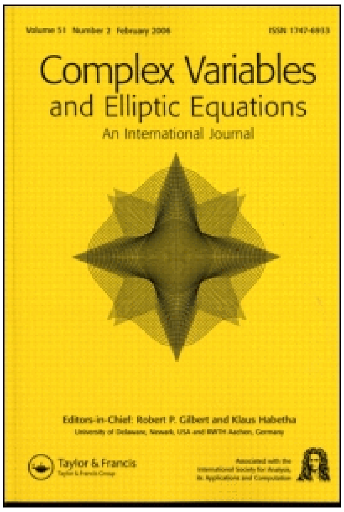

Complex Variables and Elliptic Equations
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713455999

On the stable rank of Q<i>p</i> H

Jordi Pau [a]; Daniel Suárez [b]
[a] Departament de Matemàtica Aplicada i Anàlisi, Universitat de Barcelona, Barcelona, Spain [b] Departament de Matemàtiques, Universitat Autònoma de Barcelona, Bellaterra, Spain

Online Publication Date: 01 October 2008

To cite this Article Pau, Jordi and Suárez, Daniel(2008)'On the stable rank of Q<i>p</i> H',Complex Variables and Elliptic Equations,53:10,957 — 968

To link to this Article: DOI: 10.1080/17476930802272986

URL: http://dx.doi.org/10.1080/17476930802272986

Complex Variables and Elliptic Equations
Vol. 53, No. 10, October 2008, 957–968

Taylor & Francis
Taylor & Francis Group

On the stable rank of $Q_p \cap H^\infty$

Jordi Pau[a*] and Daniel Suárez[b]

[a]*Departament de Matemàtica Aplicada i Anàlisi, Universitat de Barcelona, Barcelona, Spain;* [b]*Departament de Matemàtiques, Universitat Autònoma de Barcelona, Bellaterra, Spain*

Communicated by R.P. Gilbert

(*Final version received 9 June 2008*)

Let $0<p<1$. The stable rank of the Banach algebra $Q_p \cap H^\infty$ is 1 if given f_1, f_2 in $Q_p \cap H^\infty$ such that

$$\inf_{z\in\mathbb{D}} (|f_1(z)| + |f_2(z)|) > 0,$$

there exists g in $Q_p \cap H^\infty$ such that $f_1 + gf_2$ is invertible in $Q_p \cap H^\infty$. As a partial answer to this problem, we prove the result when f_1 is an inner function in Q_p.

Keywords: Q_p-spaces; stable rank; Carleson measures

AMS Subject Classifications: 30H05; 32A37; 46J15

1. Introduction

Let B be a commutative ring with identity. An element $a=(a_1,\ldots,a_n)\in B^n$ is *unimodular* if there is $b=(b_1,\ldots,b_n)\in B^n$ with $\sum_{k=1}^n b_k a_k = 1$. Denote the set of unimodular elements of B^n by $U_n(B)$. An element $a\in U_n(B)$ is *reducible* if there are $x_1,\ldots,x_{n-1}\in B$ such that

$$(a_1 + x_1 a_n,\ a_2 + x_2 a_n, \ldots,\ a_{n-1} + x_{n-1} a_n) \in U_{n-1}(B).$$

The *stable rank* of B, denoted by $\mathrm{sr}(B)$, is the smallest positive integer n such that each $a\in U_{n+1}(B)$ is reducible. This notion was introduced by Bass in [1] to study the stabilization of certain algebraic groups associated to a given ring. It was shown to be a useful concept for analysis after Vaserstein proved that if X is a compact Hausdorff space, the stable rank of the algebra of continuous complex-valued functions on X is $[\dim X/2]+1$, where $\dim X$ is the covering dimension of X and $[t]$ is the integer that satisfies $[t]\le t<[t]+1$ [2].

The clear relation between the condition of stable rank 1 and corona-type theorems made this concept especially suitable for study by the specialists in algebras of analytic functions. One of the first results on this direction was the first proof that the stable rank of the disc algebra is 1 [3]. Maybe the most successful accomplishment in this vein of thought is Treil's proof of $\mathrm{sr}\, H^\infty = 1$ [4]. In the present article, we explore this possibility

*Corresponding author. Email: jordi.pau@ub.edu

ISSN 1747–6933 print/ISSN 1747–6941 online

DOI: 10.1080/17476930802272986
http://www.informaworld.com

for the Banach algebra $Q_p \cap H^\infty$, and give a partial result, by showing that if the first coordinate of a unimodular pair is an inner function, then the pair is reducible.

