



An elliptic singular system with nonlocal boundary conditions

Pablo Amster, Manuel Maurette*

*Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Ciudad Universitaria, Pabellón I, (1428) Buenos Aires, Argentina
IMAS - Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET), Argentina*

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ABSTRACT

We study the existence of solutions for the nonlinear second order elliptic system $\Delta u + g(u) = f(x)$, where $g \in C(\mathbb{R}^N \setminus \mathcal{S}, \mathbb{R}^N)$ with $\mathcal{S} \subset \mathbb{R}^N$ bounded. Using topological degree methods, we prove an existence result under a geometric condition on g . Moreover, we analyze the particular case of an isolated repulsive singularity: under a Nirenberg type condition, we prove the existence of a sequence of solutions of appropriate approximated problems that converges to a generalized solution.

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1. Introduction

Let $\Omega \subset \mathbb{R}^d$ be a smooth bounded domain. We consider the following elliptic system:

$$\begin{cases} \Delta u + g(u) = f(x) & \text{in } \Omega \\ u = \text{const} & \text{on } \partial\Omega \\ \int_{\partial\Omega} \frac{\partial u}{\partial \nu} dS = 0 \end{cases} \quad (1)$$

$f : \overline{\Omega} \rightarrow \mathbb{R}^N$ continuous and $g : \mathbb{R}^N \setminus \mathcal{S} \rightarrow \mathbb{R}^N$ continuous, with $\mathcal{S} \subset \mathbb{R}^N$ bounded. Without loss of generality we will assume that $\bar{f} := \frac{1}{|\Omega|} \int_{\Omega} f(x) dx = 0$. We may also refer to the constant value of u at the boundary as C .

The particular case $\mathcal{S} = \{0\}$ was extensively studied in the literature: for example, several results when $d = 1$ can be found in [1–3], among other works.

The nonlocal boundary conditions in (1) have been studied by Berestycki and Brézis in [4] and also by Ortega in [5]. They arise from certain models in plasma physics: specifically, a model describing the equilibrium of a plasma confined in a toroidal cavity, called a Tokamak machine. A detailed description of this problem can be found in the Appendix of [6].

Note that when $d = 1$ and $\Omega = (a, b)$, the system reads:

$$u'' + g(u) = p(t), \quad t \in (a, b).$$

* Corresponding author at: Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Ciudad Universitaria, Pabellón I, (1428) Buenos Aires, Argentina. Tel.: +54 11 4576 3335; fax: +54 11 4576 3335.

E-mail addresses: pamster@dm.uba.ar (P. Amster), maurette@dm.uba.ar (M. Maurette).

In this framework, the boundary conditions can be interpreted as follows:

$$u = \text{const} \quad \text{on } \partial\Omega \Rightarrow u(a) = u(b); \quad \int_{\partial\Omega} \frac{\partial u}{\partial \nu} dS = 0 \Rightarrow u'(a) = u'(b).$$

Hence, for $d > 1$ the nonlocal boundary condition in (1) can be seen as a generalization of the well known periodic conditions.

The case $d = 1$ has been studied by the authors in [7]. Using topological degree methods it was proved that if the nonlinearity $g : \mathbb{R}^N \setminus \{0\} \rightarrow \mathbb{R}^N$ is continuous, repulsive at the origin and bounded at infinity, and an appropriate Nirenberg type condition [8] holds, then either the problem has a classical solution, or else there exists a family of solutions of perturbed problems that converges uniformly and weakly in H^1 to some limit function u . Furthermore, if the singularity is *strong* (in a sense that will be explained below), then u is nontrivial and it can be shown, under extra assumptions, that the problem has always a classical solution.

In this work, we shall consider two different problems. In the next section we shall allow the (bounded) set \mathcal{S} of singularities to be arbitrary and focus our attention on the behavior of the nonlinear term g over the boundary of an appropriate domain $D \subset \mathbb{R}^N \setminus \mathcal{S}$. More precisely, we shall assume the boundedness condition

$$(B) \limsup_{|u| \rightarrow \infty} |g(u)| < \infty$$

and introduce a condition of geometric nature that involves the geodesic distance on Ω , namely:

$$d(x, y) := \inf\{\text{length}(\gamma) : \gamma \in C^1([0, 1], \Omega) : \gamma(0) = x, \gamma(1) = y\}.$$

Indeed, we shall fix a compact neighborhood \mathcal{C} of \mathcal{S} and a number

$$r := k \text{diam}_d(\Omega)(\|f\|_\infty + \sup_{u \notin \mathcal{C}} |g(u)|), \quad (2)$$

where k is a constant such that

$$\|\nabla u\|_\infty \leq k \|\Delta u\|_\infty$$

for all $u \in C^2(\overline{\Omega}, \mathbb{R}^N)$ satisfying the nonlocal boundary conditions of (1). Then we shall assume, for a certain $D \subset \mathbb{R}^N \setminus (\mathcal{C} + \overline{B_r}(0))$:

(D₁) For all $v \in \partial D$, $0 \notin \text{co}(g(B_r(v)))$, where ‘co(X)’ stands for the convex hull of a set $X \subset \mathbb{R}^N$.

