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The Killing–Yano equation on Lie groups

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Abstract
In this paper we study 2-forms which are solutions of the Killing–Yano equation on Lie groups endowed with a left invariant metric having various curvature properties. We prove a general result for 2-step nilpotent Lie groups and as a corollary we obtain a nondegenerate solution of the Killing–Yano equation on the Iwasawa manifold with its half-flat metric.

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1. Introduction

Killing–Yano tensors are natural generalizations of ordinary Killing vectors. They were first introduced by Yano [25] and have been intensively studied by physicists since the work of Penrose–Walker [22]. Killing–Yano tensors give rise to quadratic first integrals of the geodesic equation and allow us to define operators which commute or anti-commute with the Dirac operator on the manifold [7]. Both the wave and the Dirac operators are fundamental in the study of black holes in four and higher dimensions, and the presence of these true symmetries makes the separation of variables and their explicit resolution [10–12, 16, 20] more plausible. Also Killing–Yano and conformal Killing–Yano tensors have been applied to define symmetries of field equations (see [3, 4]). Semmelmann was one of the first to study in a systematic way conformal Killing–Yano tensors [23] obtaining many global results (see also [19, 5]). Other physical applications of Killing–Yano tensors may be found in [2, 9, 12, 15] and references therein.

A skew symmetric $(0, l+1)$ tensor ω on a Riemannian manifold (M, g) is called *Killing–Yano* if it satisfies the *Killing–Yano equation*

$$\nabla\omega(X; Y, Z_1, \ldots, Z_l) + \nabla\omega(Y; X, Z_1, \ldots, Z_l) = 0, \quad (1)$$

where ∇ is the Lévi-Civitá connection (see [25]) and $X, Y, Z_1, Z_2, \ldots, Z_l$ are arbitrary vector fields on M. When ω is a one-form then the associated vector field is a Killing vector field.

In [21], Papadopoulos studied the Killing–Yano equation for the fundamental forms defining a G-structure for $G = SO(n), SU(n), U(n), Sp(n) \times Sp(1), Sp(n), G_2$ and $Spin(7)$. He proves that if the fundamental form satisfies equation (1) then, in most cases, it is parallel with respect to the Lévi-Cività connection. On the other hand, he shows that the forms defining nearly Kähler, nearly parallel (weak) G_2 and balanced $SU(n)$ manifolds are solutions of equation (1). Also it was shown in [23] that there are Killing–Yano tensors on nearly Kähler manifolds and on manifolds with a weak G_2-structure.

The case of a compact simply connected symmetric space M has been considered in [5] were it is shown that M carries a non-parallel Killing p-form, $p \geqslant 2$, if and only if it is isometric to a Riemannian product $S^k \times N$, where S^k is a round sphere and $k > p$.

The goal of this paper is to study left invariant 2-forms satisfying the Killing–Yano equation on Lie groups endowed with a left invariant metric.

In section 2, after recalling some basic identities, we observe that if (M, g, T) is a Riemannian manifold with a skew-adjoint endomorphism T of TM, the associated 2-form $\omega(X, Y) = g(TX, Y)$ satisfies the Killing–Yano equation if and only if $\nabla T(X; X) = 0$. In other words, 2-forms ω on a Riemannian manifold satisfying (1) give rise to manifolds (M, g, T) which generalize nearly Kähler manifolds. We analyze in proposition 2.4 the Killing–Yano equation in the invariant setting in order to construct Killing–Yano 2-forms ω on Lie groups (G, g) where g is a left invariant metric.

In section 3, we give a characterization of the skew symmetric endomorphisms of 2-step nilpotent metric Lie algebras which give rise to left invariant 2-forms satisfying (1) on the corresponding simply connected Lie groups. As a main corollary, it turns out that the Iwasawa manifold with the standard half-flat metric carries a nondegenerate Killing–Yano 2-form, induced from a unique (up to constant multiple) left invariant 2-form on the complex Heisenberg Lie group. We recall that nilpotent Lie groups have, for every left invariant metric, directions of positive Ricci curvature and directions of negative Ricci curvature (see [18]).

In section 4, we characterize the Killing–Yano tensors on a Lie group with a flat metric (theorem 4.1). For compact semisimple Lie groups with a bi-invariant metric, it turns out that the Killing–Yano equation has no non-trivial left invariant solutions (lemma 4.6). Finally, we prove that for any left invariant metric on $SU(2)$, there are no non-trivial solutions of the Killing–Yano equation (theorem 4.7). We note that $SU(2)$ is the only simply connected Lie group with left invariant metrics of positive sectional curvature [24].

