\| :\ i \in \mathbb{I}_n\}$ then local minimizers of FP in $\mathbb{F}_{\mathbb{C}^d}(\mathbf{a})$ are actually optimal designs i.e. that local minimizers are global minimizers. This generalization of the results from [3] motivated the study of perturbation problems related with gradient descent method of the (smooth function) FP in the (smooth) manifold $\mathbb{F}_{\mathbb{C}^d}(\mathbf{a})$.

It turns out that the frame potential can be considered within the general class of convex potentials introduced in [27] (see Definition 2.5). Moreover, in [27] it was shown that the optimal frame designs in $\mathbb{F}_{\mathbb{C}^d}(\mathbf{a})$ obtained in [8] were actually global minimizers of every convex potential within this class. In [30] the authors showed further that local minimizers of any convex potential induced by a strictly convex function are global minimizers of every convex potential within $\mathbb{F}_{\mathbb{C}^d}(\mathbf{a})$, settling in the affirmative a conjecture from [27].

In [19], given an initial sequence of vectors $\mathcal{F}_0$ in $\mathbb{C}^d$ and a sequence of positive numbers $\mathbf{a} = (a_i)_{i\in\mathbb{I}_k}$, the authors posed the problem of computing the completions $\mathcal{F} = (\mathcal{F}_0, \mathcal{G})$ obtained by appending a sequence $\mathcal{G} = \{g_i\}_{i\in\mathbb{I}_k}$ in $\mathbb{C}^d$ with norms prescribed by $\mathbf{a}$, such that these completions minimize the so-called mean squared error (MSE). This is known as the optimal completion problem with prescribed norms for the MSE, and contains the optimal design problem with prescribed norms for the MSE as a particular case (i.e. if $\mathcal{F}_0 = \emptyset$). It turns out that the MSE is also a convex potential (see the comments before Definition 2.5). In the series of papers [28, 29, 30] a complete solution to the optimal completion problem with prescribed norms was obtained with respect to every convex potential; explicitly, the authors showed that there exists a class of completions with prescribed norms, determined by certain spectral conditions, such that the members of this class minimize simultaneously every convex potential among such completions. This fact was independently re-obtained in [20], in terms of a generalized Schur-Horn theorem. Notice that there is a natural metric in the set of completions given by $d(\mathcal{F}\,,\,\tilde{\mathcal{F}}) = \max\{\|g_i - \tilde{g}_i\| : \ i \in \mathbb{I}_k\}$ for $\mathcal{F} = (\mathcal{F}_0, \mathcal{G})$ and $\tilde{\mathcal{F}} = (\mathcal{F}_0, \tilde{\mathcal{G}})$. Yet, the structure of local minimizers of convex potentials of frame completions with prescribed norms was not obtained in these works, not even for Benedetto-Fickus' frame potential.

In (the seemingly unrelated paper) [31], Strawn considered an approximate gradient descent algorithm for the *frame operator distance* (FOD), in the smooth manifold $\mathbb{T}_{\mathbb{C}^d}(\mathbf{a})$ of sequences with norms prescribed by $\mathbf{a}$. The algorithm essentially searches for critical points of the FOD. In case the sequence of iterates constructed with Strawn's method converges, one expects to reach - at best - local minimizers of the objective function. It is then relevant to understand the nature of local minima, as this exhibits some aspects of the numerical performance of the algorithm. Based on computational evidence, Strawn conjectured that - under some technical assumptions - local minimizers of the FOD are also global minimizers. As a motivation for studying the FOD, the author observed in [31] that in some cases minimization of the FOD is equivalent to minimization of the frame potential in $\mathbb{T}_{\mathbb{C}^d}(\mathbf{a})$.

In this paper, given an initial sequence of vectors $\mathcal{F}_0$ in $\mathbb{C}^d$ and a sequence of positive numbers $\mathbf{a} = (a_i)_{i\in\mathbb{I}_k}$, we show that any completion $\mathcal{F} = (\mathcal{F}_0, \mathcal{G}_0)$ of $\mathcal{F}_0$ obtained by appending a sequence in $\mathcal{G}_0 \in \mathbb{T}_{\mathbb{C}^d}(\mathbf{a})$ (i.e. with norms prescribed by the sequence of positive numbers $\mathbf{a}$) that is a local minimizer of some (strictly) convex potential, is a global minimizer of every convex potential among such completions. Thus, our results generalize those of [3, 8, 18, 19] for the frame potential and MSE, and those from [20, 27, 28, 29, 30] related with optimal designs/completions with prescribed norms with respect to arbitrary convex potentials. These results suggest the implementation of (approximate) gradient descent algorithms for computing (optimal) solutions to frame perturbation problems. As a tool we develop a local version of Lidskii's additive inequality, that is of independent interest. We apply these results to settle in the affirmative Strawn's conjecture on the structure of local minimizers of the frame operator distance from [31]. We approach this conjecture by means of a translation between frame completion problems and FOD problems. Moreover, we compute distances between certain sets of positive semidefinite matrices that generalize the FOD, in terms of arbitrary unitarily invariant norms.

The paper is organized as follows. In Section 2 we introduce the notation and terminology as well as some results from matrix analysis and frame theory used throughout the paper. We have included section 2.3 in which we summarize several results from [28, 30] for the benefit of the reader. In Section 3 we show some features of local minimizers of convex potentials within the set of frame completions with prescribed norms. Our approach is based on a local Lidskii's theorem that we describe in Section 3.1 (we delay its proof to Section 6 - Appendix). We use this result in Sections 3.2 and 3.3 to obtain geometrical and spectral properties of local minimizers. In section 4 we prove our main result, namely that local minimizers of strictly convex potentials within the set of frame completions with prescribed norms are also global minimizers. In Section 5 we apply the main result to prove (a generalized version of) Strawn's conjecture on local minima of the frame operator distance. The paper ends with Section 6 (Appendix) in which we show a local version of Lidskii's inequality.

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2 Preliminaries

In this section we introduce the notation, terminology and results from matrix analysis and frame theory that we will use throughout the paper. General references for these results are the texts [4] and [6, 11, 13].

2.1 Preliminaries from matrix analysis

In what follows we adopt the following

Notation and terminology. We let $\mathcal{M}_{k,d}(\mathbb{C})$ be the space of complex $k \times d$ matrices and write $\mathcal{M}_{d,d}(\mathbb{C}) = \mathcal{M}_d(\mathbb{C})$ for the algebra of $d \times d$ complex matrices. We denote by $\mathcal{H}(d) \subset \mathcal{M}_d(\mathbb{C})$ the real subspace of selfadjoint matrices and by $\mathcal{M}_d(\mathbb{C})^+ \subset \mathcal{H}(d)$ the cone of positive semidefinite matrices. We let $\mathcal{U}(d) \subset \mathcal{M}_d(\mathbb{C})$ denote the group of unitary matrices. For $d \in \mathbb{N}$, let $\mathbb{I}_d = \{1, \dots, d\}$. Given $x = (x_i)_{i\in\mathbb{I}_d} \in \mathbb{R}^d$ we denote by $x^\downarrow = (x_i^\downarrow)_{i\in\mathbb{I}_d}$ (respectively $x^\uparrow = (x_i^\uparrow)_{i\in\mathbb{I}_d}$) the vector obtained by rearranging the entries of x in non-increasing (respectively non-decreasing) order. We denote by $(\mathbb{R}^d)^\downarrow = \{x^\downarrow : \ x \in \mathbb{R}^d\}$, $(\mathbb{R}^d_{\geq 0})^\downarrow = \{x^\downarrow : \ x \in \mathbb{R}^d_{\geq 0}\}$ and analogously $(\mathbb{R}^d)^\uparrow$ and $(\mathbb{R}^d_{\geq 0})^\uparrow$. Given a matrix $A \in \mathcal{H}(d)$ we denote by $\lambda(A) = \lambda^\downarrow(A) = (\lambda_i(A))_{i\in\mathbb{I}_d} \in (\mathbb{R}^d)^\downarrow$ the eigenvalues of A counting multiplicities and arranged in non-increasing order, and by $\lambda^\uparrow(A)$ the same vector but ordered in non-decreasing order. For $B \in \mathcal{M}_d(\mathbb{C})$ we let $s(B) = \lambda(|B|)$ denote the singular values of B, i.e. the eigenvalues of $|B| = (B^*B)^{1/2} \in \mathcal{M}_d(\mathbb{C})^+$. If $x, y \in \mathbb{C}^d$ we denote by $x \otimes y \in \mathcal{M}_d(\mathbb{C})$ the rank-one matrix given by $(x \otimes y)\, z = \langle z\,,\, y\rangle\, x$, for $z \in \mathbb{C}^d$.

Next we recall the notion of majorization between vectors, that will play a central role throughout our work.

Definition 2.1. Let $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^d$. We say that x is *submajorized* by y, and write $x \prec_w y$, if

$$\sum_{i=1}^{j} x_i^\downarrow \leq \sum_{i=1}^{j} y_i^\downarrow \quad \text{for every} \quad 1 \leq j \leq \min\{k\,,\, d\}\,.$$

If $x \prec_w y$ and $\operatorname{tr} x = \sum_{i=1}^k x_i = \sum_{i=1}^d y_i = \operatorname{tr} y$, then x is *majorized* by y, and write $x \prec y$. △

Given $x, y \in \mathbb{R}^d$ we write $x \leqslant y$ if $x_i \leq y_i$ for every $i \in \mathbb{I}_d$. It is a standard exercise to show that $x \leqslant y \implies x^\downarrow \leqslant y^\downarrow \implies x \prec_w y$.

Although majorization is not a total order in $\mathbb{R}^d$, there are several fundamental inequalities in matrix theory that can be described in terms of this relation. As an example of this phenomenon we can consider Lidskii's (additive) inequality (see [4]). In the following result we also include the characterization of the case of equality obtained in [29].

Theorem 2.2 (Lidskii's inequality)**.** Let $A,\, B \in \mathcal{H}(d)$ with eigenvalues $\lambda(A),\, \lambda(B) \in (\mathbb{R}^d)^\downarrow$ respectively. Then

1. $\lambda^\uparrow(A) + \lambda^\downarrow(B) \prec \lambda(A+B)$.
2. If $\lambda(A+B) = \big(\lambda(A) + \lambda^\uparrow(B)\big)^\downarrow$ then there exists $\{v_i\}_{i\in\mathbb{I}_d}$ an ONB of $\mathbb{C}^d$ such that

$$A = \sum_{i\in\mathbb{I}_d} \lambda_i(A)\ v_i \otimes v_i \quad \text{ and } \quad B = \sum_{i\in\mathbb{I}_d} \lambda_i^\uparrow(B)\ v_i \otimes v_i\ . \qquad \square$$

Recall that a norm $|||\cdot|||$ in $\mathcal{M}_d(\mathbb{C})$ is unitarily invariant if

$$|||\, U\,A\,V\,||| = |||A||| \quad \text{ for every } \quad A \in \mathcal{M}_d(\mathbb{C}) \quad \text{ and } \quad U,\, V \in \mathcal{U}(d)\,.$$

Examples of unitarily invariant norms (u.i.n.) are the spectral norm $\|\cdot\|$ and the Schatten p-norms $\|\cdot\|_p$, for $p \geq 1$. It is well known that majorization is intimately related with tracial inequalities of convex functions and also with inequalities with respect to u.i.n's. The following result summarizes these relations (see for example [4]):

Theorem 2.3. *Let $x,\, y \in \mathbb{R}^d$ and let $A,\, B \in \mathcal{M}_d(\mathbb{C})$. If $\varphi : I \to \mathbb{R}$ is a convex function defined on an interval $I \subseteq \mathbb{R}$ such that $x,\, y,\, s(A),\, s(B) \in I^d$ then:*

1. *If $x \prec y$, then* $\operatorname{tr}\varphi(x) \stackrel{def}{=} \sum\limits_{i\in\mathbb{I}_d} \varphi(x_i) \leq \sum\limits_{i\in\mathbb{I}_d} \varphi(y_i) = \operatorname{tr}\varphi(y)$.
2. *If only $x \prec_w y$, but φ is an increasing function, then still* $\operatorname{tr}\varphi(x) \leq \operatorname{tr}\varphi(y)$.
3. *If $x \prec y$ and φ is a strictly convex function such that* $\operatorname{tr}\,\varphi(x) = \operatorname{tr}\,\varphi(y)$ *then there exists a permutation σ of $\mathbb{I}_d$ such that $y_i = x_{\sigma(i)}$ for $i \in \mathbb{I}_d$, i.e. $x^\downarrow = y^\downarrow$.*
4. *If $s(A) \prec_w s(B)$ then $|||A||| \leq |||B|||$, for every u.i.n. $|||\cdot|||$ defined on $\mathcal{M}_d(\mathbb{C})$.*

$\square$

2.2 Frames and frame completions with prescribed norms

In what follows we adopt the following notation and terminology from frame theory.

Notation and terminology: let $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_k}$ be a finite sequence in $\mathbb{C}^d$. Then,

1. $T_\mathcal{F} \in \mathcal{M}_{d,k}(\mathbb{C})$ denotes the synthesis operator of $\mathcal{F}$ given by $T_\mathcal{F} \cdot (\alpha_i)_{i\in\mathbb{I}_k} = \sum_{i\in\mathbb{I}_k} \alpha_i\, f_i$.
2. $T_\mathcal{F}^* \in \mathcal{M}_{k,d}(\mathbb{C})$ denotes the analysis operator of $\mathcal{F}$ and it is given by $T_\mathcal{F}^* \cdot f = (\langle f, f_i\rangle)_{i\in\mathbb{I}_k}$.
3. $S_\mathcal{F} \in \mathcal{M}_d(\mathbb{C})^+$ denotes the frame operator of $\mathcal{F}$ and it is given by $S_\mathcal{F} = T_\mathcal{F} T_\mathcal{F}^*$. Hence, $Sf = \sum_{i\in\mathbb{I}_k} \langle f, f_i\rangle f_i = \sum_{i\in\mathbb{I}_k} f_i \otimes f_i(f)$ for $f \in \mathbb{C}^d$.
4. We say that $\mathcal{F}$ is a frame for $\mathbb{C}^d$ if it spans $\mathbb{C}^d$; equivalently, $\mathcal{F}$ is a frame for $\mathbb{C}^d$ if $S_\mathcal{F}$ is a positive invertible operator acting on $\mathbb{C}^d$.

In several applied situations it is desired to construct a sequence $\mathcal{G}$ in $\mathbb{C}^d$, in such a way that the frame operator of $\mathcal{G}$ is given by some positive operator $B \in \mathcal{M}_d(\mathbb{C})^+$ and the squared norms of the frame elements are prescribed by a sequence of positive numbers $\mathbf{a} = (a_i)_{i\in\mathbb{I}_k}$. This is known as the classical frame design problem and it has been studied by several research groups (see for example [2, 5, 7, 12, 15, 16, 17, 25]). The following result characterizes the existence of such frame design in terms of majorization relations.

Proposition 2.4 ([2, 26])**.** Let $B \in \mathcal{M}_d(\mathbb{C})^+$ with eigenvalues $\lambda(B) = (\lambda_i)_{i\in\mathbb{I}_d} \in (\mathbb{R}^d_{\geq 0})^\downarrow$ and consider $\mathbf{a} = (a_i)_{i\in\mathbb{I}_k} \in (\mathbb{R}^k_{>0})^\downarrow$. Then there exists a sequence $\mathcal{G} = \{g_i\}_{i\in\mathbb{I}_k}$ in $\mathbb{C}^d$ with frame operator $S_\mathcal{G} = B$ such that $\|g_i\|^2 = a_i$ for $i \in \mathbb{I}_k$ if and only if $\mathbf{a} \prec \lambda(B)$ i.e.

$$\sum_{i\in\mathbb{I}_j} a_i \leq \sum_{i\in\mathbb{I}_j} \lambda_i \ , \qquad \text{for} \quad 1 \leq j \leq \min\{k\,,\, d\} \qquad \text{and} \qquad \sum_{i\in\mathbb{I}_k} a_i = \sum_{i\in\mathbb{I}_d} \lambda_i \,. \qquad \square$$

Recently, researchers have asked about the structure of *optimal frame completions with prescribed norms.* Explicitly, let $\mathcal{F}_0 = \{f_i\}_{i\in\mathbb{I}_{n_0}}$ be a fixed (finite) sequence of vectors in $\mathbb{C}^d$, consider a sequence $\mathbf{a} = (a_i)_{i\in\mathbb{I}_k} \in (\mathbb{R}^k_{>0})^\downarrow$ and denote by $n = n_0 + k$. Then, with this fixed data, the problem is to construct a sequence

$$\mathcal{G} = \{g_i\}_{i=1}^k \quad \text{with} \quad \|g_i\|^2 = a_i \quad \text{for} \quad i \in \mathbb{I}_k \ ,$$

such that the resulting completed sequence $\mathcal{F} = (\mathcal{F}_0\,,\,\mathcal{G})$ - obtained by appending the sequence $\mathcal{G}$ to $\mathcal{F}_0$ - is such that the eigenvalues of the frame operator of $\mathcal{F}$ are as concentrated as possible: thus, ideally, we would search for completions $\mathcal{G}$ such that $\mathcal{F} = (\mathcal{F}_0\,,\,\mathcal{G})$ is a tight frame i.e. such that $S_\mathcal{F} = c\,I$ for some $c > 0$. Unfortunately, it is well known that there might not exist such completions (see [17, 18, 19, 26, 28, 29, 30]). We could measure optimality in terms of the frame potential i.e., we search for a frame $\mathcal{F} = (\mathcal{F}_0\,,\,\mathcal{G})$, with $\|g_i\|^2 = a_i$ for $1 \leq i \leq k$, and such that its frame potential $\text{FP}\,(\mathcal{F}) = \text{tr}\ S_\mathcal{F}^2$ is minimal among all possible such completions; alternatively, we could measure optimality in terms of the so-called mean squared error (MSE) of the completed sequence $\mathcal{F}$ i.e. $\text{MSE}(\mathcal{F}) = \text{tr}(S_\mathcal{F}^{-1})$ (see [19]). More generally, we can measure stability of the completed frame $\mathcal{F} = (\mathcal{F}_0, \mathcal{G})$ in terms of general convex potentials. In order to introduce these potentials we consider the sets

$$\text{Conv}(\mathbb{R}_{\geq 0}) = \{\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \ : \ \varphi \ \text{ is a convex function }\}$$

and $\text{Conv}_\text{s}(\mathbb{R}_{\geq 0}) = \{\varphi \in \text{Conv}(\mathbb{R}_{\geq 0}) : \varphi \text{ is strictly convex }\}$.

Definition 2.5. Following [27] we consider the (generalized) convex potential P_φ associated to $\varphi \in \text{Conv}(\mathbb{R}_{\geq 0})$, given by

$$\text{P}_\varphi(\mathcal{F}) \ = \text{tr}\ \varphi(S_\mathcal{F}) = \textstyle\sum_{i\in\mathbb{I}_d} \varphi(\lambda_i(S_\mathcal{F})\,) \quad \text{for} \quad \mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n} \in (\mathbb{C}^d)^n \ ,$$

where the matrix $\varphi(S_\mathcal{F})$ is defined by means of the usual functional calculus. △

Convex potentials allow us to model several well known measures of stability considered in frame theory. For example, in case $\varphi(x) = x^2$ for $x \in \mathbb{R}_{\geq 0}$ then P_φ is the Benedetto-Fickus frame potential; in case $\varphi(x) = x^{-1}$ for $x \in \mathbb{R}_{>0}$ then P_φ is known as the mean squared error (MSE).

We can now give a detailed description of the optimal completion problem with prescribed norms with respect to convex potentials.

Notation 2.6. Let $\mathcal{F}_0 = \{f_i\}_{i\in\mathbb{I}_{n_0}} \in (\mathbb{C}^d)^{n_0}$ and $\mathbf{a} = (a_i)_{i\in\mathbb{I}_k} \in (\mathbb{R}^k_{>0})^\downarrow$.