(D₂) $\deg(g, D, 0) \neq 0$.

Condition (D₁) was introduced by Ruiz and Ward in [9] and extended in [10] by the first author and Clapp. It generalizes a classical condition given by Nirenberg in [8] which, in particular, implies that g cannot rotate around the origin when $|u|$ is large. Condition (D₁) is weaker: it allows g to rotate, although not too fast since r cannot be arbitrarily small.

The main result in Section 2 reads as follows:

Theorem 1.1. *Let $g \in C(\mathbb{R}^N \setminus \mathcal{S}, \mathbb{R}^N)$ satisfying (B) and $f \in C(\overline{\Omega}, \mathbb{R}^N)$ such that $\bar{f} = 0$. Let \mathcal{C} be a compact neighborhood of \mathcal{S} and let r be as in (2). If there exists a domain $D \subset \mathbb{R}^N \setminus (\mathcal{C} + \overline{B_r}(0))$ such that (D₁) and (D₂) hold, then (1) has at least one solution u with $\bar{u} \in \overline{D}$ and $\|u - \bar{u}\|_\infty \leq r$.*

In Section 3 we study the case in which \mathcal{S} consists in a single point; without loss of generality, it may be assumed $\mathcal{S} = \{0\}$. We shall focus our attention on the way g behaves near the singular point. In first place, we shall assume that g is repulsive, namely:

(Rep) There exists $c > 0$ such that $\langle g(u), u \rangle < 0$ for $0 < |u| < c$.

Furthermore, it will be assumed that g is *sequentially strongly repulsive*, in the following sense:

(Seq) There exists a sequence $r_n \searrow 0$ such that.

$$\sup_{|u|=r_n} \left\langle g(u), \frac{u}{|u|} \right\rangle \rightarrow -\infty \quad \text{as } n \rightarrow \infty.$$

We shall proceed as follows: firstly, we shall prove existence of at least one solution of an approximated problem. Next, we shall obtain accurate estimates and deduce the existence of a convergent sequence of these solutions.

In order to define the approximated problems, fix a sequence $\varepsilon_n \rightarrow 0$ and consider the problem

$$\Delta u + g_n(u) = f(x) \quad \text{in } \Omega \quad (3)$$

together with the nonlocal boundary conditions of (1). Although more general perturbations are admitted, for convenience we shall define g_n by

$$g_n(u) = \begin{cases} g(u) & |u| \geq \varepsilon_n \\ \rho_n(|u|)g\left(\varepsilon_n \frac{u}{|u|}\right) & 0 < |u| < \varepsilon_n \\ 0 & u = 0, \end{cases} \quad (4)$$

with $\rho_n : [0, \varepsilon_n] \rightarrow [0, +\infty)$ continuous such that $\rho_n(0) = 0$, $\rho_n(\varepsilon_n) = 1$.

The conditions on g shall be, as before, of geometric nature. However, a stronger assumption is needed in order to obtain uniform estimates. A similar condition has been introduced by one of the authors and De Nápoli in [11] and has been employed also in [7] for a system of singular periodic ordinary differential equations:

(P₁) There exists a family $\mathcal{F} = \{(U_j, w_j)\}_{j=1,\dots,J}$, where $\{U_j\}_{j=1,\dots,J}$ is an open cover of S^{N-1} , constants $c_j > 0$ and $w_j \in S^{N-1}$, such that for $j = 1, \dots, K$:

$$\limsup_{r \rightarrow +\infty} \langle g(ru), w_j \rangle \leq -c_j$$

uniformly for $u \in U_j$.

Where S^{N-1} is the unit sphere of \mathbb{R}^N . On the other hand, we shall take advantage of the repulsiveness condition (Seq), which ensures that the degree over certain small balls centered at the origin is $(-1)^N$. Thus, (D₂) shall be replaced by

(P₂) There exists $R_0 > 0$ such that $\deg(g, B_R(0), 0) \neq (-1)^N$ for $R \geq R_0$.

We remark that, although g is not defined in 0, we may still use the expression $\deg(g, B_R(0), 0)$ as a notation to refer to the Brouwer degree $\deg(\hat{g}, B_R(0), 0)$, where $\hat{g} : B_R(0) \rightarrow \mathbb{R}^N$ is any continuous function such that $\hat{g} = g$ on $\partial B_R(0)$.