2. Killing–Yano (0, 2) tensors

Given a torsion-free connection ∇ on a manifold M and a 2-form ω on M, the exterior derivative $d\omega$ of ω can be computed in terms of $\nabla\omega$ as follows:

$$d\omega(X, Y, Z) = \nabla\omega(X; Y, Z) + \nabla\omega(Y; Z, X) + \nabla\omega(Z; X, Y) \tag{2}$$

for all vector fields X, Y, Z on M. This identity is useful in proving the following fact.

Lemma 2.1. *Let (M, g) be a Riemannian manifold, ∇ the Lévi-Cività connection and ω a 2-form on M. The following conditions are equivalent:*

(i) $\nabla\omega(X; Y, Z) + \nabla\omega(Y; X, Z) = 0$;
(ii) $\nabla\omega(X; Y, Z) + \nabla\omega(Z; Y, X) = 0$;
(iii) $d\omega(X, Y, Z) = 3\nabla\omega(X; Y, Z)$.

Proof. The proof is straightforward. One has to observe that the equivalence between (i) and (ii) follows since $\nabla\omega(X; Y, Z) = -\nabla\omega(X; Z, Y)$ and the equivalence between (i) and (iii) is a consequence of (2). □

In what follows, a 2-form ω on a Riemannian manifold (M, g) satisfying any of the above conditions will be called a *Killing–Yano 2-form* [25]. In this case, the skew-adjoint section T

of End TM, characterized by $g(TX, Y) = \omega(X, Y)$, will also be called a *Killing–Yano tensor.* Since $\nabla\omega(X; Y, Z) = g(\nabla T(X; Y), Z)$, it follows that T will be a Killing–Yano tensor if and only if

$$\nabla T(X; Y) + \nabla T(Y; X) = 0. \tag{3}$$

As a consequence, one has

Theorem 2.2. *Let (M, g, T) be a Riemannian manifold with a skew-adjoint endomorphism T of TM. The associated 2-form $\omega(X, Y) = g(TX, Y)$ is Killing–Yano if and only if $\nabla T(X; X) = 0$.*

A particular case is that of an almost Hermitian manifold (M, g, J), that is, a manifold M with a Riemannian metric g and a compatible almost complex structure $J : TM \to TM$. In this case, $\omega(X, Y) = g(JX, Y)$ is known as the Kähler form. We recall that the almost Hermitian manifold (M, g, J) is nearly Kähler if $\nabla J(X; X) = 0$. Nearly Kähler manifolds were introduced by Gray [13] and much studied since then.

As a consequence of the above theorem, the following corollary, proved in [23, 21], holds.

Corollary 2.3. *Let M be a $2n$-dimensional manifold with an almost Hermitian structure (g, J) and $\omega(X, Y) = g(JX, Y)$ the corresponding Kähler form. Then, ω is a Killing–Yano 2-form on M if and only if (M, g, J) is nearly Kähler.*

Next we give conditions on an endomorphism T of the tangent bundle of M to satisfy $\nabla T(X; X) = 0$, when M is a Lie group with a left invariant Riemannian metric.

2.1. Left invariant Killing–Yano tensors

Let G be an n-dimensional Lie group and let $\mathfrak{g}$ be the associated Lie algebra of all left invariant vector fields on G. If T_eG is the tangent space of G at e, the identity of G, the correspondence $X \to X_e := x$ from $\mathfrak{g} \to T_eG$ is an isomorphism so alternatively we could take as the Lie algebra if G the tangent space T_eG with the bracket defined to make the map above an isomorphism of Lie algebras, that is, $[x, y] = [X, Y]_e$ where X, Y are the left invariant vector fields defined by x, y respectively on T_eG. Q3

A left invariant metric on G is a Riemannian metric such that left multiplication for every $a \in G$ is an isometry. Every inner product on T_eG gives rise, by left translating, to a left invariant metric. Thus, each n-dimensional Lie group possesses a $\frac{1}{2}n(n+1)$- dimensional family of distinct left invariant metrics. Different left invariant metrics on a fixed Lie group can give rise to non-isometric Riemannian manifolds.

The Riemannian manifold consisting of a Lie group with a left invariant metric is a homogeneous manifold where many geometric invariants can be computed at the Lie algebra level. In particular, the Lévi Cività connection, associated with a left invariant metric, ∇_x is a skew-symmetric endomorphism of $\mathfrak{g}$ for any $x \in \mathfrak{g}$. If $\langle\ ,\ \rangle$ stands for the left invariant metric at T_eG, then the Lévi Cività connection on left invariant vector fields is given by

$$2\langle\nabla_x y, z\rangle = \langle[x, y], z\rangle - \langle[y, z], x\rangle + \langle[z, x], y\rangle, \qquad x, y, z \in \mathfrak{g}. \tag{4}$$

A left invariant 2-form ω on G will satisfy the Killing–Yano equation if the associated left invariant endomorphism T of the tangent bundle of G satisfies the condition in the following proposition.

