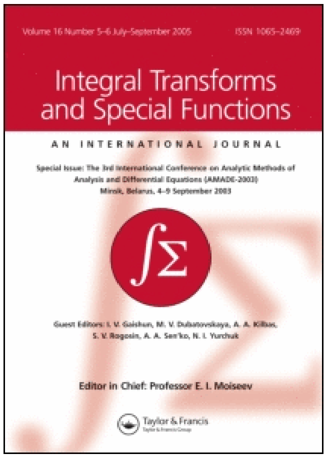
On the n-dimensional Hankel transforms of arbitrary order

Article *in* Integral Transforms and Special Functions · May 2008
DOI: 10.1080/10652460701856029



Integral Transforms and Special Functions
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713643686

On the **n**-dimensional Hankel transforms of arbitrary order

Sandra Molina [a]; Susana Elena Trione [b]
[a] Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad Nacional de Mar del Plata, Funes, Mar del Plata, Argentina
[b] Departmento de Matemática, Facultad de Ciencias Exactas y Naturales, UBA, IAM-CONICET, Saavedra, Piso, Buenos Aires, Argentina

Online Publication Date: 01 January 2008

To cite this Article: Molina, Sandra and Trione, Susana Elena (2008) 'On the **n**-dimensional Hankel transforms of arbitrary order', Integral Transforms and Special Functions, 19:5, 327 — 332

To link to this article: DOI: 10.1080/10652460701856029
URL: http://dx.doi.org/10.1080/10652460701856029

Integral Transforms and Special Functions
Vol. 19, No. 5, May 2008, 327–332

On the n-dimensional Hankel transforms of arbitrary order

Sandra Molina[a*] and Susana Elena Trione[b]

[a] *Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad Nacional de Mar del Plata, Funes, Mar del Plata, Argentina;* [b] *Departmento de Matemática, Facultad de Ciencias Exactas y Naturales, UBA, IAM-CONICET, Saavedra, Piso, Buenos Aires, Argentina*

(*Received 24 August 2007*)

In S.E. Trione and S. Molina, *The n-dimensional Hankel transform and complex powers of Bessel operator*, to appear in Integral Transforms and Spec. Funct. we study a version of the n-dimensional Hankel transform h_μ for $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n$ and $\mu_i \geq -1/2$ for all i. In this paper, we extend the n-dimensional Hankel transform to arbitrary values of $\mu \in \mathbb{R}^n$. Moreover, we obtain some results for the inverse of this extension.

Keywords: Hankel transform; inversion theorem

2000 Mathematics Subject Classification: 46F12; 44A15

1. Introduction

In [3], we study the n-dimensional Hankel transformation on the spaces $\mathcal{H}_\mu$, defined by

$$(h_\mu \phi)(y) = \int_{\mathbb{R}^n_+} \phi(x_1, \ldots, x_n) \prod_{i=1}^{n} \{\sqrt{x_i y_i} J_{\mu_i}(x_i y_i)\} \, \mathrm{d}x_1 \cdots \mathrm{d}x_n, \tag{1}$$

where J_{μ_i} is the Bessel function of the first kind and of order μ_i, given by

$$J_{\mu_i}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{\mu_i + 2k}}{k!\Gamma(\mu_i + k + 1)}.$$

$\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n$, $\mu \geq -1/2$, ($\mu_i \geq -1/2$ for $i = 1, \ldots, n$). In [4], Zemanian extended the one-dimensional Hankel transform to order μ for negative real values of μ by

$$h_{\mu,k}(\phi) = (-1)^k x^{-k} h_{\mu+k} N_{\mu+k-1} \cdots N_{\mu+1} N_\mu \phi, \tag{2}$$

where k is a positive integer; its inverse is extended by

$$h^{-1}_{\mu,k}(\phi) = (-1)^k N^{-1}_\mu N^{-1}_{\mu+1} \cdots N^{-1}_{\mu+k-1} h_{\mu+k} x^k \phi.$$

*Corresponding author. Email: smolina@mdp.edu.ar

ISSN 1065-2469 print/ISSN 1476-8291 online

DOI: 10.1080/10652460701856029
http://www.informaworld.com

Moreover, the transformation (2) verifies the same property of the classic Hankel transformation as $h_{\mu,k} = h^{-1}_{\mu,k}$, (see [1]).

