J. Math. Anal. Appl. 348 (2008) 581–588

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Bessel fusion multipliers

M. Laura Arias [a,*], Miriam Pacheco [b]

[a] Instituto Argentino de Matemática, Saavedra 15 piso 3 (1083), Buenos Aires, Argentina
[b] Facultad de Ingeniería, Universidad Nacional de la Patagonia San Juan Bosco, Comodoro Rivadavia, Argentina

ARTICLE INFO

Article history:
Received 5 May 2008
Available online 30 July 2008
Submitted by P.G. Lemarie-Rieusset

Keywords:
Bessel multiplier
Bessel fusion sequence
Bessel fusion multiplier

ABSTRACT

In this paper we study properties of a Bessel multiplier when the symbol involved belongs to l^p. Furthermore, we introduce the concept of Bessel fusion multiplier which generalizes a Bessel multiplier for Bessel fusion sequences. We study their behavior when the symbol belongs to l^p and some continuity properties.

1. Introduction

Let $\mathcal{H}$ be a separable Hilbert space. A sequence $(\psi_k)_{k\in\mathbb{N}}$ in $\mathcal{H}$ is called a *Bessel sequence* if there exists a positive constant B for which

$$\sum_{k=1}^{\infty}\left|\langle\xi,\psi_k\rangle\right|^2\leqslant B\|\xi\|^2,$$

for all $\xi\in\mathcal{H}$. A Bessel sequence $(\psi_k)_{k\in\mathbb{N}}$ is called a *frame* if there exists a constant $A>0$ such that

$$A\|\xi\|^2<\sum_{k=1}^{\infty}\left|\langle\xi,\psi_k\rangle\right|^2,$$

for every $\xi\in\mathcal{H}$. In the sequel, B_ψ will denote the optimal bound of the Bessel sequence $\psi=(\psi_k)_{k\in\mathbb{N}}$. We will use frequently the fact that $\|\psi_k\|<B_\psi^{1/2}$ for every $k\in\mathbb{N}$. Bessel sequences and, in particular, frames have been extensively studied during the last decades due to their multiple applications in different areas, e.g., signal and image processing, filter bank theory, and so on. The reader will find many relevant results and facts on frame theory in the books by I. Daubechies [8] and by O. Christensen [7]. For the relationship between Bessel sequences and bounded linear operators the reader is referred to [1].

Associated to any Bessel sequence $\psi=(\psi_k)_{k\in\mathbb{N}}$ are the *analysis operator* defined by

$$C_\psi:\mathcal{H}\to l^2,\quad C_\psi(\xi)=\big(\langle\xi,\psi_k\rangle\big)_{k\in\mathbb{N}},$$

and the *synthesis operator* defined by

$$D_\psi:l^2\to\mathcal{H},\quad D_\psi\big((c_k)_{k\in\mathbb{N}}\big)=\sum_{k=1}^{\infty}c_k\psi_k.$$

* Corresponding author.
E-mail address: ml_arias@uolsinectis.com.ar (M.L. Arias).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2008.07.056

These are everywhere-defined bounded linear operators, each adjoint to each other. Moreover, $\|C_\psi\| = \|D_\psi\| \leqslant B_\psi^{1/2}$.

Given two Bessel sequences, $(\phi_k)_{k\in\mathbb{N}}$ and $(\psi_k)_{k\in\mathbb{N}}$ in $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, and a fixed sequence, $m = (m_k)_{k\in\mathbb{N}} \in l^\infty$, the following operator can be defined:

$$M_{m(\psi_k)(\phi_k)} : \mathcal{H}_1 \to \mathcal{H}_2, \quad M_{m(\psi_k)(\phi_k)}(\xi) = \sum_{k\in\mathbb{N}} m_k \langle \xi, \phi_k\rangle \psi_k.$$

This operator, in which the analysis coefficients are rescaled by the fixed weight before resynthesis, is called the *Bessel multiplier* for $(\phi_k)_{k\in\mathbb{N}}$ and $(\psi_k)_{k\in\mathbb{N}}$. It was introduced in [2] and its properties when $m \in c_0, l^1$ and l^2 were also studied there. In the present paper we extend these results for $m \in l^p$ with $1 \leqslant p < \infty$. Bessel multipliers and, in particular, frame multipliers have useful applications. For example, in [3], frame multipliers are used to solve approximation problems.

Furthermore, we introduce the concept of Bessel fusion multiplier. This notion results in a generalization of the Bessel multiplier when Bessel fusion sequences are considered instead of Bessel sequences. Several applications of Bessel fusion sequences (see [4] and [11]) and multipliers have been studied, which suggest that these notions may be successfully applied. Bessel fusion sequences and, in particular, fusion frames have been intensely studied during the last years. Even though many properties of Bessel sequences still hold for Bessel fusion sequences, Bessel fusion theory is much more delicate. The reader is referred to [5,6,12] and the references therein for a theoretical treatment on this topic.

In the present paper we study the behavior of the Bessel fusion multiplier when $m \in c_0$ or $m \in l^p$. In order to get similar results to Bessel multiplier, extras hypotheses regarding the dimension of the subspaces involved are required. Finally, we prove that the Bessel fusion multiplier depends continuously, in a certain sense, on the weight and on the Bessel fusion sequences involved if some extra hypotheses are included.

2. Bessel multiplier

During these notes, $\mathcal{H}$ and $\mathcal{H}_i$ with $i \in \mathbb{N}$ shall denote separable Hilbert spaces and $L(\mathcal{H}_i, \mathcal{H}_j)$ denotes the space of bounded linear operators from $\mathcal{H}_i$ to $\mathcal{H}_j$. Let c_0 be the space of all sequences in $\mathbb{C}$ which tend to zero. For $1 \leqslant p < \infty$ let l^p be the space of all sequences $(\psi_i)_{i\in\mathbb{N}}$ such that $\sum_{i=1}^\infty |\psi_i|^p$ is finite, and l^∞ be the space of all sequences $(\psi_i)_{i\in\mathbb{N}}$ such that, for some $M > 0$, it holds $|\psi_i| < M$ for every $i \in \mathbb{N}$. In what follows, given $1 \leqslant p < \infty$ we will denote its conjugate by q, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. Given $\phi \in \mathcal{H}_1$ and $\eta \in \mathcal{H}_2$, $\phi \otimes_i \eta$ denotes the inner tensor product, i.e., $\phi \otimes_i \eta$ is the operator from $\mathcal{H}_2$ to $\mathcal{H}_1$ defined by $(\phi \otimes_i \eta)(\xi) = \langle \xi, \eta\rangle \phi$.

Let us introduce the definition of Bessel multiplier.