The preceding conditions will allow us to construct a sequence $\{u_n\}$ of solutions of the approximated problems that converges weakly in H^1 to some function u . It is easy to see that if u does not vanish on Ω , then u is a classical solution of the problem. If $u \not\equiv 0$ but possibly vanishes in Ω , then we shall call it a *generalized solution*. With this idea in mind, let us introduce a stronger repulsiveness condition:

(SR) $\lim_{u \rightarrow 0} \langle g(u), u \rangle = -\infty$.

We now state the main result of Section 3:

Theorem 1.2. *Let $g : \mathbb{R}^N \setminus \{0\} \rightarrow \mathbb{R}^N$ be continuous satisfying (B), (Rep), (Seq) and let $f \in C(\overline{\Omega}, \mathbb{R}^N)$ with $\bar{f} = 0$. Suppose that (P₁) and (P₂) hold and let $\{g_n\}$ be as in (4). Then there exist $\{u_n\}_n$ solutions of (3), a positive constant \tilde{r} such that $\|u_n\|_\infty \geq \tilde{r}$ and a subsequence of $\{u_n\}$ that converges weakly in H^1 to some function u . If furthermore (SR) is assumed, then u is a generalized solution of the problem.*

Remark 1.3. All the preceding results can be reproduced similarly for the Neumann boundary conditions.

It is worth observing that the passage from ODEs as in [7] to PDEs is not straightforward when one uses the approach of Section 3. The main difficulty in the case $d > 1$ consists in the fact that the Sobolev space $H^1(\Omega, \mathbb{R}^N)$ is no longer embedded in the space of continuous functions; thus, H^1 bounds do not guarantee the existence of uniform bounds and extra assumptions are needed in order to ensure that solutions of the approximated problems converge to a generalized solution. Also, some of the results of [7] seem difficult to extend to the case $d > 1$: for example, when $g = \nabla G$ with $\lim_{u \rightarrow 0} G(u) = +\infty$ it is proved in [7] that every generalized solution is classical. It is not clear whether an analogous result may hold or not for the case $d > 1$, even with stronger assumptions.

On the contrary, the proof of Theorem 1.1 is essentially the same both for the cases $d = 1$ and $d > 1$. When $\mathcal{S} = \emptyset$, it can be regarded as an improvement of the auxiliary result in [7] for the nonsingular case. Also, when \mathcal{S} is a single point the result does not depend on the fact that the singularity is repulsive, and no strong force condition is assumed.

It might be interesting to study possible generalizations to PDEs of some of the existing results for the case $d = 1$. For example, the recent work [12] contains new results for the case $d = N = 1$ that can be extended in an appropriate way for $d > 1$, although the generalization for $N > 1$ is not obvious. For example, Corollary 2.3 in [12] implies, in particular, the existence of a positive solution for (1) with $d = N = 1$ and $\Omega = (0, T)$, provided that

1. $\lim_{u \rightarrow 0^+} g(u) = -\infty$, $\int_0^1 g(u) du = -\infty$
2. $0 \leq \frac{g(u)}{u} \leq \left(\frac{\pi}{T}\right)^2 - \varepsilon$ for $u \gg 0$ and some $\varepsilon > 0$.

Following the ideas of Theorem 1.1, it is not difficult to prove that an analogous result holds if the first condition is replaced by

- 1'. $g(u) \leq 0$ on $(0, \delta)$ for some $\delta > 0$ large enough.

2. The general case. Proof of Theorem 1.1

Let $U = \{u \in C(\overline{\Omega}, \mathbb{R}^N) : \|u - \bar{u}\|_\infty < r, \bar{u} \in \bar{D}\}$ and consider, for $\lambda \in (0, 1]$, the problem

$$\begin{cases} \Delta u + \lambda \hat{g}(u) = \lambda f(x) & \text{in } \Omega \\ u = C & \text{on } \partial\Omega \\ \int_{\partial\Omega} \frac{\partial u}{\partial \nu} dS = 0, \end{cases} \quad (5)$$

where $\hat{g} : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is continuous and bounded with $\hat{g} = g$ over $\overline{D + B_r(0)}$. It is clear that if $u \in \bar{U}$ solves (5) for $\lambda = 1$ then u is a solution of (1).

For the reader's convenience, let us briefly describe how the standard continuation methods [13] can be adapted to our problem.

Let $\tilde{C} := \{u \in C(\bar{\Omega}, \mathbb{R}^N) : \bar{u} = 0\}$ and $K : \tilde{C} \rightarrow \tilde{C}$ be defined as a right inverse of Δ ; namely, for $\varphi \in \tilde{C}$ we define $u := K\varphi$ as the unique solution of the linear problem

$$\begin{cases} \Delta u = \varphi & \text{in } \Omega \\ u = C & \text{on } \partial\Omega \\ \int_{\partial\Omega} \frac{\partial u}{\partial \nu} dS = 0, \\ \bar{u} = 0. \end{cases} \quad (6)$$