In this note, we extend the n-dimensional Hankel transform to arbitrary values of $\mu \in \mathbb{R}^n$. Moreover, we obtain that $h_{\mu,k} = h^{-1}_{\mu,k}$ for the n-dimensional case.

2. Notation and preliminares

We use $\mathbb{N}_0$ to denote the set $\mathbb{N} \cup \{0\}$. For $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ we put $\|x\| = (x_1^2 + \cdots + x_n^2)^{1/2}$, and $x \leq y$ means $x_i \leq y_i, i = 1, \dots, n$. If $k \in \mathbb{N}_0^n, k = (k_1, \dots, k_n)$, we put $\|k\| = k_1 + \cdots + k_n$.

If $x, m \in \mathbb{R}^n$, $x = (x_1, \dots, x_n)$ and $m = (m_1, \dots, m_n)$, we set $x^m = x_1^{m_1} \cdots x_n^{m_n}$. If $k \in \mathbb{N}_0^n$, we use D^k and T_i to denote the operators

$$D^k = \frac{\partial^{|k|}}{\partial x_1^{k_1} \partial x_2^{k_2} \cdots \partial x_n^{k_n}}$$

and

$$T_i = x_i^{-1} \frac{\partial}{\partial x_i}$$

for $i = 1, \dots, n$. T_i verifies that $T_i T_j = T_j T_i$ and $T_i^n T_j^m = T_j^m T_i^n$ for any positive integers n, m. For $k \in \mathbb{N}_0^n$ we shall write $T^k = T_n^{k_n} T_{n-1}^{k_{n-1}} \cdots T_1^{k_1}$.

The n-dimensional Hankel transform (1) was studied over the spaces $\mathcal{H}_\mu$ and $\mathcal{H}'_\mu$ in [2,3], and we repeat the relevant material. Let $\mathbb{R}_+^n = (0, \infty) \times (0, \infty) \times \cdots \times (0, \infty)$ and μ an n-tuple of real numbers $\mu = (\mu_1, \mu_2, \dots, \mu_n)$. We define the space $\mathcal{H}_\mu$ as follows:

$$\mathcal{H}_\mu = \left\{ \phi \in C^\infty(\mathbb{R}_+^n) : \sup_{x \in \mathbb{R}_+^n} \left| x^m T^k \left\{ x^{-\mu-1/2} \phi(x) \right\} \right| < \infty, \ \forall m, k \in \mathbb{N}_0^n \right\},$$

where $-\mu - 1/2 = (-\mu_1 - 1/2, -\mu_2 - 1/2, \dots, -\mu_n - 1/2)$.

$\mathcal{H}_\mu$ endowed with the topology generated by the family of seminorms $\{\gamma^\mu_{m,k}\}_{m,k \in \mathbb{N}_0^n}$ defined by $\gamma^\mu_{m,k}(\phi) = \sup_{x \in \mathbb{R}_+^n} | x^m T^k \{x^{-\mu-1/2} \phi(x)\} |$ is a Frechét space. Thus, the dual space $\mathcal{H}'_\mu$ is also complete.

THEOREM 2.1 *The Hankel transform defined by* (1) *is an automorphism onto* $\mathcal{H}_\mu$ *for* $\mu \geq -1/2$.