Proposition 2.4. *A left invariant skew-adjoint endomorphism T of $TG = G \times \mathfrak{g}$ satisfies $\nabla T(X; X) = 0$ for all left invariant vector fields X if and only if $\alpha_T(x, y, z) = 0$ for all $x, y, z \in \mathfrak{g}$ where*

$$\begin{aligned}\alpha_T(x, y, z) := {}& \langle[Tx, y] - [x, Ty], z\rangle + \langle-T[y, z] + [Ty, z] + 2[y, Tz], x\rangle \\ & + \langle-T[x, z] + [Tx, z] + 2[x, Tz], y\rangle.\end{aligned} \tag{5}$$

Proof. A skew symmetric left invariant endomorphism T of the tangent bundle TG is a Killing–Yano tensor if and only if $\langle(\nabla_x T)y + (\nabla_y T)x, z\rangle = 0$ for all $x, y, z \in \mathfrak{g}$. Thus, using equation (4), we obtain, for all $x, y, z \in \mathfrak{g}$,

$$\begin{aligned}
\langle(\nabla_x T)y + (\nabla_y T)x, z\rangle &= \langle\nabla_x Ty - T\nabla_x y + \nabla_y Tx - T\nabla_y x, z\rangle \\
&= \langle[x, Ty], z\rangle - \langle[Ty, z], x\rangle + \langle[z, x], Ty\rangle + \langle[x, y], Tz\rangle \\
&\quad - \langle[y, Tz], x\rangle + \langle[Tz, x], y\rangle + \langle[y, Tx], z\rangle - \langle[Tx, z], y\rangle \\
&\quad + \langle[z, y], Tx\rangle + \langle[y, x], Tz\rangle - \langle[x, Tz], y\rangle + \langle[Tz, y], x\rangle \\
&= \langle[x, Ty], z\rangle - \langle[Ty, z], x\rangle - \langle T[z, x], y\rangle + 2\langle[Tz, y], x\rangle + 2\langle[Tz, x], y\rangle \\
&\quad + \langle-[Tx, y], z\rangle - \langle[Tx, z], y\rangle - \langle T[z, y], x\rangle = -\alpha_T(x, y, z).
\end{aligned}$$

Thus, the assertion follows. □

Remark 2.5. Note that $\alpha_T(x, y, z) = \alpha_T(y, x, z)$ and $\alpha_T(x, x, z) = (-2)\alpha_T(z, x, x)$.

3. Two-step nilpotent Lie groups

In this section, we give a characterization of the skew symmetric endomorphisms of 2-step nilpotent metric Lie algebras which give rise to left invariant 2-forms satisfying (1) on the corresponding simply connected Lie groups. As a main corollary, it turns out that the Iwasawa manifold with the standard half-flat metric carries a nondegenerate Killing–Yano 2-form.

Let $\mathfrak{n}$ be a 2-step nilpotent Lie algebra, that is, $[\mathfrak{n}', \mathfrak{n}] = 0$, where $\mathfrak{n}' = [\mathfrak{n}, \mathfrak{n}]$ is the commutator ideal. Fix an inner product $\langle\cdot\,,\cdot\rangle$ on $\mathfrak{n}$ and let $\mathfrak{z}$ be the center and $\mathfrak{v}$ its orthogonal complement.

Theorem 3.1. *Let T be a skew-symmetric endomorphism of $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$. Then, T is a Killing–Yano tensor on $\mathfrak{n}$ if and only if T preserves the center $\mathfrak{z}$ and for all $x, y \in \mathfrak{v}$ it holds that*

$$[Tx, y] = [x, Ty], \qquad T[x, y] = 3[Tx, y], \tag{6}$$

or equivalently

$$ad(x) \circ T = ad(Tx) = \tfrac{1}{2}[T, ad(x)].$$

Proof. According to proposition 5, a skew-symmetric endomorphism T of $\mathfrak{n}$ is a Killing–Yano tensor if and only if $\alpha_T(x, y, z) = 0$ for all $x, y, z \in \mathfrak{n}$. But

