Phase shift formulas in uniaxial media: an application to waveplates

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The calculation of phase shift and optical path difference in birefringent media is related to a wide range of applications and devices. We obtain an explicit formula for the phase shift introduced by an anisotropic uniaxial plane-parallel plate with arbitrary orientation of the optical axis when the incident wave has an arbitrary direction. This allows us to calculate the phase shift introduced by waveplates when considering oblique incidence as well as optical axis misalignments. The expressions were obtained by using Maxwell's equations and boundary conditions without any approximation. They can be applied both to single plane wave and space-limited beams.

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1. Introduction

The internal structure of a great variety of devices allows modeling as uniaxial media: waveplates, LCDs, birefringent filters, birefringent lenses, birefringent interferometers, and nonlinear optical effect generators. Particularly, one of the effects that result from the properties of these materials is the appearance of two refracted waves from an incident wave (i.e., birefringence). These waves are linearly polarized and propagate through material with different velocities, in such a way that, generally, there will be a phase shift between both waves. Different authors have performed phase difference calculations using different methods [1–5], depending on the application [6,7], approximation, or particular case of study. Some authors advise following the trajectory of each wave along the wavefront normals and using the refraction indices associated with each one to calculate the phase. On the other hand, the phase can be calculated using the optical path followed by the light. The definition of optical path was extended to birefringent media by applying Fermat's principle in 1998 [8]. In 2006, Avendaño-Alejo [1] calculated ordinary and extraordinary optical path difference correctly. In that case, the incidence was produced in the principal plane that contains the optical axis. These calculations were applied to a uniaxial plane-parallel plate whose optical axis formed an arbitrary angle with the interface, but it was always contained in the plane of incidence. This work shows that there are two ways of calculating the optical path: along the direction of propagation of the energy, or along the direction of the wavefront normals. We develop the general phase calculations for the case of the plane waves, and we see that it is convenient to follow the ray path (Poynting vector's temporal average direction) for calculating both the optical path and the phase that is related to it. The procedures developed in this work can be applied both to single plane wave approximation and plane

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waves superposition (i.e., Fourier) since many results obtained by means of models where plane waves are used coincide with the results obtained when considering space-limited beams. This also occurs in uniaxial media where the average direction of the beam energy coincides with the direction of the ray associated with the plane wave in the direction of the beam average wave [9]. Regarding the cases that are usually studied, the results obtained coincide with those in the bibliography. Particularly, we consider the case of the uniaxial plane-parallel plates, which corresponds with the model used for waveplates. A generalized way of calculating the phase shift between the emerging waves for any angle of incidence, plane of incidence, and direction of the optical axis is presented. This phase shift, in addition, is explicitly expressed as a function of the variables involved. The results obtained make it possible to analyze the effects of using waveplates under oblique incidence [10–12], which is useful for the characterization of wave plates, the determination of their linear birefringent parameters, and design and usage.

2. Theory and Preliminaries

We consider harmonic plane waves propagating through the media involved, that is, [13]

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}\exp\left[i\omega\left(\frac{\mathbf{r}\cdot\check{\mathbf{N}}}{u} - t\right)\right], \tag{1}$$

where $\mathbf{E}(\mathbf{r},t)$ is the electric field associated with the wave, $\check{\mathbf{N}}$ is a unit vector in the direction of propagation of the wave, and u is the phase velocity.

In uniaxial media, a wave traveling in a given direction can propagate with two different phase velocities: either u' or u''. The velocity u' is independent of the direction of propagation and coincides with one of the principal velocities of the crystal (the ordinary velocity), which is defined as $u_o = c/n_o$, where c is the velocity of the light in vacuum and n_o is the principal ordinary index. If the wave is propagated with this velocity, it is called an "ordinary wave." The other velocity, u'', depends on the relation between the direction of propagation and the direction of the optical axis $\check{\mathbf{z}}_3$, and it is given by [14]

$$u' = [u_e^2 + (u_o^2 - u_e^2)(\check{\mathbf{N}}\cdot\check{\mathbf{z}}_3)^2]^{\frac{1}{2}}, \tag{2}$$

where u_e is the other principal velocity (the extraordinary velocity) related to the principal extraordinary index by $u_e = c/n_e$. When the propagation has these characteristics, the wave is called an "extraordinary wave."

The propagating waves in uniaxial media are linearly polarized, as shown in Fig. 1. In the case of the ordinary waves, the direction of propagation of the wave $\check{\mathbf{N}}_\mathbf{o}$ coincides with the direction of the flow of energy that is referred to as "ordinary ray" $\check{\mathbf{R}}_\mathbf{o}$. In the case of the extraordinary wave, it propagates in a direction $\check{\mathbf{N}}_\mathbf{e}$ that is different from the direction of the flow of energy, $\check{\mathbf{R}}_\mathbf{e}$, which is called an "extraordinary ray." The relation between $\check{\mathbf{R}}_\mathbf{e}$ and $\check{\mathbf{N}}_\mathbf{e}$ is given by [14]

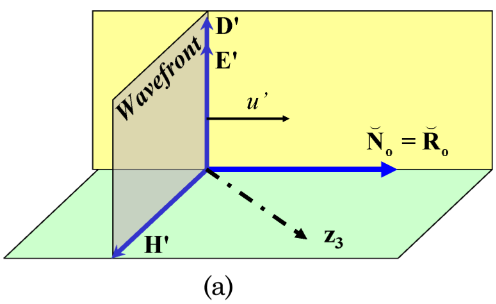
$$\check{\mathbf{R}}_\mathbf{e} = \frac{1}{f_e}[n_o^2\check{\mathbf{N}}_\mathbf{e} + (n_e^2 - n_o^2)(\check{\mathbf{N}}_\mathbf{e}\cdot\check{\mathbf{z}}_3)\check{\mathbf{z}}_3], \tag{3}$$

where f_e is a normalization factor. If we are modeling a limited beam using a single plane wave, this difference between the directions will become of notable significance.