For $\mu \geq -1/2$, we extend the de Hankel transform to the space $\mathcal{H}'_\mu$ as the adjoint of h_μ, as follows:

$$(h'_\mu f, \phi) = (f, h_\mu \phi), \quad (f \in \mathcal{H}'_\mu), \ (\phi \in \mathcal{H}_\mu). \tag{3}$$

Then, h'_μ is an automorphism onto $\mathcal{H}'_\mu$ and $h'_\mu = (h'_\mu)^{-1}$ because $h_\mu = (h_\mu)^{-1}$.

LEMMA 2.1 *If* $r \in \mathbb{Z}^n$ *and* $\mu \in \mathbb{R}^n$, *the function given by* $h(\phi) = x^r \phi$ *is an isomorphism from* $\mathcal{H}_\mu$ *onto* $\mathcal{H}_{\mu+r}$. *Moreover, the operator given by* $h'(f) = x^r f$ *for* $f \in \mathcal{H}'_{\mu+r}$, *where* $x^r f \in \mathcal{H}'_\mu$ *is defined by*

$$(x^r f, \phi) = (f, x^r \phi), \quad (\phi \in \mathcal{H}_\mu)$$

is an isomorphism from $\mathcal{H}'_{\mu+r}$ *onto* $\mathcal{H}'_\mu$.

Let $\mu \in \mathbb{R}^n$, $\mu = (\mu_1, \dots, \mu_n)$. We define the operators $N_{\mu,i}$, $M_{\mu,i}$ and $N_{\mu,i}^{-1}$ as

$$N_{\mu,i}\phi = x_i^{\mu_i+1/2}\frac{\partial}{\partial x_i}\left\{x_i^{-\mu_i-1/2}\phi(x)\right\},$$

$$M_{\mu,i}\phi = x_i^{-\mu_i-1/2}\frac{\partial}{\partial x_i}\left\{x_i^{\mu_i+1/2}\phi(x)\right\},$$

$$N_{\mu,i}^{-1}\phi = x_i^{\mu_i+1/2}\int_\infty^{x_i} t^{-\mu_i-1/2}\phi(x_1, \dots, t, \dots, x_n)\,\mathrm{d}t.$$

LEMMA 2.2 *For $i = 1, \dots, n$, $N_{\mu,i} : \mathcal{H}_\mu \to \mathcal{H}_{\mu+e_i}$ are continuous linear mappings. Also, $N_{\mu,i}$ is an isomorphism from $\mathcal{H}_\mu$ onto $\mathcal{H}_{\mu+e_i}$.*

LEMMA 2.3 *$M_{\mu,i} : \mathcal{H}_{\mu+e_i} \to \mathcal{H}_\mu$ are linear continuous operators.*

LEMMA 2.4 *Let $\mu \geq -1/2$, $\phi \in \mathcal{H}_\mu$ and let $e_1, \dots, e_n$ be the canonical basis of $\mathbb{R}^n$ and h_μ given by* (1). *Then*

(a) $h_{\mu+e_i}(-x_i\phi) = N_{\mu,i}h_\mu(\phi)$,
(b) $h_{\mu+e_i}(N_{\mu,i}\phi) = -y_i h_\mu(\phi)$,
(c) $h_\mu(-x_i^2\phi) = M_{\mu,i}N_{\mu,i}h_\mu(\phi)$,
(d) $h_\mu(M_{\mu,i}N_{\mu,i}\phi) = -y_i^2 h_\mu(\phi)$.
If $\phi \in \mathcal{H}_{\mu+e_i}$, then
(e) $h_\mu(x_i\phi) = M_{\mu,i}h_{\mu+e_i}(\phi)$,
(f) $h_\mu(M_{\mu,i}\phi) = y_i h_{\mu+e_i}(\phi)$.

Remark 2.1 The last lemma is valid for h'_μ given by (3) and $f \in \mathcal{H}'_\mu$.