(i) if $x, y, z \in \mathfrak{z}$ then always $\alpha_T(x, y, z) = 0$,
(ii) if $x, y, z \in \mathfrak{v}$ then $\alpha_T(x, y, z) = 0$ if and only if $\langle T[z, y], x\rangle + \langle T[z, x], y\rangle = 0$,
(iii) if $x, y \in \mathfrak{z}$ and $z \in \mathfrak{v}$ then $\alpha_T(x, y, z) = 0$ if and only if $\langle[Ty, z], x\rangle + \langle[Tx, z], y\rangle = 0$,
(iv) if $x, y \in \mathfrak{v}$ and $z \in \mathfrak{z}$ then $\alpha_T(x, y, z) = 0$ if and only if $[Tx, y] = [x, Ty]$,
(v) if $x \in \mathfrak{z}$ and $y, z \in \mathfrak{v}$ then $\alpha_T(x, y, z) = 0$ if and only if $\langle-T[y, z]+[Ty, z]+2[y, Tz], x\rangle = 0$,
(vi) if $x \in \mathfrak{v}$ and $y, z \in \mathfrak{z}$ then $\alpha_T(x, y, z) = 0$ if and only if $\langle-[x, Ty], z\rangle + \langle 2[x, Tz], y\rangle = 0$.

If T is a Killing–Yano tensor, then from (iii) one has $\langle[Tz_2, v], z_1\rangle + \langle[Tz_1, v], z_2\rangle = 0$, and from (vi) $\langle-[v, Tz_1], z_2\rangle + \langle 2[v, Tz_2], z_1\rangle = 0$ for all $v \in \mathfrak{v}$ and $z_1, z_2 \in \mathfrak{z}$. Thus, $[Tz, v] = 0$ for $z \in \mathfrak{z}$ and $v \in \mathfrak{v}$ showing that T preserves $\mathfrak{z}$. Furthermore, $[Tx, y] = [x, Ty]$ for all $x, y \in \mathfrak{v}$ (see (iv)). It remains to show that $T[x, y] = 3[Tx, y]$ for all $x, y \in \mathfrak{v}$. Note that (v) and (iv) imply $\langle-T[v_1, v_2] + 3[Tv_1, v_2], z\rangle = 0$ for all $z \in \mathfrak{z}$ and $v_1, v_2 \in \mathfrak{v}$. But the $\mathfrak{v}-$ component $\langle-T[v_1, v_2] + 3[Tv_1, v_2], v\rangle$ is also trivial since T preserves $\mathfrak{v}$. Thus, the assertion follows.

Assume next that T is a skew endomorphism preserving the center and satisfying the two identities in (6); then, it follows easily that $\alpha_T(x, y, z) = 0$ in all cases (i) through (vi). □

We recall from [14] that the Lie bracket on $\mathfrak{n}$ is completely determined by the linear operator $j : \mathfrak{z} \to \mathfrak{so}(\mathfrak{v})$ defined by

$$\langle j_z x, y\rangle = \langle z, [x, y]\rangle \qquad \text{for} \quad z \in \mathfrak{z}, \qquad x, y \in \mathfrak{v}. \tag{7}$$

In terms of the operator j, the condition of T being Killing–Yano is contained in the following corollary.

Corollary 3.2. *Let T be a skew-symmetric endomorphism of $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$. Then, T is a Killing–Yano tensor if and only if T preserves the center $\mathfrak{z}$ and the map j satisfies*

$$\tfrac{1}{3}\, j_{Tz} = T \circ j_z = -j_z \circ T \qquad \text{for every} \quad z \in \mathfrak{z}. \tag{8}$$

Let H_{2n+1} be the $(2n+1)$-dimensional Heisenberg group, given by real matrices:

$$H_{2n+1} = \left\{ \begin{pmatrix} 1 & x_1 & \dots & x_n & z \\ & 1 & & & y_1 \\ & & \ddots & & \vdots \\ & & & & y_n \\ & & & & 1 \end{pmatrix} : x_j, y_j, z \in \mathbb{R} \right\}.$$

The next corollary shows that given any left invariant metric on H_{2n+1}, there are no non-trivial solutions to the Killing–Yano equation.

Corollary 3.3. *If T is a left invariant Killing–Yano tensor on the $(2n+1)$-dimensional Heisenberg group H_{2n+1}, then T is trivial.*

Proof. Let $z \neq 0$ generate the center of the Lie algebra of H_{2n+1}. As a consequence of corollary 3.2, one has $j_{Tz} = 0 = j_z \circ T$ and since j_z is an isomorphism, $T = 0$. □

We consider next the complex Heisenberg group N with the standard left invariant Riemannian metric and complex structure. It is given by complex matrices

$$N = \left\{ \begin{pmatrix} 1 & x_1 + \mathrm{i}x_2 & x_5 + \mathrm{i}x_6 \\ & 1 & x_3 + \mathrm{i}x_4 \\ & & 1 \end{pmatrix} : x_j \in \mathbb{R} \right\},$$

with the left invariant metric

$$g = (\mathrm{d}x_5 - x_1\mathrm{d}x_3 + x_2\mathrm{d}x_4)^2 + (\mathrm{d}x_6 - x_2\mathrm{d}x_3 - x_1\mathrm{d}x_4)^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 + \mathrm{d}x_4^2 \tag{9}$$

and the bi-invariant complex structure, given by multiplication by i.