Definition 2.1. Let $(\phi_k)_{k\in\mathbb{N}}$ and $(\psi_k)_{k\in\mathbb{N}}$ be Bessel sequences in $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively and $m \in l^\infty$. The operator $M_{m(\psi_k)(\phi_k)} : \mathcal{H}_1 \to \mathcal{H}_2$, called the *Bessel multiplier* for the Bessel sequences $(\phi_k)_{k\in\mathbb{N}}$ and $(\psi_k)_{k\in\mathbb{N}}$, is defined by

$$M_{m(\psi_k)(\phi_k)}(\xi) = \sum_{k\in\mathbb{N}} m_k \langle \xi, \phi_k\rangle \psi_k = \sum_{k\in\mathbb{N}} m_k \big(\psi_k \otimes_i \phi_k(\xi)\big).$$

The sequence m is called the *symbol* of $M_{m(\psi_k)(\phi_k)}$. Observe that $M_{m(\psi_k)(\phi_k)} = D_\psi \mathcal{M}_m C_\phi$ where $\mathcal{M}_m : l^2 \to l^2$ is defined by $\mathcal{M}_m((c_k)_{k\in\mathbb{N}}) = (c_k m_k)_{k\in\mathbb{N}}$. It is clear that as $m \in l^\infty$, then $\mathcal{M}_m \in L(l^2)$. Moreover, $\|\mathcal{M}_m\| = \|m\|_\infty$ and so $\|M_{m(\psi_k)(\phi_k)}\| \leqslant B_\psi^{1/2} B_\phi^{1/2} \|m\|_\infty$. Furthermore, $M^*_{m(\psi_k)(\phi_k)} = M_{\bar{m}(\phi_k)(\psi_k)}$.

In [2], the behavior of the Bessel multiplier is studied when $m \in c_0, l^1$ and l^2. In this section we extend these results for $m \in l^p$ with $1 \leqslant p < \infty$. Before that let us recall the concept of Schatten p-class and some results that we will require throughout the paper.

Definition 2.2. $T \in L(\mathcal{H}_i, \mathcal{H}_j)$ is said to be in the Schatten p-class if $(\lambda_n)_{n\in\mathbb{N}} \in l^p$ where $(\lambda_n)_{n\in\mathbb{N}}$ is the sequence of positive eigenvalues of $|T| = (T^*T)^{1/2}$ arranged in decreasing order and repeated according to multiplicity. Given $1 \leqslant p < \infty$, $\mathfrak{S}_p(\mathcal{H}_i, \mathcal{H}_j)$ denotes the Schatten p-class.

The next identity will be useful in the sequel:

$$\lambda_n = \inf\big\{\|T - B\|:\ B \in L(\mathcal{H}_i, \mathcal{H}_j) \text{ and } \dim\big(B(\mathcal{H})\big) \leqslant n - 1\big\}.$$

In the next proposition we collect some useful properties of $\mathfrak{S}_p(\mathcal{H}_i, \mathcal{H}_j)$. For their proofs and more details on this topic the reader is referred to [10].

Proposition 2.3. *Let* $1 \leqslant p < \infty$. *Hence,*

(1) $\mathfrak{S}_p(\mathcal{H}_i, \mathcal{H}_j)$ *is a Banach space with the norm* $\|T\|_p = \|(\lambda_n)_{n\in\mathbb{N}}\|_p$.
(2) *If* $T \in \mathfrak{S}_p(\mathcal{H}, \mathcal{H}_1)$ *and* $V \in L(\mathcal{H}_1, \mathcal{H}_2)$ *then* $VT \in \mathfrak{S}_p(\mathcal{H}, \mathcal{H}_2)$. *Analogously, if* $S \in L(\mathcal{H}_2, \mathcal{H})$ *then* $TS \in \mathfrak{S}_p(\mathcal{H}_2, \mathcal{H}_1)$. *Furthermore,* $\|TS\|_p \leqslant \|T\|_p\|S\|$ *and* $\|VT\|_p \leqslant \|T\|_p\|V\|$.

(3) *$T \in \mathfrak{S}_p(\mathcal{H}_1, \mathcal{H}_2)$ if and only if the sequence $(\langle Te_n, f_n\rangle)_{n\in\mathbb{N}} \in l^p$ for every orthonormal sequences $(e_n)_{n\in\mathbb{N}}$ in $\mathcal{H}_1$ and $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{H}_2$.*

An easy computations shows that $\|\phi \otimes_i \eta\|_p = \|\phi\|\|\eta\|$ for $p \geqslant 1$.

In the next proposition we include the proof of the case $m \in c_0$ for completeness.

Proposition 2.4. *Let $m \in l^\infty$, $(\phi_k)_{k\in\mathbb{N}}$ and $(\psi_k)_{k\in\mathbb{N}}$ be Bessel sequences in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Then:*

(1) *If $m \in c_0$ then $M_{m(\psi_k)(\phi_k)}$ is compact.*

(2) *If $m \in l^p$ then $M_{m(\psi_k)(\phi_k)} \in \mathfrak{S}_p(\mathcal{H}_1, \mathcal{H}_2)$. Furthermore, $\|M_{m(\psi_k)(\phi_k)}\|_p \leqslant B_\phi^{1/2} B_\psi^{1/2} \|m\|_p$.*

Proof. Note that since $M_{m(\psi_k)(\phi_k)} = D_\psi \mathcal{M}_m C_\phi$, if we prove that if $m \in c_0$ (respectively $m \in l^p$) then $\mathcal{M}_m$ is compact (respectively $\mathcal{M}_m$ belongs to the p-Schatten class), then the assertion follows.

Now, consider $m \in c_0$ and let $m_N = (m_1, m_2, \ldots, m_N, 0, \ldots) \in l^\infty$. Then, for every $c \in l^2$ we have $\|\mathcal{M}_m(c) - \mathcal{M}_{m_N}(c)\| \leqslant \|m - m_N\|_\infty \|c\| \xrightarrow[N\to\infty]{} 0$. So, there exists a sequence, $(\mathcal{M}_{m_N})_{N\in\mathbb{N}}$, of operators of finite rank such that $\|\mathcal{M}_m - \mathcal{M}_{m_N}\| \xrightarrow[N\to\infty]{} 0$, i.e., $\mathcal{M}_m$ is compact.

On the other hand, let $m \in l^p$ and $(e_k)_{k\in\mathbb{N}}$ be the canonical basis of l^2. Then $\mathcal{M}_m(c) = \sum_{k\in\mathbb{N}} m_k \langle c, e_k\rangle e_k$. Now, let $\sigma : \mathbb{N} \to \mathbb{N}$ be the permutation such that $0 \leqslant |m_{\sigma(k+1)}| \leqslant |m_{\sigma(k)}|$ for $k \in \mathbb{N}$. Hence,

$$\begin{aligned}\mathcal{M}_m(c) &= \sum_{k\in\mathbb{N}} m_k \langle c, e_k\rangle e_k = \sum_{k\in\mathbb{N}} m_{\sigma(k)} \langle c, e_{\sigma(k)}\rangle e_{\sigma(k)} = \sum_{k\in\mathbb{N}} \operatorname{sign}(m_{\sigma(k)}) m_{\sigma(k)} \langle c, e_{\sigma(k)}\rangle \operatorname{sign}(m_{\sigma(k)}) e_{\sigma(k)} \\ &= \sum_{k\in\mathbb{N}} |m_{\sigma(k)}| \langle c, e_{\sigma k}\rangle U(e_{\sigma k}),\end{aligned}$$

where U is an unitary operator in $L(l^2)$. So, as $(|m_{\sigma(k)}|)_{k\in\mathbb{N}} \in l^p$, we get that $\mathcal{M}_m \in \mathfrak{S}_p(l^2)$. Furthermore, $\|\mathcal{M}_m\|_p = \|m\|_p$ and then $\|M_{m(\psi_k)(\phi_k)}\|_p \leqslant B_\phi^{1/2} B_\psi^{1/2} \|m\|_p$. □

In the next proposition we study the continuity of $M_{m(\psi_k)(\phi_k)}$.