For $\mu \in \mathbb{R}^n$ and $m \in \mathbb{N}$, we consider the following operators:

$$N_{\mu,i}^m = N_{\mu+(m-1)e_i,i}\;\; N_{\mu+(m-2)e_i,i}\cdots N_{\mu+e_i,i}\;\; N_{\mu,i}. \tag{4}$$

We put

$$N_{\mu,i}^0\phi = \phi, \quad \phi \in \mathcal{H}_\mu, \tag{5}$$

and

$$(N_{\mu,i}^m)^{-1} = (N_{\mu,i})^{-1}(N_{\mu+e_i,i})^{-1}\cdots(N_{\mu+(m-2)e_i,i})^{-1}(N_{\mu+(m-1)e_i,i})^{-1}. \tag{6}$$

Remark 2.2 By Lemma 2.2, $N_{\mu,i}^m\phi \in \mathcal{H}_{\mu+me_i}$ for $\phi \in \mathcal{H}_\mu$. Moreover, $N_{\mu,i}^m$ is an isomorphism onto $\mathcal{H}_{\mu+me_i}$.

For $\mu \in \mathbb{R}^n$, $m \in \mathbb{N}_0^n$, and $m = (m_1, \dots, m_n)$, we define

$$N_\mu^m = N_{\mu+m_1e_1+\cdots+m_{n-1}e_{n-1},n}^{m_n}\dots N_{\mu+m_1e_1,2}^{m_2}\,N_{\mu,1}^{m_1} \tag{7}$$

and

$$(N_\mu^m)^{-1} = (N_{\mu,1}^{m_1})^{-1}(N_{\mu+m_1e_1,2}^{m_2})^{-1}\cdots(N_{\mu+m_1e_1+\cdots+m_{n-1}e_{n-1},n}^{m_n})^{-1}, \tag{8}$$

where $N_{\mu',i}^{m'}$ is defined by (4) and (5), and $(N_{\mu',i}^{m'})^{-1}$ is defined by (6).

Remark 2.3 By Remark 2.2 we obtain that for $\phi \in \mathcal{H}_\mu$ and $m \in \mathbb{N}_0^n$, $N_\mu^m\phi \in \mathcal{H}_{\mu+m}$. Moreover, N_μ^m is an isomorphism onto $\mathcal{H}_{\mu+m}$.

3. The Hankel transformation of arbitrary order

Let $\mu \in \mathbb{R}^n$ and $k \in \mathbb{N}_0^n$ such that $\mu + k \geq -1/2$. We define the following transformations on $\mathcal{H}_\mu$:

$$h_{\mu,k}(\phi) = (-1)^{|k|} y^{-k} h_{\mu+k}(N_\mu^k \phi), \quad (\phi \in \mathcal{H}_\mu) \tag{9}$$

and

$$h_{\mu,k}^{-1}(\phi) = (-1)^{|k|} (N_\mu^k)^{-1} h_{\mu+k}(x^k \phi), \quad (\phi \in \mathcal{H}_\mu), \tag{10}$$

where $h_{\mu+k}$ is defined by (1).

Under the conditions stated above we have the following lemma.

LEMMA 3.1 (1) *$h_{\mu,k}(\phi)$ defined by* (9) *is an automorphism in $\mathcal{H}_\mu$, for all $\mu \in \mathbb{R}^n$,*
(2) *Its inverse $h_{\mu,k}^{-1}(\phi)$ is defined by* (10),
(3) *If $\mu \geq -1/2$ then h_μ defined by* (1) *coincides with $h_{\mu,k}$.*

Proof (1) The statement follows immediately from Remark 2.3, Lemma 2.1 and Theorem 2.1.
(2) The statement follows immediately from (9) and (10).
(3) Let $\mu \geq -1/2$, $\phi \in \mathcal{H}_\mu$ and $k \in \mathbb{N}_0^n$, $k = (k_1, \ldots, k_n)$. Then

$$h_{\mu+k}(N_\mu^k \phi) = (-y_1)^{k_1} \cdots (-y_n)^{k_n} h_\mu(\phi). \tag{11}$$