In the following section, we show how to calculate the phase shift between ordinary and extraordinary waves. In order to do so, we have to take into account the meaning of the equal phase planes. Figure 2 shows arbitrarily separated equal phase planes associated with an extraordinary wave. It can be observed that the phase difference between points T_2 and T_3 equals zero, since both points belong to the same plane. On the other hand, the phase difference between the points T_1 and T_2 is nonzero and equals the phase difference between T_1 and T_3.

3. Phase Shift between Ordinary and Extraordinary Waves

In the case of a plane-parallel uniaxial plate that is immersed in a medium of index n and considering only the first transmissions given at each interface, we could draw the equal phase planes associated with the waves in the respective media for the extraordinary case. These surfaces are seen in Fig. 3(a), where we represent the case in which the optical axis is in the plane of incidence. However, this



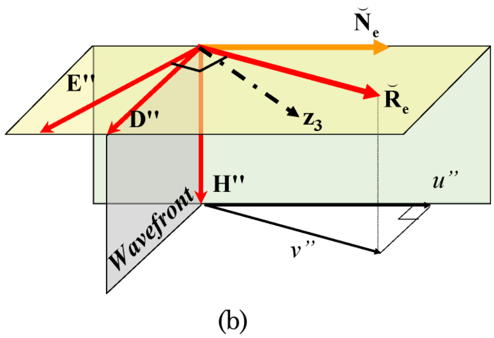


Fig. 1. (Color online) (a) Ordinary wave. (b) Extraordinary wave.

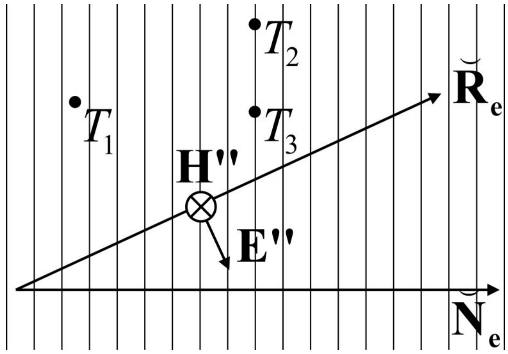


Fig. 2. Equal phase planes in a uniaxial medium associated with an extraordinary plane wave.

development covers the general situation where the optical axis has an arbitrary direction for which the ray $\breve{\mathbf{R}}_\mathbf{e}$ is not necessarily in the plane of incidence. Considering an incident wave as described in Eq. (1), we can see that the phase difference between the field evaluated at the point of incidence O and the field at any points P_1, P_2, or P_3 (distributed on the second interface) is different in each case. In order to obtain the phase difference between the point of incidence of the ray on the plate and the point where the light emerges, we must consider the phase difference between points O and P_1. The distance between the surfaces of equal phase containing O and P_1, respectively, is $\overline{OQ}$ [Fig. 3(b)]. Thus, when multiplying this by $(2\pi/\lambda_v)(c/u'')$, we obtain the phase difference between points O and P_1 (λ_v is the wavelength in vacuum). This way of calculating the phase shift coincides with that of applying the definition of extraordinary optical path (OPL_e) proposed in [8]

$$\mathrm{OPL}_e = \frac{c}{u''}\overline{OQ} = \frac{c}{v''}\overline{OP_1}, \tag{4}$$

since the ray velocity v'' and the phase velocity u'' are related,

$$v'' = \frac{u''}{\breve{\mathbf{R}}_\mathbf{e} \cdot \breve{\mathbf{N}}_\mathbf{e}}. \tag{5}$$

This quantity corresponds to the ratio of the energy per unit of time that crosses a surface that is perpendicular to the flow of energy and the volumetric energy density. The calculation of the phase shift through the distance $\overline{OP_1}$ is directly related to the ray's velocity, since O and P_1 are defined by the intersection of the beam with the surfaces. Moreover, this concept allows solving problems with other geometries as is the case of crystal prisms where the backward wave phenomenon can take place [15]. Similarly, but in a simpler way, we can calculate the phase difference between the point of incidence over the first interface O and the point where the ordinary ray emerges, since in this case the wavefront normal $\breve{\mathbf{N}}_\mathbf{o}$ and the ray $\breve{\mathbf{R}}_\mathbf{o}$ coincide.

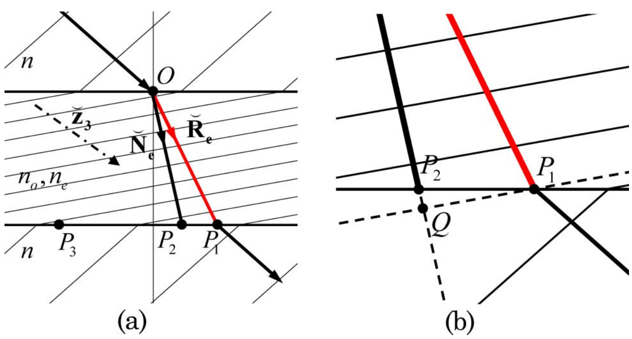


Fig. 3. (Color online) (a) Equal phase planes in a system formed by a uniaxial plate immersed in an isotropic medium. (b) Detail of Fig. 3(a).