Proposition 2.5. *Let $(\phi_k)_{k\in\mathbb{N}}$ and $(\psi_k)_{k\in\mathbb{N}}$ be Bessel sequences in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.*

(1) *If $m^{(n)} \xrightarrow[n\to\infty]{} m$ in l^p then $\|M_{m^{(n)}(\psi_k)(\phi_k)} - M_{m(\psi_k)(\phi_k)}\|_p \xrightarrow[l\to\infty]{} 0$.*

(2) *If $m \in l^p$ and $(\phi_k^{(n)})_{k\in\mathbb{N}} \xrightarrow[n\to\infty]{} (\phi_k)_{k\in\mathbb{N}}$ in l^q then $\|M_{m(\psi_k^{(n)})(\phi_k)} - M_{m(\psi_k)(\phi_k)}\|_p \xrightarrow[n\to\infty]{} 0$ and $\|M_{m(\phi_k)(\psi_k^{(n)})} - M_{m(\phi_k)(\psi_k)}\|_p \xrightarrow[n\to\infty]{} 0$.*

Proof. (1) Let $m^{(n)} \xrightarrow[n\to\infty]{} m$ in l^p. Hence, by the proof of Proposition 2.4, $\mathcal{M}_{m^{(n)}-m} \in \mathfrak{S}_p(l^2)$ and $\|\mathcal{M}_{m^{(n)}-m}\|_p = \|m^{(n)} - m\|_p$ for every $n \in \mathbb{N}$. Therefore,

$$\begin{aligned}\|M_{m^{(n)}(\psi_k)(\phi_k)} - M_{m(\psi_k)(\phi_k)}\|_p &= \|M_{(m^{(n)}-m)(\psi_k)(\phi_k)}\|_p = \|D_\psi \mathcal{M}_{m^{(n)}-m} C_\phi\|_p \leqslant B_\psi^{1/2} B_\phi^{1/2} \|\mathcal{M}_{m^{(n)}-m}\|_p \\ &= (B_\psi B_\phi)^{1/2} \big\|m^{(n)} - m\big\|_p \xrightarrow[n\to\infty]{} 0.\end{aligned}$$

(2) If $m \in l^p$ then $M_{m(\psi_k^{(n)})(\phi_k)}, M_{m(\psi_k)(\phi_k)} \in \mathfrak{S}_p(\mathcal{H}_1, \mathcal{H}_2)$. Therefore,

$$\begin{aligned}\|M_{m(\psi_k^{(n)})(\phi_k)} - M_{m(\psi_k)(\phi_k)}\|_p &= \left\| \sum_{k=1}^\infty m_k . \big(\psi_k^{(n)} - \psi_k\big) \otimes_i \phi_k \right\|_p \\ &\leqslant \sum_{k=1}^\infty |m_k| \big\| \big(\psi_k^{(n)} - \psi_k\big) \otimes_i \phi_k \big\|_p \\ &= \sum_{k=1}^\infty |m_k| \big\| \big(\psi_k^{(n)} - \psi_k\big) \big\| \|\phi_k\| \\ &\leqslant B_\phi^{1/2} \left(\sum_{k=1}^\infty |m_k|^p \right)^{1/p} \left(\sum_{k=1}^\infty \big\| \big(\psi_k^{(n)} - \psi_k\big) \big\|^q \right)^{1/q} \\ &= B_\phi^{1/2} \|m\|_p \big\| \big(\psi_k^{(n)}\big) - (\psi_k) \big\|_q \xrightarrow[n\to\infty]{} 0\end{aligned}$$

where the last inequality holds by Holder inequality.

The other limit can be proved analogously. □

Corollary 2.6. *Let $m^{(n)} \xrightarrow[n\to\infty]{} m$ in l^p, $(\psi_k^{(n)})$, $(\phi_k^{(n)})$ be Bessel sequences and $\mathbf{B}_1, \mathbf{B}_2$ such that $B_{\psi^{(n)}} < \mathbf{B}_1$ and $B_{\phi^{(n)}} < \mathbf{B}_2$. Then, if $(\psi_k^{(n)})_{k\in\mathbb{N}}$ and $(\phi_k^{(n)})_{k\in\mathbb{N}}$ converge to $(\psi_k)_{k\in\mathbb{N}}$ and $(\phi_k)_{k\in\mathbb{N}}$ in l^q, respectively, then*

$$\|M_{m^{(n)},(\psi_k^{(n)}),(\phi_k^n)} - M_{m(\psi_k)(\phi_k)}\|_p \xrightarrow[n\to\infty]{} 0.$$

Proof. Joining items (1) and (2) of the above proposition we get:

$$\begin{aligned}\|M_{m^{(n)}(\psi_k^{(n)})(\phi_k^n)} - M_{m(\psi_k)(\phi_k)}\|_p &\leqslant \|M_{m^{(n)}(\psi_k^{(n)})(\phi_k^{(n)})} - M_{m(\psi_k^{(n)})(\phi_k^{(n)})}\|_p + \|M_{m(\psi_k^{(n)})(\phi_k^{(n)})} - M_{m(\psi_k)(\phi_k^n)}\|_p \\ &\quad + \|M_{m(\psi_k)(\phi_k^{(n)})} - M_{m(\psi_k)(\phi_k)}\|_p \\ &< (\mathbf{B}_1\cdot\mathbf{B}_2)^{1/2}\|m^{(n)} - m\|_p + \mathbf{B}_2^{1/2}\|m\|_p\|(\phi_k^{(n)}) - (\phi_k)\|_q \\ &\quad + \mathbf{B}_1^{1/2}\|m\|_p\|(\psi_k^{(n)}) - (\psi_k)\|_q \xrightarrow[n\to\infty]{} 0. \qquad \square\end{aligned}$$

3. Bessel fusion multiplier

We start this section by stating the definition of a Bessel fusion sequence. In what follows, I denotes a set which is finite or countable and $P_{\mathcal{W}}$ denotes the orthogonal projection onto a closed subspace $\mathcal{W}$.

Definition 3.1. Let $\{\mathcal{W}_i\}_{i\in I}$ be a family of closed subspaces of $\mathcal{H}$ and $(\omega_i)_{i\in I}$ be a family of weights, i.e., $\omega_i > 0$ for every $i \in I$. The family $\{(\mathcal{W}_i, \omega_i)\}_{i\in I}$ is a *Bessel fusion sequence* if there exists a constant $B > 0$ such that

$$\sum_{i\in I}\omega_i^2\|P_{\mathcal{W}_i}\xi\|^2 \leqslant B\|\xi\|^2,$$

for every $\xi \in \mathcal{H}$.