The Lie algebra $\mathfrak{n}$ of N has an orthonormal basis $\{e_1, \dots, e_6\}$ with Lie brackets given by

$$[e_1, e_3] = -[e_2, e_4] = e_5, \qquad [e_1, e_4] = [e_2, e_3] = e_6.$$

We recall (see [8]) that an $SU(3)$-structure (F, ψ_+) on a six-dimensional manifold is half-flat if $F \wedge \mathrm{d}F = 0$ and $\mathrm{d}\psi_+ = 0$. In the case of the Lie group N, the $SU(3)$-structure given by the Kähler form F and the 3-form ψ_+,

$$F = e^{12} + e^{34} + e^{56}, \qquad \psi_+ = \mathrm{d}e^5 \wedge e^5 - \mathrm{d}e^6 \wedge e^6,$$

is half-flat.

The skew-symmetric endomorphisms T of $\mathfrak{n}$ which give a solution of the Killing–Yano equation (equivalently, satisfy the hypotheses of theorem 3.1) are given as follows in the ordered basis $\{e_5, e_6, e_1, \dots, e_4\}$:

$$T = \begin{pmatrix} 0 & -3a & & & & \\ 3a & 0 & & & & \\ & & 0 & -a & & \\ & & a & 0 & & \\ & & & & 0 & -a \\ & & & & a & 0 \end{pmatrix}, \qquad a \in \mathbb{R}. \tag{10}$$

Corollary 3.4. *The nondegenerate* 2*-form given by*

$$\omega = 3(\mathrm{d}x_5 - x_1\,\mathrm{d}x_3 + x_2\,\mathrm{d}x_4) \wedge (\mathrm{d}x_6 - x_2\,\mathrm{d}x_3 - x_1\,\mathrm{d}x_4) + \mathrm{d}x_1 \wedge \mathrm{d}x_2 + \mathrm{d}x_3 \wedge \mathrm{d}x_4 \tag{11}$$

is a nondegenerate Killing–Yano form on the complex Heisenberg Lie group. Any other left invariant solution of equation (1) is of the form $a\omega$ with $a \in \mathbb{R}$.

The subgroup Γ of matrices in N with $x_j \in Z$ is discrete and co-compact. The six-dimensional compact manifold $M = \Gamma\backslash N$ is known as the Iwasawa manifold. Since both the metric g and the 2-form ω are left invariant, they descend to M and one has

Corollary 3.5. *The Iwasawa manifold with its standard half-flat metric carries a nondegenerate Killing–Yano tensor.*

Remark 3.6. We observe that the solution to the Killing–Yano equation given in (10) has an eigenvalue with multiplicity 2. Thus, the results in [17] cannot be applied in a direct manner. On the other hand, left invariant solutions of the Killing–Yano equation on a nilpotent Lie group are not parallel [1]. This is in contrast with results of [19] where it is proved that every Killing p-form on a compact quaternion-Kähler manifold has to be parallel for $p \geqslant 2$.

A family of 2-step nilpotent Lie algebras generalizing the real and complex Heisenberg Lie algebras is the class of Heisenberg-type Lie algebras first considered in [14]. We recall that a metric two-step nilpotent Lie algebra $(\mathfrak{n}, \langle\cdot,\cdot\rangle)$ with $\mathfrak{n}' = \mathfrak{z}$ is of Heisenberg type if the following condition is satisfied:

$$j_{z_1} j_{z_2} + j_{z_2} j_{z_1} = -\langle z_1, z_2\rangle Id \qquad \text{for any} \quad z_1, z_2 \in \mathfrak{z},$$

where j_z are given in (7). The corresponding simply connected Lie group with Lie algebra of Heisenberg type will be called the Heisenberg type Lie group.

In [6], the Riemann curvature tensor R, the Ricci tensor Ric and the scalar curvature τ of the Riemannian metric on a Heisenberg type Lie group are obtained in terms of the maps j_z given in (7). As a consequence, one can derive an expression for the Weyl tensor C. Moreover, it is shown in [6] that the Ricci tensor of any Heisenberg type Lie group is a Killing tensor, that is, $\nabla\,\mathrm{Ric}(X; X, X) = 0$ for all vector fields.