Let $0 \leq p < \infty$. A function f belongs to Q_p if it is analytic on the unit disc $\mathbb{D}$ and

$$\|f\|_{Q_p}^2 = \sup_{w \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 \, g(z,w)^p \, \mathrm{d}A(z) < +\infty,$$

where $g(z,w) = -\log|\varphi_w(z)|$ is the Green's function on the unit disc with pole at $w \in \mathbb{D}$, $\mathrm{d}A$ is the normalized area measure, and

$$\varphi_w(z) = \frac{w - z}{1 - \bar{w}z}$$

is a Möbius map. The Q_p spaces are conformally invariant, in the sense that if $f \in Q_p$, then

$$\|f \circ \phi\|_{Q_p} = \|f\|_{Q_p}$$

for each automorphism ϕ of the disc. It turns out that $Q_0 = \mathcal{D}$, the Dirichlet space, $Q_1 = BMOA$ and for $p > 1$, $Q_p = \mathcal{B}$, the Bloch space. These spaces have attracted a lot of attention in the past years, and the theory of Q_p functions has been extensively developed. We refer to [5] for more properties of these spaces.

We ask if $\mathrm{sr}(Q_p \cap H^\infty) = 1$. That is, given functions $f_1, f_2 \in Q_p \cap H^\infty$ with

$$\inf_{z \in \mathbb{D}} (|f_1(z)| + |f_2(z)|) > 0, \tag{1}$$

does there exist $g \in Q_p \cap H^\infty$ such that $f_1 + gf_2$ is invertible in $Q_p \cap H^\infty$?

As a partial answer to that question we prove the following result.

THEOREM 1.1 *Let $0 < p < 1$. Let f_1 be an inner function in Q_p and $f_2 \in Q_p \cap H^\infty$ that satisfy* (1). *Then there exists $g \in Q_p \cap H^\infty$ such that $f_1 + gf_2$ is invertible in $Q_p \cap H^\infty$.*

This article is organized as follows. In Section 3, we study solutions of the $\bar{\partial}$-equation, in Section 4 we study some properties of the p-interpolating Blaschke products and we prove Theorem 1.1 in Section 5.

We use the notation $a \lesssim b$ to indicate that there is a constant $C > 0$ such that $a \leq Cb$. Also, we use the symbol C to denote a positive constant whose value may change from line to line. For any arc $I \subset \mathbb{T}$, we denote by $|I|$ its normalized Lebesgue measure and by $S(I)$ the Carleson box based on I:

$$S(I) = \{re^{i\theta} \in \mathbb{D} : 1 - r \leq |I|; \quad e^{i\theta} \in I\}.$$

We also denote by $\rho(z,w)$ the pseudohyperbolic distance between two points z, w of the unit disc, that is $\rho(z,w) = |\varphi_w(z)|$.

2. Preliminary facts

On $Q_p \cap H^\infty$ consider the norm given by

$$\|f\|^2 = \|f\|_\infty^2 + \|f\|_{Q_p}^2.$$

With this norm, $Q_p \cap H^\infty$ is a Banach algebra with invertible group

$$(Q_p \cap H^\infty)^{-1} = Q_p \cap (H^\infty)^{-1}.$$

The Blaschke product with zeros $\{a_n\}$ is

$$B(z) = z^m \prod_{a_n \neq 0} \frac{\bar{a}_n}{a_n} \frac{a_n - z}{1 - \bar{a}_n z},$$

where $\{a_n\}$ is a sequence of points in the unit disc $\mathbb{D}$ satisfying the Blaschke condition $\sum_n (1 - |a_n|^2) < \infty$, and m is the number of indexes n with $a_n = 0$. We denote by $Z(B)$ the sequence $\{a_n\}$ of zeros of B. Given a set $E \subset \mathbb{D}$, the angular and radial projections of E are

$$E_{\text{ang}} = \{|z|: z \in E\} \quad \text{and} \quad E_{\text{rad}} = \{z/|z|: z \in E \setminus \{0\}\}.$$