As shown in Fig. 4, the points of incidence on the second interface of the ordinary ray and the extraordinary ray are denominated as P' and P'' respectively. For the ordinary case, the phase difference between the points of incidence on the plate O and P' is calculated through the ordinary optical path OPL_o

$$\mathrm{OPL}_o = \frac{c}{u_o}\overline{OP'} = n_o\overline{OP'}. \tag{6}$$

For the extraordinary case, the phase difference between the points of incidence on the plate O and P'' are calculated through the extraordinary optical path OPL_e, by replacing P'' by P_1 in Eq. (4).

For the particular case of a plane-parallel uniaxial plate with a thickness L, with arbitrary orientation of the optical axis and principal indices n_o and n_e, explicit equations can be obtained from the optical paths OPL_o and OPL_e in terms of constitutive parameters of both the plate and the surrounding isotropic medium and the direction of incidence. If the angle of incidence is α (Fig. 4), the expression for the ordinary optical path can be calculated by Eq. (6) and by Snell's law,

$$\mathrm{OPL}_o = L\frac{n_o^2}{[n_o^2 - n^2\sin^2\alpha]^{\frac{1}{2}}}. \tag{7}$$

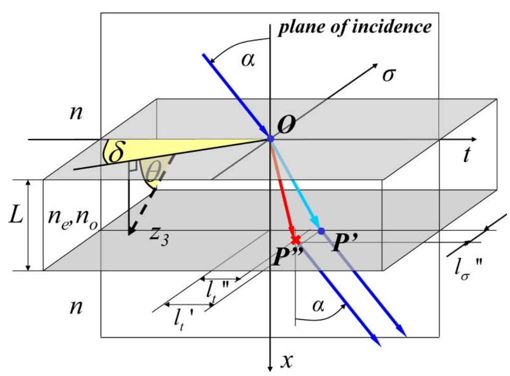


Fig. 4. (Color online) Ordinary and extraordinary transmission through a uniaxial plane-parallel plate immersed in an isotropic medium. (x, σ, t) is the coordinate system. x, t is the plane of incidence. θ is the angle between the optical axis and the interface. δ is the angle between the plane of incidence and the optical axis projection on the interface. l_t', l_t'', and l_σ'' are the coordinates of the points of incidence on the second interface for the ordinary and extraordinary rays, respectively.

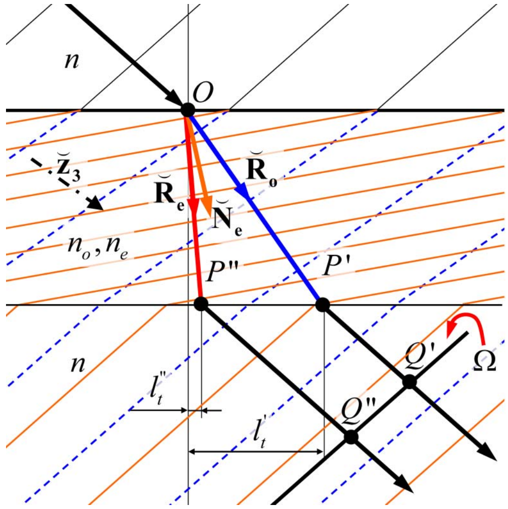


Fig. 5. (Color online) Ordinary (dashed line) and extraordinary (solid line) equal phase planes for a uniaxial plane-parallel plate immersed in an isotropic medium.

Numerous studies have been made to trace extraordinary rays in uniaxial crystals, and different approaches have been used [14,16–18]. In particular, we follow the line of authors who use Maxwell's equations and boundary conditions for the resolution [14,19,16]. Substituting Eq. (5) into Eq. (4) and using the relations $(\breve{\mathbf{R}}_\mathbf{e} \cdot \breve{\mathbf{x}}) = L/\overline{OP''}$ and $n'' = c/u''$ yields

$$\mathrm{OPL}_e = n'' L \frac{\breve{\mathbf{R}}_\mathbf{e} \cdot \breve{\mathbf{N}}_\mathbf{e}}{\breve{\mathbf{R}}_\mathbf{e} \cdot \breve{\mathbf{x}}}, \tag{8}$$

where n'' is the extraordinary refractive index. Substituting Eq. (37) and Eq. (61) from [19] to express $\breve{\mathbf{R}}_\mathbf{e} \cdot \breve{\mathbf{N}}_\mathbf{e}$ and $\breve{\mathbf{R}}_\mathbf{e} \cdot \breve{\mathbf{x}}$, respectively, and then by replacing Eqs. (32), (49), (53), (54), and (56) from [19] after a little algebra yields the explicit expression of the extraordinary optical path,

$$\mathrm{OPL}_e = \frac{L n_o n_e^2}{\{n_e^2(n_e^2 \sin^2\theta + n_o^2 \cos^2\theta) - [n_e^2 - (n_e^2 - n_o^2)\cos^2\theta \sin^2\delta] n^2 \sin^2\alpha\}^{\frac{1}{2}}}. \tag{9}$$