1. Let $\mathbb{T}_{\mathbb{C}^d}(\mathbf{a}) = \Big\{\mathcal{G} = \{g_i\}_{i\in\mathbb{I}_k} \in (\mathbb{C}^d)^k \ : \|g_i\|^2 = a_i \text{ for every } i \in \mathbb{I}_k\Big\}$
2. We consider the set $\mathcal{C}_\mathbf{a}\,(\mathcal{F}_0)$ of completions of $\mathcal{F}_0$ with norms prescribed by the sequence $\mathbf{a}$ given by
$$\mathcal{C}_\mathbf{a}\,(\mathcal{F}_0) = \Big\{\mathcal{F} = (\mathcal{F}_0\,,\,\mathcal{G}) \in (\mathbb{C}^d)^{n_0+k} \ : \mathcal{G} \in \mathbb{T}_{\mathbb{C}^d}(\mathbf{a})\Big\}\ .$$
3. For $\varphi \in \text{Conv}(\mathbb{R}_{\geq 0})$ we consider the optimal frame completions $\mathcal{F}^\text{op} = (\mathcal{F}_0, \mathcal{G}^\text{op}) \in \mathcal{C}_\mathbf{a}\,(\mathcal{F}_0)$ with respect to the convex potential P_φ i.e. such that
$$\text{P}_\varphi(\mathcal{F}^\text{op}) = \min\{\ \text{P}_\varphi(\mathcal{F}) : \ \mathcal{F} \in \mathcal{C}_\mathbf{a}\,(\mathcal{F}_0)\ \}\,. \qquad \triangle$$

Consider the Notation 2.6 above. In the series of papers [28, 29, 30] the spectral and geometrical structure of optimal frame completions $\mathcal{F}^{\rm op} = (\mathcal{F}_0, \mathcal{G}^{\rm op}) \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$ was completely described, in case $\varphi \in \text{Conv}_s(\mathbb{R}_{\geq 0})$ (see Theorem 4.8 below); indeed, in this case it was shown that if $\mathcal{F}^{\rm op} \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$ is optimal with respect to P_φ in $\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$, then $\mathcal{F}^{\rm op}$ is also optimal in $\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$ with respect to any other convex potential.

2.3 Optimal frame completions with prescribed norms: feasible cases

In this subsection we consider several concepts related with the notion of feasible index introduced in [30]. The feasible indexes (see Definition 4.2) will play a key role in our study of frame completions that are local minima of strictly convex potentials.

Definition 2.7. Let $\lambda = (\lambda_i)_{i\in\mathbb{I}_d} \in (\mathbb{R}^d_{\geq 0})^\uparrow$ and fix $t \in \mathbb{R}_{>0}$.

1. Consider the function $h_\lambda : [\lambda_1 \, , \, \infty) \to [0 \, , \, \infty)$ given by

$$h_\lambda(x) = \sum_{i\in\mathbb{I}_d} (x - \lambda_i)^+ \ , \qquad \text{for every} \qquad x \in [\lambda_1 \, , \, \infty) \ ,$$

where $y^+ = \max\{y, 0\}$ stands for the positive part of $y \in \mathbb{R}$. It is easy to see that h_λ is continuous, strictly increasing, such that $h_\lambda(\lambda_1) = 0$ and $\lim\limits_{x\to+\infty} h_\lambda(x) = +\infty$.

2. Therefore $h_\lambda^{-1} : [0 \, , \, \infty) \to [\lambda_1 \, , \, \infty)$ is well defined and bijective; hence, there exists a unique

$$c = c(t) > \lambda_1 \geq 0 \qquad \text{such that} \qquad h_\lambda(c(t)) = t > 0 \ . \tag{4}$$

3. Let $c = c(t) > \lambda_1 \geq 0$ be as in Eq. (4). Then we set

$$\nu(\lambda \, , \, t) \stackrel{\text{def}}{=} \big(\, (c - \lambda_1)^+ + \lambda_1 \, , \, \ldots \, , \, (c - \lambda_d)^+ + \lambda_d \, \big) \in (\mathbb{R}^d_{>0})^\uparrow \ . \tag{5}$$

4. We now consider the vector

$$\mu(\lambda \, , \, t) \stackrel{\text{def}}{=} \big(\, (c - \lambda_1)^+ \, , \, \ldots \, , \, (c - \lambda_d)^+ \, \big) = \nu(\lambda \, , \, t) - \lambda \in (\mathbb{R}^d_{\geq 0})^\downarrow \ . \tag{6}$$

We note the fact that, since $\lambda \in (\mathbb{R}^d_{\geq 0})^\uparrow$, then $\mu(\lambda \, , \, t) \in (\mathbb{R}^d_{\geq 0})^\downarrow$. △

Remark 2.8. Let $\lambda = (\lambda_i)_{i\in\mathbb{I}_d} \in (\mathbb{R}^d_{\geq 0})^\uparrow$, let $t > 0$ and let $c = c(t) > \lambda_1 \geq 0$ be as in Eq. (4). Notice that by construction, we have that

$$\operatorname{tr} \nu(\lambda \, , \, t) = \sum_{i\in\mathbb{I}_d} (c - \lambda_i)^+ + \lambda_i = \operatorname{tr} \lambda + h_\lambda(c) = \operatorname{tr} \lambda + t \ .$$

On the other hand, since $(c - a)^+ + a = \max\{c, a\}$ we see that:

1. If $c < \lambda_d$ then there exists a unique $r \in \mathbb{I}_{d-1}$ such that, if we let $\mathbb{1}_r = (1, \ldots, 1) \in \mathbb{R}^r$ then

$$\nu(\lambda \, , \, t) = (c \, \mathbb{1}_r \, , \, \lambda_{r+1} \, , \, \ldots \, , \, \lambda_d) \in (\mathbb{R}^d_{>0})^\uparrow \qquad \text{with} \qquad \lambda_r \leq c < \lambda_{r+1} \, . \tag{7}$$

In this case, $\operatorname{tr} \lambda + t = \operatorname{tr} \nu(\lambda \, , \, t) < d \, \lambda_d$ and then $\lambda_d > \frac{\operatorname{tr} \lambda + t}{d}$.

2. Otherwise, $c \geq \lambda_d$ and therefore $\nu(\lambda \, , \, t) = c \, \mathbb{1}_d \in (\mathbb{R}^d_{>0})^\uparrow$. In this case

$$\operatorname{tr} \lambda + t = \operatorname{tr} \nu(\lambda \, , \, t) = d \, c \geq d \, \lambda_d \qquad \text{and then} \qquad \lambda_d \leq \frac{\operatorname{tr} \lambda + t}{d} \ .$$

The previous remarks show that if $\rho = (e\,\mathbb{1}_s\,,\,\lambda_{s+1}\,,\,\dots\,,\,\lambda_d)$ or $\rho = e\,\mathbb{1}_d$ for some $e > 0$ is such that

$$\rho \in (\mathbb{R}^d_{>0})^\uparrow \quad , \quad \rho \geq \lambda \quad \text{ and } \quad \operatorname{tr}\rho = \operatorname{tr}\lambda + t \implies \rho = \nu(\lambda\,,\,t)\ . \qquad \triangle$$

Next we introduce the notion of a feasible pair. Then, we clarify the relation between this notion and the frame completion problem.

Definition 2.9. Let $\lambda = (\lambda_i)_{i\in\mathbb{I}_d} \in (\mathbb{R}^d_{\geq 0})^\uparrow$ and let $\mathbf{a} = (a_i)_{i\in\mathbb{I}_k} \in (\mathbb{R}^k_{>0})^\downarrow$.

1. Let $r = \min\{k, d\}$, let $\tilde{\lambda} = (\lambda_i)_{i\in\mathbb{I}_r}$ and $t = \sum_{i\in\mathbb{I}_k} a_i > 0$. Let $\nu(\lambda\,,\,\mathbf{a}) \in \mathbb{R}^d_{>0}$ be given by:

$$\nu(\lambda\,,\,\mathbf{a}) = \begin{cases} \nu(\lambda\,,\,t) & \text{if} \quad r = d \leq k \\ \big(\,\nu(\tilde{\lambda}\,,\,t)\,,\,\lambda_{r+1}\,,\,\dots\,,\,\lambda_d\,\big) & \text{if} \quad r = k < d\ . \end{cases} \tag{8}$$

   Observe that in the second case $\nu(\lambda\,,\,\mathbf{a})$ could be a non ordered vector (if $c(t) > \lambda_{r+1}$).

2. Consider the vector $\mu(\lambda\,,\,\mathbf{a}) \overset{\text{def}}{=} \nu(\lambda\,,\,\mathbf{a}) - \lambda \in \mathbb{R}^d_{\geq 0}$. By inspection of Definition 2.7 and item 1 above we see that $\mu(\lambda\,,\,\mathbf{a}) = \mu(\lambda\,,\,\mathbf{a})^\downarrow$ and $\operatorname{tr}\,\mu(\lambda\,,\,\mathbf{a}) = \operatorname{tr}\,\mathbf{a} = t$.

3. We say that the pair $(\lambda\,,\,\mathbf{a})$ is **feasible** if $\mathbf{a} \prec \mu(\lambda\,,\,\mathbf{a})$ that is, if

$$\sum_{i\in\mathbb{I}_j} a_i \leq \sum_{i\in\mathbb{I}_j} \mu_i(\lambda\,,\,\mathbf{a}) \quad \text{ for } \quad j \in \mathbb{I}_{r-1}\ , \tag{9}$$

   where the equivalence follows from the properties of $\mu(\lambda\,,\,\mathbf{a})$ given in item 2. Notice that in the case that $k < d$ then $\mu_{k+1}(\lambda\,,\,\mathbf{a}) = 0$. $\triangle$

We point out that the computation of $\nu(\lambda\,,\,\mathbf{a})$ and $\mu(\lambda\,,\,\mathbf{a})$ as in Definition 2.9, as well as the verification of the inequalities in Eq. (9) above can be implemented by a finite step algorithm.

The following result is taken from [30] (see also [28]) and it describes, in the feasible case, the spectral structure of global minimizers of the convex potentials P_φ in $\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$, for $\varphi \in \operatorname{Conv_s}(\mathbb{R}_{\geq 0})$. As already mentioned, this structure does not depend on φ.

Theorem 2.10. Let $\mathcal{F}_0 = \{f_i\}_{i\in\mathbb{I}_{n_0}} \in (\mathbb{C}^d)^{n_0}$ and let $\mathbf{a} = (a_i)_{i\in\mathbb{I}_k} \in (\mathbb{R}^k_{>0})^\downarrow$. Let $\lambda = \lambda(S_{\mathcal{F}_0})^\uparrow$ and assume that the pair $(\lambda\,,\,\mathbf{a})$ is feasible. Let $\nu(\lambda\,,\,\mathbf{a}) = (\nu_i(\lambda\,,\,\mathbf{a}))_{i\in\mathbb{I}_d} \in \mathbb{R}^d_{\geq 0}$ be as in Definition 2.9. Then, for every $\varphi \in \operatorname{Conv_s}(\mathbb{R}_{\geq 0})$ we have that

$$\min\{\mathrm{P}_\varphi(\mathcal{F}) :\ \mathcal{F} = (\mathcal{F}_0\,,\,\mathcal{G}) \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)\} = \sum_{i\in\mathbb{I}_d} \varphi(\nu_i(\lambda\,,\,\mathbf{a}))\ .$$

Moreover, given $\mathcal{F} = (\mathcal{F}_0\,,\,\mathcal{G}) \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$ then

$$\mathrm{P}_\varphi(\mathcal{F}) = \sum_{i\in\mathbb{I}_d} \varphi(\nu_i(\lambda\,,\,\mathbf{a})) \iff \lambda(S_{\mathcal{F}}) = \nu(\lambda\,,\,\mathbf{a})^\downarrow\ . \qquad \square$$

3 Local minima of frame completions with prescribed norms

We begin with a brief description of our main problem (for a detailed description see Section 3.2). Let $\mathcal{F}_0 = \{f_i\}_{i\in\mathbb{I}_{n_0}}$ be a fixed family in $(\mathbb{C}^d)^{n_0}$ and $\mathbf{a} = (a_i)_{i\in\mathbb{I}_k} \in (\mathbb{R}^k_{>0})^\downarrow$; consider $\mathbb{T}_{\mathbb{C}^d}(\mathbf{a})$ (see Notation 2.6) endowed with the $d(\mathcal{G}\,,\,\tilde{\mathcal{G}}) = \max\{\,\|g_i - \tilde{g}_i\| :\ i \in \mathbb{I}_k\,\}$ for $\mathcal{G} = \{g_i\}_{i\in\mathbb{I}_k}$, $\tilde{\mathcal{G}} = \{\tilde{g}_i\}_{i\in\mathbb{I}_k} \in \mathbb{T}_{\mathbb{C}^d}(\mathbf{a})$. Our main goal is to study the structure of local minimizers of the map

$$\mathbb{T}_{\mathbb{C}^d}(\mathbf{a}) \ni \mathcal{G} \mapsto \mathrm{P}_\varphi(\mathcal{F}_0\,,\,\mathcal{G}) = \operatorname{tr}(\varphi(S_{(\mathcal{F}_0\,,\,\mathcal{G})})) = \operatorname{tr}(\varphi(S_{\mathcal{F}_0} + S_{\mathcal{G}}))\,, \tag{10}$$

where $\varphi \in \operatorname{Conv_s}(\mathbb{R}_{\geq 0})$ is a strictly convex function and $\mathcal{F} = (\mathcal{F}_0\,,\,\mathcal{G}) \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$ is a completion of $\mathcal{F}_0$ with a sequence of vectors in $\mathbb{C}^d$ with norms prescribed by the sequence $\mathbf{a}$.

In this section we describe the first structural features of local minimizers of the map in Eq. (10), for general strictly convex potentials. These results are applied in the next section to prove that local minima are also global minima.

3.1 On a local Lisdkii's theorem

The result in this subsection lies in the context of matrix analysis, and it is of independent interest. It will be systematically used in the rest of the paper. Since its proof is rather technical, we shall present it in the Appendix (see Section 6). In order to put our result in perspective, we consider the following

Remark 3.1 (Lidskii's (global) inequality). Fix $S \in \mathcal{M}_d(\mathbb{C})^+$ and $\mu \in (\mathbb{R}^d_{\geq 0})^\downarrow$. Consider the unitary orbit

$$\mathcal{O}_\mu = \{G \in \mathcal{M}_d(\mathbb{C})^+ : \lambda(G) = \mu\} = \{U^* D_\mu U : \ U \in \mathcal{U}(d)\} \tag{11}$$

where $D_\mu \in \mathcal{M}_d(\mathbb{C})^+$ denotes the diagonal matrix with main diagonal μ.

Given $\varphi \in \operatorname{Conv_s}(\mathbb{R}_{\geq 0})$ we define the map $\Phi = \Phi_{S,\varphi} : \mathcal{O}_\mu \to \mathbb{R}_{\geq 0}$ given by

$$\Phi(G) = \operatorname{tr}(\varphi(S+G)) = \sum_{j \in \mathbb{I}_d} \varphi(\lambda_j(S+G)) \quad \text{for} \quad G \in \mathcal{O}_\mu \,. \tag{12}$$

(Notice that Eq. (10) motivates the consideration of the map Φ as defined above). Let $(\lambda_i)_{i \in \mathbb{I}_d} = \lambda(S)^\uparrow \in (\mathbb{R}^d_{\geq 0})^\uparrow$. Then, Lidskii's (additive) inequality together with the characterization of the case of equality given in Theorem 2.2 imply that

$$\min_{G \in \mathcal{O}_\mu} \Phi(G) = \sum_{i \in \mathbb{I}_d} \varphi(\lambda_i + \mu_i)$$

and, if $G_0 \in \mathcal{O}_\mu$ then G_0 is a global minimum of Φ on $\mathcal{O}_\mu$ if and only if there exists $\{v_i\}_{i \in \mathbb{I}_d}$ an ONB of $\mathbb{C}^d$ such that

$$S = \sum_{i \in \mathbb{I}_d} \lambda_i \ v_i \otimes v_i \quad \text{and} \quad G_0 = \sum_{i \in \mathbb{I}_d} \mu_i \ v_i \otimes v_i \,.$$

Indeed, Lidskii's inequality states that $\lambda(S)^\uparrow + \lambda(G) \prec \lambda(S+G)$ for every $G \in \mathcal{O}_\mu$; since $\varphi \in \operatorname{Conv_s}(\mathbb{R}_{\geq 0})$, the previous majorization relation implies that

$$\Phi(G) = \operatorname{tr}(\varphi(S+G)) \geq \sum_{i \in \mathbb{I}_d} \varphi(\lambda_i + \mu_i) \,.$$

If we assume that $G_0 \in \mathcal{O}_\mu$ is a global minimum then (see Theorem 2.3),

$$\lambda(S)^\uparrow + \lambda(G_0) \prec \lambda(S+G_0) \quad \text{and} \quad \operatorname{tr}(\varphi(\lambda(S)^\uparrow + \lambda(G_0))) = \operatorname{tr}(\varphi(\lambda(S+G_0)))$$
$$\implies \lambda(S+G_0) = (\lambda(S)^\uparrow + \lambda(G_0))^\downarrow \,,$$

so equality holds in Lidskii's inequality. Hence we can apply Theorem 2.2. Notice that in particular, S and G_0 commute. △

Let $\mu \in (\mathbb{R}^d_{\geq 0})^\downarrow$ and consider the unitary orbit $\mathcal{O}_\mu$ from Eq. (11). In what follows we consider $\mathcal{O}_\mu$ endowed with the metric induced by the operator norm. The next result states that given $\varphi \in \operatorname{Conv_s}(\mathbb{R}_{\geq 0})$ then the **local** minimizers of the map $\Phi = \Phi_{S,\varphi} : \mathcal{O}_\mu \to \mathbb{R}_{\geq 0}$ given by Eq. (12) - in the metric space $\mathcal{O}_\mu$ - are also global minimizers.

Theorem 3.2 (Local Lidskii's theorem). *Let $S \in \mathcal{M}_d(\mathbb{C})^+$ and $\mu = (\mu_i)_{i \in \mathbb{I}_d} \in (\mathbb{R}^d_{\geq 0})^\downarrow$. Assume that $\varphi \in \operatorname{Conv_s}(\mathbb{R}_{\geq 0})$ and that $G_0 \in \mathcal{O}_\mu$ is a local minimizer of $\Phi = \Phi_{S,\varphi}$ on $\mathcal{O}_\mu$. Then, there exists $\{v_i\}_{i \in \mathbb{I}_d}$ an ONB of $\mathbb{C}^d$ such that, if we let $(\lambda_i)_{i \in \mathbb{I}_d} = \lambda^\uparrow(S) \in (\mathbb{R}^d_{\geq 0})^\uparrow$ then*

$$S = \sum_{i \in \mathbb{I}_d} \lambda_i \ v_i \otimes v_i \quad \text{and} \quad G_0 = \sum_{i \in \mathbb{I}_d} \mu_i \ v_i \otimes v_i \ . \tag{13}$$

In particular, $\lambda(S+G_0) = (\lambda(S)^\uparrow + \lambda(G_0)^\downarrow)^\downarrow$ so G_0 is also a global minimizer of Φ on $\mathcal{O}_\mu$.

Proof. See Theorem 6.5 in the Appendix. □

3.2 Geometrical properties of local minima

In the following two sections we study the relative geometry (of both the frame vectors and the eigenvectors of their frame operator) of frame completions with prescribed norms that are local minima of strictly convex potentials. We begin by introducing the basic notation used throughout these sections.

Notation 3.3.

