The Gap Between Local Multiplier Algebras of C*-algebras*

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Abstract

The local multiplier algebra $M_{\mathrm{loc}}(A)$ of a C*-algebra A has the property that $M_{\mathrm{loc}}(A) \subseteq M_{\mathrm{loc}}(M_{\mathrm{loc}}(A))$. In this paper we show that there is a separable liminal C*-algebra A such that the inclusion is proper.

The local multiplier algebra of a C*-algebra A is the C*-algebra

$$M_{\mathrm{loc}}(A) = \lim_{\rightarrow} M(K)\,,$$

where the direct limit is considered with respect to the directed system of multiplier algebras $M(K)$ of the essential ideals K of A. If $I(A)$ denotes the injective envelope [11] of A and if $\overline{A}$ denotes the regular monotone completion [12] of A, then

$$A \subseteq M_{\mathrm{loc}}(A) \subseteq M_{\mathrm{loc}}\left(M_{\mathrm{loc}}(A)\right) \subseteq \overline{A} \subseteq I(A)\,, \tag{1}$$

where each inclusion is as a C*-subalgebra [9, Theorem 4.6].

A question posed by G.K. Pedersen in connection with his work on derivations [17] asks whether $M_{\mathrm{loc}}(A) = M_{\mathrm{loc}}(M_{\mathrm{loc}}(A))$, for every C*-algebra A. This question has been answered only recently: P. Ara and M. Mathieu [3] found the first example of a C*-algebra A for which $M_{\mathrm{loc}}(A) \neq$

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$M_{\text{loc}}(M_{\text{loc}}(A))$. The Ara–Mathieu example A is a prime AF C*-algebra. Because $M_{\text{loc}}(A) \neq M_{\text{loc}}(M_{\text{loc}}(A))$, for this particular A, one concludes from [5, Corollary 2.4] that the injective envelope of A is a wild type III AW*-factor. Furthermore, since A is prime and thus every nonzero ideal of A is essential, similar reasoning shows that A cannot have any nonzero liminal ideals; hence, A is an antiliminal C*-algebra.

The purpose of the present paper is to give a new example of a separable C*-algebra A for which $M_{\text{loc}}(A) \neq M_{\text{loc}}(M_{\text{loc}}(A))$. The example occurs with $A = C([0,1]) \otimes K(H)$, which, in contrast to the C*-algebra in the Ara–Mathieu example, is liminal. The proof is achieved, in part, via M. Hamana's theory of the monotone complete tensor product [13, 14], as well as by using his seminal work on injective envelopes and regular monotone completions [12, 15].

The work of D. Somerset [20, 21] will also be important in our study. In particular, Somerset shows that if A is any separable postliminal C*-algebra, then $M_{\text{loc}}(M_{\text{loc}}(A))$ is a type I AW*-algebra [21]. Hence, if A is separable and postliminal, every derivation of $M_{\text{loc}}(M_{\text{loc}}(A))$ is inner. Pedersen's main question from [17] is: if A is separable, then is every derivation of $M_{\text{loc}}(A)$ inner ? In the case where A is separable and postliminal, the answer would be yes if it were true that $M_{\text{loc}}(A) = M_{\text{loc}}(M_{\text{loc}}(A))$. Therefore, in light of our example herein, Pedersen's derivation problem remains open even in the interesting special case of separable postliminal C*-algebras.

We now review the notations used throughout the paper. Let $B(H)$ denote the C*-algebra of bounded operators on a Hilbert space H and let $K(H)$ be the ideal of compact operators. For a locally compact Hausdorff space X, let βX denote its Stone–Čech compactification and $C_b(X)$ denote the C*-algebra of all bounded continuous maps from X into $\mathbb{C}$. If X is compact, then we write $C(X)$ for $C_b(X)$. The C*-subalgebra $C_0(X) \subseteq C_b(X)$ consists of all $f \in C_b(X)$ that vanish at infinity. The multiplier algebra $M(C_0(X))$ of $C_0(X)$ is given by $M(C_0(X)) = C_b(X) \cong C(\beta X)$. More generally, the multiplier algebra of $C_0(X) \otimes K(H)$ is $C_b(X, B(H)_{*-\text{st}})$, the C*-algebra of all bounded functions $f : X \to B(H)$ that are continuous with respect to the strong* operator topology [1]. For a compact Hausdorff space Δ, $C(\Delta, B(H)_{\sigma-\text{wk}})$ denotes the set of all bounded functions $f : \Delta \to B(H)$ that are continuous with respect to the σ-weak operator topology. Under pointwise operations and the supremum norm, $C(\Delta, B(H)_{\sigma-\text{wk}})$ is an involutive Banach space whose positive cone consists of all $f \in C(\Delta, B(H)_{\sigma-\text{wk}})$ for which $f(t) \in B(H)^+$ for every $t \in \Delta$.