Our first auxiliary result is

LEMMA 2.1 *Let $f \in H^\infty$ with $\|f\|_\infty \leq 1$. Given $0 < \delta, \ \varepsilon < 1$, there exists $\alpha = \alpha(\delta, \varepsilon)$, with $0 < \alpha < 1$, such that for any Carleson box $S(I)$,*

$$\sup\{|f(z)|: 1 - |z| \geq |I|/4, \quad z \in S(I)\} \geq \delta$$

implies

$$|(E_\alpha)_{\text{ang}}| < \varepsilon |I| \quad \textit{and} \quad |(E_\alpha)_{\text{rad}}| < \varepsilon |I|,$$

where

$$E_\alpha = \{z \in S(I): |f(z)| \leq \alpha\}.$$

Proof The result follows immediately from Lemma 2.1 of [6]. We note also that the estimate for the radial projection can also be deduced from [7, Chapter VIII, Theorem 3.2]. ■

3. p-Carleson measures and the $\bar{\partial}$-equation

Let $p > 0$. We say that a positive Borel measure μ on $\mathbb{D}$ is a p-Carleson measure if

$$\|\mu\|_p = \sup_{I \subset \mathbb{T}} \frac{\mu(S(I))}{|I|^p} < \infty.$$

When $p = 1$, we get the standard definition of a Carleson measure. Also, p-Carleson measures can be described in terms of conformal invariants as those positive measures μ for which

$$\sup_{w \in \mathbb{D}} \int_{\mathbb{D}} \left(\frac{1 - |w|^2}{|1 - \bar{w} z|^2} \right)^p \mathrm{d}\mu(z) < \infty,$$

and this quantity is equivalent to $\|\mu\|_p$ (see [5, Chapter IV]). It is well known that an analytic function f is in Q_p if and only if the measure

$$\mathrm{d}\mu_{f,p}(z) = |f'(z)|^2 (1 - |z|^2)^p \, \mathrm{d}A(z)$$

is p-Carleson. Moreover, this is equivalent for f to have a radial boundary function $f \in Q_p(\mathbb{T})$, where $Q_p(\mathbb{T})$ are the functions $f \in L^2(\mathbb{T})$ with

$$\sup_{I \subset \mathbb{T}} |I|^{-p} \int_I \int_I \frac{|f(\zeta) - f(\eta)|^2}{|\zeta - \eta|^{2-p}} \, |\mathrm{d}\zeta| \, |\mathrm{d}\eta| < \infty.$$

Reciprocally, if $f \in Q_p(\mathbb{T})$ and $\tilde{f}$ is the Poisson integral of f, then

$$|\nabla \tilde{f}(z)|^2 (1 - |z|^2)^p \, \mathrm{d}A(z) \tag{2}$$

is a p-Carleson measure. We refer to [5] for the properties stated before. Let ∂ and $\overline{\partial}$ be the Cauchy–Riemann operators and $\Delta = \partial\overline{\partial}$ (that is, a quarter of the standard Laplacian operator). The following two results were proved in [8] (see also Corollary 7.1.1 of [5]).

LEMMA 3.1 *Let $f \in L^2(\mathbb{T})$. If there is $F \in C^1(\mathbb{D})$ such that*

$$\lim_{r \to 1} F(re^{it}) = f(e^{it}) \ \textit{ for almost every } e^{it} \in \mathbb{T}$$

and $|\nabla F(z)|^2 \, (1 - |z|^2)^p \, \mathrm{d}A(z)$ is a p-Carleson measure, then $f \in Q_p(\mathbb{T})$.

LEMMA 3.2 *Let $0 < p < 1$ and g be a function on $\mathbb{D}$ such that $\mu = |g(z)|^2 \, (1 - |z|^2)^p \, \mathrm{d}A(z)$ is a p-Carleson measure. Then there is a function $f \in C^2(\mathbb{D})$ with boundary values in $Q_p(\mathbb{T}) \cap L^\infty(\mathbb{T})$ such that $\overline{\partial} f(z) = g(z)$ for $z \in \mathbb{D}$ and $\max\{\|f\|_{L^\infty(\mathbb{T})}, \|f\|_{Q_p(\mathbb{T})}\} \le C(\|\mu\|_p)$.*

We will also use several times the following well-known lemma [8]. For completeness we give a proof here.