1. Let $\mathcal{F}_0 = \{f_i\}_{i\in\mathbb{I}_{n_0}}$ be a fixed family in $(\mathbb{C}^d)^{n_0}$ and $\mathbf{a} = (a_i)_{i\in\mathbb{I}_k} \in (\mathbb{R}^k_{>0})^{\downarrow}$;
2. Given a subspace $V \subseteq \mathbb{C}^d$ we denote by
$$\mathbb{T}_V(\mathbf{a}) = \Big\{ \mathcal{G} = \{g_i\}_{i\in\mathbb{I}_k} \in V^k : \ \|g_i\|^2 = a_i \, , \ i \in \mathbb{I}_k \ \Big\}$$
endowed with the product topology - of the usual topology in $\mathbb{T}_V(a_i)$ for $i \in \mathbb{I}_k$ - i.e. induced by the metric
$$d(\mathcal{G} \, , \, \tilde{\mathcal{G}}) = \max \{ \, \|g_i - \tilde{g}_i\| : \ i \in \mathbb{I}_k \, \} \, .$$
3. Given $\varphi \in \text{Conv}_s(\mathbb{R}_{\geq 0})$, let $\Psi_\varphi = \Psi_{\varphi \, , \, \mathcal{F}_0} : \mathbb{T}_{\mathbb{C}^d}(\mathbf{a}) \to [0, \infty)$ be given by
$$\Psi_\varphi(\mathcal{G}) = \mathrm{P}_\varphi(\mathcal{F}_0 \, , \, \mathcal{G}) = \mathrm{tr} \, \big(\varphi(\mathrm{S}_{\mathcal{F}_0} + \mathrm{S}_{\mathcal{G}}) \big) \quad \text{for every} \quad \mathcal{G} \in \mathbb{T}_{\mathbb{C}^d}(\mathbf{a}), \tag{14}$$
the convex potential induced by φ of the completed sequence $\mathcal{F} = (\mathcal{F}_0 \, , \, \mathcal{G}) \in \mathcal{C}_{\mathbf{a}} \, (\mathcal{F}_0)$.
4. We shall fix $\mathcal{G}_0 = \{g_i\}_{i\in\mathbb{I}_k} \in \mathbb{T}_{\mathbb{C}^d}(\mathbf{a})$ which is a **local minimum** of Ψ_φ in $\mathbb{T}_{\mathbb{C}^d}(\mathbf{a})$. △

In the following sections we shall see that $\mathcal{G}_0$ is actually a global minimum in $\mathbb{T}_{\mathbb{C}^d}(\mathbf{a})$ for Ψ_φ or, in other words, that $\mathcal{F} = (\mathcal{F}_0 \, , \, \mathcal{G}_0)$ is a global minimum in $\mathcal{C}_{\mathbf{a}} \, (\mathcal{F}_0)$ for the convex potential P_φ .

The following result, which is based on the Local Lidskii's Theorem 3.2, depicts the first structural properties of local minimizers of strictly convex potentials.

Theorem 3.4. *Consider the Notation 3.3 and fix a local minimum* $\mathcal{G}_0 = \{g_i\}_{i\in\mathbb{I}_k} \in \mathbb{T}_{\mathbb{C}^d}(\mathbf{a})$ *of* Ψ_φ *in* $\mathbb{T}_{\mathbb{C}^d}(\mathbf{a})$*. Then,*

1. *For every* $j \in \mathbb{I}_k$*,* $S_\mathcal{F} = S_{\mathcal{F}_0} + S_{\mathcal{G}_0}$ *commutes with* $g_j \otimes g_j$ *or equivalently,* g_j *is an eigenvector of* $S_\mathcal{F} = S_{\mathcal{F}_0} + S_{\mathcal{G}_0}$*.*
2. *There exists* $\{v_i\}_{i\in\mathbb{I}_d}$ *an ONB of* $\mathbb{C}^d$ *such that*
$$S_{\mathcal{F}_0} = \sum_{i\in\mathbb{I}_d} \lambda_i^{\uparrow}(S_{\mathcal{F}_0}) \ v_i \otimes v_i \quad \text{and} \quad S_{\mathcal{G}_0} = \sum_{i\in\mathbb{I}_d} \lambda_i^{\downarrow}(S_{\mathcal{G}_0}) \ v_i \otimes v_i \ .$$
In particular, we have that $\lambda(S_{\mathcal{F}_0} + S_{\mathcal{G}_0}) = \big[\lambda^{\uparrow}(S_{\mathcal{F}_0}) + \lambda(S_{\mathcal{G}_0})\big]^{\downarrow}$*.*

Proof. For each $j \in \mathbb{I}_k$, let $S_j = S_{\mathcal{F}_0} + \sum_{i\in\mathbb{I}_n\setminus\{j\}} g_i \otimes g_i \in \mathcal{M}_d(\mathbb{C})^+$ and $\mu_{[j]} = a_j \, e_1 \in \mathbb{R}^d_{\geq 0}$. Notice that, as in Eq.(11), the orbit $\mathcal{O}_{\mu_{[j]}} = \{g \otimes g : \|g\|^2 = a_j\}$. By hypothesis, it is clear (comparing the maps Ψ_φ and $\Phi_{S_j \, , \varphi}$) that the matrix $G_j = g_j \otimes g_j$ is a local minimum for the map $\Phi_{S_j \, , \varphi}$ on $\mathcal{O}_{\mu_{[j]}}$. Using Theorem 3.2, we conclude that S_j and G_j commute, which implies item 1.

By hypothesis, there exists $\varepsilon > 0$ such that every $U \in B_{(I \, , \, \varepsilon)} \overset{\text{def}}{=} \{U \in \mathcal{U}(d) : \|I - U\| < \epsilon\}$ satisfies that $U \cdot \mathcal{G}_0 = \{U \, g_i\}_{i\in\mathbb{I}_k} \in \mathbb{T}_{\mathbb{C}^d}(\mathbf{a})$, $S_{U\cdot\mathcal{G}_0} = U \, S_{\mathcal{G}_0} \, U^*$ and that $U \cdot \mathcal{G}_0$ is close enough to $\mathcal{G}_0$ so that
$$\Phi_{S_{\mathcal{F}_0} \, , \varphi}(U \, S_{\mathcal{G}_0} \, U^*) = \mathrm{tr} \, \big(\varphi(S_{\mathcal{F}_0} + U \, S_{\mathcal{G}_0} \, U^*) \big) = \Psi_\varphi(U \cdot \mathcal{G}_0) \geq \Psi_\varphi(\mathcal{G}_0) = \Phi_{S_{\mathcal{F}_0} \, , \varphi}(S_{\mathcal{G}_0}) \ .$$
Let $\mu = \lambda(S_{\mathcal{G}_0}) \in (\mathbb{R}^d_{\geq 0})^{\downarrow}$. Notice that the map $\pi : \mathcal{U}(d) \to \mathcal{O}_\mu$ given by $\pi(U) = U \, S_{\mathcal{G}_0} \, U^*$ is open (see [1, Thm 4.1]), so that $\pi(B_{(I \, , \, \varepsilon)})$ is an open neighborhood of $S_{\mathcal{G}_0}$ in $\mathcal{O}_\mu$, and $S_{\mathcal{G}_0}$ is a local minimum for the map $\Phi_{S_{\mathcal{F}_0} \, , \varphi}$ on $\mathcal{O}_\mu$. Item 2 now follows from Theorem 3.2. □

Notation 3.5. Consider the Notation 3.3. Then, Theorem 3.4 allows us to introduce the following notions and notation:

1. We denote by $\lambda = (\lambda_i)_{i\in\mathbb{I}_d} = \lambda_i^\uparrow(S_{\mathcal{F}_0}) \in (\mathbb{R}^d_{\geq 0})^\uparrow$ and $\mu = (\mu_i)_{i\in\mathbb{I}_d} = \lambda_i^\downarrow(S_{\mathcal{G}_0}) \in (\mathbb{R}^d_{\geq 0})^\downarrow$.
2. We fix $\mathcal{B} = \{v_i\}_{i\in\mathbb{I}_d}$ an ONB of $\mathbb{C}^d$ as in Theorem 3.4. Hence,
$$S_{\mathcal{F}_0} = \sum_{i\in\mathbb{I}_d} \lambda_i \ v_i \otimes v_i \quad \text{and} \quad S_{\mathcal{G}_0} = \sum_{i\in\mathbb{I}_d} \mu_i \ v_i \otimes v_i \ , \tag{15}$$
3. We denote by $\nu(\mathcal{G}_0) = \lambda + \mu \in \mathbb{R}^d_{\geq 0}$ so that $S_\mathcal{F} = \sum_{i\in\mathbb{I}_d} \nu_i(\mathcal{G}_0) \ v_i \otimes v_i$. Notice that $\nu(\mathcal{G}_0)$ is constructed by pairing the entries of ordered vectors (since $\lambda = \lambda^\uparrow$ and $\mu = \mu^\downarrow$) but $\nu(\mathcal{G}_0)$ is not necessarily an ordered vector. Nevertheless, we have that $\lambda(S_\mathcal{F}) = \nu(\mathcal{G}_0)^\downarrow$. In what follows we obtain some properties of (the unordered vector) $\nu(\mathcal{G}_0)$
4. Let $s_\mathcal{F} = \max\{i \in \mathbb{I}_d : \mu_i \neq 0\} = \operatorname{rk} S_{\mathcal{G}_0}$. Denote by $W = R(S_{\mathcal{G}_0})$, which reduces $S_\mathcal{F}$.
5. Let $S = S_\mathcal{F}\big|_W \in L(W)$ and $\sigma(S) = \{c_1 , \dots , c_p\}$ (where $c_1 > c_2 > \cdots > c_p > 0$).
6. For each $j \in \mathbb{I}_p$, we consider the following sets of indexes:
$$K_j = \{i \in \mathbb{I}_{s_\mathcal{F}} : \nu_i(\mathcal{G}_0) = \lambda_i + \mu_i = c_j\} \quad \text{and} \quad J_j = \{i \in \mathbb{I}_k : S\, g_i = c_j\, g_i\} \ .$$
Theorem 3.4 assures that $\mathbb{I}_{s_\mathcal{F}} = \bigsqcup_{j\in\mathbb{I}_p} K_j$ and $\mathbb{I}_k = \bigsqcup_{j\in\mathbb{I}_p} J_j$.
7. Since $R(S_{\mathcal{G}_0}) = \operatorname{span}\{g_i : i \in \mathbb{I}_k\} = W = \bigoplus_{i\in\mathbb{I}_p} \ker(S - c_i\, I_W)$ then, for every $j \in \mathbb{I}_p$,
$$W_j \overset{\text{def}}{=} \operatorname{span}\{g_i : i \in J_j\} = \ker(S - c_j\, I_W) = \operatorname{span}\{v_i : i \in K_j\} \ , \tag{16}$$
because $g_i \in \ker(S - c_j\, I_W)$ for every $i \in J_j$. Note that, by Theorem 3.4, each W_j reduces both $S_{\mathcal{F}_0}$ and $S_{\mathcal{G}_0}$. △

The next remark allow us to consider reduction arguments when computing different aspects of the structure of local minima of the completion problem with prescribed norms.

Remark 3.6 (Two reduction arguments for local minima)**.** Consider the data, assumptions and terminology fixed in the Notation 3.3 and 3.5.

a) For any $j \leq p-1$ denote by
$$I_j = \mathbb{I}_d \setminus \bigcup_{i\leq j} K_i \ , \ L_j = \mathbb{I}_k \setminus \bigcup_{i\leq j} J_i \ , \ \lambda^{I_j} = (\lambda_i)_{i\in I_j} \ , \ \mathcal{G}_0^{(j)} = \{g_i\}_{i\in L_j} \ , \ \mathbf{a}^{L_j} = (a_i)_{i\in L_j}$$
and take some sequence $\mathcal{F}_0^{(j)}$ in $\mathcal{H}_j = \big[\bigoplus_{i\leq j} W_i\big]^\perp$ such that $S_{\mathcal{F}_0^{(j)}} = S_{\mathcal{F}_0}|_{\mathcal{H}_j}$ (notice that, by construction, $\mathcal{H}_j$ reduces $S_{\mathcal{F}_0}$). Then, it is straightforward to show that $\mathcal{G}_0^{(j)}$ is a local minimizer of $\Psi^j_{\varphi\,,\,\mathcal{F}_0^{(j)}} : \mathbb{T}_{\mathcal{H}_j}(\mathbf{a}^{L_j}) \to \mathbb{R}_{\geq 0}$. Indeed, if $\mathcal{M}_j$ is any sequence of $|L_j|$ vectors in $\mathcal{H}_j$ with norms prescribed by $\mathbf{a}^{L_j}$ then $\mathcal{M} = (\{g_i\}_{i\in\mathbb{I}_k\setminus L_j} , \mathcal{M}_j) \in \mathbb{T}_{\mathbb{C}^d}(\mathbf{a})$ (in some order) and
$$\Psi_\varphi(\mathcal{F}_0 , \mathcal{M}) = \sum_{i=1}^{j} \varphi(c_i) \ \dim W_i + \Psi^j_\varphi(\mathcal{M}_j) \ ,$$
where the last equation is a consequence of the orthogonality relations between the families $\{g_i\}_{i\in\mathbb{I}_n\setminus L_j}$ and $\mathcal{M}_j$. Also notice that the distance between $\mathcal{M}_j$ and $\mathcal{G}_0^{(j)}$ is the same as the distance between $\mathcal{M}$ and $\mathcal{G}_0$.

The importance of the previous remark lies in the fact that it provides a reduction method to compute the structure of the sets $\mathcal{G}_0^{(i)}$, K_i and J_i for $1 \le i \le p$, as well as the set of constants $c_1 > \ldots > c_p \ge 0$. Indeed, assume that we are able to describe the sets $\mathcal{G}_0^{(1)}$, K_1 , J_1 and the constant c_1 in some structural sense, using the fact that these sets are extremal (e.g. these sets are built on $c_1 > c_j$ for $2 \le j \le p$). Then, we can the apply the same argument to compute, for example, the sets $\mathcal{G}_0^{(2)}$, K_2 , J_2 using that these are extremal for the reduced problem described above for $j = 1$.

b) Assume that $k \in \mathbb{I}_{d-1}$ and let $\mathcal{F} = (\mathcal{F}_0, \mathcal{G}_0) \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$. Fix a sequence $\tilde{\mathcal{F}}_0 = \{\tilde{f}_i\}_{i\in\mathbb{I}_{n_0}}$ in $W = R(S_{\mathcal{G}_0})$ such that $S_{\tilde{\mathcal{F}}_0} = S_{\mathcal{F}_0}|_W$. Then, for every $\mathcal{M} \in \mathbb{T}_W(\mathbf{a})$,

$$\Psi_{\varphi\,,\,\mathcal{F}_0}(\mathcal{M}) = \mathrm{P}_\varphi(\mathcal{F}_0\,,\,\mathcal{M}) = \mathrm{P}_\varphi(\tilde{\mathcal{F}}_0\,,\,\mathcal{M}) + \sum_{i=k+1}^{d} \varphi(\lambda_i) = \Psi_{\varphi\,,\,\tilde{\mathcal{F}}_0}(\mathcal{M}) + \sum_{i=k+1}^{d} \varphi(\lambda_i)\ ,$$

even for $\mathcal{M} = \mathcal{G}_0$. The identity above shows that $\mathcal{G}_0$ is a local minimizer of $\Psi_{\varphi\,,\,\tilde{\mathcal{F}}_0}$ in $\mathbb{T}_W(\mathbf{a})$. In this setting we have that $d' = \dim W = \mathrm{rk}(S_{\mathcal{G}_0}) \le k$. So that in order to compute the structure of $\mathcal{G}_0$ we can assume, as we sometimes do, that $k \ge d$. △

3.3 Inner structure of local minima

Throughout this section we consider the Notation 3.3 and 3.5. Recall that we have fixed $\varphi \in \mathrm{Conv_s}(\mathbb{R}_{\ge 0})$ and a sequence $\mathcal{G}_0 = \{g_i\}_{i\in\mathbb{I}_k} \in \mathbb{T}_{\mathbb{C}^d}(\mathbf{a})$ which is a local minimum of the potential $\Psi_{\varphi\,,\,\mathcal{F}_0}$ in $\mathbb{T}_{\mathbb{C}^d}(\mathbf{a})$. The following result is inspired by some ideas from [8].

Proposition 3.7. *Let $\mathcal{F} = (\mathcal{F}_0\,,\,\mathcal{G}_0) \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$ be as in Notation 3.5 and assume that there exist $j \in \mathbb{I}_p$ and $c \in \sigma(S_\mathcal{F})$ such that $c < c_j$. Then, the family $\{g_i\}_{i\in J_j}$ is linearly independent.*

Proof. Suppose that for some $j \in \mathbb{I}_p$ the family $\{g_i\}_{i\in J_j}$ is linearly dependent. Hence there exist coefficients $z_l \in \mathbb{C}$, $l \in J_j$ (not all zero) such that every $|z_l| \le 1/2$ and

$$\sum_{l\in J_j} \overline{z_l}\ a_l^{1/2}\ g_l = 0\ . \tag{17}$$

Let $I_j \subseteq J_j$ be given by $I_j = \{l \in J_j :\ z_l \neq 0\}$. Assume that there exists $c \in \sigma(S_\mathcal{F})$ such that $c < c_j$ and let $h \in \mathbb{C}^d$ be such that $\|h\| = 1$ and $S_\mathcal{F} h = c\,h$. For $t \in (-1/2, 1/2)$ let $\mathcal{F}(t) = (\mathcal{F}_0\,,\,\mathcal{G}(t))$ where $\mathcal{G}(t) = \{g_i(t)\}_{i\in\mathbb{I}_k}$ is given by

$$g_l(t) = \begin{cases} (1 - t^2\,|z_l|^2)^{1/2} g_l + t\, z_l\, a_l^{1/2}\, h & \text{if } \ l \in I_j\,; \\ \qquad g_l & \text{if } \ l \in K \setminus I_j\ . \end{cases}$$

Notice that $\mathcal{G}(t) \in \mathbb{T}_{\mathbb{C}^d}(\mathbf{a})$ for $t \in (-1/2, 1/2)$. Let $\mathrm{Re}(A) = \frac{A+A^*}{2}$ denote the real part of $A \in \mathcal{M}_d(\mathbb{C})$. For $l \in I_j$ then

$$g_l(t) \otimes g_l(t) = (1 - t^2\,|z_l|^2)\ g_l \otimes g_l + t^2\,|z_l|^2\,a_l\ h \otimes h + 2\,(1 - t^2\,|z_l|^2)^{1/2}\,t\ \ \mathrm{Re}(h \otimes \overline{z_l}\,a_l^{1/2}\,g_l)$$

Let $S(t)$ denote the frame operator of $\mathcal{F}(t) = (\mathcal{F}_0\,,\,\mathcal{G}(t)) \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$, so that $S(0) = S_\mathcal{F}$. Note that

$$S(t) = S_\mathcal{F} + t^2 \sum_{l\in I_j} |z_l|^2\,(-g_l \otimes g_l + a_l\ h \otimes h) + R(t)$$

where $R(t) = 2 \sum\limits_{l\in I_j} (1 - t^2\,|z_l|^2)^{1/2}\,t\ \ \mathrm{Re}(h \otimes a_l^{1/2}\,\overline{z_l}\,g_l)$. Then $R(t)$ is a smooth function such that

$$R(0) = 0 \quad , \quad R'(0) = \sum_{l\in I_j} \mathrm{Re}(h \otimes \overline{z_l}\,a_l^{1/2}\,g_l) = \mathrm{Re}(h \otimes \sum_{l\in I_j} \overline{z_l}\,a_l^{1/2}\,g_l) \overset{(17)}{=} 0\ ,$$

and such that $R''(0) = 0$. Therefore $\lim\limits_{t\to 0} \ t^{-2} \ R(t) = 0$. We now consider

$$V = \text{span} \ \big(\{g_l : \ l \in I_j\} \cup \{h\} \big) = \text{span} \ \big\{ \ g_l : \ l \in I_j \ \big\} \overset{\perp}{\oplus} \mathbb{C} \cdot h \ .$$

Then $\dim V = s + 1$, for $s = \dim \text{span}\{g_l : \ l \in I_j\} \geq 1$. By construction, the subspace V reduces $S_\mathcal{F}$ and $S(t)$ for $t \in \mathbb{R}$, in such a way that $S(t)|_{V^\perp} = S_\mathcal{F}|_{V^\perp}$ for $t \in \mathbb{R}$. On the other hand

$$S(t)|_V = S_\mathcal{F}|_V + t^2 \sum_{l\in I_j} |z_l|^2 \left(-g_l \otimes g_l + a_l \ h \otimes h\right) + R(t) = A(t) + R(t) \in L(V) \ , \qquad (18)$$

where we use the fact that the ranges of the selfadjoint operators in the second and third term in the formula above clearly lie in V. Then $\lambda\big(S_\mathcal{F}|_V \big) = \big(c_j \, \mathbb{1}_s \, , \, c \big) \in (\mathbb{R}_{>0}^{s+1})^\downarrow$ and

$$\lambda\Big(\textstyle\sum_{l\in I_j} |z_l|^2 g_l \otimes g_l \Big) \quad = (\gamma_1 \, , \, \dots \, , \, \gamma_s \, , \, 0) \in (\mathbb{R}_{\geq 0}^{s+1})^\downarrow \quad \text{with} \quad \gamma_s > 0 \ ,$$

where we have used the definition of s and the fact that $|z_l| > 0$ for $l \in I_j$ (and the known fact that if $S \, , \, T \in \mathcal{M}_d(\mathbb{C})^+ \implies R(S + T) = R(S) + R(T)$). Hence, for sufficiently small t, the spectrum of the operator $A(t) \in L(V)$ defined in (18) is

$$\lambda\big(A(t) \big) \quad = \big(c_j - t^2 \, \gamma_s \, , \, \dots \, , \, c_j - t^2 \, \gamma_1 \, , \, c + t^2 \, \textstyle\sum_{l\in I_j} a_l \, |z_l|^2 \big) \in (\mathbb{R}_{\geq 0}^{s+1})^\downarrow \ ,$$

where we have used the fact that $\langle g_l \, , \, h \rangle = 0$ for every $l \in I_j$. Let us now consider

$$\lambda\big(R(t) \big) = \big(\delta_1(t) \, , \, \dots \, , \, \delta_{s+1}(t) \big) \in (\mathbb{R}_{\geq 0}^{s+1})^\downarrow \quad \text{for} \quad t \in \mathbb{R} \ .$$