Overviews of the theory of local multiplier algebras and injective operator systems can be found in the monographs [2, 8, 16].

1 An Embedding of $M_{\mathrm{loc}}(A)$ into $I(A)$, where $A = C_0(Y) \otimes K(H)$

Inspired by the inclusions (1) of M. Frank and V. Paulsen, our first task is to exhibit an explicit embedding of $M_{\mathrm{loc}}(A)$ as a C*-subalgebra of $I(A)$.

Henceforth assume that Y denotes a locally compact Hausdorff space and that Δ is the maximal ideal space of $M_{\mathrm{loc}}(C_0(Y))$. Since $M_{\mathrm{loc}}(C_0(Y))$ is an abelian AW*-algebra [2, Proposition 3.1.5], [9, Theorem 4.5], the compact Hausdorff space Δ is Stonean.

In [13], Hamana introduces the notion of a monotone complete tensor product of an AW*-algebra and a von Neumann algebra. (See [18] for additional information about this tensor product.) We are interested in a particular case of Hamana's construction, namely $C(\Delta)\overline{\otimes}B(H)$; thus, we give below a brief description of what it represents and how we shall work with it.

Assume that $C(\Delta)$ is represented faithfully and nondegenerately as a C*-algebra of operators acting on a Hilbert space K. Fix an orthonormal basis $\{e_i\}_{i\in\mathbb{I}}$ of H and let $\{e_{ij}\}_{(i,j)\in\mathbb{I}\times\mathbb{I}}$ be the system of matrix units associated with this basis. By [13, Lemma 3.4], the elements of $C(\Delta)\overline{\otimes}B(H)$ are all operators $x \in B(K \otimes H)$ that can be written as strong limits of nets $\{\sum_{i,j} f_{ij} \otimes e_{ij}\}_{i,j}$, where $f_{ij} \in C(\Delta) \subset B(K)$. We use the notation $x = \mathrm{st} - \sum_{i,j} f_{ij} \otimes e_{ij} \in B(K \otimes H)$ to denote such $x \in C(\Delta)\overline{\otimes}B(H)$. The operator system $C(\Delta)\overline{\otimes}B(H)$ is injective [13]. On every injective operator system I there is a product $\odot$ such that $(I, \odot)$ is a C*-algebra and is completely isometrically order isomorphic to I. For the case of interest here, the C*-algebra $(C(\Delta)\overline{\otimes}B(H), \odot)$ is in fact a type I AW*-algebra [13, Corollary 4.11].

To explain the meaning of the product $\odot$ in $C(\Delta)\overline{\otimes}B(H)$, we require the Banach space $C(\Delta, B(H)_{\sigma-\mathrm{wk}})$. By [14, Lemma 1.1], there is an isometric $*$-preserving order isomorphism $\delta : C(\Delta)\overline{\otimes}B(H) \to C(\Delta, B(H)_{\sigma-\mathrm{wk}})$, which is defined as follows. If $x \in C(\Delta)\overline{\otimes}B(H)$ is given by $x = \mathrm{st} - \sum_{i,j} f_{ij} \otimes e_{ij} \in B(K \otimes H)$, then $\delta(x) : \Delta \to B(H)$ is the σ-weakly continuous function given by $\delta(x)(t) = \mathrm{st} - \sum_{i,j} f_{ij}(t)\, e_{ij} \in B(H)$. Now if $f, g \in C(\Delta)\overline{\otimes}B(H)$, then $f \odot g \in C(\Delta)\overline{\otimes}B(H)$ is determined uniquely by the following property: for every $\rho \in B(H)_*$ there exists a meager set $M_\rho \subset \Delta$ such that

$$\rho(\delta(f \odot g)(t)) \;=\; \rho(\delta(f)(t)\, \delta(g)(t)) \quad \forall\, t \in \Delta \setminus M_\rho\,. \tag{2}$$

Henceforth, we shall consider the induced product in $C(\Delta, B(H)_{\sigma-\mathrm{wk}})$ given by the identification δ and satisfying (2), which we still denote by $\odot$.