The compactness of K follows from the standard Sobolev embeddings. Next, let $Nu = f - \hat{g}(u)$ and define the homotopy $h(u, \lambda)$ as

$$h(u, \lambda) = u - [\bar{u} + \bar{Nu} + \lambda K(Nu - \bar{Nu})].$$

For $\lambda > 0$, it is easy to check that $u \in C(\bar{\Omega}, \mathbb{R}^N)$ is a solution of (5) if and only if $h(\lambda, u) = 0$. Thus, it suffices to prove that (5) has no solutions on ∂U for $0 < \lambda < 1$. Indeed, in this case problem (1) has a solution on ∂U or either

$$\deg(h(1, \cdot), U, 0) = \deg(h(0, \cdot), U, 0) = \deg(g, D, 0) \neq 0.$$

Let $u \in \partial U$ be a solution of (5), then $\bar{u} \in \bar{D}$ and $\|u - \bar{u}\|_\infty \leq r$, so $\hat{g} \circ u = g \circ u$. As $\text{dist}(\bar{u}, \mathcal{C}) \geq r$, we deduce that $u(x) \in \mathbb{R}^N - \mathcal{C}$ and hence $|g(u(x))| \leq \sup_{z \notin \mathcal{C}} |g(z)|$ for all x . This implies

$$\|\nabla u\|_\infty \leq k \|\Delta u\|_\infty < k(\|f\|_\infty + \sup_{z \notin \mathcal{C}} |g(z)|),$$

and thus

$$\|u - \bar{u}\|_\infty \leq \text{diam}_d(\Omega) \|\nabla u\|_\infty < r.$$

Hence, $\bar{u} \in \partial D$. Moreover, it follows from the mean value theorem for vector integrals that

$$\frac{1}{|\Omega|} \int_\Omega g(u(x)) dx \in \text{co}(g(u(\bar{\Omega}))) \subset \text{co}(g(B_r(\bar{u}))).$$

On the other hand, simple integration shows that

$$\int_\Omega g(u(x)) dx = 0,$$

so $0 \in \text{co}(g(B_r(\bar{u})))$, a contradiction. \square

Remark 2.1. In this framework, taking $\mathcal{S} = \emptyset$ we obtain the main result in [9] for the nonsingular case, conveniently adapted to our problem.

It is worth noticing that the previous result can be extended for g sublinear, that is:

$$\lim_{|u| \rightarrow \infty} \frac{g(u)}{|u|} = 0.$$

Indeed, for any given $\varepsilon > 0$, there exist a constant $M_{\varepsilon, \mathcal{C}}$ such that

$$|g(u)| \leq \varepsilon |u| + M_{\varepsilon, \mathcal{C}} \quad \forall u \in \mathbb{R}^N \setminus \mathcal{C}.$$

Thus, if u is a solution of problem (1), then

$$\|\nabla u\|_\infty \leq k \|\Delta u\|_\infty \leq k(\|f\|_\infty + \varepsilon \|u\|_\infty + M_{\varepsilon, \mathcal{C}})$$

and hence

$$\|\nabla u\|_\infty \leq k(\|f\|_\infty + M_{\varepsilon, \mathcal{C}} + \varepsilon(\text{diam}_d(\Omega) \|\nabla u\|_\infty + |\bar{u}|)).$$

Suppose that $|\bar{u}| = R < \alpha K \text{diam}_d(\Omega)$ for some constants $\alpha > 1$, $K > 0$. If $\|\nabla u\| \geq K$, then:

$$K(1 - k\varepsilon \text{diam}_d(\Omega)(1 + \alpha)) \leq k(\|f\|_\infty + M_{\varepsilon, \mathcal{C}}).$$

Consequently, taking

$$\varepsilon < \frac{1}{k \text{diam}_d(\Omega)(1 + \alpha)}, \quad K > \frac{k(\|f\|_\infty + M_{\varepsilon, \mathcal{C}})}{1 - k\varepsilon \text{diam}_d(\Omega)(1 + \alpha)}, \quad r := K \text{diam}_d(\Omega), \quad (7)$$

it follows that any solution u such that $|\bar{u}| = R < \alpha r$ satisfies:

$$\|\nabla u\|_\infty < K, \quad \|u - \bar{u}\|_\infty < r.$$

Then we have:

Corollary 2.2. Let $g \in C(\mathbb{R}^N \setminus \mathcal{S}, \mathbb{R}^N)$ be sublinear and $f \in C(\overline{\Omega}, \mathbb{R}^N)$ such that $\bar{f} = 0$. Let \mathcal{C} be a compact neighborhood of \mathcal{S} and assume that for some α, ε, K and r satisfying (7), there exists a set $D \subset B_{\alpha r}(0) \setminus (\mathcal{C} + \overline{B_r(0)}) \subset \mathbb{R}^N$ such that (D_1) and (D_2) hold. Then (1) has at least one solution u with $\bar{u} \in \overline{D}$ and $\|u - \bar{u}\|_\infty \leq r$.