Recall that in this case $\lim\limits_{t\to 0} t^{-2} \delta_j(t) = 0$ for $1 \leq j \leq s + 1$. Using Weyl's inequality on Eq. (18), we now see that $\lambda\big(S(t)|_V \big) \prec \lambda\big(A(t) \big) + \lambda\big(R(t) \big) \overset{\text{def}}{=} \rho(t) \in (\mathbb{R}_{\geq 0}^{s+1})^\downarrow$. We know that

$$\begin{aligned} \rho(t) \quad &= \big(c_j - t^2 \, \gamma_s + \delta_1(t) \, , \, \dots \, , \, c_j - t^2 \, \gamma_1 + \delta_s(t) \, , \, c + t^2 \, \textstyle\sum_{l\in I_j} a_l \, |z_l|^2 + \delta_{s+1}(t) \big) \\ &= \Big(c_j - t^2 \, (\gamma_s - \tfrac{\delta_1(t)}{t^2}) \, , \, \dots \, , \, c_j - t^2 \, (\gamma_1 - \tfrac{\delta_s(t)}{t^2}) \, , \, c + t^2 \, (\textstyle\sum_{l\in I_j} a_l \, |z_l|^2 + \tfrac{\delta_{s+1}(t)}{t^2}) \Big) \ . \end{aligned}$$

Since by hypothesis $c_j > c$ then, the previous remarks show that there exists $\varepsilon > 0$ such that if $t \in (0, \varepsilon)$ then, for every $i \in \mathbb{I}_s$

$$c_j > c_j - t^2(\gamma_{s-i+1} - \frac{\delta_i(t)}{t^2}) > c + t^2 \, (\sum_{l\in I_j} a_l \, |z_l|^2 + \frac{\delta_{s+1}(t)}{t^2}) \, .$$

The previous facts show that for $t \in (0, \varepsilon)$ then $\rho(t) \prec \lambda(S_\mathcal{F}|_V) = \big(c_j \, \mathbb{1}_s \, , \, c \big)$ strictly. Since φ is strictly convex, for every $t \in (0, \varepsilon)$ we have that

$$\Psi_\varphi\big(\mathcal{G}(t) \big) \leq \text{tr} \ \varphi\big(\lambda(S_\mathcal{F}|_{V^\perp}) \big) + \text{tr} \ \varphi\big(\rho(t) \big) < \text{tr} \ \varphi\big(\lambda(S_\mathcal{F}|_{V^\perp}) \big) + \text{tr} \ \varphi\big(\lambda(S_\mathcal{F}|_V) \big) = \Psi_\varphi(\mathcal{G}_0) \ .$$

This last fact contradicts the assumption that $\mathcal{G}_0$ is a local minimizer of Ψ_φ in $\mathbb{T}_{\mathbb{C}^d}(\mathbf{a})$. □

Recall that, according to Notation 3.5, $c_1 > \ldots > c_p$. Thus, the following result is an immediate consequence of Proposition 3.7 above.

Corollary 3.8. Let $\mathcal{F} = (\mathcal{F}_0 \, , \, \mathcal{G}_0) \in \mathcal{C}_\mathbf{a} \, (\mathcal{F}_0)$ be as in Notation 3.5 and assume that $p > 1$. Then, the family $\{g_i\}_{i\in J_j}$ is linearly independent for every $j \in \mathbb{I}_{p-1}$. In particular, by Eq. (16),

$$\dim(W_j) = |K_j| = |J_j| \quad \text{for} \quad j \in \mathbb{I}_{p-1} \ . \qquad \square$$

Corollary 3.9. *Consider the Notation 3.5 and assume that $k \geq d$. Let $\mathcal{F} = (\mathcal{F}_0\,,\,\mathcal{G}_0) \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$ and assume that $S_{\mathcal{F}}|_W \in L(W)$ has one eigenvalue, where $W = R(S_{\mathcal{G}_0})$. Then:*

1. *$(\lambda,\,\mathbf{a})$ is feasible;*
2. *$\lambda^{\uparrow}(S_{\mathcal{F}}) = \nu(\lambda\,,\,\mathbf{a})$ and $\lambda(S_{\mathcal{G}_0}) = \mu(\lambda\,,\,\mathbf{a})$ (see Definition 2.9);*
3. *$\mathcal{F}$ is a global minimum of Ψ_φ in $\mathbb{T}_{\mathbb{C}^d}(\mathbf{a})$.*

Proof. Assume first that $s_{\mathcal{F}} = d$, i.e. that $W = \mathbb{C}^d$. In this case $S_{\mathcal{F}} = S_{\mathcal{F}_0} + S_{\mathcal{G}_0} = c_1\, I$ and $\mathcal{F}$ is a tight frame. Using the comments at the end of Remark 2.8 and Definition 2.9 (notice that in this case $\min\{d,k\} = d$) we see that $\nu(\lambda\,,\,\mathbf{a}) = c_1\,\mathbb{1}_d$. Hence,

$$\mu(\lambda\,,\,\mathbf{a}) = c_1\,\mathbb{1}_d - \lambda = \lambda^{\downarrow}(S_{\mathcal{G}_0})\ .$$

Since $S_{\mathcal{G}_0}$ is the frame operator of $\mathcal{G}_0 \in \mathbb{T}_{\mathbb{C}^d}(\mathbf{a})$, Proposition 2.4 shows that the majorization relation $\mathbf{a} \prec \mu(\lambda\,,\,\mathbf{a})$ holds, so that the pair $(\lambda\,,\,\mathbf{a})$ is feasible. The fact that $\mathcal{G}_0$ is a global minimizer of Ψ_φ in $\mathbb{T}_{\mathbb{C}^d}(\mathbf{a})$ now follows from Theorem 2.10 (or directly, being a tight completion).

We now consider the case $s_{\mathcal{F}} < d$. Hence, $\mu_i > 0$ for $1 \leq i \leq s_{\mathcal{F}}$ and

$$S_{\mathcal{F}} = \sum_{i\in\mathbb{I}_d} (\lambda_i + \mu_i)\ v_i \otimes v_i = \sum_{i\in\mathbb{I}_{s_{\mathcal{F}}}} c_1\ v_i \otimes v_i + \sum_{i=s_{\mathcal{F}}+1}^{d} \lambda_i\ v_i \otimes v_i\ .$$

In particular, $c_1 = \lambda_{s_{\mathcal{F}}} + \mu_{s_{\mathcal{F}}} > \lambda_{s_{\mathcal{F}}}$. On the other hand, $k \geq d > \dim W$, and thus $\{g_i\}_{i\in\mathbb{I}_k} = \{g_i\}_{i\in J_1}$ is a linearly dependent family. Hence, Proposition 3.7 implies that $c_1 \leq \lambda_i$ for $s_{\mathcal{F}}+1 \leq i \leq d$; in particular, $c_1 \leq \lambda_{s_{\mathcal{F}}+1}$.

The previous facts together with Remark 2.8 show that $\lambda(S_{\mathcal{F}})^{\uparrow} = (c_1\,\mathbb{1}_{s_{\mathcal{F}}}, \lambda_{s_{\mathcal{F}}+1}, \ldots, \lambda_d) = \nu(\lambda\,,\,\mathbf{a})$, according to Definition 2.9. Moreover, we also get that $\lambda(S_{\mathcal{G}_0}) = \nu(\lambda\,,\,\mathbf{a}) - \lambda = \mu(\lambda\,,\,\mathbf{a})$. Again, since $\mathcal{G}_0 \in \mathbb{T}_{\mathbb{C}^d}(\mathbf{a})$ we conclude that the majorization relation $\mathbf{a} \prec \mu(\lambda\,,\,\mathbf{a})$ holds, and therefore the pair $(\lambda\,,\,\mathbf{a})$ is feasible. As before, Theorem 2.10 shows that $\mathcal{G}_0$ is a global minimizer of Ψ_φ in $\mathbb{T}_{\mathbb{C}^d}(\mathbf{a})$. □

The next result is [30, Proposition 4.5]. Although the result is stated for a global minimum in [30], the inspection of its proof (for $\varphi \in \mathrm{Conv_s}(\mathbb{R}_{\geq 0})$, so that the previous results hold) reveals that it also holds for a local minimum as well. Recall from Notation 3.5 that, if $p > 1$ then

$$K_j = \{i \in \mathbb{I}_{s_{\mathcal{F}}} : \lambda_i + \mu_i = c_j\} \quad \text{ and } \quad J_j = \{i \in \mathbb{I}_k : S_{\mathcal{F}}\, g_i = c_j\, g_i\}$$

for each $j \in \mathbb{I}_p$, where $\lambda = \lambda^{\uparrow}(S_{\mathcal{F}_0})$ and $\mu = \mu^{\downarrow} = \lambda(S_{\mathcal{G}_0})$.

Proposition 3.10. Consider the Notation 3.5 with $\mathcal{F} = (\mathcal{F}_0\,,\,\mathcal{G}_0) \in \mathcal{C}^{\rm op}_{\mathbf{a}}(\mathcal{F}_0)$ and assume that $k \geq d$ and $p > 1$. Given $i\,,\,r \in \mathbb{I}_p$, $h \in J_i$ and $l \in J_r$ then

$$i < r \implies a_h - a_l \geq c_i - c_r > 0 \implies h < l\ .$$

In particular, there exist $s_0 = 0 < s_1 < \ldots < s_{p-1} < s_{\mathcal{F}} \leq d$ such that

$$J_j = \{s_{j-1} + 1\,,\,\ldots\,,\,s_j\}\ ,\quad j \in \mathbb{I}_{p-1} \quad \text{ and } \quad J_p = \{s_{p-1} + 1\,,\,\ldots\,,\,k\}\ . \qquad \square$$

Proposition 3.11. *Consider the Notation 3.5 with $\mathcal{F} = (\mathcal{F}_0\,,\,\mathcal{G}_0) \in \mathcal{C}^{\rm op}_{\mathbf{a}}(\mathcal{F}_0)$ and assume that $k \geq d$ and $p > 1$. We have that:*

$$i \in K_1 \implies i < j\ (\implies \lambda_i \leq \lambda_j) \quad \textit{for every} \quad j \in \bigcup_{r>1} K_r = \mathbb{I}_{s_{\mathcal{F}}} \setminus K_1\ .$$

Inductively, by means of Remark 3.6, we deduce that all sets K_j consist of consecutive indexes. Therefore, if $s_0 = 0 < s_1 < \ldots < s_{p-1} < s_p \stackrel{\text{def}}{=} s_{\mathcal{F}}$ are as in Proposition 3.10 then

$$K_j = \{s_{j-1} + 1\,,\,\ldots\,,\,s_j\}\ ,\quad j \in \mathbb{I}_{p-1} \quad \textit{and} \quad K_p = \{s_{p-1} + 1, \ldots, s_p\}\ .$$

Proof. Assume that there exist $i \in K_1$ and $j \in K_r$ for $1 < r$ such that $j < i$. In this case,

$$\mu_i \le \mu_j \quad , \quad \lambda_j \le \lambda_i \quad \text{and} \quad c_1 = \lambda_i + \mu_i > c_r = \lambda_j + \mu_j \ .$$

Consider $\mathcal{B} = \{v_l\}_{l \in \mathbb{I}_d}$ as in Notation 3.5. For $t \in [0,1)$ we let

$$g_l(t) = g_l + \big((1-t^2)^{1/2} - 1 \big) \, \langle g_l, v_i \rangle \, v_i + t \, \langle g_l, v_i \rangle \, v_j \quad \text{for} \quad l \in \mathbb{I}_k \, . \tag{19}$$

Notice that, if $l \in J_1$, then $S_{\mathcal{F}} \, g_l = c_1 \, g_l \implies \langle g_l \, , \, v_j \rangle = 0$. Similarly, if $l \in \mathbb{I}_k \setminus J_1$ then $\langle g_l \, , \, v_i \rangle = 0$ (so that $g_l(t) = g_l$). Therefore the sequence $\mathcal{G}(t) = \{g_l(t)\}_{l \in \mathbb{I}_k} \in \mathbb{T}_{\mathbb{C}^d}(\mathbf{a})$ for $t \in [0,1)$. Let $P_i = v_i \otimes v_i$ and $P_{ji} = v_j \otimes v_i$ (so that $P_{ji} \, x = \langle x \, , \, v_i \rangle \, v_j$). Then, for every $t \in [0 \, , \, 1)$,

$$g_l(t) = \big(I + ((1-t^2)^{1/2} - 1) \; P_i + t \; P_{ji} \big) \; g_l \quad \text{for every} \quad l \in \mathbb{I}_k \ .$$

That is, if $V(t) = I + ((1-t^2)^{1/2} - 1) \; P_i + t \; P_{ji} \in \mathcal{M}_d(\mathbb{C})$ then $g_l(t) = V(t) \; g_l$ for every $l \in \mathbb{I}_k$ and $t \in [0,1)$. Therefore, we get that

$$\mathcal{G}(t) = V(t) \, G = \{V(t) \, g_l\}_{l \in \mathbb{I}_n} \implies S_{G(t)} = V(t) \, S_{\mathcal{G}} \, V(t)^* \quad \text{for} \quad t \in [0,1) \ .$$

Hence, we obtain the representation

$$S_{\mathcal{G}(t)} = \sum_{\ell \in \mathbb{I}_d \setminus \{i,j\}} \mu_\ell \; v_\ell \otimes v_\ell + \gamma_{11}(t) \; v_j \otimes v_j + \gamma_{12}(t) \; v_j \otimes v_i + \gamma_{21}(t) \; v_i \otimes v_j + \gamma_{22}(t) \; v_i \otimes v_i \ ,$$

where the functions $\gamma_{rs}(t)$ are the entries of $A(t) = \big(\, \gamma_{rs}(t) \, \big)_{r\,,\,s=1}^{2} \in \mathcal{H}(2)$ defined by

$$A(t) = \begin{pmatrix} 1 & t \\ 0 & (1-t^2)^{1/2} \end{pmatrix} \begin{pmatrix} \mu_j & 0 \\ 0 & \mu_i \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t & (1-t^2)^{1/2} \end{pmatrix} \quad \text{for every} \quad t \in [0 \, , \, 1) \ .$$

It is straightforward to check that $\text{tr}(A(t)) = \mu_i + \mu_j$ and that $\det(A(t)) = (1-t^2) \, \mu_j \, \mu_i$. These facts imply that if we consider the continuous function $L(t) = \lambda_{\max}(A(t))$ then $L(0) = \mu_j$ and $L(t)$ is strictly increasing in $[0,1)$. More straightforward computations show that we can consider continuous curves $x_i(t) : [0,1) \to \mathbb{C}^2$ which satisfy that $\{x_1(t), \, x_2(t)\}$ is ONB of $\mathbb{C}^2$ such that

$$A(t) \, x_1(t) = L(t) \, x_1(t) \quad \text{for} \quad t \in [0,1) \quad \text{and} \quad x_1(0) = e_1 \ , \ \ x_2(0) = e_2 \, .$$

For $t \in [0,1)$ we let $X(t) = (u_{r,s}(t))_{r,s=1}^2 \in \mathcal{U}(2)$ with columns $x_1(t)$ and $x_2(t)$. By construction, $X(t) = [0,1) \to \mathcal{U}(2)$ is a continuous curve such that $X(0) = I_2$ and such that

$$X(t)^* \, A(t) \, X(t) = \begin{pmatrix} L(t) & 0 \\ 0 & \mu_i + \mu_j - L(t) \end{pmatrix} \, .$$

Finally, consider the continuous curve $U(t) : [0,1) \to \mathcal{U}(d)$ given by

$$U(t) = u_{11}(t) \, v_j \otimes v_j + u_{12}(t) \, v_j \otimes v_i + u_{21}(t) \, v_i \otimes v_j + u_{22}(t) \, v_i \otimes v_i + \sum_{\ell \in \mathbb{I}_d \setminus \{i,j\}} v_l \otimes v_l \ .$$

Notice that $U(0) = I$; also, let $\tilde{\mathcal{G}}(t) = U(t)^* \, \mathcal{G}(t) \in \mathbb{T}_{\mathbb{C}^d}(\mathbf{a})$ for $t \in [0,1)$, which is a continuous curve such that $\tilde{\mathcal{G}}(0) = \mathcal{G}_0$. In this case, for $t \in [0,1)$ we have that

$$S_{\tilde{\mathcal{G}}(t)} = U(t)^* \, S_{\mathcal{G}(t)} \, U(t) = L(t) \, v_j \otimes v_j + (\mu_i + \mu_j - L(t)) \, v_i \otimes v_i + \sum_{\ell \in \mathbb{I}_d \setminus \{i,j\}} \mu_\ell \; v_\ell \otimes v_\ell \ .$$

In other words, $U(t)$ is constructed in such a way that $\mathcal{B} = \{v_l\}_{i\in\mathbb{I}_d}$ consists of eigenvectors of $S_{\tilde{\mathcal{G}}(t)}$ for every $t \in [0,1)$. Hence, if $\tilde{\mathcal{F}}(t) = (\mathcal{F}_0\,,\,\tilde{\mathcal{G}}(t))$ and $E(t) = L(t) - \mu_j \geq 0$ for $t \in [0\,,\,1)$, we get that

$$S_{\tilde{\mathcal{F}}(t)} = (c_r + E(t)\,)\, v_j \otimes v_j + (c_1 - E(t)\,)\, v_i \otimes v_i + \sum_{\ell\in\mathbb{I}_d\setminus\{i,j\}} (\lambda_\ell + \mu_\ell)\; v_\ell \otimes v_\ell \;.$$

Let $\varepsilon > 0$ be such that $E(t) = L(t) - \mu_j \leq \frac{c_1 - c_r}{2}$ for $t \in [0, \varepsilon]$. (recall that $L(0) = \mu_j$ and that $c_1 > c_r$). Since $L(t)$ (and hence $E(t)$) is strictly increasing in $[0,1)$, we see that

$$(c_1 - E(t)\,,\, c_r + E(t)) \prec (c_1\,,\, c_r) \implies \lambda(S_{\tilde{\mathcal{F}}(t)}) \prec \lambda(S_{\mathcal{F}}) \quad \text{for} \quad t \in (0\,,\,\varepsilon]\;,$$

where the majorization relations above are strict. Hence, since $\varphi \in \operatorname{Conv_s}(\mathbb{R}_{\geq 0})$ then

$$\Psi_\varphi(\tilde{\mathcal{G}}(t)) = \operatorname{tr}(\varphi(\lambda(S_{\tilde{\mathcal{F}}(t)}))) < \operatorname{tr}(\varphi(\lambda(S_{\mathcal{F}}))) = \Psi_\varphi(\mathcal{G}_0) \quad \text{for} \quad t \in (0\,,\,\varepsilon]\;.$$

This last fact contradicts the local minimality of $\mathcal{G}_0$ and the result follows. The description of the sets K_i's now follows from Corollary 3.8. □

4 Local minima are global minima

Throughout this section we adopt Notation 3.3 and 3.5. Recall that we have fixed a map $\varphi \in \operatorname{Conv_s}(\mathbb{R}_{\geq 0})$ and a sequence $\mathcal{G}_0 = \{g_i\}_{i\in\mathbb{I}_k} \in \mathbb{T}_{\mathbb{C}^d}(\mathbf{a})$ which is a local minimum of the potential Ψ_φ in $\mathbb{T}_{\mathbb{C}^d}(\mathbf{a})$, among several other specific notation.