In this way, $(C(\Delta, B(H)_{\sigma-\text{wk}}), \odot)$ becomes a C*-algebra—in fact a type I AW*-algebra—that is compatible with the involutive ordered vector space structure of $C(\Delta, B(H)_{\sigma-\text{wk}})$ described above.

The C*-algebra $C_0(Y) \otimes K(H)$ is isomorphic to the C*-algebra of all norm-continuous functions $f : Y \to K(H)$ that vanish at infinity; we shall make this identification throughout this paper. Furthermore, we will determine an embedding, which is rigid in the sense of [16, Corollary 15.7], of $M_{\text{loc}}(C_0(Y) \otimes K(H))$ into the injective C*-algebra $(C(\Delta, B(H)_{\sigma-\text{wk}}), \odot)$.

To describe the embedding, we need to consider the space Δ in some detail. To this end, let

$$I_e(Y) \;=\; \{X \subseteq Y \;:\; X \text{ is open and dense in } Y\}\,.$$

For each $X \in I_e(Y)$, let $\iota_X : X \to \beta X$ be the continuous embedding of X as a dense subset of βX. Because each $X \in I_e(Y)$ is open and, hence, locally compact [7, Theorem XI.6.5], the embedding $\iota_X : X \to \beta X$ is an open map [7, Theorem VII.7.3]; therefore $\iota_X(X)$ is a dense open subset of βX. If $X, Z \in I_e(Y)$ satisfy $X \subset Z$, then ι_Z embeds X into βZ as a dense subset. Thus, βZ is a compactification of X and so, by the Stone–Čech Theorem [7, Theorem 8.2], there is a unique continuous function $\Phi_{Z,X} : \beta X \to \beta Z$ for which $\Phi_{Z,X} \circ \iota_X = \iota_Z|_X$. Because $\iota_Z(X)$ is dense in βZ, $\Phi_{Z,X}$ is a surjection. Note that if $X \subset W \subset Z$, for $X, W, Z \in I_e(Y)$, then $\Phi_{Z,X} = \Phi_{Z,W} \circ \Phi_{W,X}$. Hence $(\{\beta X \;:\; X \in I_e(Y)\}, \Phi_{Z,X})$ is an inverse spectrum over $I_e(Y)$ endowed with the order of reversed inclusion. The maximal ideal space Δ of $M_{\text{loc}}(C_0(Y))$ is precisely the inverse limit space that arises from this inverse spectrum [19]; that is,

$$\Delta \;=\; \lim_{\leftarrow} \beta X\,. \tag{3}$$

Since Δ is an inverse limit space, there is a family $\{\Phi_X :\; X \in I_e(Y)\}$ of continuous functions $\Phi_X : \Delta \to \beta X$ satisfying $\Phi_Z = \Phi_{Z,X} \circ \Phi_X$ whenever $Z \in I_e(Y)$ is such that $X \subset Z$ [7, Appendix Two, p. 433]. Such functions Φ_X are surjective because every $\Phi_{Z,X}$ is surjective.

Lemma 1.1. *For every $X \in I_e(Y)$, $\Phi_X^{-1}(\beta X \setminus \iota_X(X))$ is a nowhere dense subset of Δ.*

Proof. Let $X \in I_e(Y)$ and let $M = \Phi_X^{-1}(\beta X \setminus \iota_X(X)) \subset \Delta$. Since $\iota_X(X)$ is an open subset of βX and Φ_X is continuous, M is closed.

Assume, contrary to what we aim to prove, that the interior U of M is nonempty. Select $t \in U$. By [7, Proposition 2.3 in Appendix Two], there is a $Z \in I_e(Y)$ and an open set $V \subseteq \beta Z$ such that $t \in \Phi_Z^{-1}(V) \subseteq U$. Because $\iota_Z(Z)$ is a dense open subset of βZ, the set $W = V \cap \iota_Z(Z)$ is a nonempty open subset of βZ. Thus, $\iota_Z^{-1}(W)$ is a nonempty open subset of Z.