Let us show an example that illustrates the possibility of obtaining multiple solutions. For convenience, let us call $B_\rho := B_\rho(0) = \{u \in \mathbb{R}^N : |u| < \rho\}$.

Example 2.3. Let $A : \mathbb{R}^N \rightarrow \mathbb{R}^N$ be continuous and bounded, $a = \|A\|_\infty$ and $b > 0$. Define $g(u) = \frac{A(u)}{|u|(b-|u|)}$, so $\mathcal{S} = \{0\} \cup \partial B_b$. Let $\eta > 0$ and consider the following compact set:

$$\mathcal{C} = \overline{B_\eta} \cup (\overline{B_{b+\eta}} \setminus B_{b-\eta}).$$

Hence, $\mathbb{R}^N \setminus \mathcal{C} = (B_{b-\eta} \setminus \overline{B_\eta}) \cup (\mathbb{R}^N \setminus \overline{B_{b+\eta}})$. From the previous computations, the following estimate holds:

$$\|\nabla u\|_\infty \leq K := k \left(\|f\|_\infty + \frac{a}{\eta(b+\eta)} \right).$$

Thus,

$$r = \text{diam}_d(\Omega) k \left(\|f\|_\infty + \frac{a}{\eta(b+\eta)} \right).$$

If also $b > 2(r + \eta)$, then we might be able to obtain two disjoint sets $D^1, D^2 \subset \mathbb{R}^N \setminus (\mathcal{C} + B_r)$ such that:

$$D^1 \subset B_{b-\eta-r} \setminus B_{\eta+r}, \quad D^2 \subset \mathbb{R}^N \setminus B_{b+\eta+r}$$

leading to two different solutions u_1, u_2 with $\bar{u}_1 \in \overline{D^1}$ and $\bar{u}_2 \in \overline{D^2}$ respectively.

In order to apply our previous result, observe that condition (D_1) requires $\eta + 2r < b - \eta - 2r$, that is: $b > 4r + 2\eta$.

For example, let $T > 0$ be large enough and define $g : B_{b+T} \setminus \mathcal{S} \rightarrow \mathbb{R}^N$ by

$$g(u) := \frac{(|u| - x_1)(|u| - x_2)u}{|u|(|u| - b)}$$

for some numbers $x_1, x_2 > 0$. The numerator of this function can be extended continuously to $\mathbb{R}^N \setminus \mathcal{S}$ in such a way that $a \leq (b+T)^3$. Taking $\text{diam}(\Omega)$ small enough, the preceding inequalities for r are satisfied, so we may fix $x_1 \in (\eta + 2r, b - \eta - 2r)$ and $x_2 \in (b + \eta + 2r, b + T - 2r)$.

Thus, all the assumptions are satisfied for D^1 and D^2 ; hence, by Theorem 1.1 we deduce the existence of classical solutions $u^1 \neq u^2$ of problem (1) such that $\bar{u}^i \in \overline{D^i}$, for $i = 1, 2$.

Remark 2.4. This example shows that if the assumptions of Theorem 1.1 are verified, then the distance between different connected components of \mathcal{S} cannot be too small.

3. The case $\mathcal{S} = \{0\}$

Before giving a proof of Theorem 1.2, let us make some comments on the concept of generalized solution. Let u_n be a weak solution of (3) such that $u_n \rightarrow u$ weakly in H^1 . From the equality

$$\int_\Omega \Delta u_n \varphi + \int_\Omega g_n(u_n) \varphi = \int_\Omega f \varphi \quad \forall \varphi \in H$$

we deduce that the operator $A : H \rightarrow \mathbb{R}^N$ given by

$$A\varphi = \lim_{n \rightarrow \infty} \int_\Omega g_n(u_n) \varphi$$

is well defined and continuous, that is: $A \in H^{-1}$. In fact,

$$A\varphi = \int_\Omega f \varphi dx + \sum_{j=1}^N \nabla u^j \nabla \varphi^j dx$$

so we may regard it as a pair $(f, \nabla u) \in H^{-1}$, namely

$$A\varphi := (f, \nabla u)[\varphi].$$

Thus, we are able to define the operator $\mathcal{G} : H \rightarrow H^{-1}$ by

$$\mathcal{G}(u) := (f, \nabla u); \quad \text{i.e. } \mathcal{G}(u)[\varphi] = A\varphi.$$

(8)