In what follows, we show that local minimizers (as $\mathcal{G}_0$) of Ψ_φ in $\mathbb{T}_{\mathbb{C}^d}(\mathbf{a})$ are global minimizers (see Theorem 4.10 below). In order to do this, we develop a detailed study of the inner structure of local minimizers, based on the results from Section 3.

Remark 4.1 (Case $k \geq d$)**.** Consider the Notation 3.3 and 3.5 and assume that $k \geq d$. Then, according to Propositions 3.10 and 3.11, there exist $p \in \mathbb{I}_d$ and $s_0 = 0 < s_1 < \ldots < s_{p-1} < s_p = s_{\mathcal{F}} \leq d$, where $s_{\mathcal{F}} = \operatorname{rk}(S_{\mathcal{G}_0})$, such that

$$\begin{aligned} K_j &= J_j = \{s_{j-1}+1\,,\,\ldots\,,\,s_j\}\;, \qquad \text{for} \quad j \in \mathbb{I}_{p-1}\;, \\ K_p &= \{s_{p-1}+1\,,\,\ldots\,,\,s_p\}\;,\;\; J_p = \{s_{p-1}+1\,,\,\ldots\,,\,k\}\;. \end{aligned} \tag{20}$$

In terms of these indexes we also get that:

$$\lambda(S_{\mathcal{F}}) = \big(\, c_1\, \mathbb{1}_{s_1}\,,\,\ldots\,,\, c_p\, \mathbb{1}_{s_p - s_{p-1}}\,,\, \lambda_{s_p+1}\,,\,\ldots\,,\,\lambda_d\,\big)^{\downarrow} \in (\mathbb{R}^d_{>0})^{\downarrow} \quad \text{if} \quad s_p < d \tag{21}$$

or

$$\lambda(S_{\mathcal{F}}) = \big(\, c_1\, \mathbb{1}_{s_1}\,,\,\ldots\,,\, c_p\, \mathbb{1}_{s_p - s_{p-1}}\,\big)^{\downarrow} \in (\mathbb{R}^d_{>0})^{\downarrow} \quad \text{if} \quad s_p = d \tag{22}$$

In what follows, we describe an algorithm that computes both the constants $c_1 > \ldots > c_p$ as well as the indexes $s_1 < \ldots < s_p$ in terms of the index s_{p-1}. △

In order to show the role of the index s_{p-1} as described in Remark 4.1 above, we consider the following

Definition 4.2. Let $\lambda = (\lambda_i)_{i\in\mathbb{I}_d} \in (\mathbb{R}^d_{\geq 0})^{\uparrow}$ and $\mathbf{a} = (a_i)_{i\in\mathbb{I}_k} \in (\mathbb{R}^k_{>0})^{\downarrow}$, with $k \geq d$.

1. Given $0 \leq s \leq d-1$ denote by

$$\lambda^s = (\lambda_{s+1}\,,\,\ldots\,,\,\lambda_d) \in \mathbb{R}^{d-s} \quad \text{and} \quad \mathbf{a}^s = (a_{s+1}\,,\,\ldots\,,\,a_k) \in \mathbb{R}^{k-s}\;,$$

   the truncations of the vectors λ and $\mathbf{a}$.

2. We say that the index s is feasible (for the pair $(\lambda\,,\,\mathbf{a})$) if $(\lambda^s\,,\,\mathbf{a}^s)$ is a feasible pair (see Definition 2.9) i.e. if $\mathbf{a}^s \prec \nu(\lambda^s\,,\,\mathbf{a}^s) - \lambda^s$. △

Notice that, with the notation and terminology from Definition 4.2 above, the pair $(\lambda\,,\,\mathbf{a})$ is feasible (according to Definition 2.9) if and only if the index $s = 0$ is feasible (according to Definition 4.2).

In the following statements we shall use the Notation 3.3 and 3.5. Recall that we have fixed a map $\varphi \in \mathrm{Conv_s}(\mathbb{R}_{\geq 0})$ and a sequence $\mathcal{G}_0 = \{g_i\}_{i\in\mathbb{I}_k} \in \mathbb{T}_{\mathbb{C}^d}(\mathbf{a})$ which is a local minimum of the potential Ψ_φ in $\mathbb{T}_{\mathbb{C}^d}(\mathbf{a})$, among several other specific notation.

Proposition 4.3. *Consider the Notation 3.3 and 3.5 and assume that $k \geq d$. Let $s_0 = 0 < s_1 < \ldots < s_{p-1} < s_p \leq d$ be as in Remark 4.1. Then:*

1. *The index $s_{p-1} \geq 0$ is feasible;*
2. *The constant c_p and the index s_p are determined by:*

$$c_p = (d-s_{p-1})^{-1}\,(\operatorname{tr}\lambda^{s_{p-1}} + \operatorname{tr}\mathbf{a}^{s_{p-1}}) \quad \text{and} \quad s_p = d \quad \text{if} \quad (d-s_{p-1})^{-1}\,(\operatorname{tr}\lambda^{s_{p-1}} + \operatorname{tr}\mathbf{a}^{s_{p-1}}) \geq \lambda_d$$

   or otherwise, if $(d-s_{p-1})^{-1}\,(\operatorname{tr}\lambda^{s_{p-1}} + \operatorname{tr}\mathbf{a}^{s_{p-1}}) < \lambda_d$ by the identity

$$\nu(\lambda^{s_{p-1}}\,,\,\mathbf{a}^{s_{p-1}}) = \big(\,c_p\,\mathbb{1}_{s_p - s_{p-1}}\,,\,\lambda_{s_p+1}\,,\,\ldots\,,\,\lambda_d\,\big) \quad \text{and} \quad s_p < d\,. \tag{23}$$

3. *If we let $\mathcal{G}_0^{(p-1)} = \{g_i\}_{i=s_{p-1}+1}^k$ then $\lambda(S_{\mathcal{G}_0^{(p-1)}}) = ((\mu_i)_{i=s_{p-1}+1}^{s_p}, 0_{d-(s_p-s_{p-1})}) \in (\mathbb{R}^d_{\geq 0})^\downarrow$; hence*

$$(a_i)_{i=s_{p-1}+1}^k \prec (\mu_i)_{i=s_{p-1}+1}^{s_p}\,.$$

Proof. By Remark 3.6 (item a) with $j = p-1$) the family $\mathcal{G}_0^{(p-1)} = \{g_i\}_{i=s_{p-1}+1}^k$ is a local minimum of the map (set $k' = k - s_{p-1} \geq 1$)

$$\{\mathcal{K} = \{k_i\}_{i\in\mathbb{I}_{k'}} \in (\mathcal{H}_{p-1})^{k'}\ ,\ \|k_i\|^2 = a_{s_{p-1}+i}\,,\, i \in \mathbb{I}_{k'}\} \ni \mathcal{K} \mapsto \mathrm{P}_\varphi(\mathcal{F}_0^{(p-1)}, \mathcal{K})$$

where, using the notation from Remark 3.6, $\mathcal{H}_{p-1} = \big[\,\bigoplus_{i\leq p-1} W_i\big]^\perp$ and $\mathcal{F}_0^{(p-1)}$ is a sequence in $\mathcal{H}_{p-1}$ such that $S_{\mathcal{F}_0^{p-1}} = S_{\mathcal{F}_0}|_{\mathcal{H}_{p-1}}$. Moreover, by construction of the subspace $\mathcal{H}_{p-1}$ we see that if we let $\mathcal{F}^{(p-1)} = (\mathcal{F}_0^{(p-1)}, \mathcal{G}_0^{(p-1)}) \in \mathcal{C}_{\mathbf{a}^{s_{p-1}}}(\mathcal{F}_0^{(p-1)})$ then $W_p = R(S_{\mathcal{G}_0^{(p-1)}})$ and

$$S_{\mathcal{F}^{(p-1)}}\ P_{W_p} = c_p\ P_{W_p}\,.$$

Therefore, by Corollary 3.9, we see that the pair $(\lambda^s\,,\,\mathbf{a}^s)$ is feasible. The other claims follow from Remark 2.8, Corollary 3.9 and Proposition 2.4. □

Remark 4.4. Observe that, under the assumptions of Proposition 4.3 then item 2 implies that

$$s_p = \max\{j \in \mathbb{I}_d :\ \lambda_j < c_p\} \in \mathbb{I}_d\,.$$

△

Notation 4.5. Consider the Notation 3.3 and 3.5, and assume that $k \geq d$.

1. We let $h_i := \lambda_i + a_i$ for every $i \in \mathbb{I}_d$.
2. Given $j \leq r \leq d$, let

$$P_{j\,,\,r} = \frac{1}{r-j+1}\ \sum_{i=j}^r\ h_i = \frac{1}{r-j+1}\ \sum_{i=j}^r\ \lambda_i + a_i\,.$$

   We abbreviate $P_{1\,,\,r} = P_r$ for the initial averages. △

The following result will allow us to obtain several relations between the indexes and constants describing $\lambda(S_\mathcal{F})$ as in Remark 4.1. We point out that the ideas behind its proof are derived from [30].

Lemma 4.6. *Consider the Notation 3.3 and 3.5 and assume that $k \geq d$, $p > 1$. With the notation of Remark 4.1 we have that*

1. *If $1 \leq r \leq d$ then*

$$(a_j)_{j\in\mathbb{I}_r} \prec (P_r - \lambda_j)_{j\in\mathbb{I}_r} \iff P_r \geq P_i \ , \quad i \in \mathbb{I}_r \iff P_r = \max\{P_i : \ i \in \mathbb{I}_r\} \, .$$

2. *$c_1 = P_{s_1} = \max\{P_j : \ j \leq s_{p-1}\}$. Moreover, if $s_1 < t \leq s_{p-1} \implies P_t < c_1$.*

Proof. 1. Since $\lambda = \lambda^\uparrow$ and $\mathbf{a} = \mathbf{a}^\downarrow$ then $(P_r - \lambda_j)_{j\in\mathbb{I}_r} = (P_r - \lambda_j)^\downarrow_{j\in\mathbb{I}_r}$ and $(a_j)_{j\in\mathbb{I}_r} = (a_j)^\downarrow_{j\in\mathbb{I}_r}$. On the other hand, $\sum_{j\in\mathbb{I}_r} a_j = \sum_{r\in\mathbb{I}_r} P_r - \lambda_j$ by definition of P_r. Therefore, $(a_j)_{j\in\mathbb{I}_r} \prec (P_r - \lambda_j)_{j\in\mathbb{I}_r}$ if and only if for $k \in \mathbb{I}_r$

$$\sum_{j\in\mathbb{I}_k} a_j \leq \sum_{j\in\mathbb{I}_k} (P_r - \lambda_j) \iff P_k = \frac{1}{k} \sum_{j\in\mathbb{I}_k} a_j + \lambda_j \leq P_r \, .$$

2. By Propositions 3.10 and 3.11 we see that the sequence $\{g_j\}_{j\in\mathbb{I}_{s_1}}$ is such that its frame operator has eigenvalues given by $(\mu_1, \ldots, \mu_{s_1}, 0, \ldots, 0) \in (\mathbb{R}^d_{\geq 0})^\downarrow$ and their norms are given by $\|g_j\|^2 = a_j$ for $j \in \mathbb{I}_{s_1}$. By Proposition 2.4 we get that $(a_j)_{j\in\mathbb{I}_{s_1}} \prec (\mu_j)_{j\in\mathbb{I}_{s_1}}$. On the other hand, Proposition 3.11 implies that $\lambda_j + \mu_j = c_1$ for $j \in \mathbb{I}_{s_1}$. Then

$$s_1 \, c_1 = \sum_{j\in\mathbb{I}_{s_1}} \lambda_j + \mu_j = \sum_{j\in\mathbb{I}_{s_1}} \lambda_j + a_j \implies c_1 = \frac{1}{s_1} \sum_{j\in\mathbb{I}_{s_1}} \lambda_j + a_j = P_{s_1} \, .$$

Hence $(a_j)_{j\in\mathbb{I}_{s_1}} \prec (c_1 - \lambda_j)_{j\in\mathbb{I}_{s_1}} = (P_{s_1} - \lambda_j)_{j\in\mathbb{I}_{s_1}} \implies P_{s_1} = \max\{P_j : \ j \in \mathbb{I}_{s_1}\}$. Consider now $s_1 < t \leq s_{p-1}$ and let $2 \leq r \leq p-1$ be such that $s_{r-1} < t \leq s_r$. Then

$$\begin{aligned} P_t &= \frac{s_1}{t} \left(\frac{1}{s_1} \sum_{j\in\mathbb{I}_{s_1}} h_j \right) + \frac{t-s_1}{t} \left(\frac{1}{t-s_1} \sum_{j=s_1+1}^{t} h_j \right) \\ &= \frac{s_1}{t} \, c_1 + \frac{t-s_1}{t} \left(\frac{1}{t-s_1} \, (\sum_{\ell=2}^{r-1} c_\ell \, (s_\ell - s_{\ell-1}) + \sum_{\ell=s_{r-1}+1}^{t} \lambda_\ell + a_\ell \,) \right) \, , \end{aligned}$$

that represents P_t as a convex combination, where we have used the identities

$$\sum_{i=s_{\ell-1}+1}^{s_\ell} h_i = \sum_{i=s_{\ell-1}+1}^{s_\ell} \lambda_i + \mu_i = (s_\ell - s_{\ell-1}) \, c_\ell$$

that follow from the majorization relation $(a_i)_{i=s_{\ell-1}+1}^{s_\ell} \prec (\mu_i)_{i=s_{\ell-1}+1}^{s_\ell}$ for $2 \leq \ell \leq p-1$, which are a consequence of Propositions 3.10, 3.11 and 2.4; using the relation $(a_i)_{i=s_{r-1}+1}^{s_r} \prec (\mu_i)_{i=s_{r-1}+1}^{s_r}$, together with the fact that the entries of these two vectors are downwards ordered, we conclude that

$$\frac{1}{t-s_1} \, (\sum_{\ell=2}^{r-1} c_\ell \, (s_\ell - s_{\ell-1}) + \sum_{\ell=s_{r-1}+1}^{t} \lambda_\ell + a_\ell \,) \leq \frac{1}{t-s_1} \, (\sum_{\ell=2}^{r-1} c_\ell \, (s_\ell - s_{\ell-1}) + c_r \, (t - s_{r-1}) \,) < c_1$$

since the expression to the left is a convex combination of $c_2, \ldots, c_r < c_1$. Finally, we can deduce that $P_t < \frac{s_1}{t} \, c_1 + \frac{t-s_1}{t} \, c_1 = c_1$. □

Proposition 4.7. Consider the Notation 3.3 and 3.5, and assume that $k \geq d$. Let p, $s_0 = 0 < s_1 < \ldots < s_{p-1} < s_p \leq d$ and $c_1 > \ldots > c_p$ be as in Remark 4.1, and assume that $p > 1$. If we let $\mathcal{F} = (\mathcal{F}_0, \mathcal{G}_0)$ then, we have the following relations between these indexes and constants:

1. The index $s_1 = \max \big\{ j \leq s_{p-1} \,:\, P_{1\,,\,j} = \max\limits_{i \leq s_{p-1}} P_{1\,,\,i} \big\}$, and $c_1 = P_{1\,,\,s_1}$.

2. If $s_j < s_{p-1}$, then
$$s_{j+1} = \max \big\{ s_j < r \leq s_{p-1} \,:\, P_{s_j+1\,,\,j} = \max_{s_j < i \leq s_{p-1}} P_{s_j+1\,,\,i} \big\} \quad \text{and} \quad c_{j+1} = P_{s_j+1\,,\,s_{j+1}} \, .$$

3. s_{p-1} is a feasible index and c_p and s_p are determined by (Definition 4.2)
$$c_p = (d - s_{p-1})^{-1} \, (\operatorname{tr} \lambda^{s_{p-1}} + \operatorname{tr} \mathbf{a}^{s_{p-1}}) \quad \text{and} \quad s_p = d \quad \text{if} \quad (d - s_{p-1})^{-1} \, (\operatorname{tr} \lambda^{s_{p-1}} + \operatorname{tr} \mathbf{a}^{s_{p-1}}) \geq \lambda_d$$
or otherwise, if $(d - s_{p-1})^{-1} \, (\operatorname{tr} \lambda^{s_{p-1}} + \operatorname{tr} \mathbf{a}^{s_{p-1}}) < \lambda_d$ by the identity
$$\nu(\lambda^{s_{p-1}}, \mathbf{a}^{s_{p-1}}) = \big(c_p \, \mathbb{1}_{s_p - s_{p-1}} \, , \, \lambda_{s_p+1} \, , \, \ldots \, , \, \lambda_d \big) \quad \text{and} \quad s_p < d \, . \tag{24}$$
Moreover, the following inequalities hold:
$$c_p \geq \frac{1}{\ell - s_{p-1}} \sum_{i=s_{p-1}+1}^{\ell} h_i \quad \text{for} \quad s_{p-1} + 1 \leq \ell \leq s_p \, . \tag{25}$$

Proof. Item 1 is contained in Lemma 4.6. Item 2 above also follows from Lemma 4.6 applied to the reduced families $\mathcal{G}_0^{(j)}$ as defined in Remark 3.6. Notice that, as a consequence of Propositions 3.10 and 3.11 then - using the notation from Remark 3.6 - we have that, for $1 \leq j \leq s_{p-1}$, then
$$I_j = \{i : \ s_j + 1 \leq i \leq d\} \quad \text{and} \quad L_j = \{i : \ s_j + 1 \leq i \leq k\} \, ,$$
which imply that $\lambda^{I_j} = (\lambda_i)_{i=s_j+1}^{d} \in (\mathbb{R}_{\geq 0}^{d-s_j})^{\uparrow}$, $\mathcal{G}_0^{(j)} = \{g_i\}_{i=s_j+1}^{k}$ and $\mathbf{a}^{L_j} = (a_i)_{i=s_j+1}^{k} \in (\mathbb{R}_{\geq 0}^{k-s_j})^{\downarrow}$. Proposition 4.3 shows that s_{p-1} is a feasible index and that the constant c_p and the index s_p are determined as described above. Finally, notice that Proposition 4.3 shows the majorization relation $(a_i)_{i=s_{p-1}+1}^{k} \prec (\mu_i)_{i=s_{p-1}+1}^{s_p}$, where $1 \leq s_p \leq d \leq k$. Hence,
$$\sum_{i=s_{p-1}+1}^{\ell} a_i \leq \sum_{i=s_{p-1}+1}^{\ell} \mu_i \quad \text{for every } \ell \text{ such that} \quad s_{p-1} + 1 \leq \ell \leq s_p \, .$$
Using this inequality and the fact that $\lambda_i + \mu_i = c_p$ for $s_{p-1} + 1 \leq i \leq s_p$ we get (25). □

The following are the two main results of [30]. We will need the detailed structure of **global** minima described in both results in order to prove Theorem 4.10 below.