From Eqs. (7) and (9), we obtain the phase differences of the ordinary and extraordinary waves from the point of incidence on the first interface, O, to the respective points of incidence of the rays on the second interface, P' and P'' (Fig. 4). One of the main characteristics of the emerging light resulting from the superposition of the two waves that emerge from the plate is given by the phase shift between the waves at each point of the space. In order to evaluate it, we place an equal phase plane Ω (Fig. 5) that is perpendicular to the direction of propagation of both waves. This plane is located at an arbitrary distance from the second interface. The wave originated by the ordinary ray will travel a distance $\overline{P'Q'}$, and the one originated by the extraordinary ray will travel along $\overline{P''Q''}$. Therefore, the optical path difference Δ_{o-e} between the waves up to the plane Ω is

$$\Delta_{o-e} = (\mathrm{OPL}_o + n\overline{P'Q'}) - (\mathrm{OPL}_e + n\overline{P''Q''}). \tag{10}$$

Therefore, Eq. (10) indicates the difference between the ordinary and the extraordinary optical paths traveled by waves inside and outside the crystal.

Since the analysis performed corresponds to a single incident plane wave, we use a coordinate system (x, σ, t) associated with it (Fig. 4). Thus, we consider the point of incidence of light on the first interface as the origin of the coordinate system, the plane of incidence being the plane x, t. The points of incidence of both ordinary and extraordinary rays on the second interface are $P' = (L, 0, l'_t)$ and $P'' = (L, l''_\sigma, l''_t)$. The lateral displacement of the ordinary ray l'_t can be calculated as in isotropic media. On the other hand, l''_t and l''_σ are obtained from l_t and l_σ in Eqs. (83) to (87) from [19]. Figure 5 shows the particular case of an optical axis in the plane of incidence, where $l''_\sigma = 0$. However, the value of $\overline{P''Q''}$ does not depend on whether the extraordinary ray lies on the plane of incidence or not, and Eq. (10) is valid for any orientation of the optical axis.

If we group together the terms that correspond to the paths in the isotropic medium in Eq. (10), we can see that the displacements l''_σ, which are perpendicular to the plane of incidence, do not affect the result. This is because these displacements are also perpendicular to the direction of propagation, which only has x and t components. Since the second and third terms of Eq. (10) depend on the coordinates of the points of incidence on the second interface l'_t and l''_t, we obtain

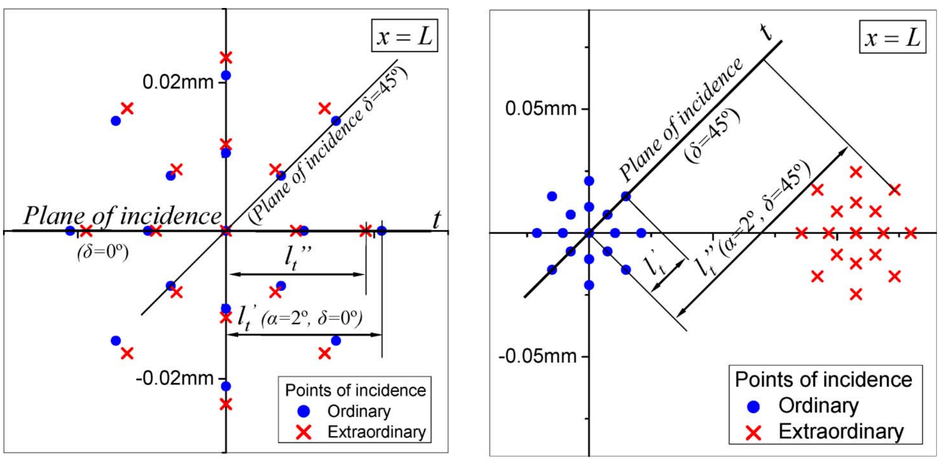
$$n(\overline{P'Q'}) - n(\overline{P''Q''}) = n(l''_t - l'_t)\sin\alpha. \tag{11}$$

Substituting Eqs. (7), (9), and (11) into Eq. (10) and using the relation between the optical path difference and the phase shift $\Delta\phi = 2\pi\Delta_{o-e}/\lambda_v$ yields

$$\Delta\phi = \frac{2\pi L}{\lambda_v}\left((n_o^2 - n^2\sin^2\alpha)^{\frac{1}{2}} + \frac{n(n_o^2 - n_e^2)\sin\theta\cos\theta\cos\delta\sin\alpha}{n_e^2\sin^2\theta + n_o^2\cos^2\theta} + \frac{-n_o\{n_e^2(n_e^2\sin^2\theta + n_o^2\cos^2\theta) - [n_e^2 - (n_e^2 - n_o^2)\cos^2\theta\sin^2\delta]n^2\sin^2\alpha\}^{\frac{1}{2}}}{n_e^2\sin^2\theta + n_o^2\cos^2\theta}\right). \tag{12}$$

This equation is the explicit general expression for the phase shift $\Delta\phi$ introduced by a uniaxial plane-parallel plate with arbitrary orientation θ of the optical axis when the incident wave has an arbitrary direction, i.e., $0° \le \alpha < 90°$ and $0° \le \delta < 360°$. From this phase shift we can recover the optical path difference obtained in [1], where the particular case of optical axis contained in the plane of incidence was considered ($\delta = 0°$).