Now let $W' = \iota_Z^{-1}(W) \cap X$. Note that $\emptyset \neq W' \subseteq R$, where $R = Z \cap X \in I_e(Y)$. Therefore, $\emptyset \neq \iota_Z(W') \subseteq W \subset \beta Z$. Because ι_Z is an open map, $\Phi_Z^{-1}(\iota_Z(W'))$ is a nonempty open subset contained in $\Phi_Z^{-1}(W) \subseteq \Phi_Z^{-1}(V) \subseteq U$. Therefore, $\Phi_Z^{-1}(\iota_Z(W')) \subseteq U$ implies that $\iota_Z(W') \cap \Phi_Z(U) \neq \emptyset$, which in turn implies that $\iota_Z(R) \cap \Phi_Z(U) \neq \emptyset$. Because $R \subseteq Z$, $\Phi_Z = \Phi_{Z,R} \circ \Phi_R$, and so

$$\iota_Z(R) \cap \Phi_{Z,R}(\Phi_R(U)) \neq \emptyset \,. \tag{4}$$

Furthermore, because $\Phi_{Z,R} \circ \iota_R = \iota_{Z|R}$, $\Phi_{Z,R}$ is a homeomorphism when restricted to the dense subset $\iota_R(R)$ of βR. Hence, by [10, Lemma 6.11], $\Phi_{Z,R}$ maps $\beta R \setminus \iota_R(R)$ into $\beta Z \setminus \iota_Z(R)$. This means, by (4), that

$$\Phi_R(U) \cap \iota_R(R) \neq \emptyset \,. \tag{5}$$

However, $\Phi_{X,R}(\iota_R(R)) = \iota_X(R) \subseteq \iota_X(X)$ and (5) imply that $\Phi_X(U) = \Phi_{X,R} \circ \Phi_R(U)$ intersects $\iota_X(X)$, which is in contradiction to $U \subset \Phi_X^{-1}(\beta X \setminus \iota_X(X))$. □

Every essential ideal J of $C_0(Y) \otimes K(H)$ has the form $J = C_0(X) \otimes K(H)$ for some $X \in I_e(Y)$, and the multiplier algebra of J is $M(J) = C_b(X, B(H)_{*-\mathrm{st}})$ [1]. Therefore if $f \in M(J)$, then $f \in C_b(X, B(H)_{\sigma-\mathrm{wk}})$. The Stone–Čech Theorem implies then that f extends uniquely to an element $\tilde{f} \in C(\beta X, B(H)_{\sigma-\mathrm{wk}})$ with the same (uniform) norm as f and such that

$$\tilde{f} \circ \iota_X = f \,.$$

(The Stone–Čech Theorem applies because norm-closed balls are compact in the σ-weak operator topology.) Thus, the map $f \mapsto \tilde{f}$ is an isometric embedding of $M(J)$ into $C(\beta X, B(H)_{\sigma-\mathrm{wk}})$.

Let $\Phi_X : \Delta \to \beta X$ be the continuous surjection considered before and consider the map $\pi_X : C_b(X) \to C(\Delta)$ given by $\pi_X(f) = \tilde{f} \circ \Phi_X$. In a similar fashion define a map $\tilde{\pi}_X : C_b(X, B(H)_{*-\mathrm{st}}) \to C(\Delta, B(H)_{\sigma-\mathrm{wk}})$ by

$$\tilde{\pi}_X(f) = \tilde{f} \circ \Phi_X \,.$$

The uniqueness in the Stone–Čech Theorem guarantees the following equations:

$$(\tilde{\pi}_X f)_{ij} = \pi_X(f_{ij}) \,, \quad \forall\, i, j \in \mathbb{I} \,. \tag{6}$$

Note that π_X and $\tilde{\pi}_X$ agree when the dimension of H is one.

Since Φ_X is continuous and surjective, π_X and $\tilde{\pi}_X$ are well-defined linear isometries of $C_b(X)$ into $C(\Delta)$ and of the C*-algebra $C_b(X, B(H)_{*-\mathrm{st}})$ into $C(\Delta, B(H)_{\sigma-\mathrm{wk}})$ respectively.