Theorem 4.8 ([30]). Let $\mathcal{F}_0 = \{f_i\}_{i \in \mathbb{I}_{n_0}} \in (\mathbb{C}^d)^{n_0}$, let $\lambda = \lambda(S_{\mathcal{F}_0})^{\uparrow} \in (\mathbb{R}_{\geq 0}^d)^{\uparrow}$ and let $\mathbf{a} = (a_i)_{i \in \mathbb{I}_k} \in (\mathbb{R}_{>0}^k)^{\downarrow}$. In order to construct the optimal spectra of frame completions we consider the following two cases:

C.1. In case $k \geq d$, define
$$s^* = \min \, \{ \ 0 \leq s \leq d - 1 : \ s \text{ is a feasible index for the pair } (\lambda \, , \, \mathbf{a}) \ \}$$
and let $q \in \mathbb{I}_d$, $s_0^* = 0 < s_1^* < \ldots < s_{q-1}^* = s^* < s_q \leq d$ and $c_1^* < \ldots < c_{q-1}^* < c_q^*$ be computed according to the following recursive algorithm:

1. The index $s_1^* = \max \left\{ j \leq s^* \ : \ P_{1\,,\,j} = \max\limits_{i\leq s^*} P_{1\,,\,i} \right\}$, and $c_1^* = P_{1\,,\,s_1^*}$.
2. If the index s_j^* is already computed and $s_j^* < s^*$, then

$$s_{j+1}^* = \max \left\{ s_j^* < r \leq s^* \ : \ P_{s_j^*+1\,,\,j} = \max_{s_j^* < i \leq s^*} P_{s_j^*+1\,,\,i} \right\} \quad \text{and} \quad c_{j+1}^* = P_{s_j^*+1\,,\,s_{j+1}^*} \ .$$

3. Set $s_{q-1}^* = s^*$, and let c_q^* and $s_{q-1}^* < s_q^* \leq d$ be such that (see Definition 4.2)

$$(c_q^* \, \mathbb{1}_{s_q^* - s_{q-1}^*} \, , \, \lambda_{s_q^*+1} \, , \, \ldots \, , \, \lambda_d) = \nu(\lambda^{s^*} \, , \, \mathbf{a}^{s^*}) \in (\mathbb{R}_{>0}^{d-s^*})^{\uparrow} \ .$$

Finally, set

$$\nu^{\rm op}(\lambda \, , \, \mathbf{a}) := (c_1^* \, \mathbb{1}_{s_1^*} \, , \, c_2^* \, \mathbb{1}_{s_2^* - s_1^*} \, , \, \ldots \, , \, c_{q-1}^* \, \mathbb{1}_{s_{q-1}^* - s_{q-2}^*} \, , \, \nu(\lambda^{s^*} \, , \, \mathbf{a}^{s^*})) \in \mathbb{R}_{>0}^d \ ,$$

C.2. In case $k < d$, define

$$\tilde{\lambda} = (\lambda_i)_{i\in\mathbb{I}_k} \in (\mathbb{R}_{\geq 0}^k)^{\uparrow} \quad \text{and} \quad \tilde{\nu} = \nu^{\rm op}(\tilde{\lambda} \, , \, \mathbf{a}) \in (\mathbb{R}_{>0}^k)^{\downarrow} \, ,$$

where the second vector is constructed according case C.1. above, and set

$$\nu^{\rm op}(\lambda \, , \, \mathbf{a}) := (\tilde{\nu} \, , \, \lambda_{k+1} \, , \, \ldots \, , \, \lambda_d) \in \mathbb{R}_{\geq 0}^d \ .$$

Then, in any case, there exists $\mathcal{F}^{\rm op} = (\mathcal{F}_0, \, \mathcal{G}^{\rm op}) \in \mathcal{C}_{\mathbf{a}}\,(\mathcal{F}_0)$ with $\lambda(S_{\mathcal{F}^{\rm op}}) = \nu^{\rm op}(\lambda \, , \, \mathbf{a})^{\downarrow}$ and such that for every $\varphi \in {\rm Conv_s}(\mathbb{R}_{\geq 0})$

$$\mathrm{P}_\varphi(\mathcal{F}_0, \, \mathcal{G}) \geq \mathrm{P}_\varphi(\mathcal{F}^{\rm op}) \quad \text{for every} \quad (\mathcal{F}_0, \, \mathcal{G}) \in \mathcal{C}_{\mathbf{a}}\,(\mathcal{F}_0) \ . \tag{26}$$

Moreover, given $(\mathcal{F}_0, \, \mathcal{G}) \in \mathcal{C}_{\mathbf{a}}\,(\mathcal{F}_0)$, equality holds in Eq. (26) $\iff \lambda(S_{(\mathcal{F}_0, \mathcal{G})}) = \nu^{\rm op}(\lambda \, , \, \mathbf{a})^{\downarrow}$. □

Consider the notation and terminology from Theorem 4.8 above, and let $\mathcal{F}^{\rm op} = (\mathcal{F}_0, \, \mathcal{G}^{\rm op}) \in \mathcal{C}_{\mathbf{a}}\,(\mathcal{F}_0)$ be such that $\lambda(S_{\mathcal{F}^{\rm op}}) = \nu^{\rm op}(\lambda \, , \, \mathbf{a})^{\downarrow}$. If $\varphi \in {\rm Conv_s}(\mathbb{R}_{\geq 0})$ then it follows that $\mathcal{G}^{\rm op}$ is a global minimum of Ψ_φ so, in particular, $\mathcal{G}^{\rm op}$ is a local minimum. Hence, we can apply Proposition 4.7 to $\mathcal{G}^{\rm op}$ and deduce some of the information contained in the case C.1. of Theorem 4.8 with one notable exception, namely that $s_{q-1} = s^*$ is the minimal feasible index of the pair $(\lambda \, , \, \mathbf{a})$.

Remark 4.9. Let $\mathcal{F}_0 = \{f_i\}_{i\in\mathbb{I}_{n_0}} \in (\mathbb{C}^d)^{n_0}$, let $\lambda = \lambda(S_{\mathcal{F}_0})^{\uparrow} \in (\mathbb{R}_{\geq 0}^d)^{\uparrow}$ and let $\mathbf{a} = (a_i)_{i\in\mathbb{I}_k} \in (\mathbb{R}_{>0}^k)^{\downarrow}$. Let $\nu^{\rm op}(\lambda \, , \, \mathbf{a})$ be constructed according to Theorems 4.8. The fact that $\nu^{\rm op}(\lambda \, , \, \mathbf{a})$ is the optimal spectrum for every convex potential $\varphi \in {\rm Conv_s}(\mathbb{R}_{\geq 0})$ is equivalent to the assertion that

$$\nu^{\rm op}(\lambda \, , \, \mathbf{a}) \prec \lambda(S_{(\mathcal{F}_0, \mathcal{G})}) \quad \text{for every} \quad (\mathcal{F}_0, \, \mathcal{G}) \in \mathcal{C}_{\mathbf{a}}\,(\mathcal{F}_0) \ . \tag{27}$$

See [30] or [20] for an independent proof of this fact. △

The following is our main result:

Theorem 4.10. *Consider the Notation 3.3 and 3.5. Then the local minimizer $\mathcal{G}_0 \in \mathbb{T}_{\mathbb{C}^d}(\mathbf{a})$ is also a global minimizer of Ψ_φ in $\mathbb{T}_{\mathbb{C}^d}(\mathbf{a})$.*

Proof. We adopt the terminology of Notation 3.3 and 3.5. We first assume that $k \geq d$ and argue by induction on $p \geq 1$ i.e. the number of constants $c_1 > \ldots > c_p > 0$.

Indeed, if $p = 1$ then Corollary 3.9 shows that $\mathcal{G}_0$ is a global minimum of Ψ_φ in $\mathbb{T}_{\mathbb{C}^d}(\mathbf{a})$ and we are done. Hence, assume that $p > 1$ and that the inductive hypothesis holds for $p - 1$. By Proposition 4.3 the index s_{p-1} is feasible and then,

$$s_{p-1} \geq s^* = \min\{ \, 0 \leq s \leq d-1 : \ s \ \text{ is a feasible index of the pair } \ (\lambda \, , \, \mathbf{a}) \ \} \ .$$

Consider now the notation and terminology from the case C.1. of Theorem 4.8, describing the optimal spectra $\nu^{\rm op}(\lambda\,,\,\mathbf{a})$ (notice that $q\geq 1$).

Assume first that $q=1$. In this case $\nu=\nu^{\rm op}(\lambda\,,\,\mathbf{a})=(c_1^*\,\mathbb{1}_{s_1^*}\,,\,\lambda_{s_1^*+1}\,,\,\dots\,,\,\lambda_d)\in(\mathbb{R}^d_{\geq 0})^\uparrow$. In particular the majorization relation $\lambda(S_\mathcal{F})\prec\nu$ in Eq. (27) shows that

$$c_1^*=\min\{\nu_j:\ j\in\mathbb{I}_d\}\geq\min\{\lambda_j(S_\mathcal{F}):\ j\in\mathbb{I}_d\}=c_p\ .$$

Hence, by Remark 4.4 and the fact that $\lambda\leqslant\nu\in(\mathbb{R}^d_{\geq 0})^\uparrow$, we deduce that

$$s_1^*=\max\{j\in\mathbb{I}_d:\ \lambda_j<c_1^*\}\geq\max\{j\in\mathbb{I}_d:\ \lambda_j<c_p\}=s_p>s_1\ .$$

Hence, by Theorem 4.8, $c_1^*\geq P_{1\,,\,j}$ for $1\leq j\leq s_1^*\implies c_1^*\geq P_{1\,,\,s_1}\overset{4.7}{=}c_1$. Using these facts it is easy to check that $\operatorname{tr}(\nu)>\operatorname{tr}(\lambda(S_\mathcal{F}))$, which contradicts the majorization relation in Eq. (27).

We now assume that $q>1$. In this case, we have that

$$s_1=\max\{1\leq j\leq s_{p-1}:\ P_{1,j}=\max_{1\leq i\leq s_{p-1}}P_{1,i}\}\quad\text{ and }\quad c_1=P_{1,s_1}\tag{28}$$

and

$$s_1^*=\max\{1\leq j\leq s_{q-1}^*=s^*:\ P_{1,j}=\max_{1\leq i\leq s_{q-1}^*}P_{1,i}\}\quad\text{ and }\quad c_1^*=P_{1,s_1^*}\tag{29}$$

Assume that $s_1^*\neq s_1$. Using that $s^*\leq s_{p-1}$ we see that $s^*=s_{q-1}^*<s_1$. Since $\nu^{\rm op}(\lambda\,,\,\mathbf{a})$ corresponds to the spectra of a global minimum which, in particular is a local minimum, we can apply item 3 in Proposition 4.7 (see Eq. (25)) and get:

$$c_q^*\geq\frac{1}{\ell-s_{q-1}^*}\sum_{i=s_{q-1}^*+1}^{\ell}h_i\quad\text{ for }\quad s_{q-1}^*+1\leq\ell\leq s_q^*\,.\tag{30}$$

We consider the following two sub-cases:

Sub-case a: $s_q^*\geq s_1$. In this case, since $s^*=s_{q-1}^*<s_1$ we get that

$$c_1=\frac{1}{s_1}\sum_{i=1}^{s_1}h_i=\frac{s^*}{s_1}\left(\frac{1}{s^*}\sum_{i=1}^{s^*}h_i\right)+\frac{(s_1-s^*)}{s_1}\left(\frac{1}{(s_1-s^*)}\sum_{i=s^*+1}^{s_1}h_i\right)\tag{31}$$

that represents c_1 as a convex combination of averages. The first average satisfies (by construction of c_1 and $s^*\leq s_{p-1}$)

$$\frac{1}{s^*}\sum_{i=1}^{s^*}h_i\leq c_1\quad\implies\quad\frac{1}{(s_1-s^*)}\sum_{i=s^*+1}^{s_1}h_i\geq c_1$$

since otherwise, Eq. (31) can not hold. Using the hypothesis $s_q^*\geq s_1>s^*=s_{q-1}^*$, Eq. (30) and the previous inequality

$$c_q^*\geq\frac{1}{s_1-s_{q-1}^*}\sum_{i=s_{q-1}^*+1}^{s_1}h_i\geq c_1\geq c_1^*\,,$$

where we have used Eqs. (28) and (29) and the fact that $s_{q-1}^*\leq s_{p-1}$. Hence $q=1$ contradicting our assumption $q>1$. Therefore, Sub-case a is not possible.

Sub-case b: $s_q^*<s_1$. Recall that $s_{p-1}\geq s^*=s_{q-1}^*$ which, by Eqs. (28) and (29), implies that $c_1\geq c_1^*$. Thus, $c_1^*\,s_q^*\leq c_1\,s_q^*<c_1\,s_1$ and hence,

$$\begin{aligned}\operatorname{tr}(\nu^{\rm op}(\lambda\,,\,\mathbf{a}))&=\sum_{i\in\mathbb{I}_q}c_i^*\,(s_i^*-s_{i-1}^*)+\sum_{i=s_q^*+1}^{d}\lambda_i\leq c_1\,s_q^*+\sum_{i=s_q^*+1}^{d}\lambda_i\\&<\sum_{i\in\mathbb{I}_p}c_i\,(s_i-s_{i-1})+\sum_{i=s_p+1}^{d}\lambda_i=\operatorname{tr}(S_\mathcal{F})\,.\end{aligned}$$

This last fact contradicts the majorization relation in Eq. (27). We conclude that Sub-case b is not possible.

Therefore, we should have that $s_1^* = s_1$ and hence $c_1 = c_1^*$. We prove that $\mathcal{F} = (\mathcal{F}_0, \mathcal{G}_0)$ is a global minimum by showing that $\lambda(S_{\mathcal{F}}) = \nu^{\rm op}(\lambda\,,\,\mathbf{a})$. Indeed, by applying the reduction argument described in Remark 3.6 we deduce, setting $k' = k - s_1$, that $\mathcal{G}_0^{(1)} = \{g_{i+s_1}\}_{i\in I_{k'}}$ is a local minimum of the map

$$\{\mathcal{K} = (k_i)_{i\in\mathbb{I}_{k'}} \in (\mathcal{H}_1)^{k'}\ ,\ \|g_i\|^2 = a_{s_1+i}\,, i\in\mathbb{I}_{k'}\} \ni \mathcal{K} \mapsto \mathrm{P}_\varphi(\mathcal{F}_0^{(1)}, \mathcal{K}) \tag{32}$$

where $\mathcal{H}_1 = W_1^\perp$ for $W_1 = \operatorname{span}\{g_i\}_{i\in\mathbb{I}_{s_1}}$ and $\mathcal{F}_0^{(1)}$ is a sequence in $\mathcal{H}_1$ such that $S_{\mathcal{F}_0^{(1)}} = S_{\mathcal{F}_0}|_{\mathcal{H}_1}$. In this case, by Corollary 3.8 and the fact that $p > 1$, $d' = \dim \mathcal{H}_1 = d - s_1 \le k - s_1 = k'$. Moreover, by construction of $\mathcal{H}_1$, if we let $\tilde{\mathcal{F}} = (\mathcal{F}_0^{(1)}, \mathcal{G}_0^{(1)})$ then

$$\lambda(S_{\mathcal{F}_0^{(1)}}) = (\lambda_{s_1+i})^\downarrow_{i\in\mathbb{I}_{d'}} \quad \text{ and } \quad \lambda(S_{\tilde{\mathcal{F}}}) = (c_2\,\mathbb{1}_{s_2-s_1}, \ldots, c_p\,\mathbb{1}_{s_p-s_{p-1}}, \lambda_{s_p+1}, \ldots, \lambda_d)^\downarrow\,.$$

Hence, the induction hypothesis applies to $\mathcal{G}_0^{(1)}$ and we conclude that $\mathcal{G}_0^{(1)}$ is a global minimizer of the map in Eq. (32). Therefore, with the notation of Definition 4.2 and case C.1. of Theorem 4.8,

$$\lambda(S_{\tilde{\mathcal{F}}}) = \nu^{\rm op}(\lambda^{s_1}\,,\,\mathbf{a}^{s_1})\,.$$

Now, an inspection of the construction in case C.1. of Theorem 4.8 reveals that

$$\nu^{\rm op}(\lambda\,,\,\mathbf{a}) = (c_1^*\,\mathbb{1}_{s_1^*}\,,\,\nu^{\rm op}(\lambda^{s_1}\,,\,\mathbf{a}^{s_1})) \in \mathbb{R}^d_{\ge 0}\,. \tag{33}$$

Indeed, since the notion of feasible index depends on the tail of the sequences of eigenvalues and norms we see that

$$s^* = s_1^* + \min\{0 \le s \le d'-1 :\ s \ \text{ is a feasible index for the pair } \ (\lambda^{s_1}\,,\,\mathbf{a}^{s_1})\}\,.$$

Eq. (33) now follows using that $s_1 = s_1^*$, the identity above and the formulas for the indexes s_i^* both for $\nu^{\rm op}(\lambda\,,\,\mathbf{a})$ and $\nu^{\rm op}(\lambda^{s_1}\,,\,\mathbf{a}^{s_1})$ from case C.1. of Theorem 4.8. Now, we see that

$$\lambda(S_{\mathcal{F}}) = (c_1\,\mathbb{1}_{s_1}\,,\,\lambda(S_{\tilde{\mathcal{F}}})) = (c_1^*\,\mathbb{1}_{s_1^*}\,,\,\nu^{\rm op}(\lambda^{s_1}\,,\,\mathbf{a}^{s_1})) = \nu^{\rm op}(\lambda\,,\,\mathbf{a})\,. \tag{34}$$

Eq. (34) together with Theorem 4.8 show that $\mathcal{F} = (\mathcal{F}_0\,,\,\mathcal{G}_0)$ is a global minimizer in this case.

Finally, in case $k < d$ we argue as in the second part of Remark 3.6. Using Notation 3.5 and the fact that $\operatorname{rk}(S_{\mathcal{G}_0}) \le k$, we see that $\mu_i = 0$ for $k+1 \le i \le d$ and therefore

$$S_{\mathcal{G}_0} = \sum_{i\in\mathbb{I}_k} \mu_i\ v_i \otimes v_i \implies \lambda(S_{\mathcal{F}}) = (\lambda_1+\mu_1, \ldots, \lambda_k+\mu_k, \lambda_{k+1}, \ldots, \lambda_d)^\downarrow \tag{35}$$

and $W := R(S_{\mathcal{G}_0}) \subset \mathcal{H} = \operatorname{span}\{v_i :\ i\in\mathbb{I}_k\}$. Notice that $\mathcal{H}$ reduces $S_{\mathcal{F}_0}$; then, we can consider a sequence $\tilde{\mathcal{F}}_0$ in $\mathcal{H}$ such that $S_{\tilde{\mathcal{F}}_0} = S_{\mathcal{F}_0}|_{\mathcal{H}}$. In this case $\mathcal{G}_0$ is a local minimizer of the map

$$\mathbb{T}_{\mathcal{H}}(\mathbf{a}) = \{\mathcal{G} = \{g_i\}_{i\in\mathbb{I}_k} \in \mathcal{H}^k :\ \|g_i\|^2 = a_i\ ,\ i\in\mathbb{I}_k\ \} \ni \mathcal{G} \mapsto \mathrm{P}_\varphi(\tilde{\mathcal{F}}_0, \mathcal{G})\,. \tag{36}$$

Since $\dim\mathcal{H} = k$ then, by the first part of this proof, we conclude that $\mathcal{G}_0$ is a global minimizer of the map in Eq. (36) and that, by Theorem 4.8,

$$(\lambda_1+\mu_1, \ldots, \lambda_k+\mu_k)^\downarrow = \lambda(S_{(\tilde{\mathcal{F}}_0\,,\,\mathcal{G}_0)}) = \nu^{\rm op}(\tilde\lambda\,,\,\mathbf{a})\,, \tag{37}$$

where $\tilde\lambda = \lambda(S_{\tilde{\mathcal{F}}_0})^\uparrow = (\lambda_i)_{i\in\mathbb{I}_k}$. Finally, notice that by the case C.2. of Theorem 4.8 and Eqs. (35), (37) we now conclude that

$$\nu^{\rm op}(\lambda\,,\,\mathbf{a}) = (\nu^{\rm op}(\tilde\lambda\,,\,\mathbf{a}), \lambda_{k+1}, \ldots, \lambda_d)^\downarrow = \lambda(S_{\mathcal{F}})\,. \tag{38}$$

Eq. (38) together with Theorem 4.8 show that $\mathcal{F} = (\mathcal{F}_0\,,\,\mathcal{G}_0)$ is a global minimizer in this case. □

5 An application: generalized frame operator distances in $\mathbb{T}_{\mathbb{C}^d}(\mathbf{a})$

We begin with a brief description of Strawn's work [31] on the frame operator distance. Thus, we consider a positive semidefinite $d\times d$ complex matrix $S_0 \in \mathcal{M}_d(\mathbb{C})^+$ and $\mathbf{a} = (a_i)_{i\in\mathbb{I}_k} \in (\mathbb{R}^k_{>0})^\downarrow$ such that $\mathbf{a} \prec \lambda(S_0)$. The frame operator distance (FOD) is defined as the function $\Theta_2 = \Theta_{2\,,\,S_0\,,\,\mathbf{a}} : \mathbb{T}_{\mathbb{C}^d}(\mathbf{a}) \to \mathbb{R}_{\geq 0}$ given by

$$\Theta_2(\mathcal{G}) = \|S_0 - S_{\mathcal{G}}\|_2 \quad \text{for} \quad \mathcal{G} \in \mathbb{T}_{\mathbb{C}^d}(\mathbf{a})\,,$$

where $\|A\|_2^2 = \operatorname{tr}(A^*A)$ for $A \in \mathcal{M}_d(\mathbb{C})$ denotes the Frobenius norm in $\mathcal{M}_d(\mathbb{C})$. By Proposition 2.4, the relation $\mathbf{a} \prec \lambda(S_0)$ implies that there exists $\mathcal{G}^{\rm op} \in \mathbb{T}_{\mathbb{C}^d}(\mathbf{a})$ such that $S_{\mathcal{G}^{\rm op}} = S_0$. In this case,

$$\min\{\,\Theta_2(\mathcal{G}) :\ \mathcal{G} \in \mathbb{T}_{\mathbb{C}^d}(\mathbf{a})\,\} = 0\,.$$

In [31], after noticing the minimum value of Θ_2 above, an algorithm based on approximate gradient descent is presented. This algorithm exploits some geometrical aspects of the differential geometry of the manifold $\mathbb{T}_{\mathbb{C}^d}(\mathbf{a})$ obtained by Strawn in [32] (some of which are of a similar nature to those considered in [27]). Based on numerical evidence, the author then poses the following:

Conjecture (Strawn [31]): Let $S_0 \in \mathcal{M}_d(\mathbb{C})^+$, $\mathbf{a} = (a_i)_{i\in\mathbb{I}_k} \in (\mathbb{R}^k_{>0})^\downarrow$ with $k \geq d$ and assume that $\mathbf{a} \prec \lambda(S_0)$. Then local minimizers of Θ_2 in $\mathbb{T}_{\mathbb{C}^d}(\mathbf{a})$ are also global minimizers. △

As remarked in [31], proving this conjecture would provide a beneficial theoretical guarantee for the performance of the frame operator distance algorithm based on approximate gradient descent presented in that paper, from a numerical perspective.