There are two important cases of symmetry associated with two different optical axis directions: parallel to the interfaces ($\theta = 0°$) and perpendicular to the interfaces ($\theta = 90°$). In these cases, the second term of Eq. (12) is zero for every value of δ. Thus, the phase shift dependence on δ is due to the third term, which depends on the square sine of this angle and causes the phase shift to be repeated by quadrants. For other optical axis orientations, this symmetry is broken by the addition of a cosine dependence on δ. Moreover, the third term is related to the difference between the paths traveled by the waves within the plate, i.e., $\mathrm{OPL}_o - \mathrm{OPL}_e$, and depends on the square sine of δ. Therefore, the cosine dependence on δ comes from the path difference in the isotropic medium [Eq. (11)]. The influence of this term can be seen by comparing the examples of Figs. 6(a) and 6(b), where we have plotted the location of the points of incidence of rays on the second interface for two calcite plane-parallel plates 1 mm thick, one with $\theta = 0°$ and the other one with $\theta = 45°$, for different directions of incidence. In these graphs the x axis intersection with the second interface corresponds to the origin, and the projection of the optical axis on the interface corresponds to the horizontal axis. Different planes of incidence and angles of incidence were set in order to observe the effects of different orientations of the optical axis. In both figures, we can see that the points of incidence of the ordinary rays (round dots) and extraordinary rays (crosses) do not match. In the case of $\theta = 0°$ (optical axis parallel to the interfaces), the pattern that corresponds to the points of incidence of the extraordinary rays (described by crosses) is centered on the coordinate system (as is always the case with the ordinary pattern) [Fig. 6(a)]. This high symmetry causes the differences $l_t'' - l_t'$, for a given angle of incidence, to be repeated by quadrants. Mathematically, this means that the cosine dependence on δ has been canceled and the dependence is only within the square sine of δ. In the case of the optical axis nonparallel to the interfaces, the pattern that corresponds to the extraordinary rays has moved from the center of the graph $(L, 0, 0)$ in the direction of the optical axis [20]. This shift leads to a symmetry with respect to the horizontal axis instead of that of the previous case. In this case, for the same angle of incidence, the differences $l_t'' - l_t'$ are repeated for $0° < \delta < 180°$ and $180° < \delta < 360°$. This is reflected in



(a) $\theta = 0°$. (b) $\theta = 45°$.

Fig. 6. (Color online) Diagram of the points of incidence on the second interface for a calcite plate: (a) $\theta = 0°$ and (b) $\theta = 45°$, for $|\alpha| = 0°$, 1°, and 2°, and $\delta = 0°$, 45°, 90°, and 135°. $L = 1\,\mathrm{mm}$, $n_o = 1.66$, $n_e = 1.49$, and $\lambda_v = 632.8\,\mathrm{nm}$.

the cosine dependence on δ (even function) present in Eq. (12).

4. Alternative Approach

Another way to obtain the phase difference is to study the problem from the fields associated with the waves in each medium. The space is divided into three regions with their respective interfaces. The first region corresponds to the isotropic incidence medium, the second one is formed by a plane-parallel uniaxial plate, and the third one is an isotropic medium of the same characteristics as that of the first region. If we consider only the first transmissions on each interface, in the first region we will obtain an incident plane wave with an associated electric field as in Eq. (1). This wave will give rise to two linearly polarized waves that will superpose in the medium consisting of a uniaxial plane-parallel plate. Thus, the electric field in the uniaxial medium $\mathbf{E}^{\mathbf{II}}(\mathbf{r},t)$ can be written as the vector addition of two electric fields, one associated with the ordinary wave $\mathbf{E}^{\mathbf{II}}_{\mathbf{o}}(\mathbf{r},t)$ and another one associated with the extraordinary wave $\mathbf{E}^{\mathbf{II}}_{\mathbf{e}}(\mathbf{r},t)$:

$$\mathbf{E}^{\mathbf{II}}(\mathbf{r},t) = \mathbf{E}^{\mathbf{II}}_{\mathbf{o}} \exp\left[i\omega\left(\frac{\mathbf{r}\cdot\check{\mathbf{N}}_{\mathbf{o}}}{u_o} - t\right)\right] + \mathbf{E}^{\mathbf{II}}_{\mathbf{e}} \exp\left[i\omega\left(\frac{\mathbf{r}\cdot\check{\mathbf{N}}_{\mathbf{e}}}{u''} - t\right)\right]. \tag{13}$$

Wavefront normals $\check{\mathbf{N}}_{\mathbf{o}}$ and $\check{\mathbf{N}}_{\mathbf{e}}$ as well as the phase velocity of the extraordinary wave u'' are obtained by using the boundary conditions on the first interface (isotropic-uniaxial). In this case $\mathbf{r}$ corresponds to any point of the second region (uniaxial medium). In the third region we will have the superposition of waves that comes from the ordinary and extraordinary waves that were refracted in the second interface (uniaxial-isotropic). Analogously we write the electric field in this region $\mathbf{E}^{\mathbf{III}}(\mathbf{r},t)$ as a vector addition,

$$\mathbf{E}^{\mathrm{III}}(\mathbf{r},t) = \mathbf{E}^{\mathbf{III}}_{\mathbf{o}} \exp\left(i\left\{\omega\left[\frac{(\mathbf{r}-\mathbf{r}_{2\mathbf{o}})\cdot\check{\mathbf{N}}}{u} - t\right] + \phi_o(\mathbf{r}_{2\mathbf{o}})\right\}\right) + \mathbf{E}^{\mathbf{III}}_{\mathbf{e}} \exp\left(i\left\{\omega\left[\frac{(\mathbf{r}-\mathbf{r}_{2\mathbf{e}})\cdot\check{\mathbf{N}}}{u} - t\right] + \phi_e(\mathbf{r}_{2\mathbf{e}})\right\}\right). \tag{14}$$