Using the universal property of multiplier algebras [2, 1.2.20], we also define connecting maps, for $X, Z \in I_e(Y)$ with $X \subset Z$, as follows. The inclusion $C_0(X) \subset C_0(Z)$ of essential ideals induces a unique injective homomorphism $\pi_{X,Z} : C_b(Z) \to C_b(X)$ of their multiplier algebras such that $\pi_Z = \pi_X \circ \pi_{X,Z}$. In fact, by the uniqueness of the homomorphism, $\pi_{X,Z}$ is given by $\pi_{X,Z} f = f|_X$, using the fact that $\iota_Z(X) \subseteq \beta Z$ is open and dense in βZ and the relation $\Phi_Z = \Phi_{Z,X} \circ \Phi_X$. Likewise, the inclusion $C_0(X) \otimes K(H) \subset C_0(Z) \otimes K(H)$ of essential ideals of $C_0(Y) \otimes K(H)$ induces a unique embedding $\tilde{\pi}_{X,Z} : C_b(Z, B(H)_{*-\mathrm{st}}) \to C_b(X, B(H)_{*-\mathrm{st}})$ of multiplier algebras, namely (again by the uniqueness of the embedding of multiplier algebras) $\tilde{\pi}_{X,Z} f = f|_X$, with compatibility relations

$$\tilde{\pi}_Z = \tilde{\pi}_X \circ \tilde{\pi}_{X,Z} .$$

Lemma 1.2. *For every $X \in I_e(Y)$, the map $\tilde{\pi}_X : C_b(X, B(H)_{*-\mathrm{st}}) \to (C(\Delta, B(H)_{\sigma-\mathrm{wk}}), \odot)$ is a $*$-monomorphism.*

Proof. Since $\tilde{\pi}_X$ is clearly isometric and positive, all we need to check is that it is a homomorphism. Suppose that $X \in I_e(Y)$ and $f,\, g \in C_b(X, B(H)_{*-\mathrm{st}})$. Let $\tilde{f}$, $\tilde{g}$, $(\tilde{fg})$ denote the σ-weakly continuous extensions of f, g, and fg to βX. Thus, $\tilde{f}(\iota_X(x)) = f(x)$, $\tilde{g}(\iota_X(x)) = g(x)$, and $(\tilde{fg})(\iota_X(x)) = (fg)(x) = f(x)\, g(x)$ for every $x \in X$. Therefore, we conclude that

$$(\tilde{fg})(\iota_X(x)) = \tilde{f}(\iota_X(x))\, \tilde{g}(\iota_X(x)), \quad \forall x \in X .$$

Hence, for every $t \in \Phi_X^{-1}(\iota_X(X))$,

$$\tilde{\pi}_X(fg)(t) = (\tilde{fg})(\Phi_X(t)) = \tilde{f}(\Phi_X(t))\, \tilde{g}(\Phi_X(t)) = \tilde{\pi}_X(f)(t)\, \tilde{\pi}_X(g)(t) .$$

By Lemma 1.1, $\Phi_X^{-1}(\beta X \setminus \iota_X(X))$ is nowhere dense (and, hence, meager); thus, $\tilde{\pi}_X(fg) = \tilde{\pi}_X(f) \odot \tilde{\pi}_X(g)$ by (2). □

The maps $\tilde{\pi}_X$ allow us to realise $M_{\mathrm{loc}}(C_0(Y) \otimes K(H))$ in $C(\Delta, B(H)_{\sigma-\mathrm{wk}})$.

Theorem 1.3. *If* $A = C_0(Y) \otimes K(H)$*, then*

$$M_{\rm loc}(A) \;=\; \left[\bigcup_{X \in I_e(Y)} \tilde{\pi}_X(C_b(X, B(H)_{*-{\rm st}}))\right]^{-\|\,\|} \;\subseteq\; (C(\Delta, B(H)_{\sigma-{\rm wk}}), \odot)\;.$$

Proof. The following result concerning direct limit C*-algebras B is standard. Assume that $B = \lim\limits_{\rightarrow}(B_{ij}, \varrho_{ij})$, where the homomorphisms $\varrho_{ij} : B_i \to B_j$, for $i \leq j$, are injective, and let D be any C*-algebra. If a family of injective homomorphisms $\varrho_i : B_i \to D$ satisfies $\varrho_i = \varrho_j \circ \varrho_{ij}$, for $i \leq j$, then B is isomorphic to the C*-subalgebra of D given by the norm closure of $\bigcup_i \varrho_i(B_i)$ in D. To apply this result, we use the compatibility relations $\tilde{\pi}_Z = \tilde{\pi}_X \circ \tilde{\pi}_{X,Z}$. □

Theorem 1.4. $(C(\Delta, B(H)_{\sigma-{\rm wk}}), \tilde{\pi}_Y)$ *is an injective envelope of* $C_0(Y) \otimes K(H)$.