If in addition there exists a constant $A > 0$ such that

$$A\|\xi\|^2 \leqslant \sum_{i\in I}\omega_i^2\|P_{\mathcal{W}_i}\xi\|^2,$$

for every $\xi \in \mathcal{H}$, then $\{(\mathcal{W}_i, \omega_i)\}_{i\in I}$ is called a *fusion frame*.

It is clear that a Bessel sequence (respectively frame) is a special case of Bessel fusion sequence (respectively fusion frame). Indeed, if $(\psi_k)_{k\in\mathbb{N}}$ is a Bessel sequence (respectively frame) then $\{(\mathrm{span}\{\psi_k\}, \|\psi_k\|)\}_{k\in\mathbb{N}}$ is a Bessel fusion sequence (respectively fusion frame). This Bessel fusion sequence will be called the Bessel fusion sequence related to $(\psi_k)_{k\in\mathbb{N}}$.

The notions of synthesis and analysis operator also can be defined for Bessel fusion sequences. For this, a new Hilbert space must be considered. Given $\{(\mathcal{W}_i, \omega_i)\}_{i\in I}$ a Bessel fusion sequence on $\mathcal{H}$, define the Hilbert space

$$\left(\sum_{i\in I}\bigoplus\mathcal{W}_i\right)_{l^2} = \left\{(\xi_i)_{i\in I}\colon\ \xi_i \in \mathcal{W}_i \text{ and } \left(\|\xi_i\|\right)_{i\in I} \in l^2(I)\right\}$$

with the inner product $\langle(\xi_i)_{i\in I}, (\eta_i)_{i\in I}\rangle = \sum_{i\in I}\langle\xi_i, \eta_i\rangle$. So, $\|(\xi_i)_{i\in I}\|^2 = \sum_{i\in I}\|\xi_i\|^2$.

Therefore, the analysis operator $C_{\mathcal{W}} : \mathcal{H} \to (\sum_{i\in I}\bigoplus\mathcal{W}_i)_{l^2}$ is defined by

$$C_{\mathcal{W}}(\xi) = (\omega_i P_{\mathcal{W}_i}\xi)_{i\in I}$$

and the synthesis operator $D_{\mathcal{W}} : (\sum_{i\in I}\bigoplus\mathcal{W}_i)_{l^2} \to \mathcal{H}$ is defined by

$$D_{\mathcal{W}}\big((\xi_i)_{i\in I}\big) = \sum_{i\in I}\omega_i\xi_i.$$

Many properties of Bessel fusion sequences can be studied using these operators, as well as for Bessel sequences (see [5]). However, several known results of Bessel theory are not valid in the Bessel fusion setting. For example, not every surjective operator is the synthesis operator of a fusion frame. For this, and for more details on the relationship between operators and fusion frames the reader is referred to [12].

Given two Bessel fusion sequences, $\mathcal{W} = (\mathcal{W}_i, \omega_i)_{i\in I}$ and $\mathcal{V} = (\mathcal{V}_i, \upsilon_i)_{i\in I}$, and a fixed $m \in l^\infty(I)$ we are interested on the operator $S_{m\mathcal{V}\mathcal{W}} : \mathcal{H} \to \mathcal{H}$ defined by

$$S_{m\mathcal{V}\mathcal{W}}(\xi) = \sum_{i\in I} m_i\upsilon_i\omega_i P_{\mathcal{V}_i}P_{\mathcal{W}_i}\xi.$$

We shall call $S_{m\mathcal{V}\mathcal{W}}$ the *Bessel fusion multiplier* of the Bessel fusion sequences $\mathcal{W} = (\mathcal{W}_i, \omega_i)_{i\in I}$ and $\mathcal{V} = (\mathcal{V}_i, \upsilon_i)_{i\in I}$. Observe that if $\mathcal{W}$ and $\mathcal{V}$ are the Bessel fusion sequences related to the Bessel sequences $(\phi_k)_{k\in N}$ and $(\psi_k)_{k\in\mathbb{N}}$, respectively, then $\mathcal{S}_{m'\mathcal{V}\mathcal{W}} = M_{m(\psi_k)(\phi_k)}$ where $m_k = m'_k\frac{\langle\psi_k, \phi_k\rangle}{\|\psi_k\|\|\phi_k\|}$.

In order to express $S_{m\mathcal{V}\mathcal{W}}$ as composition of $C_{\mathcal{V}}$ and $D_{\mathcal{W}}$, a new operator which relates $(\sum_{i\in I}\bigoplus\mathcal{W}_i)_{l^2}$ and $(\sum_{i\in I}\bigoplus\mathcal{V}_i)_{l^2}$ is needed. Namely, let

$$P:\left(\sum_{i\in I}\bigoplus\mathcal{W}_i\right)_{l^2}\to\left(\sum_{i\in I}\bigoplus\mathcal{V}_i\right)_{l^2}\quad\text{defined by } P\big((\xi_i)_{i\in I}\big)=(P_{\mathcal{V}_i}\xi_i)_{i\in I}.$$

P is a well defined linear bounded operator. In fact, $P_{\mathcal{V}_i}\xi_i\in\mathcal{V}_i$ for every $i\in I$ and $\|(P_{\mathcal{V}_i}\xi_i)_{i\in I}\|^2=\sum_{i\in I}\|P_{\mathcal{V}_i}\xi_i\|^2\leqslant\sum_{i\in I}\|\xi_i\|^2=\|(\xi_i)_{i\in I}\|^2$. Therefore, it is easy to check that

$$S_{m\mathcal{V}\mathcal{W}}=D_{\mathcal{V}}\mathcal{S}_mPC_{\mathcal{W}},$$

where $\mathcal{S}_m:(\sum_{i\in I}\bigoplus\mathcal{V}_i)_{l^2}\to(\sum_{i\in I}\bigoplus\mathcal{V}_i)_{l^2}$ is defined by $\mathcal{S}_m((\xi_i)_{i\in I})=(m_i\xi_i)_{i\in I}$. $S_{m\mathcal{V}\mathcal{W}}$ is bounded. In fact, as $m\in l^\infty$, for every $(\xi_i)_{i\in I}\in(\sum_{i\in I}\bigoplus\mathcal{V}_i)_{l^2}$ we have that $\|\mathcal{S}_m((\xi_i)_{i\in I})\|^2=\|(m_i\xi_i)_{i\in I}\|^2=\sum_{i\in I}\|m_i\xi_i\|^2\leqslant\|m\|_\infty^2\|(\xi_i)_{i\in I}\|^2$. So, $\mathcal{S}_m$ is bounded and $\|\mathcal{S}_m\|\leqslant\|m\|_\infty$. Thus, since $S_{m\mathcal{V}\mathcal{W}}=D_{\mathcal{V}}\mathcal{S}_mPC_{\mathcal{W}}$, then $S_{m\mathcal{V}\mathcal{W}}$ is bounded. It is easy to check that $S^*_{m\mathcal{V}\mathcal{W}}=S_{\bar{m}\mathcal{W}\mathcal{V}}$.

In the next proposition we study the behavior of the Bessel fusion multiplier when the symbol belongs to c_0 or l^p with $p\geqslant 1$.