The value of $\mathbf{r}$ corresponds to the position of any point belonging to the third region where, as we saw earlier, the refracted waves propagate in the direction of $\check{\mathbf{N}}$ with the characteristic phase velocity of the medium u. Phases $\phi_o(\mathbf{r}_{2\mathbf{o}})$ and $\phi_e(\mathbf{r}_{2\mathbf{e}})$ have been referred to as arbitrary points on the second interface: $\mathbf{r}_{2\mathbf{o}}$ for the one coming from the ordinary wave and $\mathbf{r}_{2\mathbf{e}}$ for the extraordinary case. From Eqs. (13) and (14) and by the equal-phase condition in the second interface ($x = L$) we obtain

$$\phi_o(\mathbf{r}_{2\mathbf{o}}) = \omega\frac{\mathbf{r}_{2\mathbf{o}}\cdot\check{\mathbf{N}}_{\mathbf{o}}}{u_o}, \tag{15}$$

$$\phi_e(\mathbf{r}_{2\mathbf{e}}) = \omega\frac{\mathbf{r}_{2\mathbf{e}}\cdot\check{\mathbf{N}}_{\mathbf{e}}}{u''}. \tag{16}$$

If the choice of $\mathbf{r}_{2\mathbf{o}}$ and $\mathbf{r}_{2\mathbf{e}}$ coincides with the points of incidence of the rays on the second interface P' and P'', then $\phi_o(\mathbf{r}_{2\mathbf{o}})$ and $\phi_e(\mathbf{r}_{2\mathbf{e}})$ correspond to $2\pi\mathrm{OPL}_o/\lambda_v$ and $2\pi\mathrm{OPL}_e/\lambda_v$, respectively [Fig. 7(a)]. However, if we choose for both plane waves the same point on the second interface $\mathbf{r}_2 = \mathbf{r}_{2\mathbf{o}} = \mathbf{r}_{2\mathbf{e}}$ [Fig. 7(b)] and replace in Eq. (14), we obtain

$$\mathbf{E}^{\mathbf{III}}(\mathbf{r},t) = \mathbf{E}^{\mathbf{III}}_{\mathbf{o}} \exp\left(i\left\{\omega\left[\frac{(\mathbf{r}-\mathbf{r}_2)\cdot\check{\mathbf{N}}}{u} - t\right] + \phi_o(\mathbf{r}_2)\right\}\right) + \mathbf{E}^{\mathbf{III}}_{\mathbf{e}} \exp\left(i\left\{\omega\left[\frac{(\mathbf{r}-\mathbf{r}_2)\cdot\check{\mathbf{N}}}{u} - t\right] + \phi_e(\mathbf{r}_2)\right\}\right). \tag{17}$$

This equation allows us to clearly see that the phase shift between both emerging waves in any position of the third region of coordinates $\mathbf{r}$ is given by the difference $\Delta\phi(\mathbf{r}) = \phi_o(\mathbf{r}_2) - \phi_e(\mathbf{r}_2)$. Equations (15) and (16) show that this phase difference is independent of the coordinates of the point $\mathbf{r}$. In addition, when developing this expression from Eqs. (15) and (16), we obtain that the phase difference is even independent of the chosen point $\mathbf{r}_2$, and it also coincides with the expression for the phase shift found in the former section [Eq. (12)]. This implies that the phase shift calculation can be done not only along normals to the wavefronts or along rays [1], but can be calculated by any other path represented by the choice of an arbitrary point on the second interface. Alternatively, it can be seen qualitatively in Fig. 7(b), assuming an arbitrary separation of 2π between equal phase planes. In order to calculate the phase at any point in space we can count the number of equal phase planes (which is not necessarily an integer) that were crossed to connect the reference point O with the point being studied (located in $\mathbf{r}$) and multiply it by 2π. In the ordinary case, we count

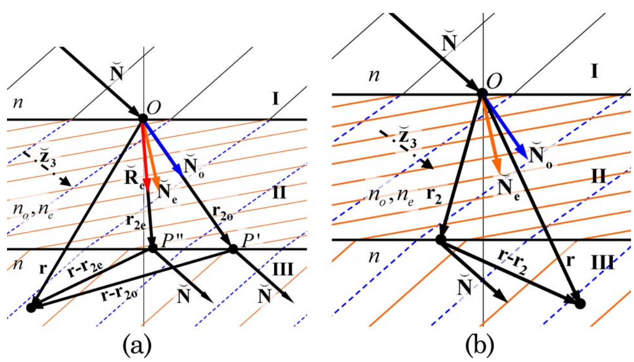


Fig. 7. (Color online) Equal phase planes: (a) the arbitrary points on the second interface, $\mathbf{r}_{2\mathbf{o}}$ and $\mathbf{r}_{2\mathbf{e}}$, respectively, coincide with the coordinates of P' and P''. (b) $\mathbf{r}_2 = \mathbf{r}_{2\mathbf{o}} = \mathbf{r}_{2\mathbf{e}}$.

the planes represented by a dotted line, and in the extraordinary case, we count the ones represented by a solid line.