In the particular case of the complex Heisenberg Lie group, there are two natural left invariant Killing tensors. One is given by the square of the Killing–Yano tensor, and the other by the Ricci tensor. While the first one is positive definite, the second one has positive and negative eigenvalues. Indeed

$$g(T(z+v), T(z'+v')) = 9g(z,z') + g(v,v'),$$
$$\mathrm{Ric}(z+v, z'+v') = 2g(z,z') - g(v,v').$$

3.1. Higher-dimensional 2-step nilpotent examples

The next examples, obtained by applying (8), are natural extensions of the complex Heisenberg group carrying nondegenerate Killing–Yano tensors.

(1) Let $\mathfrak{n}$ be a vector space with an inner product $\langle\cdot,\cdot\rangle$ and an orthogonal decomposition $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$. Fix orthonormal bases of $\mathfrak{z}$ and $\mathfrak{v}$, respectively, $\mathfrak{z} = \text{span}\{z_1, z_2\}$, $\mathfrak{v} = \text{span}\{x_1, \ldots, x_{2n}, y_1, \ldots, y_{2n}\}$. Let T be the skew-symmetric endomorphism of $\mathfrak{n}$ whose matrix in the ordered basis $\{z_1, z_2, x_1, \ldots, x_{2n}, y_1, \ldots, y_{2n}\}$ is given by

$$T = \begin{pmatrix} 0 & -3 & & \\ 3 & 0 & & \\ & & 0 & -I \\ & & I & 0 \end{pmatrix},$$

where I is the identity $2n \times 2n$ matrix. We define a Lie bracket on $\mathfrak{n}$ as follows. Let A be a skew-symmetric $2n \times 2n$ matrix and set

$$j_{z_1} = \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} \qquad j_{z_2} = \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}.$$

It follows that j_z satisfies (8) for every $z \in \mathfrak{z}$; therefore, T is a Killing–Yano tensor on $(\mathfrak{n}, \langle\cdot,\cdot\rangle)$. Note that when $n = 1$ and $A = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, then $\mathfrak{n}$ is the Lie algebra of the complex Heisenberg group. More generally, if the matrix A above satisfies $A^2 = -I$, then $\mathfrak{n}$ is a Lie algebra of Heisenberg type (see [14]).

(2) Let $(\mathfrak{n}, \langle\cdot,\cdot\rangle)$ be a Lie algebra of Heisenberg type (see [14]) such that the center $\mathfrak{z}$ and its orthogonal complement $\mathfrak{v}$ have orthonormal bases $\{z_1, \ldots, z_m\}$ and $\{x_1, \ldots, x_n\}$, respectively. We set $\tilde{\mathfrak{n}} = \tilde{\mathfrak{z}} \oplus \tilde{\mathfrak{v}}$, where

$$\tilde{\mathfrak{z}} = \text{span}\{z_1, \ldots, z_m, w_1, \ldots, w_m\}, \qquad \tilde{\mathfrak{v}} = \text{span}\{x_1, \ldots, x_n, y_1, \ldots, y_n\},$$

and fix the inner product on $\tilde{\mathfrak{n}}$, still denoted by $\langle\cdot,\cdot\rangle$, such that $\tilde{\mathfrak{z}}$ is orthogonal to $\tilde{\mathfrak{v}}$ and the above bases are orthonormal. We define the Lie bracket on $\tilde{\mathfrak{n}}$ corresponding to the following endomorphism $\tilde{j} : \tilde{\mathfrak{z}} \to \mathfrak{so}(\tilde{\mathfrak{v}})$:

$$\widetilde{j_{z_k}} = \begin{pmatrix} j_{z_k} & 0 \\ 0 & -j_{z_k} \end{pmatrix}, \qquad \widetilde{j_{w_k}} = \begin{pmatrix} 0 & j_{z_k} \\ j_{z_k} & 0 \end{pmatrix}, \qquad 1 \leqslant k \leqslant m,$$

where $j : \mathfrak{z} \to \mathfrak{so}(\mathfrak{v})$ is the linear map defining the Lie bracket on $\mathfrak{n}$. Using (8), it turns out that the endomorphism T given by

$$T = \begin{pmatrix} 0 & -3I_m & & \\ 3I_m & 0 & & \\ & & 0 & -I_n \\ & & I_n & 0 \end{pmatrix}$$

is a Killing–Yano tensor on $(\tilde{\mathfrak{n}}, \langle\cdot,\cdot\rangle)$, where I_p is the identity $p \times p$ matrix. Note that the complex Heisenberg Lie algebra is of the form $\tilde{\mathfrak{n}}$ where $\mathfrak{n} = \mathfrak{h}_3$ is the three-dimensional Heisenberg Lie algebra.