Proposition 3.2. *Let $\mathcal{W}=(\mathcal{W}_i,\omega_i)_{i\in\mathbb{N}}$ and $\mathcal{V}=(\mathcal{V}_i,\upsilon_i)_{i\in\mathbb{N}}$ be Bessel fusion sequences. Hence,*

(1) *If $m\in c_0$ and $\dim\mathcal{V}_i$ is finite for every $i\in\mathbb{N}$, then $\mathcal{S}_m$ is compact and, in particular, $S_{m\mathcal{V}\mathcal{W}}$ is compact.*
(2) *If $m\in l^p$ and $(\dim\mathcal{V}_i)_{i\in\mathbb{N}}\in l^\infty$ then $\mathcal{S}_m\in\mathfrak{S}_p((\sum_{i\in\mathbb{N}}\oplus\mathcal{V}_i)_{l^2})$ and, in particular, $S_{m\mathcal{V}\mathcal{W}}\in\mathfrak{S}_p(\mathcal{H})$.*

Proof. (1) Let $m\in c_0$ and consider $m_N=(m_1,\dots,m_N,0,0,\dots)$. Therefore,

$$\big\|\mathcal{S}_{m_N}\big((\xi_i)_{i\in\mathbb{N}}\big)-\mathcal{S}_m\big((\xi_i)_{i\in\mathbb{N}}\big)\big\|=\big\|\mathcal{S}_{m_N-m}\big((\xi_i)_{i\in\mathbb{N}}\big)\big\|\leqslant\|m_N-m\|_\infty\big\|(\xi_i)_{i\in\mathbb{N}}\big\|\xrightarrow[N\to\infty]{}0.$$

Observe that, since $\dim\mathcal{V}_i$ is finite for every $i\in\mathbb{N}$, then $\mathcal{S}_{m_N}$ is a finite rank operator for every N. Thus, $\mathcal{S}_m$ is compact. In particular, as $S_{m\mathcal{V}\mathcal{W}}=D_{\mathcal{V}}\mathcal{S}_mPC_{\mathcal{W}}$, $S_{m\mathcal{V}\mathcal{W}}$ is compact.

(2) Let $E_i=(e^i_k)_{k\in K_i}$ be an orthonormal basis of $\mathcal{V}_i$. Define

$$F_{i,k}=\big(0,\dots,\underbrace{e^i_k}_{\text{position } i},\dots,0,\dots\big)\in\left(\sum_{i\in\mathbb{N}}\oplus\mathcal{V}_i\right)_{l^2}.$$

So, $F=((F_{i,k})_{k\in K_i})_{i\in\mathbb{N}}$ is an orthonormal basis of $(\sum_{i\in\mathbb{N}}\oplus\mathcal{V}_i)_{l^2}$. Now, let $(\hat{F}_j)_{j\in\mathbb{N}}$ be the rearrangement of F given by:

(a) if $1\leqslant j\leqslant K_1$ then $\hat{F}_j=F_{1,j}$,
(b) if $j>K_1$ then $\hat{F}_j=F_{n+1,k}$ where $n=\max\{m\in\mathbb{N}\colon\ j-(K_1+\dots+K_m)>0\}$ and $k=j-(K_1+\dots+K_n)$.

Hence, for every $f=(f_j)_{j\in\mathbb{N}}\in(\sum_{i\in\mathbb{N}}\oplus\mathcal{V}_i)_{l^2}$ it holds:

$$\mathcal{S}_m(f)=\sum_{i\in\mathbb{N}}\sum_{k\in K_i}m_i\langle f,F_{i,k}\rangle F_{i,k}=\sum_{j\in\mathbb{N}}\hat{m}_j\langle f,\hat{F}_j\rangle\hat{F}_j,$$

where $(\hat{m}_j)_{j\in\mathbb{N}}=(\underbrace{m_1,\dots,m_1}_{K_1},\underbrace{m_2,\dots,m_2}_{K_2},\dots,\underbrace{m_j,\dots,m_j}_{K_j},\dots)$.

Now, $\sum_{j\in\mathbb{N}}|\hat{m}_j|^p=\sum_{i\in\mathbb{N}}K_i|m_i|^p\leqslant\|(\dim\mathcal{V}_i)_{i\in\mathbb{N}}\|_\infty\|m\|_p^p<\infty$. Thus, $\mathcal{S}_m\in\mathfrak{S}_p((\sum_{i\in\mathbb{N}}\oplus\mathcal{V}_i)_{l^2})$. In particular, $S_{m\mathcal{V}\mathcal{W}}=D_{\mathcal{V}}\mathcal{S}_mPC_{\mathcal{W}}\in\mathfrak{S}_p(\mathcal{H})$. □

As the next examples show, the hypotheses in the above proposition regarding the dimension of $\mathcal{V}_i$ are crucial.

Example 3.3. Consider $\mathcal{H}=l^2$ and $(e_k)_{k\in\mathbb{N}}$ be an orthonormal basis of l^2.

1. Define $\mathcal{W}_0=\overline{\operatorname{span}}\{e_{2k+1}\}_{k\in\mathbb{N}}$ and $\omega_0=1$ and for $i\geqslant 1$ define $\mathcal{W}_i=\overline{\operatorname{span}}\{e_{2k}\}_{k\in\mathbb{N}-\{i\}}$, $\omega_i=\frac{1}{(\sqrt{2})^i}$. It is clear that $\mathcal{W}_i$ is infinite dimensional for every $i\in\mathbb{N}$. Let us see that $(\mathcal{W}_i,\omega_i)_{i\in\mathbb{N}}$ is a Bessel fusion sequence. By simple calculation, we get that, for $i\geqslant 1$, it holds $\omega_i^2\|P_{\mathcal{W}_i}(\xi)\|^2=\frac{1}{2^i}\sum_{k=1,\ k\neq i}^\infty|\langle\xi,e_{2k}\rangle|^2$ for every $\xi\in\mathcal{H}$. Hence, for every $\xi\in\mathcal{H}$ it holds

$$\begin{aligned}\sum_{i\in\mathbb{N}_0}\omega_i^2\big\|P_{\mathcal{W}_i}(\xi)\big\|^2&=\sum_{k=0}^\infty\big|\langle\xi,e_{2k+1}\rangle\big|^2+\left(\sum_{k=1,\ k\neq 1}^\infty\frac{1}{2^k}\right)\big|\langle\xi,e_2\rangle\big|^2+\left(\sum_{k=1,\ k\neq n}^\infty\frac{1}{2^k}\right)\big|\langle\xi,e_{2n}\rangle\big|^2+\cdots\\&=\sum_{k=0}^\infty\big|\langle\xi,e_{2k+1}\rangle\big|^2+\left(1-\frac{1}{2}\right)\big|\langle\xi,e_2\rangle\big|^2+\left(1-\frac{1}{2^n}\right)\big|\langle\xi,e_{2n}\rangle\big|^2+\cdots.\end{aligned}\tag{3.1}$$

Therefore,

$$\sum_{i\in\mathbb{N}_0} \omega_i^2 \left\| P_{\mathcal{W}_i}(\xi) \right\|^2 \leqslant \|\xi\|^2 \quad \text{for every } \xi \in \mathcal{H}.$$

Now, let us see that $S_{\omega\mathcal{W}\mathcal{W}}$ is not compact where $\omega = (\omega_i)_{i\in\mathbb{N}} = (\frac{1}{\sqrt{2}^i}) \in c_0$. For this, note that $S_{\omega\mathcal{W}\mathcal{W}}(e_{2k+1}) = \sum_{i\in\mathbb{N}_0} \omega_i^3 P_{\mathcal{W}_i} e_{2k+1} = e_{2k+1}$ and so, the sequence $(S_{\omega\mathcal{W}\mathcal{V}}(e_{2k+1}))_{k\in\mathbb{N}}$ does not possess a convergent subsequence.