Remark 3.1. As shown in [7], it is always possible to find approximations in such a way that $u \equiv 0$, this is why we need to exclude this case in the definition of generalized solution. Indeed, for $\lambda > 0$ let G_λ be the Green function associated to the operator $-\Delta u + \lambda u$ for the nonlocal boundary conditions. Let $c(\lambda) := \sup_{x \in \Omega} \|G_\lambda(x, \cdot)\|_{L^1}$, then $c(\lambda)$ is well defined and tends to 0 as $\lambda \rightarrow +\infty$. Next, define g_n in such a way that

$$g_n(u) = \begin{cases} g(u) & \text{if } |u| \geq \frac{2}{n} \\ -\lambda_n u & \text{if } |u| \leq \frac{1}{n}, \end{cases}$$

with λ_n satisfying $c(\lambda_n)\|f\|_\infty \leq \frac{1}{n}$ for all n . Let u_n be the unique solution of the linear problem $\Delta u - \lambda_n u = f$ satisfying the nonlocal boundary conditions, then

$$|u_n(x)| = \left| \int_{\Omega} G_{\lambda_n}(x, y) f(y) dy \right| \leq c(\lambda_n) \|f\|_\infty \leq \frac{1}{n}.$$

Thus, u_n is a solution of (3) and $u_n \rightarrow 0$ uniformly.

Also, observe that if u does not vanish in Ω then for any $\varphi \in H$ then

$$\mathcal{G}(u)[\varphi] = A\varphi = \lim_{n \rightarrow \infty} \int_{\Omega} g_n(u_n) \varphi dx = \int_{\Omega} g(u) \varphi dx.$$

So a generalized solution can be regarded as a nontrivial distributional solution of the equation

$$\Delta u + \mathcal{G}(u) = f.$$

In order to prove Theorem 1.2, let us state an existence result for the approximated problems:

Proposition 3.2. Let $\Omega \subset \mathbb{R}^d$ a bounded C^2 domain. Let $g : \mathbb{R}^N \setminus \{0\} \rightarrow \mathbb{R}^N$ be continuous satisfying (B), (Rep), (Seq) and let $f \in C(\overline{\Omega}, \mathbb{R}^N)$ with $\bar{f} = 0$. Suppose that (P_1) and (P_2) hold and let $\{g_n\}$ be as in (4). Then there exist $\{u_n\}_n$ solutions of (3) and a constant $\tilde{r} > 0$ such that $\|u_n\|_\infty \geq \tilde{r}$.

Proof. Fix $\tilde{r} > 0$ such that

$$\left\langle g(u), \frac{u}{|u|} \right\rangle + \|f\|_{L^\infty} < 0 \quad \text{for } u \in \mathbb{R}^N : |u| = \tilde{r} \quad (9)$$

and $n_0 \in \mathbb{N}$ such that $g_n \equiv g$ in $\mathbb{R}^N \setminus B_{\tilde{r}}$ for $n \geq n_0$. As before, we shall apply the continuation method, now over the set

$$U := \{u \in C(\overline{\Omega}, \mathbb{R}^N) : \tilde{r} < \|u\|_\infty < R\}$$

for some $R > \tilde{r}$ to be specified.

Suppose that for some $\lambda \in (0, 1)$ there exists $u \in \partial U$ a solution of (5) with $\hat{g} = g_n$.

If $\|u\|_\infty = \tilde{r}$, then we may fix x_0 such that $\|u\|_\infty = |u(x_0)| = \tilde{r}$ and define $\phi(x) := \frac{|u(x)|^2}{2}$.

As $g_n(u(x_0)) = g(u(x_0))$, if $x_0 \in \Omega$ then it is seen that

$$\begin{aligned} \Delta \phi(x_0) &= |\nabla u(x_0)|^2 + \langle u(x_0), \Delta u(x_0) \rangle \geq \langle u(x_0), \lambda(f(x_0) - g(u(x_0))) \rangle \\ &= \lambda \left[\langle u(x_0), f(x_0) \rangle - |u(x_0)| \left\langle g(u(x_0)), \frac{u(x_0)}{|u(x_0)|} \right\rangle \right] \\ &\geq \lambda \tilde{r} \left[-\|f\|_\infty - \left\langle g(u(x_0)), \frac{u(x_0)}{|u(x_0)|} \right\rangle \right] > 0, \end{aligned}$$

a contradiction.

If $x_0 \in \partial\Omega$, then $\tilde{r} = |C|$. Moreover,

$$\int_{\partial\Omega} \frac{\partial \phi}{\partial \nu} dS = \int_{\partial\Omega} \left\langle u, \frac{\partial u}{\partial \nu} \right\rangle dS = \left\langle C, \int_{\partial\Omega} \frac{\partial u}{\partial \nu} dS \right\rangle = 0. \quad (10)$$

From the continuity of ϕ , arguing as before we deduce that, $\Delta \phi > 0$ in $B_{2\delta}(x_0) \cap \Omega$ for some $\delta > 0$.