4. The Killing–Yano equation on Lie groups with a non-negative curvature

In this section, we characterize in theorem 4.1 the Killing–Yano tensors on a Lie group with a flat metric. We also show that for any left invariant metric on $SU(2)$ there are no nontrivial solutions of the Killing–Yano equation (theorem 4.7). We note that it was shown in [24] that $SU(2)$ is the only simply connected Lie group with left invariant metrics of positive sectional curvature.

4.1. Flat Lie groups

Let G be a Lie group with Lie algebra $\mathfrak{g}$. We recall from [18] that an inner product $\langle \cdot , \cdot \rangle$ on $\mathfrak{g}$ induces a flat left invariant metric on G if and only if the following conditions are satisfied:

(i) there exists an Abelian ideal $\mathfrak{u}$ of $\mathfrak{g}$ such that its orthogonal complement $\mathfrak{a} = \mathfrak{u}^{\perp}$ is an Abelian subalgebra,

(ii) $ad(x)$ is skew-symmetric for any $x \in \mathfrak{a}$.

Theorem 4.1. *Let G be a Lie group with a flat left invariant metric whose Lie algebra $\mathfrak{g}$ decomposes $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{u}$ satisfying (i) and (ii) above. If T is a skew-symmetric endomorphism of $\mathfrak{g}$, then T is a Killing–Yano tensor if and only if $[Tx, z] = 0$ and $[ad(x), T] = 0$ for any $x, z \in \mathfrak{a}$.*

Proof. Let T be a skew-symmetric endomorphism of $\mathfrak{g}$. Studying the various cases to obtain the vanishing of α_T, one can show that

(1) $\alpha_T(x, y, z) = 0$ if $x, y, z \in \mathfrak{a}$ since $\mathfrak{a}$ is Abelian and $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{u}$,

(2) if $x, y, z \in \mathfrak{u}$ then $\alpha_T(x, y, z) = 0$ since $ad(a)$ is skew-symmetric for any $a \in \mathfrak{a}$,

(3) if $x, y \in \mathfrak{a}$ and $z \in \mathfrak{u}$, then $\alpha_T(x, y, z) = 0$ if and only if $[Tx, y] = [x, Ty]$,

(4) if $x, y \in \mathfrak{u}$ and $z \in \mathfrak{a}$, then $\alpha_T(x, y, z) = 0$ since $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{u}$ and $ad(a)$ is skew-symmetric for any $a \in \mathfrak{a}$,

(5) if $x \in \mathfrak{a}$ and $y, z \in \mathfrak{u}$, then $\alpha_T(x, y, z) = 0$ if and only if $(T[x, z])_{\mathfrak{u}} = [x, Tz]$,

(6) if $x \in \mathfrak{u}$ and $y, z \in \mathfrak{a}$, then $\alpha_T(x, y, z) = 0$ if and only if $[y, Tz] = 0$.

Note that if (6) holds, that is, $[y, Tz] = 0$ for all $y, z \in \mathfrak{a}$, then (3) holds. Moreover, if (6) holds then $(T[x, z])_{\mathfrak{a}} = 0$, if $x \in \mathfrak{a}$ and $z \in \mathfrak{u}$, since $\langle T[x, z], y\rangle = \langle z, [x, Ty]\rangle = 0$. Thus, a solution T of the Killing–Yano equation exists if and only if $[Tx, z] = 0$ and $[ad(x), T] = 0$ for any $x, z \in \mathfrak{a}$ as claimed. □

Remark 4.2. As a consequence of theorem 4.1, it turns out that any $T = T_1 + T_2$, with T_1 a skew-symmetric endomorphism of $\mathfrak{a}$ and T_2 a skew-symmetric endomorphism of $\mathfrak{u}$ commuting with $ad(x)$ for all $x \in \mathfrak{a}$ (take, for example, $T_2 = ad(y)$ for a fixed $y \in \mathfrak{a}$) will provide a left invariant 2-form which gives a solution of equation (1).

Corollary 4.3. *Any flat left invariant metric on a Lie group G admits a Killing–Yano 2-form.*

Remark 4.4. As a consequence of corollary 4.3, it turns out that the Euclidean Lie algebra $\mathfrak{e}(2)$ admits a solution of the Killing–Yano equation, which is degenerate since this Lie algebra is three dimensional.