2. Let $\mathcal{W}_i = \overline{\operatorname{span}}\{e_1, e_2, \ldots, e_{2^i}\}$, $\omega_i = \frac{1}{(\sqrt[3]{2})^i}$, $i \geqslant 1$. It is clear that there does not exist $A > 0$ such that $\dim \mathcal{W}_i < A$ for every $i \in \mathbb{N}$. Let us show that $\mathcal{W} = (\mathcal{W}_i, \omega_i)_{i\in\mathbb{N}}$ is a Bessel fusion sequence. It is straightforward that $\omega_i^2 \|P_{\mathcal{W}_i}(\xi)\|^2 = \frac{1}{4^{i/3}} \sum_{j=1}^{2^i} |\langle \xi, e_j\rangle|^2$ for every $\xi \in \mathcal{H}$. Hence,

$$\sum_{i=1}^{\infty} \omega_i^2 \left\| P_{\mathcal{W}_i}(\xi) \right\|^2 = \left(\sum_{j=1}^{\infty} \frac{1}{4^{j/3}} \right) \left(\left|\langle \xi, e_1\rangle\right|^2 + \left|\langle \xi, e_2\rangle\right|^2 \right) + \left(\sum_{j=2}^{\infty} \frac{1}{4^{j/3}} \right) \left(\left|\langle \xi, e_3\rangle\right|^2 + \left|\langle \xi, e_4\rangle\right|^2 \right) + \cdots.$$

Now, as $\sum_{j=1}^{\infty} \frac{1}{4^{j/3}} < 2$, then $\sum_{i=1}^{\infty} \omega_i^2 \|P_{\mathcal{W}_i}(\xi)\|^2 \leqslant 2\|\xi\|^2$ for every $\xi \in \mathcal{H}$.

Observe that $\sum_{i=1}^{\infty} \omega_i = \sum_{i=1}^{\infty} \frac{1}{(\sqrt[3]{2})^i} < \infty$, i.e., $\omega = (\omega_i)_{i\in\mathbb{N}} \in l^1$. However, let us see that $S_{\omega\mathcal{W}\mathcal{W}} \notin \mathfrak{S}_1(\mathcal{H})$. For this, it suffices to show that $(\langle S_{\omega\mathcal{W}\mathcal{W}}(e_n), e_n\rangle)_{n\in\mathbb{N}} \notin l^1$. Now, by simple calculation, we get that $\sum_{n=1}^{\infty} \langle S_{\omega\mathcal{W}\mathcal{W}}(e_n), e_n\rangle = \sum_{n=1}^{\infty} \alpha_n \frac{1}{2^{n-1}}$ where $\alpha_1 = 2$ and $\alpha_n = 2^{n-1}$ for $n > 1$, then $\sum_{n=1}^{\infty} \langle S_{\omega\mathcal{W}\mathcal{W}}(e_n), e_n\rangle$ diverges.

Corollary 3.4. *Let $\mathcal{W} = (\mathcal{W}_i, \omega_i)_{i\in\mathbb{N}}$ and $\mathcal{V} = (\mathcal{V}_i, \upsilon_i)_{i\in\mathbb{N}}$ be Bessel fusion sequences such that $(\dim \mathcal{V}_i)_{i\in\mathbb{N}} \in l^\infty$. Hence, if $m^{(k)} \xrightarrow[k\to\infty]{} m$ in l^p then $\|S_{m^{(k)}\mathcal{V}\mathcal{W}} - S_{m\mathcal{V}\mathcal{W}}\|_p \xrightarrow[k\to\infty]{} 0$.*

Proof. Let $m^{(k)} \xrightarrow[k\to\infty]{} m$ in l^p. By Proposition 3.2, we have that $\mathcal{M}_{m^{(k)}-m} \in \mathfrak{S}_p((\sum_{i\in I} \bigoplus \mathcal{V}_i)_{l^2})$ and $\|\mathcal{M}_{m^{(k)}-m}\|_p \leqslant \|m^{(k)} - m\|_p$ for every $k \in \mathbb{N}$. Therefore,

$$\left\| S_{m^{(k)}\mathcal{V}\mathcal{W}} - S_{m\mathcal{V}\mathcal{W}} \right\|_p = \left\| S_{(m^{(k)}-m)\mathcal{V}\mathcal{W}} \right\|_p = \left\| D_{\mathcal{V}} \mathcal{S}_{m^{(k)}-m} P C_{\mathcal{W}} \right\|_p \leqslant \|D_{\mathcal{V}}\| \left\| m^{(k)} - m \right\|_p \|C_{\mathcal{W}}\| \xrightarrow[k\to\infty]{} 0. \qquad \square$$

Proposition 3.5. *Let $\mathcal{W} = (\mathcal{W}_i, \omega_i)_{i\in\mathbb{N}}$, $\mathcal{W}_k = (\mathcal{W}_i, \omega_i^k)_{i\in\mathbb{N}}$, and $\mathcal{V} = (\mathcal{V}_i, \upsilon_i)_{i\in\mathbb{N}}$ be Bessel fusion sequences such that $(\dim \mathcal{W}_i)_{i\in\mathbb{N}}$ and $(\upsilon_i)_{i\in\mathbb{N}} \in l^\infty$. Hence, if $m \in l^p$ and $(\omega_i^k)_{i\in\mathbb{N}} \xrightarrow[k\to\infty]{} (\omega_i)_{i\in\mathbb{N}}$ in l^q then*

$$\|S_{m\mathcal{V}\mathcal{W}_k} - S_{m\mathcal{V}\mathcal{W}}\|_p \xrightarrow[k\to\infty]{} 0 \quad \text{and} \quad \|S_{m\mathcal{W}_k\mathcal{V}} - S_{m\mathcal{W}\mathcal{V}}\|_p \xrightarrow[k\to\infty]{} 0.$$

Proof.

$$\begin{aligned}
\left\| S_{m\mathcal{V}\mathcal{W}_k} - S_{m\mathcal{V}\mathcal{W}} \right\|_p &= \left\| \sum_{i\in\mathbb{N}} m_i \upsilon_i \left(\omega_i^k - \omega_i\right) P_{\mathcal{V}_i} P_{\mathcal{W}_i} \right\|_p \\
&\leqslant \sum_{i\in\mathbb{N}} |m_i \upsilon_i| \left|\omega_i^k - \omega_i\right| \|P_{\mathcal{W}_i}\|_p \\
&= \sum_{i\in\mathbb{N}} |m_i \upsilon_i| \left|\omega_i^k - \omega_i\right| \dim \mathcal{W}_i \\
&\leqslant C \sum_{i\in\mathbb{N}} |m_i| \left|\omega_i^k - \omega_i\right| \\
&\leqslant C \left(\sum_{i\in\mathbb{N}} |m_i|^p \right)^{1/p} \left(\sum_{i\in\mathbb{N}} \left|\omega_i^k - \omega_i\right|^q \right)^{1/q} \\
&= C \|m\|_p \left\| \left(\omega_i^k\right)_{i\in\mathbb{N}} - (\omega_i)_{i\in\mathbb{N}} \right\|_q \xrightarrow[k\to\infty]{} 0,
\end{aligned}$$

where $C = \|(\dim \mathcal{W}_i)_{i\in\mathbb{N}}\|_\infty \|(\upsilon_i)_{i\in\mathbb{N}}\|_\infty$.