Proof. By [15, Lemma 1.1], if J is an essential ideal of a C*-algebra A, then $\overline{J} = \overline{A}$. Therefore, since $C_0(Y)\otimes K(H)$ is an essential ideal of $C_0(Y)\otimes B(H)$,

$$\overline{C_0(Y) \otimes K(H)} \;=\; \overline{C_0(Y) \otimes B(H)}\,. \tag{7}$$

Furthermore, because $C_0(Y) \otimes K(H)$ is liminal and the regular monotone completion of every postliminal C*-algebra is a type I AW*-algebra (and hence injective) [12], equation (7) becomes

$$I\left(C_0(Y) \otimes K(H)\right) \;=\; I\left(C_0(Y) \otimes B(H)\right)\;. \tag{8}$$

Now by [13, Proposition 3.11],

$$I\left(C_0(Y) \otimes B(H)\right) \;=\; I\left(C_0(Y)\right) \overline{\otimes}\, B(H)\,, \tag{9}$$

where $I\left(C_0(Y)\right) \overline{\otimes}\, B(H)$ is Hamana's monotone complete tensor product [13, Definition 3.3].

Next, represent $M_{\rm loc}(C_0(Y))$ nondegenerately and faithfully as a C*-subalgebra of $B(L)$ for some Hilbert space L. Since $M_{\rm loc}(C_0(Y))$ coincides with $I(C_0(Y))$, and since $C(\Delta) = M_{\rm loc}(\pi_Y(C_0(Y)))$, there is a complete order isomorphism $\gamma : I(C_0(Y)) \to C(\Delta) \subset B(L)$ extending π_Y. Hence, by [13, Lemma 3.5(ii)], $\gamma \otimes {\rm id}_{B(H)}$ is a complete order isomorphism between $I\left(C_0(Y)\right) \overline{\otimes} B(H)$ and $C(\Delta) \overline{\otimes} B(H) \subset B(L) \otimes_{\rm min} B(H)$, and $\gamma \otimes {\rm id}$ extends

$\pi_Y \otimes \mathrm{id}$. In other words, the injective envelope of $\pi_Y(C_0(Y)) \otimes K(H)$ is the operator system $C(\Delta)\overline{\otimes}B(H)$.

Finally, by [14, Lemma 1.1], there exists an isometric order $*$-isomorphism δ between $C(\Delta)\overline{\otimes}B(H)$ and $C(\Delta, B(H)_{\sigma-\mathrm{wk}})$. Under this isomorphism, if $\{e_{\alpha\beta}\}_{\alpha,\beta} \subset B(H)$ is a fixed system of matrix units for $B(H)$, and if $f \in C(\Delta) \subset B(L)$, then $f \otimes e_{\alpha\beta} \in C(\Delta)\overline{\otimes}B(H)$ is mapped to the σ-weakly continuous function $t \mapsto f(t)e_{\alpha\beta}$. So using (6), for $f \in C_0(Y)$ we have

$$\delta((\gamma \otimes \mathrm{id})(f \otimes e_{\alpha\beta})) = \delta(\pi_Y f \otimes e_{\alpha\beta}) = (\pi_Y f)\, e_{\alpha\beta} = \tilde{\pi}_Y(f\, e_{\alpha\beta}).$$

Since elements of the form $f \otimes e_{\alpha\beta}$, where $f \in C_0(Y)$, span a norm-dense subset of $C_0(Y) \otimes K(H)$, $\delta \circ (\gamma \otimes \mathrm{id})$ is a complete order isomorphism that extends $\tilde{\pi}_Y$. □

Using Hamana's results [13, p. 271], [14, Theorem 1.3] on the multiplicative structure of the injective operator system $C(\Delta)\overline{\otimes}B(H)$, we obtain:

Corollary 1.5. *As a C*-algebra, the injective envelope of $C_0(Y) \otimes K(H)$ is $(C(\Delta, B(H)_{\sigma-\mathrm{wk}}), \odot)$, with the embedding given by $\tilde{\pi}_Y$.*

2 An Example of A for which $M_{\mathrm{loc}}(A) \neq M_{\mathrm{loc}}(M_{\mathrm{loc}}(A))$

Suppose now that $Y = [0,1]$ and that the Hilbert space H is separable and infinite-dimensional. Thus, $A = C([0,1]) \otimes K(H)$ is separable and liminal. The AW*-algebra $C(\Delta)$, where Δ is the maximal ideal space of $M_{\mathrm{loc}}(C([0,1]))$, is known in the literature as the Dixmier algebra. Since A is separable and liminal, Somerset's theorem [21, Theorem 2.8] shows that $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A)) = I(A)$. However, as we show below, the C*-algebra $M_{\mathrm{loc}}(A)$ does not coincide with $I(A)$.

Theorem 2.1. *If $A = C([0,1]) \otimes K(H)$, where H is separable and infinite-dimensional, then $M_{\mathrm{loc}}(A) \subsetneqq C(\Delta, B(H)_{\sigma-\mathrm{wk}})$.*

Proof. Let Q denote the set of rational numbers in $[0,1]$ and, for each $r \in Q$, let $f'_r : [0,1] \to [0,1]$ be the characteristic function $f'_r = \chi_{(r,1]}$. Although each function f'_r is discontinuous in $[0,1]$, if we consider $Y_r = [0,1] \setminus \{r\}$, then $f_r = f'_r|_{Y_r} \in C_b(Y_r) = M(C_0(Y_r))$.

Let $X \in I_e([0,1])$. As X is open and Q dense in $[0,1]$, there exists $r \in Q \cap X$. Let $X(r) = X \setminus \{r\}$ and $g \in C_b(X)$. Since $\lim_{x\to r^-} f_r(x) = 0$ and $\lim_{x\to r^+} f_r(x) = 1$, we conclude that

$$\lim_{x\to r^-} (\pi_{X(r),Y_r} f_r)(x) = 0, \quad \lim_{x\to r^+} (\pi_{X(r),Y_r} f_r)(x) = 1.$$

Since g is continuous, if $\ell = g(r)$ we have

$$\lim_{x\to r^-}(\pi_{X(r),X}\, g)(x) = \ell, \quad \lim_{x\to r^+}(\pi_{X(r),X}\, g)(x) = \ell.$$

Thus,

$$\|\, \pi_{X(r),Y_r} f_r - \pi_{X(r),X}\, g\,\|_\infty \geq \frac{1}{2}\,.$$

Then, using that $\pi_{X(r)}$ is isometric and that $\pi_{Y_r} = \pi_{X(r)}\,\pi_{X(r),Y_r}$, $\pi_X = \pi_{X(r)}\,\pi_{X(r),X}$, we get

$$\|\, \pi_{Y_r} f_r - \pi_X g\,\|_\infty \geq \frac{1}{2}\,, \quad \forall\, g \in C_b(X)\,. \tag{10}$$

Finally, fix a maximal family of pairwise orthogonal minimal projections $\{p_r\}_{r\in Q} \subset B(H)$, and let $q : \Delta \to B(H)$ be the diagonal operator-valued function given by

$$q(t) \;=\; \sum_{r\in Q}(\pi_{Y_r} f_r)(t)\; p_r, \quad t \in \Delta\,.$$

Note that $q \in C(\Delta, B(H)_{\sigma-\mathrm{wk}})$. We now show that $q \notin \mathrm{M}_{\mathrm{loc}}(C([0,1]) \otimes K(H))$.