Proposition 4.5. $\mathfrak{e}(2) \times \mathfrak{e}(2)$ *carries a 3-parameter family of nondegenerate Killing–Yano 2-forms.*

Proof. We consider the Lie subgroup of the isometry group of $\mathbb{R}^4$ given by the semidirect product of a maximal torus T^2 in $SO(4)$ with $\mathbb{R}^4$. The Lie algebra is isomorphic to $\mathfrak{e}(2) \times \mathfrak{e}(2)$. Set $\{e_1, e_2, e_3, e_4\}$ as an orthonormal basis of $\mathbb{R}^4$ and $\{e_5, e_6\}$ as an orthonormal basis of $\mathbb{R}^2$, the Lie algebra of T^2. The Lie bracket is given by

$$[e_5, e_1] = e_2, \qquad [e_5, e_2] = -e_1, \qquad [e_6, e_3] = e_4, \qquad [e_6, e_4] = -e_3.$$

The family of skew-symmetric transformations, given in the basis $\{e_5, e_6, e_1, e_2, e_3, e_4\}$ by

$$T_{r,s,t} = \begin{pmatrix} 0 & -r & & & & \\ r & 0 & & & & \\ & & 0 & -s & & \\ & & s & 0 & & \\ & & & & 0 & -t \\ & & & & t & 0 \end{pmatrix},$$

provide left invariant 2-forms that are solutions of equation (1). They are non-degenerate if $rst \neq 0$. □

4.2. The 3-sphere SU(2)

We recall that a left invariant metric on a Lie group G is called bi-invariant if right translations are also isometries. This implies that $ad(x)$ is a skew-symmetric endomorphism of $\mathfrak{g}$ for all $x \in \mathfrak{g}$. It is well known that a connected Lie group G carries a bi-invariant metric if and only if it is a direct product of a connected compact Lie group with R^k (see, for example, [18]).

The proof of the next lemma follows as a direct application of equation (5).

Lemma 4.6. *Let G be a Lie group with a bi-invariant metric. If T is a skew-symmetric endomorphism of* $\mathfrak{g}$*, then* $\nabla T(X; X) = 0$ *for all left invariant vector fields if and only if* $T|[\mathfrak{g}, \mathfrak{g}] = 0$*.*

In particular, a compact semisimple Lie group has no non-trivial solution to the Killing–Yano equation.

In the particular case of $SU(2)$, we prove next a stronger version of lemma 4.6. In fact, it follows that the solutions of the Killing–Yano equation are trivial for any left invariant metric on $SU(2)$, not just the bi-invariant one.

Theorem 4.7. *For any left invariant metric g on* $SU(2)$*, the solutions of the Killing–Yano equation are trivial.*

Proof. Let g_0 be the bi-invariant metric on $SU(2)$, that is, ad_x is skew-symmetric on its Lie algebra $\mathfrak{su}(2)$, let g be any left invariant metric and P be a positive definite matrix such that $g(u, v) = g_0(Pu, v)$. Let $\{x, y, z\}$ be an orthonormal basis of $\mathfrak{su}(2)$ such that $P(x) = \alpha x,\ P(y) = \beta y$ and $P(z) = \gamma z$. One verifies easily that

$$[x, y] = cz, \qquad [y, z] = ax, \qquad [z, x] = by, \tag{12}$$

where $c = \gamma\eta$, $a = \alpha\eta$, $b = \beta\eta$ and $\eta = g_0([x, y], z) = g_0([y, z], x) = g_0([z, x], y)$.

Let T be a skew-symmetric transformation whose matrix with respect to the orthonormal basis $\{x, y, z\}$ is given by

$$T = \left\{ \begin{pmatrix} 0 & -a_{2,1} & -a_{3,1} \\ a_{2,1} & 0 & -a_{3,2} \\ a_{3,1} & a_{3,2} & 0 \end{pmatrix} : a_{i,j} \in \mathbb{R} \right\}.$$

We apply equation (5) and compute

(1) $\alpha(x, x, z) = 2a_{2,1}(a - b - c)$ and $\alpha(y, y, z) = 2a_{2,1}(-a + b - c)$. If $a_{2,1} \neq 0$, then $c = 0$, which is impossible. Hence, $a_{2,1} = 0$.

(2) $\alpha(x, x, y) = 2a_{3,1}(-a + b + c)$ and $\alpha(z, z, y) = 2a_{3,1}(a + b - c)$. If $a_{3,1} \neq 0$, then $b = 0$, which is impossible. Hence, $a_{3,1} = 0$.

(3) $\alpha(y, y, x) = 2a_{3,2}(-a + b - c)$ and $\alpha(z, z, x) = 2a_{3,2}(-a - b + c)$. If $a_{3,2} \neq 0$, then $a = 0$, which is impossible. Hence, $a_{3,2} = 0$.

Thus, $T = 0$, as claimed. □

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