Analogously, it can be proved that $\|S_{m\mathcal{W}_k\mathcal{V}} - S_{m\mathcal{W}\mathcal{V}}\|_p \to 0$. $\square$

3.1. *The case $m = (1, 1, \ldots)$*

In [9] the next operator which relates two Bessel fusion sequences $\mathcal{W} = (\mathcal{W}_i, \omega_i)_{i\in I}$ and $\mathcal{V} = (\mathcal{V}_i, \upsilon_i)_{i\in I}$ is studied:

$$S_{\mathcal{V}\mathcal{W}} : \mathcal{H} \to \mathcal{H} \quad \text{defined by } S_{\mathcal{V}\mathcal{W}}(\xi) = \sum_{i\in I} \upsilon_i \omega_i P_{\mathcal{V}_i} P_{\mathcal{W}_i} \xi.$$

Clearly, in our context, $S_{\mathcal{V}\mathcal{W}}$ is the Bessel fusion multiplier of the Bessel fusion sequences $(\mathcal{W}_i, \omega_i)_{i\in I}$ and $\mathcal{V} = (\mathcal{V}_i, \upsilon_i)_{i\in I}$ where $m = (1, 1, \ldots) \in l^\infty$. Since m does not belong to l^p neither to c_o we study under which conditions $S_{\mathcal{V}\mathcal{W}}$ is compact or it belongs to $\mathfrak{S}_p(\mathcal{H})$.

Proposition 3.6. *Let $\mathcal{W} = (\mathcal{W}_i, \omega_i)_{i\in\mathbb{N}}$ and $\mathcal{V} = (\mathcal{V}_i, \upsilon_i)_{i\in\mathbb{N}}$ be Bessel fusion sequences. Then*:

(1) *If $(\upsilon_i)_{i\in\mathbb{N}} \in c_0$, $(\omega_i)_{i\in\mathbb{N}} \in l^1$ and $\dim \mathcal{V}_i$ is finite for every $i \in \mathbb{N}$ then $S_{\mathcal{V}\mathcal{W}}$ is compact.*
(2) *If $(\upsilon_i)_{i\in\mathbb{N}} \in l^p$, $(\omega_i)_{i\in\mathbb{N}} \in l^q$ and $(\dim \mathcal{V}_i)_{i\in\mathbb{N}} \in l^\infty$ then $S_{\mathcal{V}\mathcal{W}} \in \mathfrak{S}_p(\mathcal{H})$.*

Proof. We can assume without loss of generality that $\upsilon_{k+1} \leqslant \upsilon_k$ for every $k \in \mathbb{N}$.

(1) Define $\mathcal{S}^N_{\mathcal{V}\mathcal{W}} : \mathcal{H} \to \mathcal{H}$ by $\mathcal{S}^N_{\mathcal{V}\mathcal{W}}(\xi) = \sum_{i=1}^N \upsilon_i\omega_i P_{\mathcal{V}_i} P_{\mathcal{W}_i}\xi$. Then,

$$\left\| S_{\mathcal{V}\mathcal{W}}(\xi) - \mathcal{S}^N_{\mathcal{V}\mathcal{W}}(\xi)\right\| = \left\| \sum_{i=N+1}^{\infty} \upsilon_i\omega_i P_{\mathcal{V}_i} P_{\mathcal{W}_i}\xi \right\| \leqslant \|\xi\| \sum_{i=N+1}^{\infty} \upsilon_i\omega_i \leqslant \|\xi\| \upsilon_{N+1} \sum_{i=N+1}^{\infty} \omega_i \leqslant \|\xi\|\upsilon_{N+1}\|\omega\|_1.$$

Therefore, $\|S_{\mathcal{V}\mathcal{W}} - \mathcal{S}^N_{\mathcal{V}\mathcal{W}}\| \leqslant \upsilon_{N+1}\|\omega\|_1 \xrightarrow[N\to\infty]{} 0$. So, as $\mathcal{S}^N_{\mathcal{V}\mathcal{W}}$ are finite rank operators, we get that $S_{\mathcal{V}\mathcal{W}}$ is compact.

(2) Observe that

$$\lambda_1(S_{\mathcal{V}\mathcal{W}}) = \|S_{\mathcal{V}\mathcal{W}}\| \leqslant \upsilon_1\|\omega\|_1.$$

Let $N_k = \dim \mathcal{V}_k$ and $\mathcal{S}^{N_1}_{\mathcal{V}\mathcal{W}} = \upsilon_1\omega_1 P_{\mathcal{V}_1} P_{\mathcal{W}_1}$. Then, $\dim \mathcal{S}^{N_1}_{\mathcal{V}\mathcal{W}}(\mathcal{H}) \leqslant N_1$ and $\|S_{\mathcal{V}\mathcal{W}} - \mathcal{S}^{N_1}_{\mathcal{V}\mathcal{W}}\| \leqslant \upsilon_2\|\omega\|_1$. Hence,

$$\lambda_{N_1+1}(S_{\mathcal{V}\mathcal{W}}) \leqslant \upsilon_2\|\omega\|_1.$$

Moreover,

$$\lambda_{N_1}(S_{\mathcal{V}\mathcal{W}}) \leqslant \lambda_{N_1-1}(S_{\mathcal{V}\mathcal{W}}) \leqslant \cdots \leqslant \lambda_1(S_{\mathcal{V}\mathcal{W}}) \leqslant \upsilon_1\|\omega\|_1.$$

Now, let $\mathcal{S}^{N_1+N_2}_{\mathcal{V}\mathcal{W}} = \upsilon_1\omega_1 P_{\mathcal{V}_1} P_{\mathcal{W}_1} + \upsilon_2\omega_2 P_{\mathcal{V}_2} P_{\mathcal{W}_2}$. So, $\dim \mathcal{S}^{N_1+N_2}_{\mathcal{V},\mathcal{W}}(\mathcal{H}) \leqslant N_1 + N_2$ and $\|S_{\mathcal{V}\mathcal{W}} - \mathcal{S}^{N_1+N_2}_{\mathcal{V}\mathcal{W}}\| \leqslant \upsilon_3\|\omega\|_1$. Hence,

$$\lambda_{N_1+N_2+1}(S_{\mathcal{V}\mathcal{W}}) \leqslant \upsilon_3\|\omega\|_1$$

and

$$\lambda_{N_1+N_2}(S_{\mathcal{V}\mathcal{W}}) \leqslant \lambda_{N_1+N_2-1}(S_{\mathcal{V}\mathcal{W}}) \leqslant \cdots \leqslant \lambda_{N_1+1}(S_{\mathcal{V}\mathcal{W}}) \leqslant \upsilon_2\|\omega\|_1.$$