From the standard regularity theory, it follows that $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$. Moreover, we may consider a C^2 domain $\Omega_0 \subset \Omega$ such that $B_\delta \cap \Omega \subset \Omega_0$ and $\Omega_0 \subset B_{2\delta} \cap \Omega$; then $\phi(x_0) > \phi(x)$ for every $x \in \Omega_0$, and from Hopf's Lemma we obtain

$$\frac{\partial \phi}{\partial \nu}(x_0) > 0.$$

As $u \equiv \text{const}$ on the boundary, it follows that $|u(x)| \equiv \tilde{r}$ and so $\frac{\partial \phi}{\partial \nu}(x) > 0$ for each $x \in \partial\Omega$. This contradicts (10) and thus $\|u\|_\infty = R$.

Also, $\|u - \bar{u}\|_\infty < r$, then from condition (P_1) we deduce (D_1) by putting $D = B_R(0)$ when R is sufficiently large. Indeed, assume that (P_1) holds and fix a positive constant $c < c_j$ for all j and R_0 such that

$$\langle g(Ru), w_j \rangle < -c, \quad \text{for all } u \in U_j, R \geq R_0.$$

In particular, for $|v| = R$ with $R > R_0 + r$ large enough, there exists an index $j \in \{1, \dots, J\}$ such that if $z \in \overline{B_r(v)}$ then $\frac{z}{|z|} \in U_j$, and hence $\langle g(z), w_j \rangle \leq -c$. In other words, there exists a hyperplane separating 0 and $g(\overline{B_r(v)})$. Thus, condition (D_1) holds for $D = B_R(0)$. Then, using the same arguments as in Theorem 1.1, it follows that $\|u\|_\infty < R$. Finally, observe that the repulsiveness condition implies that $\deg(g_n, B_r(0), 0) = (-1)^N$. Using the excision property of the degree, condition (P_2) ensures that the degree $\deg(g_n, U \cap \mathbb{R}^N, 0) \neq 0$ and so completes the proof. \square

The following lemma shows that the solutions of the perturbed problems are also bounded for the H^1 norm.

Lemma 3.3. *In the situation of Proposition 3.2, there exists a constant \mathfrak{C} independent of n such that $\|u_n\|_{H^1} \leq \mathfrak{C}$ for all n .*

Proof. As $\Delta u_n + g_n(u_n) = f(x)$ in Ω and let us call C_n Let us call C_n the constant value that correspond to $u_n \equiv C_n$ on $\partial\Omega$, we may multiply by $u_n - C_n$ and integrate to obtain:

$$\int_{\Omega} \langle \Delta u_n + g_n(u_n), u_n - C_n \rangle dx = \int_{\Omega} \langle p, u_n - C_n \rangle dx.$$

Integrating by parts, the left hand side is equal to:

$$- \int_{\Omega} |\nabla u_n|^2 dx + \int_{\partial\Omega} \left\langle \frac{\partial u_n}{\partial \nu}, u_n - C_n \right\rangle dS + \int_{\Omega} \langle g_n(u_n), u_n - C_n \rangle dx.$$

As $u_n \equiv C_n$ on $\partial\Omega$, it follows that

$$\|\nabla u_n\|_{L^2}^2 = \int_{\Omega} \langle g_n(u_n), u_n - C_n \rangle dx - \int_{\Omega} \langle p, u_n - C_n \rangle dx.$$

Now, taking absolute value and using the Cauchy–Schwarz inequality, we get

$$\|\nabla u_n\|_{L^2}^2 \leq \left| \int_{\Omega} \langle g_n(u_n), u_n - C_n \rangle dx \right| + \|p\|_{L^2} \|u_n - C_n\|_{L^2}.$$

Let c be the constant in condition (Rep) and write:

$$\left| \int_{\Omega} \langle g_n(u_n), u_n - C_n \rangle dx \right| \leq \left| \int_{\{|u_n| < c\}} \langle g_n(u_n), u_n - C_n \rangle dx \right| + \left| \int_{\{|u_n| \geq c\}} \langle g_n(u_n), u_n - C_n \rangle dx \right|.$$

Fix $n_0 \in \mathbb{N}$ such that $\frac{1}{n} < c$ for every $n \geq n_0$, then $g_n(u_n(x)) = g(u_n(x))$ if $|u_n(x)| > c > \frac{1}{n}$ and hence on the one hand

$$\left| \int_{\{|u_n| \geq c\}} \langle g_n(u_n), u_n - C_n \rangle dx \right| \leq |\Omega|^{1/2} \gamma_c \|u_n - C_n\|_{L^2},$$

where $\gamma_c := \sup_{|u| > c} |g(u)|$ and, on the other hand, condition (Rep) implies that $\int_{\{|u_n| < c\}} \langle g_n(u_n), u_n \rangle dx \leq 0$, so

$$\int_{\{|u_n| < c\}} \langle g_n(u_n), u_n - C_n \rangle dx \leq - \int_{\{|u_n| < c\}} \langle g_n(u_n), C_n \rangle dx.$$