In fact, from (b) of Lemma 2.4 and by induction on $l \in \mathbb{N}$ we obtain that

$$h_{\mu+le_i}(N_{\mu,i}^l \phi) = (-y_i)^l h_\mu(\phi). \tag{12}$$

Next, applying (12), we obtain

$$\begin{aligned}
&h_{\mu+k}(N_\mu^k\{\phi\}) \\
&\quad = h_{\mu+k_1e_1+\cdots+k_ne_n}(N_{\mu+k_1e_1+\cdots+k_{n-1}e_{n-1},n}^{k_n} \cdots N_{\mu+k_1e_1,2}^{k_2}\, N_{\mu,1}^{k_1}\{\phi\}) \\
&\quad = (-y_n)^{k_n} h_{\mu+k_1e_1+\cdots+k_{n-1}e_{n-1}}(N_{\mu+k_1e_1+\cdots+k_{n-2}e_{n-2},n-1}^{k_{n-1}} \cdots N_{\mu+k_1e_1,2}^{k_2}\, N_{\mu,1}^{k_1}\{\phi\}) \\
&\quad = (-y_n)^{k_n}(-y_{n-1})^{k_{n-1}} h_{\mu+k_1e_1+\cdots+k_{n-2}e_{n-2}}(N_{\mu+k_1e_1+\cdots+k_{n-3}e_{n-3},n-2}^{k_{n-2}} \cdots N_{\mu+k_1e_1,2}^{k_2}\, N_{\mu,1}^{k_1}\{\phi\}).
\end{aligned}$$

By repeated application of this step, we obtain (11). If $\mu \geq -1/2$, by (9) and (11), we conclude that

$$h_{\mu,k}(\phi) = (-1)^{|k|} y^{-k} h_{\mu+k}(N_\mu^k \phi) = (-1)^{|k|} y^{-k} (-y_1)^{k_1} \cdots (-y_n)^{k_n} h_\mu(\phi) = h_\mu(\phi). \qquad \blacksquare$$

LEMMA 3.2 *Let $\phi \in \mathcal{H}_\mu$ and $m \in \mathbb{N}_0^n$, then*

$$N_\mu^m \phi = x^{\mu+m+(1/2)} T^m \{x^{-\mu-(1/2)} \phi(x)\}. \tag{13}$$

Proof See [3]. $\blacksquare$

COROLLARY 3.1 *Let $m, k \in \mathbb{N}_0^n$. Then*

$$N_{\mu+k}^m N_\mu^k = N_\mu^{m+k}. \tag{14}$$

Proof Let $\phi \in \mathcal{H}_\mu$; then

$$\begin{aligned}N^m_{\mu+k}N^k_\mu\phi &= x^{\mu+k+m+1/2}T^m\{x^{-\mu-k-1/2}N^k_\mu\phi(x)\}\\ &= x^{\mu+k+m+1/2}T^m\{x^{-\mu-k-1/2}x^{\mu+k+1/2}T^k\{x^{-\mu-1/2}\phi(x)\}\}\\ &= N^{m+k}_\mu\phi.\end{aligned}$$ ■

LEMMA 3.3 *Let $k, m \in \mathbb{N}^n_0$ such that $\mu + k \geq -1/2$ and $\mu + m \geq -1/2$. Then $h_{\mu,k} = h_{\mu,m}$ and $h^{-1}_{\mu,k} = h^{-1}_{\mu,m}$.*

Proof Assuming that $m > k$, we have, from Lemma 3.1 and $\mu + k \geq -1/2$, that

$$h_{\mu+k,m-k} = h_{\mu+k}.$$

Let $\phi \in \mathcal{H}_\mu$; then

$$\begin{aligned}h_{\mu,m}(\phi) &= (-1)^{|m|}x^{-m}h_{\mu+m}(N^m_\mu\phi)\\ &= (-1)^{|k|}x^{-k}(-1)^{|m-k|}x^{-(m-k)}x^{-k}h_{\mu+k+m-k}(N^{m-k}_{\mu+k}N^k_\mu\phi)\\ &= (-1)^{|k|}x^{-k}h_{\mu+k}(N^k_\mu\phi) = h_{\mu,k}.\end{aligned}$$