In what follows we consider a generalized version of the frame operator distance, in terms of unitarily invariant norms (u.i.n) in $\mathcal{M}_d(\mathbb{C})$ (see Section 2.1). Moreover, we consider the general case of $S_0 \in \mathcal{M}_d(\mathbb{C})^+$ and $\mathbf{a} \in (\mathbb{R}^k_{\geq 0})^\downarrow$, without assuming that $k \geq d$ nor $\mathbf{a} \prec \lambda(S_0)$.

Definition 5.1. Let $S_0 \in \mathcal{M}_d(\mathbb{C})^+$ and let $\mathbf{a} = (a_i)_{i\in\mathbb{I}_k} \in (\mathbb{R}^k_{>0})^\downarrow$. Given a u.i.n. $|||\cdot|||$ in $\mathcal{M}_d(\mathbb{C})$ we consider the generalized frame operator distance (G-FOD) function

$$\Theta_{(\,S_0\,,\,\mathbf{a}\,,\,|||\cdot|||\,)} = \Theta : \mathbb{T}_{\mathbb{C}^d}(\mathbf{a}) \to \mathbb{R}_{\geq 0} \quad \text{given by} \quad \Theta(\mathcal{G}) = |||S_0 - S_{\mathcal{G}}|||\,,$$

where $S_{\mathcal{G}} \in \mathcal{M}_d(\mathbb{C})^+$ denotes the frame operator of $\mathcal{G} \in \mathbb{T}_{\mathbb{C}^d}(\mathbf{a})$. △

In case $S_0 \in \mathcal{M}_d(\mathbb{C})^+$ and $\mathbf{a} = (a_i)_{i\in\mathbb{I}_k} \in (\mathbb{R}^k_{>0})^\downarrow$ are arbitrary then it seems that the minimum value of Θ in $\mathbb{T}_{\mathbb{C}^d}(\mathbf{a})$ has not been computed in the literature, not even for $\|\cdot\|_2$. The following result states that there are structural solution to the G-FOD optimization problem in the sense that there are families $\mathcal{G}^{\rm op} \in \mathbb{T}_{\mathbb{C}^d}(\mathbf{a})$ such that for every u.i.n. $|||\cdot|||$ the minimum value of Θ in $\mathbb{T}_{\mathbb{C}^d}(\mathbf{a})$ is $\Theta(\mathcal{G}^{\rm op})$. In particular, this allows us to compute the minimum value of Θ in $\mathbb{T}_{\mathbb{C}^d}(\mathbf{a})$ for an arbitrary u.i.n. $|||\cdot|||$ in the general case.

In what follows, given $\tilde\lambda \in (\mathbb{R}^d_{\geq 0})^\uparrow$ and $\mathbf{a} = (a_i)_{i\in\mathbb{I}_k} \in (\mathbb{R}^k_{>0})^\downarrow$ we consider $\nu^{\rm op}(\tilde\lambda\,,\,\mathbf{a})$ constructed as in Theorem 4.8, according to the case $k \geq d$ or $k < d$.

Theorem 5.2. *Let $S_0 \in \mathcal{M}_d(\mathbb{C})^+$ and let $\mathbf{a} = (a_i)_{i\in\mathbb{I}_k} \in (\mathbb{R}^k_{>0})^\downarrow$. Let $\tilde\lambda = (\tilde\lambda_i)_{i\in\mathbb{I}_d} \in (\mathbb{R}^d_{\geq 0})^\uparrow$ be given by $\tilde\lambda = \|S_0\|\,\mathbb{1}_d - \lambda(S_0)$ and let $\delta = \delta(\lambda(S_0)\,,\,\mathbf{a}) \in \mathbb{R}^d$ be given by $\delta = (\delta_i)_{i\in\mathbb{I}_d} = \|S_0\|\,\mathbb{1}_d - \nu^{\rm op}(\tilde\lambda\,,\,\mathbf{a})$.*

1. *For every u.i.n. $|||\cdot|||$ in $\mathcal{M}_d(\mathbb{C})$ we have that*

$$\min\{\ \Theta(\mathcal{G}) = |||S_0 - S_{\mathcal{G}}||| :\ \mathcal{G} \in \mathbb{T}_{\mathbb{C}^d}(\mathbf{a})\ \} = |||D_\delta|||\,.$$

2. *If $|||\cdot|||$ is strictly convex and $\mathcal{G} \in \mathbb{T}_{\mathbb{C}^d}(\mathbf{a})$ then*

$$|||S_0 - S_{\mathcal{G}}||| = |||D_\delta||| \quad \text{if and only if} \quad \lambda(S_0 - S_{\mathcal{G}}) = \delta^\downarrow\,.$$

In this case, there exists $\{v_i\}_{i\in\mathbb{I}_d}$ an ONB for $\mathbb{C}^d$ such that

$$S_0=\sum_{i\in\mathbb{I}_d}\lambda_i(S_0)\ v_i\otimes v_i \quad \text{and} \quad S_{\mathcal{G}}=\sum_{i\in\mathbb{I}_d}(\lambda_i(S_0)-\delta_i)\ v_i\otimes v_i\,. \tag{39}$$

We prove Theorem 5.2 below, by means of a translation between the optimization problem for the frame operator distance and the optimization problem for convex potentials of frame completions with prescribed norms. It is worth pointing out that a relation between frame operator distances (for the Frobenius norm) and minimization of the frame potential of sequences with prescribed norms was already noticed in [31].

We will use the following result for the uniqueness of item 2 in Theorem 5.2 above.

Lemma 5.3. *Let $a,\,b\in\mathbb{R}^d$ be such that $a\succ b$ and $|a|^\downarrow=|b|^\downarrow$. Then $a^\downarrow=b^\downarrow$.*

Proof. We can assume that $a=a^\downarrow$ and $b=b^\downarrow$. Also that $a\neq\lambda\,\mathbb{1}$ (the case $a=\lambda\mathbb{1}$ is trivial). We argue by induction on the dimension d. If $d=1$ the result is clear.

Assume that the result holds for $d-1\geq 1$ and let $a,\,b\in\mathbb{R}^d$ be such that $a\succ b$ and $|a|^\downarrow=|b|^\downarrow$. By replacing a by $-a^\downarrow$ and b by $-b^\downarrow$ if necessary, we can assume that

$$|a_1|\geq|a_d| \implies \max_{i\in\mathbb{I}_d}|a_i|=|a_1|=a_1>0\ ,$$

where the fact that $a_1>0$ (in this case) follows easily using that $a^\downarrow=a\neq\lambda\,\mathbb{1}$.

Note that $b\prec a \implies a_1\geq b_1$. Assume that $a_1>b_1$. Then $a_d=b_d=-a_1$. Indeed, b_d must achieve the maximal modulus (since b_1 doesn't), and $a_d\leq b_d$ by majorization. Let $\tilde{a}=(a_i)_{i\in\mathbb{I}_{d-1}}$ and $\tilde{b}=(b_i)_{i\in\mathbb{I}_{d-1}}$. It is easy to see that $\tilde{a}^\downarrow=\tilde{a}\prec\tilde{b}=\tilde{b}^\downarrow$ and $|\tilde{a}|^\downarrow=|\tilde{b}|^\downarrow$. Hence, by inductive hypothesis $\tilde{a}=\tilde{b}\implies a_1=b_1$, a contradiction.

So we can assume that $a_1=b_1$. As before, we can apply the inductive hypothesis and conclude that $(a_{i+1})_{i\in\mathbb{I}_{d-1}}=(b_{i+1})_{i\in\mathbb{I}_{d-1}}$ and hence $a=b$. □

Proof of Theorem 5.2. Consider $\tilde{S}_0=\|S_0\|\,I-S_0\in\mathcal{M}_d(\mathbb{C})^+$ and let $\mathcal{F}_0=\{f_i\}_{i\in\mathbb{I}_d}$ be a sequence in $\mathbb{C}^d$ such that $S_{\mathcal{F}_0}=\tilde{S}_0$. Notice that if we let $\lambda(S_0)=(\lambda_i(S_0))_{i\in\mathbb{I}_d}\in(\mathbb{R}^d_{\geq 0})^\downarrow$ then $\lambda(S_{\mathcal{F}_0})^\uparrow=(\|S_0\|-\lambda_i(S_0))_{i\in\mathbb{I}_d}=\tilde{\lambda}$. If $\mathcal{G}\in\mathbb{T}_{\mathbb{C}^d}(\mathbf{a})$ then

$$S_0-S_{\mathcal{G}}=\|S_0\|\,I+(S_0-\|S_0\|\,I)-S_{\mathcal{G}}=\|S_0\|\,I-(\tilde{S}_0+S_{\mathcal{G}})\,. \tag{40}$$

In particular, we get that

$$\lambda(S_0-S_{\mathcal{G}})^\uparrow=\|S_0\|\,\mathbb{1}_d-\lambda(\tilde{S}_0+S_{\mathcal{G}})\in(\mathbb{R}^d)^\uparrow\,. \tag{41}$$

But notice that, since $\mathcal{G}\in\mathbb{T}_{\mathbb{C}^d}(\mathbf{a})$ then

$$\tilde{S}_0+S_{\mathcal{G}}=S_{\mathcal{F}_0}+S_{\mathcal{G}}=S_{\mathcal{F}} \quad \text{for} \quad \mathcal{F}=(\mathcal{F}_0,\,\mathcal{G})\in\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)\ .$$

Then, by Theorem 4.8 (according to the case $k\geq d$ or $k<d$)

$$\nu^{\rm op}(\tilde{\lambda}\,,\,\mathbf{a})\prec\lambda(\tilde{S}_0+S_{\mathcal{G}}) \implies \delta\overset{\rm def}{=}\|S_0\|\,\mathbb{1}_d-\nu^{\rm op}(\tilde{\lambda}\,,\,\mathbf{a})\prec\|S_0\|\,\mathbb{1}_d-\lambda(\tilde{S}_0+S_{\mathcal{G}})\,. \tag{42}$$

Using Eq. (41) and that the function $\mathbb{R}\ni x\mapsto|x|\in\mathbb{R}_{\geq 0}$ is convex we conclude that

$$|\delta|=(|\delta_i|)_{i\in\mathbb{I}_d}\prec_w|\,\|S_0\|\,\mathbb{1}_d-\lambda(\tilde{S}_0+S_{\mathcal{G}})\,|=s(S_0-S_{\mathcal{G}})^\uparrow \tag{43}$$

where $s(S_0 - S_{\mathcal{G}}) = |\lambda(S_0 - S_{\mathcal{G}})|^\downarrow \in (\mathbb{R}^d_{\geq 0})^\downarrow$ denotes the vector of singular values of $S_0 - S_{\mathcal{G}}$. By Theorem 2.3, the previous sub-majorization relation implies that for every uin $|||\cdot|||$

$$|||S_0 - S_{\mathcal{G}}||| \geq |||D_\delta||| \quad \text{for every} \quad \mathcal{G} \in \mathbb{T}_{\mathbb{C}^d}(\mathbf{a})\,.$$

In order to show that this lower bound is attained, consider $\mathcal{G}^{\rm op} \in \mathbb{T}_{\mathbb{C}^d}(\mathbf{a})$ an optimal frame completion in $\mathbb{T}_{\mathbb{C}^d}(\mathbf{a})$ of $\mathcal{F}_0$ i.e. such that $\lambda(\tilde S_0 + S_{\mathcal{G}^{\rm op}}) = \lambda(S_{\mathcal{F}_0} + S_{\mathcal{G}^{\rm op}}) = \nu^{\rm op}(\tilde\lambda\,,\,\mathbf{a})^\downarrow$. Hence, by Eq. (43) we see that

$$s(S_0 - S_{\mathcal{G}^{\rm op}})^\uparrow = |\|S_0\|\,\mathbb{1}_d - \nu^{\rm op}(\tilde\lambda\,,\,\mathbf{a})^\downarrow| \implies |\delta|^\uparrow = s(S_0 - S_{\mathcal{G}^{\rm op}})^\uparrow \quad \text{and} \quad |||S_0 - S_{\mathcal{G}^{\rm op}}||| = |||D_\delta|||\,.$$

This last fact shows that the lower bound is attained at $\mathcal{G}^{\rm op}$ and proves item 1.

Assume further that $|||\cdot|||$ is strictly convex and let $\mathcal{G} \in \mathbb{T}_{\mathbb{C}^d}(\mathbf{a})$ be such that $|||S_0 - S_{\mathcal{G}}||| = |||D_\delta|||$. The sub-majorization relation in Eq. (43) together with the previous hypothesis imply that

$$|\delta|^\uparrow = |\,\|S_0\|\,\mathbb{1}_d - \lambda(\tilde S_0 + S_{\mathcal{G}})\,| = s(S_0 - S_{\mathcal{G}})^\uparrow\,.$$

The identity above together with the majorization relation in Eq. (42) and Lemma 5.3 imply that

$$\delta^\downarrow = (\|S_0\|\,\mathbb{1}_d - \nu^{\rm op}(\tilde\lambda\,,\,\mathbf{a}))^\downarrow = (\|S_0\|\,\mathbb{1}_d - \lambda(\tilde S_0 + S_{\mathcal{G}}))^\downarrow$$

from which it follows that $\lambda(S_{\mathcal{F}_0} + S_{\mathcal{G}}) = \lambda(\tilde S_0 + S_{\mathcal{G}}) = \nu^{\rm op}(\tilde\lambda\,,\,\mathbf{a})^\downarrow$. Therefore, $\mathcal{G} \in \mathbb{T}_{\mathbb{C}^d}(\mathbf{a})$ is a global minimizer of $\Psi_\varphi(\mathcal{G}) = \operatorname{tr}(\varphi(S_{\mathcal{F}_0} + S_{\mathcal{G}}))$ for every $\varphi \in \operatorname{Conv_s}(\mathbb{R}_{\geq 0})$. By Theorem 3.4 there exists $\{v_i\}_{i\in\mathbb{I}_d}$ an ONB of $\mathbb{C}^d$ such that

$$\tilde S_0 = \sum_{i\in\mathbb{I}_d} \tilde\lambda_i\ v_i \otimes v_i \quad \text{and} \quad S_{\mathcal{G}} = \sum_{i\in\mathbb{I}_d} (\nu^{\rm op}(\tilde\lambda\,,\,\mathbf{a}) - \tilde\lambda)_i\ v_i \otimes v_i\,. \tag{44}$$

Using that $S_0 = \tilde S_0 + \|S_0\|\,I$ so that $\tilde\lambda = \|S_0\|\,\mathbb{1}_d - \lambda(S_0)$ we see that Eq. (39) holds for $\{v_i\}_{i\in\mathbb{I}_d}$. □

The following result settles in the affirmative a generalized version of Strawn's conjecture on local minimizers of the FOD for the Frobenius norm in $\mathcal{M}_d(\mathbb{C})$, since we do not assume that $k \geq d$ nor the majorization relation $\mathbf{a} \prec \lambda(S_0)$ (see the comments at the beginning of this section).

Theorem 5.4. *Let $S_0 \in \mathcal{M}_d(\mathbb{C})^+$ and let $\mathbf{a} = (a_i)_{i\in\mathbb{I}_k} \in (\mathbb{R}^k_{>0})^\downarrow$ and consider the FOD given by*

$$\Theta_2 : \mathbb{T}_{\mathbb{C}^d}(\mathbf{a}) \to \mathbb{R}_{\geq 0} \quad \textit{given by} \quad \Theta_2(\mathcal{G}) = \|S_0 - S_{\mathcal{G}}\|_2\,.$$

Then, the local minimizers of Θ_2 in $\mathbb{T}_{\mathbb{C}^d}(\mathbf{a})$ are also global minimizers.

Proof. Consider $\tilde S_0 = \|S_0\|\,I - S_0 \in \mathcal{M}_d(\mathbb{C})^+$ and let $\mathcal{F}_0 = \{f_i\}_{i\in\mathbb{I}_d}$ be a sequence in $\mathbb{C}^d$ such that $S_{\mathcal{F}_0} = \tilde S_0$. Hence $\lambda(S_{\mathcal{F}_0})^\uparrow = (\|S_0\| - \lambda_i(S_0))_{i\in\mathbb{I}_d} = \tilde\lambda$. Let $\varphi \in \operatorname{Conv_s}(\mathbb{R}_{\geq 0})$ be given by $\varphi(x) = x^2$ for $x \in \mathbb{R}_{\geq 0}$ and consider $\Psi_\varphi : \mathbb{T}_{\mathbb{C}^d}(\mathbf{a}) \to \mathbb{R}_{\geq 0}$ be given by $\Psi_\varphi(\mathcal{G}) = \operatorname{tr}(\varphi(S_{(\mathcal{F}_0\,,\,\mathcal{G})})) = \operatorname{tr}((S_{\mathcal{F}_0} + S_{\mathcal{G}})^2)$. If $\mathcal{G} \in \mathbb{T}_{\mathbb{C}^d}(\mathbf{a})$ then, using Eq. (40)

$$\begin{aligned}\Theta_2(\mathcal{G})^2 &= \|S_0 - S_{\mathcal{G}}\|_2^2 = \operatorname{tr}((\|S_0\|\,I - [S_{\mathcal{F}_0} + S_{\mathcal{G}}])^2)\\ &= \|S_0\|^2\,d - 2\,\|S_0\|\ \operatorname{tr}(S_{\mathcal{F}_0} + S_{\mathcal{G}}) + \operatorname{tr}((S_{\mathcal{F}_0} + S_{\mathcal{G}})^2) = c + \Psi_\varphi(\mathcal{G})\end{aligned}$$

where

$$c = \|S_0\|^2\,d - 2\,\|S_0\|\ \operatorname{tr}(S_{\mathcal{F}_0} + S_{\mathcal{G}}) = \|S_0\|^2\,d - 2\,\|S_0\|\,(\operatorname{tr}\tilde S_0 + \operatorname{tr}\mathbf{a})$$

is a constant (since $\mathcal{G} \in \mathbb{T}_{\mathbb{C}^d}(\mathbf{a})$). Hence $\Theta_2(\mathcal{G})^2 = \Psi_\varphi(\mathcal{G}) + c$ for every $\mathcal{G} \in \mathbb{T}_{\mathbb{C}^d}(\mathbf{a})$. In particular, local minimizers of Θ_2 and Ψ_φ coincide. The result now follows from these remarks and Theorem 4.10. □

6 Appendix: on a local Lidskii's theorem

Let $S \in \mathcal{M}_d(\mathbb{C})^+$, let $\mu \in (\mathbb{R}^d_{\geq 0})^\downarrow$ and consider $\mathcal{O}_\mu$ given by

$$\mathcal{O}_\mu = \{G \in \mathcal{M}_d(\mathbb{C})^+ : \lambda(G) = \mu\} = \{U^* D_\mu U : \ U \in \mathcal{U}(d)\} \tag{45}$$

We consider the usual metric in $\mathcal{O}_\mu$ induced by the operator norm; hence $\mathcal{O}_\mu$ is a metric space. For $\varphi \in \mathrm{Conv_s}(\mathbb{R}_{\geq 0})$, let $\Phi = \Phi_{S,\varphi} : \mathcal{O}_\mu \to \mathbb{R}_{\geq 0}$ be given by

$$\Phi(G) = \mathrm{tr}(\varphi(S+G)) = \sum_{j \in \mathbb{I}_d} \varphi(\lambda_j(S+G)) \quad \text{for} \quad G \in \mathcal{O}_\mu \,. \tag{46}$$

In what follows, we prove what we call a local Lidskii's theorem (Theorem 3.2) namely that local minimizers of Φ in $\mathcal{O}_\mu$ are also global minimizers.