Assume, on the contrary, that $q \in \mathrm{M}_{\mathrm{loc}}(C([0,1]) \otimes K(H))$. Then by Theorem 1.3, for every $\varepsilon > 0$ there exists $X \in I_e([0,1])$ and $k \in C_b(X, B(H)_{*-\mathrm{st}})$ such that

$$\|\, \tilde{\pi}_X k - q\,\| \;\leq\; \varepsilon.$$

Fixing $\varepsilon = 1/4$ and compressing with the projections p_r, we conclude that there exists $X \in I_e([0,1])$ such that

$$\sup_{r\in Q} \|\pi_X(k_{rr}) - \pi_{Y_r}(f_r)\|_\infty \;\leq\; \frac{1}{4}\,,$$

where for every $r \in Q$, $k_{rr} \in C_b(X)$. But this contradicts inequality (10) for $r \in X \cap Q \neq \emptyset$. □

Corollary 2.2. *For $A = C([0,1]) \otimes K(H)$, $M_{\mathrm{loc}}(A)$ is not isomorphic to $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A))$.*

Proof. We know—from [21, Theorem 2.8], Theorem 1.4, and Corollary 1.5—that

$$M_{\mathrm{loc}}\left(M_{\mathrm{loc}}(A)\right) = I(A) = \left(C(\Delta, B(H)_{\sigma-\mathrm{wk}}), \odot\right) .$$

So $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A))$ is injective. Since the inclusion $M_{\mathrm{loc}}(A) \subset M_{\mathrm{loc}}(M_{\mathrm{loc}}(A))$ is proper by Theorem 2.1, and we know [9] that

$$A \subset M_{\mathrm{loc}}(A) \subset M_{\mathrm{loc}}(M_{\mathrm{loc}}(A)) \subset I(A),$$

$M_{\mathrm{loc}}(A)$ cannot be injective by the minimality of the injective envelope. Thus, $M_{\mathrm{loc}}(A)$ and $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A))$ are not isomorphic. □

3 Remarks

1. Theorem 2.1 also gives a negative answer to an issue raised by Somerset [21] after his Theorem 2.8: is $M_{\mathrm{loc}}\left(M_{\mathrm{loc}}(A)\right) = M_{\mathrm{loc}}(A)$, for every separable C*-algebra A with an essential postliminal ideal? In showing that $M_{\mathrm{loc}}\left(M_{\mathrm{loc}}(A)\right) = I(A)$, for $A = C_0(Y) \otimes K(H)$, we also recover in this special case Somerset's result [21, Theorem 2.7] that $M_{\mathrm{loc}}\left[M_{\mathrm{loc}}\left(M_{\mathrm{loc}}(A)\right)\right] = M_{\mathrm{loc}}\left(M_{\mathrm{loc}}(A)\right)$.

2. By a straightforward extension of Semadeni's theorem,

$$\lim_{\rightarrow}\left(C(\beta X) \otimes M_n\right) \;\cong\; C(\lim_{\leftarrow} \beta X) \otimes M_n \,,$$

   for every full matrix algebra M_n. Therefore, $M_{\mathrm{loc}}(C_0(Y) \otimes M_n) = C(\Delta) \otimes M_n$, which is a type I AW*-algebra [6]. Hence, the gap between the local multiplier algebras of $C([0,1]) \otimes K(H)$ can be realised only with infinite-dimensional Hilbert spaces H.

3. On the other hand, notwithstanding Theorem 2.1, for infinite-dimensional H and under certain restrictions on the topology of Y, it can happen that there is no gap between the local multiplier algebras of $C_0(Y) \otimes K(H)$. For example, if Y is discrete, then $C_0(Y)$ has no nontrivial essential ideals, and neither does $C_0(Y) \otimes K(H)$; thus,

$$M_{\mathrm{loc}}(C_0(Y) \otimes K(H)) = M(C_0(Y) \otimes K(H)) = C_b(Y, B(H)_{*-\mathrm{st}}).$$

   But in this case $C_b(Y, B(H)_{*-\mathrm{st}}) = C_b(Y, B(H)) = \prod_{y \in Y} B(H)$, an injective von Neumann algebra. Therefore,

$$M_{\mathrm{loc}}(C_0(Y) \otimes K(H)) = M_{\mathrm{loc}}(M_{\mathrm{loc}}(C_0(Y) \otimes K(H))) = I(C_0(Y) \otimes K(H)).$$

However, the equality between $M_{\mathrm{loc}}(C_0(Y) \otimes K(H)$ and $I(C_0(Y) \otimes K(H))$ can fail to hold when Y is Stonean [4, Theorem 6.13].

4. Since this paper was submitted, Theorem 2.1 was also obtained by P. Ara and M. Mathieu [4, Remark 6.15(2)] by different methods.

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