Two procedures for calculating the phase shift have been explored—one in Section 3 and the other one in this section. This second approach may be regarded at first glance as elegant and direct but hides the direction of the energy flux. If we consider plane waves and ideal plane interfaces of infinite extent, as the case studied in this work, there can hardly be any discussion as to which procedure is preferred, since both lead straight to the same result. However, the latter has to be applied carefully in more complex geometries, for example, the case where the ordinary ray and the extraordinary ray emerge from different faces of a prism. In such case, the direction of the energy flux associated with each wave is fundamental for the resolution of the problem. Therefore, we consider that the first procedure is more suitable for this purpose, since it is intrinsically associated with the trajectories of the rays.

5. Waveplates

We apply the results obtained in former sections to the study of waveplates. An ideal retarder is an optical device that transforms a polarized electromagnetic wave in two polarized disturbances that are orthogonal, but leading to a phase shift between them. Ideally, retarders do not polarize or induce an intensity change in the electromagnetic beam—they simply change the polarization status. In systems where polarization control is critical, a proper characterization of the retarders is important. Problems are not exclusively related to their use; they also arise from their manufacture, and they lead to behaviors that are far from the ideal one. For the ideal case of an optical axis parallel to the surfaces ($\theta = 0°$), the expression obtained [Eq. (12)] coincides with the result we obtained in [21]

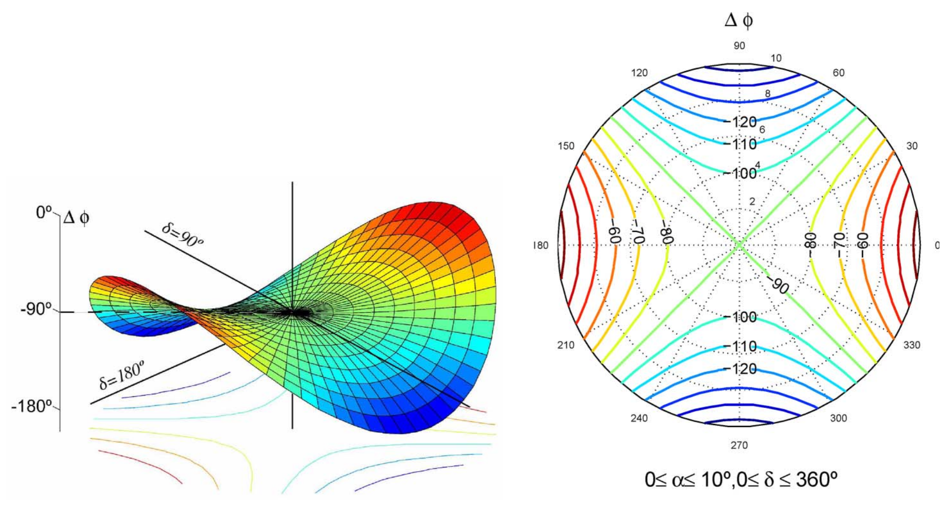
$$\Delta\phi|_{\theta=0°} = \frac{2\pi L}{\lambda_v}\left\{(n_o^2 - n^2\sin^2\alpha)^{\frac{1}{2}} - \frac{[n_o^2 n_e^2 - n^2\sin^2\alpha(n_o^2\sin^2\delta + n_e^2\cos^2\delta)]^{\frac{1}{2}}}{n_o}\right\}. \tag{18}$$

For normal incidence ($\alpha = 0°$) this equation converges to the known value $2\pi L(n_o - n_e)/\lambda_v$. Equation (18) is not limited to incidence in the principal planes ($\delta = 0°$ or $\delta = 90°$, where the phase shift extremes in terms of λ occur [10,12]), and it allows us to calculate the phase shift for every plane of incidence. If, due to manufacturing problems or other causes, the optical axis is not parallel to the interfaces, Eq. (12) must be used. In this case, if the incidence is perpendicular to the plate, the phase shift is given by

$$\Delta\phi|_{\alpha=0°} = \frac{2\pi L}{\lambda_v} n_o\left[1 - \frac{n_e}{(n_e^2\sin^2\theta + n_o^2\cos^2\theta)^{\frac{1}{2}}}\right]. \tag{19}$$

From Eq. (12), we can analyze oblique incidence in waveplates. Particularly, we will use the values in [3], where the authors consider a quartz plate with the optical axis parallel to the interfaces and a thickness L so that the phase shift is of $113\pi/2$ when the incidence is normal. This phase shift corresponds to $L = 1.973$ mm. In Fig. 8(a), the phase shift $\Delta\phi(\alpha,\delta)$ for this plate is shown. This phase shift map shows the phase shift for every plane of incidence, given by δ as an azimuthal coordinate, and is arbitrarily restricted to values of $\alpha \le 10°$ in the radial coordinate.

In the central region of Fig. 8(a), that is, the surroundings of the point that corresponds to normal incidence ($\alpha = 0°$), the phase shift has a minimum variation. This can also be observed in Fig. 8(b), since the equal phase lines are further apart from each



(a) (b)

Fig. 8. (Color online) Phase shift $\Delta\phi(\alpha,\delta)$ for a quartz plate. (a) The phase shift was expressed in degrees (vertical axis), the angle of incidence α in the radial coordinate, and the angle that forms the plane of incidence with the projection of the optical axis on the interfaces δ as an azimuthal coordinate. (b) Phase shift contour lines: $\theta = 0°$, $L = 1.973$ mm, $n_o = 1.54264$, $n_e = 1.5517$, and $\lambda_v = 632.8$ nm.