Moreover, as $\int_{\Omega} g_n(u_n) dx = 0$, we deduce that

$$\int_{\{|u_n| < c\}} \langle g_n(u_n), u_n - C_n \rangle dx \leq \left\langle C_n, \int_{\{|u_n| \geq c\}} g_n(u_n) dx \right\rangle \leq |\Omega|^{1/2} \gamma_c |C_n|.$$

Gathering all together,

$$\left| \int_{\Omega} \langle g_n(u_n), u_n - C_n \rangle dx \right| \leq |\Omega|^{1/2} \gamma_c (\|u_n - C_n\|_{L^2} + |C_n|).$$

Thus,

$$\|\nabla u_n\|_{L^2}^2 \leq \mathfrak{C}_1 \|u_n - C_n\|_{L^2} + \mathfrak{C}_2 |C_n|$$

for some constants $\mathfrak{C}_1, \mathfrak{C}_2$. Using Poincaré inequality, we deduce the existence of a constant \mathfrak{C} such that

$$\|\nabla u_n\|_{L^2}^2 \leq \mathfrak{C} |C_n|$$

and hence

$$\|u_n - C_n\|_{H^1}^2 \leq A + B|C_n| \quad \text{for some } A, B > 0.$$

Suppose that $|C_n|$ is unbounded, then taking a subsequence (still denoted C_n) we may assume that $|C_n| \rightarrow +\infty$, $\frac{C_n}{|C_n|} \rightarrow \eta \in S^{N-1}$. From the inequality

$$\left\| \frac{u_n - C_n}{\sqrt{|C_n|}} \right\|_{H^1}^2 \leq \frac{A}{|C_n|} + B \quad \forall n \geq n_0,$$

we may take again a subsequence and thus assume that $\frac{u_n - C_n}{\sqrt{|C_n|}}$ converges almost everywhere and weakly in H^1 to some $w \in H^1$.

Let $\varepsilon > 0$ and fix M large enough so that $|\Omega \setminus \Omega_M| < \varepsilon$, where

$$\Omega_M := \{x \in \Omega : |w(x)| \leq M\}.$$

Then $\frac{u_n - C_n}{|C_n|} \rightarrow 0$ and $\frac{u_n}{|u_n|} \rightarrow \eta$ almost everywhere in Ω_M .

Fix $U_k \subset S^{N-1}$ as in (P_1) such that $\eta \in U_k$, then writing

$$\langle g(u_n(x)), w_k \rangle = \left\langle g \left(|u_n(x)| \frac{u_n(x)}{|u_n(x)|} \right), w_k \right\rangle$$

we deduce that

$$\limsup_{n \rightarrow \infty} \langle g(u_n(x)), w_k \rangle \leq -c_k$$

a.e. in Ω_M . Thus we obtain, from Fatou's Lemma:

$$\limsup_{n \rightarrow \infty} \int_{\Omega_M} \langle g(u_n(x)), w_k \rangle dx \leq \int_{\Omega_M} \limsup_{n \rightarrow \infty} \langle g(u_n(x)), w_k \rangle dx \leq -c_k |\Omega_M|.$$

We may assume that $M \geq c$, then taking $\varepsilon < \frac{c_k |\Omega|}{\gamma_c}$ we conclude:

$$\begin{aligned} \limsup_{n \rightarrow \infty} \int_{\Omega} \langle g(u_n(x)), w_k \rangle dx &\leq -c_k |\Omega_M| + \limsup_{n \rightarrow \infty} \int_{\Omega \setminus \Omega_M} \langle g(u_n(x)), w_k \rangle dx \\ &\leq -c_k |\Omega_M| + \gamma_c |\Omega \setminus \Omega_M| < 0, \end{aligned}$$

which contradicts the fact that $\int_{\Omega} g(u_n(x)) dx = 0$. \square

Proof of Theorem 1.2. From the preceding results, there exists a sequence (still denoted $\{u_n\}$) of solutions of the approximated problems converging a.e. and weakly in H^1 to some function u , and also such that $\|u_n\|_{\infty} \geq \tilde{r}$. It remains to prove that if (SR) holds then $u \neq 0$.

Suppose that $u \equiv 0$, then from (3) we obtain

$$\int_{\Omega} \langle \Delta u_n(x), u_n(x) \rangle + \langle g(u_n(x)), u_n(x) \rangle dx = \int_{\Omega} \langle p(x), u_n(x) \rangle dx \rightarrow 0$$

as $n \rightarrow \infty$. Moreover,

$$\int_{\Omega} \langle \Delta u_n(x), u_n(x) \rangle dx = - \int_{\Omega} |\nabla u_n(x)|^2 dx$$

is bounded, and from (SR) an Fatou's Lemma we obtain

$$\limsup_{n \rightarrow \infty} \int_{\Omega} \langle g(u_n(x)), u_n(x) \rangle dx \leq \int_{\Omega} \limsup_{n \rightarrow \infty} \langle g(u_n(x)), u_n(x) \rangle dx = -\infty$$

a contradiction. \square

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