From the last equality, we immediately deduce that $h^{-1}_{\mu,k} = h^{-1}_{\mu,m}$ for $\mu + k \geq -1/2$ and $\mu + m \geq -1/2$. ■

Now, we can define h_μ and h^{-1}_μ on $\mathcal{H}_\mu$ for $\mu < -1/2$ as

$$h_\mu(\phi) = h_{\mu,k}(\phi), \quad (\phi \in \mathcal{H}_\mu) \tag{15}$$

and

$$h^{-1}_\mu(\phi) = h^{-1}_{\mu,k}(\phi), \quad (\phi \in \mathcal{H}_\mu), \tag{16}$$

where $k \in \mathbb{N}^n_0$ such that $\mu + k \geq -1/2$. We know that $h_\mu = h^{-1}_\mu$ for $\mu \in \mathbb{R}^n$, $\mu \geq -1/2$.

We will prove that this equality is valid for $\mu < -1/2$.

LEMMA 3.4 *Let $\phi \in \mathcal{H}_\mu$, $\mu \in \mathbb{R}^n$, and $k \in \mathbb{N}^n_0$; then we have*

(a) $N_{\mu+k,i}\{x^k\phi\} = x^kN_{\mu,i}\phi$ *for* $1 \leq i \leq n$.
(b) $N^m_{\mu+k,i}\{x^k\phi\} = x^kN^m_{\mu,i}\phi$ *for* $1 \leq i \leq n$ *and* $m \in \mathbb{N}_0$.
(c) $N^m_{\mu+k}\{x^k\phi\} = x^kN^m_\mu\phi$ *for all* $m \in \mathbb{N}^n_0$.

Proof See [3]. ■

LEMMA 3.5 *Let $\phi \in \mathcal{H}_\mu$, $k \in \mathbb{N}^n_0$, and $k = (k_1, \ldots, k_n)$ such that $\mu + k \geq -1/2$. Then*

(a) $N_{\mu,i}\, h_{\mu,k}(\phi) = h_{\mu+e_i,k}(-x_i\phi)$.
(b) $N^{k_i}_{\mu,i}\, h_{\mu,k}(\phi) = (-1)^{k_i}h_{\mu+k_ie_i,k}x^{k_i}_i\,(\phi)$.
(c) $N^k_\mu h_{\mu,k}(\phi) = (-1)^{|k|}h_{\mu+k}x^k(\phi)$.

Proof (a) By Lemma 3.4(c) and lemma 2.4(a), we obtain

$$h_{\mu+e_i,k}(-x_i\phi) = (-1)^{|k|}x^{-k}h_{\mu+e_i+k}\left(N^k_{\mu+e_i}(-x_i\phi)\right)$$
$$= (-1)^{|k|}x^{-k}h_{\mu+e_i+k}\left(-x_iN^k_\mu\phi\right) = (-1)^{|k|}x^{-k}N_{\mu+k,i}\ h_{\mu+k}\left(N^k_\mu\phi\right). \quad (17)$$

By Lemma 3.4(a), we have for $\Phi \in \mathcal{H}_{\mu+k}$ that

$$N_{\mu,i}\{x^{-k}\Phi\} = x^{-k}N_{\mu+k,i}\Phi. \quad (18)$$

Putting $\Phi = h_{\mu+k}\left(N^k_\mu\phi\right)$ we obtain from (17) and (18) that

$$(-1)^{|k|}x^{-k}N_{\mu+k,i}\ h_{\mu+k}\left(N^k_\mu\phi\right) = (-1)^{|k|}N_{\mu,i}\ \ x^{-k}h_{\mu+k}\left(N^k_\mu\phi\right) = N_{\mu,i}\ h_{\mu,k}(\phi). \quad (19)$$