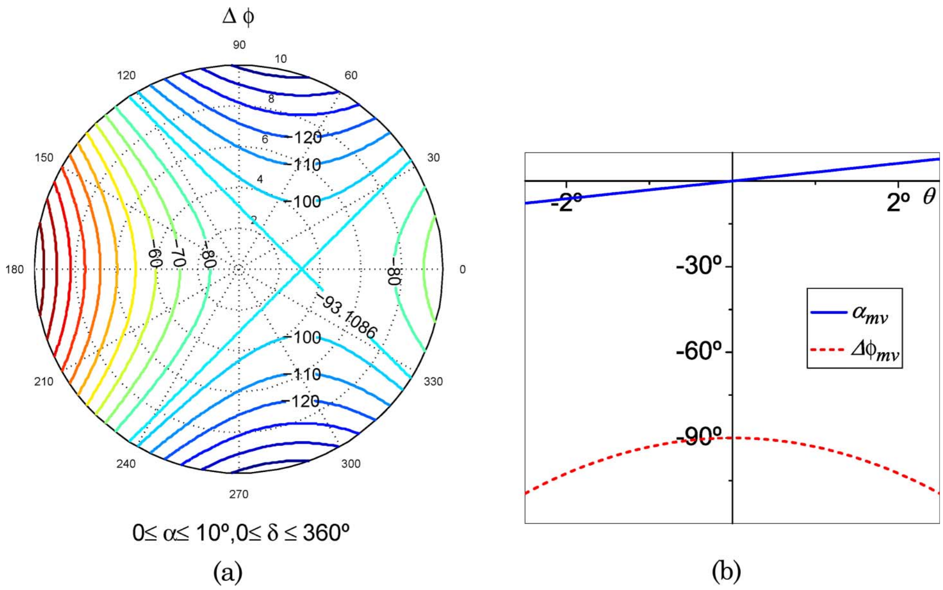


Fig. 9. (Color online) Phase shift of the region of minimum variation. (a) Phase shift contour lines $\Delta\phi(\alpha, \delta)$ for a quartz plate ($\theta = 1°$). (b) Angle of incidence, $\alpha_{\mathrm{mv}}(\theta)$, and phase shift, $\Delta\phi_{\mathrm{mv}}(\theta)$, associated with the center of the region of minimum variation of the phase shift as a function of the direction of the optical axis ($-2.5° < \theta < 2.5°$). $L = 1.973$ mm, $n_o = 1.54264$, $n_e = 1.5517$, and $\lambda_v = 632.8$ nm.

other. The equal phase lines corresponding to $\Delta\phi = -90°$ intersect in the center of the figure, resulting in two planes of incidence where the sensitivity to the angle of incidence is lower [3].

In the case of normal incidence, if the optical axis is not parallel to the surfaces ($\theta \neq 0°$), the phase shift will not correspond to a quarter-wave plate. Equation (12) shows that the region of minimum variation will appear for angles of incidence that are different from zero on the plane of incidence that contains the optical axis, which is the horizontal axis in Fig. 8(b). This is shown in Fig. 9(a) for $\theta = 1°$, where the center of the region of minimum variation has shifted to the right and corresponds to an angle of incidence $\alpha = 3.096°$, with an associated phase shift $\Delta\phi = -93.109°$. In contrast to the case of $\theta = 0°$, the lines that intersect in the center of this region do not correspond to planes of incidence, where the phase variation is minimum.

The angle of incidence corresponding to the center of the region of minimum variation, α_{mv}, is determined from the phase shift derivative and is represented in Fig. 9(b) for different inclinations of the optical axis. In this figure, the phase shift associated with these angles of incidence, $\Delta\phi_{\mathrm{mv}}$, for the different values of θ is also shown. To summarize, the expression (12) allows us to analyze the behavior of the plane-parallel plate as a function of the constructive parameters and the incidence direction, for arbitrary θ.

6. Conclusions

We obtained an explicit general expression for the phase shift introduced by a uniaxial plane-parallel plate with arbitrary orientation of the optical axis with regard to the incident wave direction. Moreover, we showed that the phase shift between waves can be calculated at any point in space by any path (even when dealing with uniaxial media), since it is associated with the distance between the equal phase planes that contain the origin of reference and the point under study, respectively. This way, we also obtained that the phase shift between the waves emerging from the plate is independent of the point under study. On the other hand, the wave propagation direction is the direction that is perpendicular to the equal phase planes, while the ray corresponds to the energy propagation direction and is associated with the observable path followed by light in any device. This is the reason why it is convenient to evaluate the phase shift by using the points of incidence of the rays. This is a generic methodology and it is applicable to more complex problems, either due to the shape of the incident beam or the geometry of the device.

In order to calculate the phase shift, it was not necessary to take into account the magnitude and the direction of the electrical fields associated with the waves in the different media. This is why, in order to obtain the polarization status—which is of particular interest in waveplates—the transmission and reflecting coefficients in the interfaces must be considered. Even in those particular cases where the incidence is oblique but the emerging waves are orthogonally polarized, the amplitude of the fields associated with both waves is not equal. Thus, this work intends to set the basis for more complex future developments where these factors, as well as multiple reflections and refractions in the interfaces, will be taken into account. By means of the maps of phase shift versus direction of incidence, we can see how the behavior of the plate changes as we modify the direction of the optical axis, its thickness, or the refractive indices. Therefore, they constitute a useful tool both for waveplate design and use.

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