Definition 6.1. *Let $A, B \in \mathcal{M}_d(\mathbb{C})^+$. We consider*

1. *The product manifold $\mathcal{U}(d) \times \mathcal{U}(d)$ endowed with the metric*

$$d((U_1, V_1), (U_2, V_2)) = \max\{\|I - U_1^* U_2\|, \|I - V_1^* V_2\|\} \,.$$

2. *$\Gamma = \Gamma_{(A,B)} : \mathcal{U}(d) \times \mathcal{U}(d) \to \mathcal{M}_d(\mathbb{C})^+_\tau \stackrel{\text{def}}{=} \{M \in \mathcal{M}_d(\mathbb{C})^+ : \ \mathrm{tr}(M) = \tau\}$ for $\tau = \mathrm{tr}(A) + \mathrm{tr}(B)$, given by*

$$\Gamma(U, V) = U^* A U + V^* B V \quad \textit{for} \quad U, V \in \mathcal{U}(d) \,.$$

3. *For a given $\varphi \in \mathrm{Conv}(\mathbb{R}_{\geq 0})$ we consider $\Delta^\varphi_{(A,B)} = \Delta : \mathcal{U}(d) \times \mathcal{U}(d) \to \mathbb{R}_{\geq 0}$ given by*

$$\Delta(U, V) = \mathrm{tr}(\varphi(\Gamma(U, V))) \quad \textit{for} \quad U, V \in \mathcal{U}(d) \,.$$

△

Our motivation for considering the previous notions comes from the following:

Lemma 6.2. *Let $A = S \in \mathcal{M}_d(\mathbb{C})^+$ let $\mu \in (\mathbb{R}^d_{\geq 0})^\downarrow$, $B = G_0 \in \mathcal{O}_\mu$ and consider the notation from Definition 6.1. Given $\varphi \in \mathrm{Conv}(\mathbb{R}_{\geq 0})$ then the following conditions are equivalent:*

1. *G_0 is a local minimizer of Φ_φ in $\mathcal{O}_\mu$;*
2. *(I, I) is a local minimizer of $\Delta^\varphi_{(S,G_0)} = \Delta$ on $\mathcal{U}(d) \times \mathcal{U}(d)$.*

Proof. 1. $\implies$ 2. Consider $(U, W) \in \mathcal{U}(d) \times \mathcal{U}(d)$ such that

$$d((U, W), (I, I)) = \max\{\, \|I - U^*\| \,, \, \|I - W^*\| \,\} := \varepsilon \,.$$

Hence,

$$U^* S U + W^* G_0 W = U^*(S + Z^* G_0 Z) U \quad \text{with} \quad Z = W U^* \in \mathcal{U}(d) \,.$$

Notice that

$$\|Z - I\| = \|W (U^* - W^*)\| \leq \|U^* - I\| + \|I - W^*\| \leq 2\,\varepsilon \,.$$

Hence,

$$\Delta(U, W) = \mathrm{tr}(\varphi(U^*(S + Z^* G_0 Z) U)) = \Phi(Z^* G_0 Z) \quad \text{with} \quad \|Z^* G_0 Z - G_0\| \leq 4\,\varepsilon \|G_0\| \,.$$

2. $\implies$ 1. This is a consequence of the fact that the map $\mathcal{U}(d) \ni Z \mapsto Z^* G\, Z \in \mathcal{O}_\mu$ is open (see, for example, [1, Thm. 4.1] or [14]). □

In what follows, given $\mathcal{S} \subset \mathcal{M}_d(\mathbb{C})^+$ we consider the commutant of $\mathcal{S}$, denoted S', that is the unital *-subalgebra of $\mathcal{M}_d(\mathbb{C})$ given by

$$\mathcal{S}' = \{ \ C \in \mathcal{M}_d(\mathbb{C}) : \ [C, D] = 0 \ \text{ for every } \ D \in \mathcal{S} \ \} \subset \mathcal{M}_d(\mathbb{C}) \, ,$$

where $[C, D] = CD - DC$ denotes the commutator of C and D.

The following result is standard.

Lemma 6.3. *Consider the notation from Definition 6.1. Then*

$$\Gamma \quad \textit{is a submersion at} \quad (I, I) \quad \iff \quad \{A, \, B\}' = \mathbb{C} \cdot I \ .$$

Proof. The (exponential) map $\mathcal{H}(d) \ni X \mapsto \exp(X)$ allows us to identify the tangent space $\mathcal{T}_I\mathcal{U}(d)$ with $i \cdot \mathcal{H}(d)$. Since we consider the product structure on $\mathcal{U}(d) \times \mathcal{U}(d)$ we conclude that the differential of Γ satisfies

$$D_{(I,I)}\Gamma(X, 0) = [A, X] \quad \text{ and } \quad D_{(I,I)}\Gamma(0, X) = [B, X] \quad \text{ for } \quad X \in i \cdot \mathcal{H}(d) \, .$$

Therefore Γ is not a submersion at (I, I) if and only if there exists $0 \neq Y \in \mathcal{T}\mathcal{M}_d(\mathbb{C})^+_\tau$ (i.e. $Y \in \mathcal{H}(d)$ such that $\text{tr } Y = 0$) such that

$$\text{tr}(Y \, [A, Z]) = \text{tr}(Y \, [B, Z]) = 0 \quad \text{ for every } \quad Z \in i \cdot \mathcal{H}(d) \ . \tag{47}$$

Since $\text{tr}(Y \, [A, Z]) = \text{tr}([Y, A] \, Z)$ and similarly $\text{tr}(Y \, [B, Z]) = \text{tr}([Y, B] \, Z)$, we see that in this case

$$[Y, A] = 0 = [Y, B] \in i \cdot \mathcal{H}(d) \ .$$

Moreover, since $Y \neq 0$ and $\text{tr } Y = 0$, then Y has some non-trivial spectral projection P which also satisfies that $[P, A] = [P, B] = 0$. Conversely, in case there exists a non-trivial projection P such that $[P, A] = [P, B] = 0$, we can construct $Y = \frac{P}{\text{tr } P} - \frac{I-P}{\text{tr}\,(I-P)}$ so that $\text{tr } Y = 0$. Then $0 \neq Y \in \mathcal{T}\mathcal{M}_d(\mathbb{C})^+_\tau$ and it satisfies Eq. (47), so that this matrix Y is orthogonal to the range of the operator $D_{(I,I)}\Gamma$. □

Proposition 6.4. *Consider the notation from Definition 6.1 and assume that $\varphi \in \text{Conv}_\text{s}(\mathbb{R}_{\geq 0})$. If (I, I) is a local minimizer of Δ in $\mathcal{U}(d) \times \mathcal{U}(d)$ then $[A, B] = 0$.*

Proof. Assume that $[A, B] \neq 0$. Then there exists a minimal projection P of the unital *-subalgebra $\mathcal{C} = \{A, \, B\}' \subseteq \mathcal{M}_d(\mathbb{C})$ such that $[PA, PB] \neq 0$. Indeed, $I \in \mathcal{C}$ is a projection such that $[IA, IB] \neq 0$. If I is not a minimal projection in $\mathcal{C}$ then there exists $P_1, \, P_2 \in \mathcal{C}$ non-zero projections such that $I = P_1 + P_2$; hence $[P_iA, P_iB] \neq 0$ for $i = 1$ or $i = 2$. If the corresponding P_i is not minimal in $\mathcal{C}$ we can repeat the previous argument (halving) applied to P_i. Since we deal with finite dimensional algebras, the previous procedure finds a minimal projection $P \in \mathcal{C}$ as above. By applying a convenient change of orthonormal basis we can assume that $R(P) = \text{span}\{e_i : i \in \mathbb{I}_r\}$, where $r = \text{rk}(P) > 1$. Since P reduces both A and B we can consider $A_1 = A|_{R(P)} \in \mathcal{M}_r(\mathbb{C})^+$ and $B_1 = B|_{R(P)} \in \mathcal{M}_r(\mathbb{C})^+$. Then, we have that $\{A_1, \, B_1\}' = \mathbb{C} \cdot I_r$: indeed, using the well known fact that $\{QA \, , \, QB\}' = Q \, \mathcal{C} \, Q \subseteq L(R(Q))$ that is valid for every projection $Q \in \mathcal{C}$ (see [24, Section 5.5]) and the minimality of P in $\mathcal{C}$ we see that

$$\{A_1, \, B_1\}' = P \, \mathcal{C} \, P = \mathbb{C} \cdot P|_{R(P)} = \mathbb{C} \cdot I_r \, .$$

Using the case of equality of Lidskii's inequality (see Theorem 2.2), we conclude that

$$b := (\lambda(A_1)^\downarrow + \lambda(B_1)^\uparrow)^\downarrow \prec a := \lambda(A_1 + B_1) \quad \text{ and } \quad a \neq b \, .$$

If we let $\sigma = \operatorname{tr}(A_1 + B_1)$ then, by Lemma 6.3 the map

$$\mathcal{U}(r) \times \mathcal{U}(r) \ni (U, V) \mapsto U^* A_1 U + V^* B_1 V \in \mathcal{M}_r(\mathbb{C})^+_\sigma$$

is a submersion at (I_r, I_r). In particular, for every open neighborhood $\mathcal{N}$ of (I_r, I_r) in $\mathcal{U}(r) \times \mathcal{U}(r)$ the set

$$\mathcal{M} := \{U^* A_1 U + V^* B_1 V : \ (U, V) \in \mathcal{N}\}$$

contains an open neighborhood of $A_1 + B_1$ in $\mathcal{M}_r(\mathbb{C})^+_\sigma$. Consider $\rho : [0, 1] \to (\mathbb{R}^r_{\geq 0})^\downarrow$ given by $\rho(t) = (1 - t)\, a + t\, b$ for $t \in [0, 1]$. Notice that $\rho(t) \prec a$ and $\rho(t) \neq a$ for $t \in (0, 1]$. If we let $A_1 + B_1 = W^* D_a W$ for $W \in \mathcal{U}(r)$ then the continuous curve $S(\cdot) : [0, 1] \to \mathcal{M}_r(\mathbb{C})^+_\sigma$ given by $S(t) = W^* D_{\rho(t)} W$ for $t \in [0, 1]$ satisfies that $S(0) = A_1 + B_1$, $\lambda(S(t)) \prec a$ and $\lambda(S(t)) \neq a$ for $t \in (0, 1]$. Therefore, there exists $t_0 \in (0, 1]$ such that $S(t) \in \mathcal{M}$ for $t \in [0, t_0]$ so, in particular, there exists $(U, V) \in \mathcal{N}$ such that

$$S(t_0) = U^* A_1 U + V^* B_1 V \implies \Delta(U \oplus P^\perp, V \oplus P^\perp) < \Delta(I_d, I_d) \ ,$$

because $\varphi \in \operatorname{Conv_s}(\mathbb{R}_{\geq 0})$, where $U \oplus P^\perp$, $V \oplus P^\perp \in \mathcal{U}(d)$ act as the identity on $R(P)^\perp \subset \mathbb{C}^d$. Since $\mathcal{N}$ was an arbitrary neighborhood of (I_r, I_r) we conclude that (I_d, I_d) is not a local minimizer of Δ in $\mathcal{U}(d) \times \mathcal{U}(d)$. □

Theorem 6.5 (Local Lidskii's theorem). *Let $S \in \mathcal{M}_d(\mathbb{C})^+$ and $\mu = (\mu_i)_{i \in \mathbb{I}_d} \in (\mathbb{R}^d_{\geq 0})^\downarrow$. Assume that $\varphi \in \operatorname{Conv_s}(\mathbb{R}_{\geq 0})$ and that $G_0 \in \mathcal{O}_\mu$ is a local minimizer of $\Phi_{S,\varphi}$ on $\mathcal{O}_\mu$. Then, there exists $\{v_i\}_{i \in \mathbb{I}_d}$ an ONB of $\mathbb{C}^d$ such that, if we let $(\lambda_i)_{i \in \mathbb{I}_d} = \lambda^\uparrow(S) \in (\mathbb{R}^d_{\geq 0})^\uparrow$ then*

$$S = \sum_{i \in \mathbb{I}_d} \lambda_i \ v_i \otimes v_i \quad \text{ and } \quad G_0 = \sum_{i \in \mathbb{I}_d} \mu_i \ v_i \otimes v_i \ . \tag{48}$$

In particular, $\lambda(S + G_0) = (\lambda(S)^\uparrow + \lambda(G_0)^\downarrow)^\downarrow$ so G_0 is also a global minimizer of Φ on $\mathcal{O}_\mu$.

Proof. By Lemma 6.2 and Proposition 6.4 we conclude that $[S, G_0] = 0$. Notice that in this case there exists $\mathcal{B} = \{v_i\}_{i \in \mathbb{I}_d}$ an ONB of $\mathbb{C}^d$ such that

$$S = \sum_{i \in \mathbb{I}_d} \lambda_i \ v_i \otimes v_i \ , \ G_0 = \sum_{i \in \mathbb{I}_d} \nu_i \ v_i \otimes v_i \ \text{ with } \ \lambda = (\lambda_i)_{i \in \mathbb{I}_d} \in (\mathbb{R}^d_{\geq 0})^\uparrow \ ,$$

for some $\nu_1, \ldots, \nu_d \geq 0$. We now show that under suitable permutations of the elements of $\mathcal{B}$ we can obtain a representation as in Eq. (48) above. Indeed, assume that $j \in \mathbb{I}_{d-1}$ is such that $\nu_j < \nu_{j+1}$. If we assume that $\lambda_j < \lambda_{j+1}$ then consider the continuous curve of unitary operators $U(t) : [0, \pi/2) \to \mathcal{U}(d)$ given by

$$U(t) = \sum_{i \in \mathbb{I}_d \setminus \{j, j+1\}} v_i \otimes v_i + \cos(t)\ (v_j \otimes v_j + v_{j+1} \otimes v_{j+1}) + \sin(t)\ (v_j \otimes v_{j+1} - v_{j+1} \otimes v_j) \ , \quad t \in [0, \pi/2) \, .$$

Notice that $U(0) = I_d$. We now define the continuous curve $G(t) = U(t)\, G_0\, U(t)^* \in \mathcal{O}_\mu$, for $t \in [0, \pi/2)$. Then $G(0) = G_0$ and we have that

$$S + G(t) = \sum_{i \in \mathbb{I}_d \setminus \{j, j+1\}} (\lambda_i + \nu_i) \ v_i \otimes v_i + \sum_{r,s=1}^{2} \gamma_{r,s}(t) \ v_{j+r} \otimes v_{j+s} \, , \tag{49}$$

where $M(t) = (\gamma_{r,s})_{r,s=1}^2$ is determined by

$$M(t) = \begin{pmatrix} \lambda_j & 0 \\ 0 & \lambda_{j+1} \end{pmatrix} + V(t) \begin{pmatrix} \nu_j & 0 \\ 0 & \nu_{j+1} \end{pmatrix} V(t)^* \quad \text{and} \quad V(t) = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \ , \quad t \in [0, \pi/2) \, .$$

Let us consider

$$R(t) = V^*(t) \begin{pmatrix} \lambda_j - \lambda_{j+1} & 0 \\ 0 & 0 \end{pmatrix} V(t) + \begin{pmatrix} \nu_j & 0 \\ 0 & \nu_{j+1} \end{pmatrix} \implies M(t) = V(t)\, R(t)\, V^*(t) + \lambda_{j+1}\, I_2 \,. \quad (50)$$

We claim that $\lambda(R(t)) \prec \lambda(R(0))$ and $\lambda(R(t)) \neq \lambda(R(0))$ for $t \in (0, \pi/2)$ (i.e., the majorization relation is strict). Indeed, since $R(t)$ is a curve in $\mathcal{M}_2(\mathbb{C})^+$ such that $\operatorname{tr}(R(t))$ is constant, it is enough to show that the function $[0, \pi/2) \ni t \mapsto \operatorname{tr}(R(t)^2)$ is strictly decreasing in $[0, \pi/2)$. Indeed, since $\lambda_j - \lambda_{j+1} > 0$ then

$$V^*(t) \begin{pmatrix} \lambda_j - \lambda_{j+1} & 0 \\ 0 & 0 \end{pmatrix} V(t) = g(t) \otimes g(t) \quad \text{where} \quad g(t) = (\lambda_j - \lambda_{j+1})^{1/2} (\cos(t), \sin(t)) \,, \ \ t \in [0, \pi/2) \,.$$

If $D \in \mathcal{M}_2(\mathbb{C})$ is the diagonal matrix with main diagonal $(\nu_j,\, \nu_{j+1})$ then $R(t) = g(t) \otimes g(t) + D$ so

$$\operatorname{tr}(R(t)^2) = \operatorname{tr}((g(t) \otimes g(t))^2) + \operatorname{tr}(D^2) + 2\, \operatorname{tr}(g(t) \otimes g(t)\ D) = c + \langle D\, g(t),\, g(t) \rangle$$

where $c = \|g(t)\|^4 + \nu_j^2 + \nu_{j+1}^2 = (\lambda_j - \lambda_{j+1})^2 + \nu_j^2 + \nu_{j+1}^2 \in \mathbb{R}$ is a constant and

$$\langle D\, g(t),\, g(t) \rangle = (\lambda_j - \lambda_{j+1})\ (\cos^2(t)\, \nu_j + \sin^2(t)\, \nu_{j+1})$$

is strictly decreasing in $[0, \pi/2)$, since $\nu_j > \nu_{j+1}$. Thus, $\lambda(R(t)) \prec \lambda(R(0))$ and $\lambda(R(t)) \neq \lambda(R(0))$ for $t \in (0, \pi/2)$. Hence, by Eq. (50), we see that

$$\lambda(M(t)) = \lambda(R(t)) + \lambda_{j+1}\, \mathbb{1}_2 \implies \lambda(M(t)) \prec \lambda(M(0)) \ , \ \lambda(M(t)) \neq \lambda(M(0)) \ , \quad t \in (0, \pi/2) \,.$$

Using Eq. (49) and that $\varphi \in \operatorname{Conv_s}(\mathbb{R}_{\geq 0})$, we see that for $t \in (0, \pi/2)$

$$\Phi(G(t)) = \sum_{i \in \mathbb{I}_d \setminus \{j, j+1\}} \varphi(\lambda_i + \nu_i) + \operatorname{tr}(\varphi(M(t))) < \sum_{i \in \mathbb{I}_d \setminus \{j, j+1\}} \varphi(\lambda_i + \nu_i) + \operatorname{tr}(\varphi(M(0))) = \Phi(G(0))$$

This last inequality, which is a consequence of the assumption $\lambda_j < \lambda_{j+1}$, contradicts the local minimality of G_0 in $\mathcal{O}_\mu$. Hence, since $\lambda_j \leq \lambda_{j+1}$ we see that $\lambda_j = \lambda_{j+1}$; in this case, we can consider the basis $\mathcal{B}' = \{v_i'\}_{i \in \mathbb{I}_d}$ obtained by transposing the vectors v_j and v_{j+1} in the basis $\mathcal{B}$. In this case $S\ v_i' = \lambda_i\ v_i'$ for $i \in \mathbb{I}_d$, $G_0\ v_i = \nu_i\ v_i'$ for $i \in \mathbb{I}_d \setminus \{j,\, j+1\}$ and $G_0\ v_j' = \nu_{j+1}\, v_j'$, $G_0 v_{j+1}' = \nu_j\, v_{j+1}'$. After performing this argument at most d times we get the desired ONB. □

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