Next, (a) follows by (17) and (19).
(b) By applying (a) $k_i - 1$ times, we obtain

$$N^{k_i}_{\mu,i}h_{\mu,k}(\phi) = N_{\mu+(k_i-1)e_i,i}\ \ N_{\mu+(k_i-2)e_i,i}\cdots N_{\mu+e_i,i}\ \ N_{\mu,i}\ h_{\mu,k}(\phi)$$
$$= N_{\mu+(k_i-1)e_i,i}\ \ N_{\mu+(k_i-2)e_i,i}\cdots N_{\mu+e_i,i}\ h_{\mu+e_i,k}(-x_i\phi)$$
$$= N_{\mu+(k_i-1)e_i,i}\ \ N_{\mu+(k_i-2)e_i,i}\cdots N_{\mu+2e_i,i}\ h_{\mu+2e_i,k}\left((-1)^2x_i\phi\right)$$
$$= \cdots = (-1)^{k_i}h_{\mu+k_ie_i,k}\left(x_i^{k_i}\phi\right).$$

(c) From (7) and applying (b) n times, we obtain

$$N^k_\mu\ h_{\mu,k}(\phi) = N^{k_n}_{\mu+k_1e_1+\cdots+k_{n-1}e_{n-1},n}\cdots N^{k_2}_{\mu+k_1e_1,2}\ \ N^{k_1}_{\mu,1}\ h_{\mu,k}(\phi)$$
$$= N^{k_n}_{\mu+k_1e_1+\cdots+k_{n-1}e_{n-1},n}\cdots N^{k_2}_{\mu+k_1e_1,2}(-1)^{k_1}h_{\mu+k_1e_1,k}\left(x_1^{k_1}\phi\right)$$
$$= N^{k_n}_{\mu+k_1e_1+\cdots+k_{n-1}e_{n-1},n}\cdots N^{k_3}_{\mu+k_1e_1+k_2e_2,3}\ (-1)^{k_1}(-1)^{k_2}h_{\mu+k_1e_1+k_2e_2,k}\left(x_1^{k_1}x_2^{k_2}\phi\right)$$
$$= \cdots = (-1)^{|k|}h_{\mu+k}\left(x^k\phi\right).$$

■

THEOREM 3.1 *Let $\mu \in \mathbb{R}^n$, and $k, m \in \mathbb{N}^n_0$ such that $\mu + k \geq -1/2$. Then $h_{\mu,k} = h^{-1}_{\mu,k}$.*

Proof In view of (c) of Lemma 3.5, we have

$$N^k_\mu h_{\mu,k}(\phi) = (-1)^{|k|}h_{\mu+k}\left(x^k\phi\right);$$

then

$$h_{\mu,k}(\phi) = (-1)^{|k|}(N^k_\mu)^{-1}h_{\mu+k}\left(x^k\phi\right) = h^{-1}_{\mu,k}(\phi).$$

■

References

[1] E.L. Koh and C.K. Li, *On the inverse of Hankel transform*, Integral Transforms Spec. Funct. 2(4) (1994), pp. 279–282.
[2] S. Molina, *A generalization of the spaces $\mathcal{H}_\mu$ and $\mathcal{H}'_\mu$ and the space of multipliers*, Actas del VII Congreso Dr. Antonio Monteiro, Departmento de Matemática - Instituto de Matemática, Universidad Nacional de Sur, Bahia Blanca, Argentina, 2003, pp. 49–56.
[3] S.E. Trione and S. Molina, *The n-dimensional Hankel transform and complex powers of Bessel operator*, to appear in Integral Transforms Spec. Funct.
[4] A.H. Zemanian, *Hankel transforms of arbitrary order*, Duke Math. J. 34(4) (1967), pp. 761–769.