Therefore,

$$\sum_{i=1}^{\infty} \lambda_i^p(S_{\mathcal{V}\mathcal{W}}) \leqslant \sum_{i=1}^{\infty} N_i\upsilon_i^p\|\omega\|_1^p \leqslant \|\omega\|_1^p \left\|(N_i)_{i\in\mathbb{N}}\right\|_\infty \|\upsilon\|_p^p < \infty.$$

So, $S_{\mathcal{V}\mathcal{W}} \in \mathfrak{S}_p(\mathcal{H})$. □

3.2. Final comments

The results presented in this paper can also be extended to symmetrically-normed ideals generated by symmetric norming functions. See [10] for a complete treatment on this topic. Following the notation used in [10], given a symmetric norming function $\Phi(\xi)$ let c_Φ be the natural domain of Φ, and $\mathfrak{S}_\Phi(\mathcal{H}_1, \mathcal{H}_2)$ be the symmetrically-normed ideal of all compact operators $T \in L(\mathcal{H}_1, \mathcal{H}_2)$ such that $(\lambda_i(T))_{i\in\mathbb{N}} \in c_\Phi$ with the norm $\|T\|_\Phi = \Phi((\lambda_i(T))_{i\in\mathbb{N}})$. Then, the previous results can be rephrased as follows:

Let $(\phi_k)_{k\in\mathbb{N}}$ and $(\psi_k)_{k\in\mathbb{N}}$ be Bessel sequences in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.

(1) If $m \in c_\Phi$ then $M_{m(\psi_k)(\phi_k)} \in \mathfrak{S}_\Phi(\mathcal{H}_1, \mathcal{H}_2)$.
(2) If $\Phi(m^{(l)} - m) \xrightarrow[l\to\infty]{} 0$ then $\|M_{m^{(l)}(\psi_k)(\phi_k)} - M_{m(\psi_k)(\phi_k)}\|_\Phi \xrightarrow[l\to\infty]{} 0$.
(3) If $m \in c_\Phi$ and $\Phi^*((\phi_k^{(l)})_{k\in\mathbb{N}} - (\phi_k)_{k\in\mathbb{N}}) \xrightarrow[l\to\infty]{} 0$, then

$$\|M_{m(\psi_k^{(l)})(\phi_k)} - M_{m(\psi_k)(\phi_k)}\|_\Phi \xrightarrow[l\to\infty]{} 0 \quad \text{and} \quad \|M_{m(\phi_k)(\psi_k^{(l)})} - M_{m(\phi_k)(\psi_k)}\|_\Phi \xrightarrow[l\to\infty]{} 0.$$

Let $\mathcal{W} = (\mathcal{W}_i, \omega_i)_{i\in\mathbb{N}}$, $\mathcal{W}_k = (\mathcal{W}_i, \omega_i^k)_{i\in\mathbb{N}}$, and $\mathcal{V} = (\mathcal{V}_i, \upsilon_i)_{i\in\mathbb{N}}$ be Bessel fusion sequences.

(1) If $m \in c_\Phi$ and $(\dim \mathcal{V}_i)_{i\in\mathbb{N}} \in l^\infty$ then $S_{m\mathcal{V}\mathcal{W}} \in \mathfrak{S}_\Phi(\mathcal{H})$.
(2) If $(\dim \mathcal{V}_i)_{i\in\mathbb{N}} \in l^\infty$ and $\Phi(m^{(l)} - m) \xrightarrow[l\to\infty]{} 0$ then $\|S_{m^{(l)}\mathcal{V}\mathcal{W}} - S_{m\mathcal{V}\mathcal{W}}\|_\Phi \xrightarrow[l\to\infty]{} 0$.
(3) Let $(\dim \mathcal{W}_i)_{i\in\mathbb{N}}$ and $(\upsilon_i)_{i\in\mathbb{N}} \in l^\infty$. Hence, if $m \in c_\Phi$ and $\Phi((\omega_i^k)_{i\in\mathbb{N}} - (\omega_i)_{i\in\mathbb{N}}) \xrightarrow[k\to\infty]{} 0$ then

$$\|S_{m\mathcal{V}\mathcal{W}_k} - S_{m\mathcal{V}\mathcal{W}}\|_\Phi \xrightarrow[k\to\infty]{} 0 \quad \text{and} \quad \|S_{m\mathcal{W}_k\mathcal{V}} - S_{m\mathcal{W}\mathcal{V}}\|_\Phi \xrightarrow[k\to\infty]{} 0.$$

Acknowledgment

The authors thank the referee for carefully reading and useful comments.

References

[1] M.L. Arias, M. Pacheco, G. Corach, Characterization of Bessel sequences, Extracta Math. 22 (1) (2007) 55–66.
[2] P. Balazs, Basic definitions and properties of Bessel multipliers, J. Math. Anal. Appl. 325 (1) (2007) 571–585.
[3] P. Balazs, Hilbert–Schmidt operators and frames—classification, approximation by multipliers and algorithms, Int. J. Wavelets Multiresolut. Inf. Process. 6 (2) (2008) 315–330.
[4] B.G. Bodmann, Optimal linear transmission by loss-insensitive packet encoding, Appl. Comput. Harmon. Anal. 22 (2007) 274–285.
[5] P.G. Casazza, G. Kutyniok, Frames of subspaces, in: Wavelets, Frames and Operator Theory, College Park, MD, 2003, in: Contemp. Math., vol. 345, Amer. Math. Soc., Providence, RI, 2004, pp. 87–113.
[6] P.G. Casazza, G. Kutyniok, S. Li, Fusion frames and distributed processing, Appl. Comput. Harmon. Anal. 25 (2008) 114–132.
[7] O. Christensen, An Introduction to Frames and Riesz Bases, Birkhäuser, Boston, 2003.
[8] I. Daubechies, Ten Lectures on Wavelets, CBMS-NSF Regional Conf. Ser. in Appl. Math., vol. 61, SIAM, Philadelphia, PA, 1992.
[9] P. Gavruta, On the duality of fusion frames, J. Math. Anal. Appl. 333 (2007) 871–879.
[10] I.C. Gohberg, M.G. Krein, Introduction to the Theory of Linear Nonselfadjoint Operators, Transl. Math. Monogr., vol. 18, Amer. Math. Soc., Providence, RI, 1969.
[11] C.J. Rozell, D.H. Johnson, Analyzing the robustness of redundant population codes in sensory and feature extraction systems, Neurocomputing 69 (2006) 1215–1218.
[12] M.A. Ruiz, D. Stojanoff, Some properties of frames of subspaces obtained by operator theory methods, J. Math. Anal. Appl. 343 (2008) 366–378.