LEMMA 3.3 *Let $0 < p < 1$. If $\mathrm{d}\mu = |V(z)|^2 \, (1 - |z|^2)^p \, \mathrm{d}A(z)$ is a p-Carleson measure, then $|V(z)| \mathrm{d}A(z)$ is a 1-Carleson measure.*

Proof For any Carleson box $S(I)$, the Cauchy–Schwarz inequality gives

$$\left(\int_{S(I)} |V(z)| \, \mathrm{d}A(z) \right)^2 \lesssim |I|^{2-p} \int_{S(I)} |V(z)|^2 (1 - |z|^2)^p \, \mathrm{d}A(z) \le |I|^2 \|\mu\|_p.$$

■

The following lemma is a modification of a result given by Treil [4].

LEMMA 3.4 *Let $0 < p < 1$, and $V \in C^\infty(\mathbb{D})$ such that*

(a) $\mathrm{d}\mu_1 = |V(z)|^2 \, (1 - |z|^2)^p \, \mathrm{d}A$ *is a p-Carleson measure with* $\|\mu_1\|_p \le K_1$,
(b) $\mathrm{d}\mu_2 = |\partial V(z)| \, (1 - |z|^2) \, \mathrm{d}A$ *is a 1-Carleson measure with* $\|\mu_2\|_1 \le K_2$,
(c) $\sup_{z \in \mathbb{D}} (1 - |z|)^2 |\partial V(z)| \le K_3$ *and* $\sup_{z \in \mathbb{D}} (1 - |z|) |V(z)| \le K_3$.

Then there exists $b \in C^2(\mathbb{D})$ with boundary values in $Q_p(\mathbb{T})$ such that $\overline{\partial} b = V$, and

$$\sup_{z \in \mathbb{D}} |b(z)| \le C,$$

for some positive constant depending only of K_1, K_2 and K_3.

Proof For any $n \ge 1$ let $\phi_n : \mathbb{D} \to [0, 1]$ be a C^∞ function such that

$$\phi_n(z) = \begin{cases} 1 & \text{if } |z| \le 1 - 2^{-n} \\ 0 & \text{if } |z| > 1 - 2^{-(n+1)} \end{cases}$$

and $|\nabla\phi_n(z)| \le C\,2^n$, for some absolute positive constant C. Therefore $(1-|z|)|\nabla\phi_n(z)| \le C$. Now, the function $\phi_n V$ is also in $C(\overline{\mathbb{D}})$, and by condition (a), we can apply Lemma 3.2 to obtain a function $b_n \in C^2(\overline{\mathbb{D}})$ with $\bar\partial b_n = \phi_n V$ and $\max\{\|b_n\|_{L^\infty(\mathbb{T})}, \|b_n\|_{Q_p(\mathbb{T})}\} \le C(K_1)$. Hence, the Poisson integral u_n of b_n is also bounded by $C(K_1)$ in the whole closed disc. We want to show that b_n is bounded on $\mathbb{D}$ independently of n. By Green's formula we have $b_n(z) = u_n(z) - g_n(z)$, where

$$g_n(z) = \int_{\mathbb{D}} \Delta b_n(w) \log\left|\frac{1-\bar z w}{z-w}\right|^2 \mathrm{d}A(w).$$

Since $\Delta b_n = \partial(\phi_n V)$, we have

$$|g_n(z)| \le C\int_{\mathbb{D}} \left(\frac{|V(w)|}{1-|w|} + |\partial V(w)|\right) \log\left|\frac{1-\bar z w}{z-w}\right|^2 \mathrm{d}A(w).$$

Split the integral into $\int_{D_z} + \int_{\mathbb{D}\setminus D_z}$, where $D_z = \{w\colon \rho(w,z) \le 1/2\}$. By condition (c), the first integral is bounded by

$$\int_{D_z} \frac{K_3}{(1-|w|)^2} \log\left|\frac{1-\bar z w}{z-w}\right|^2 \mathrm{d}A(w),$$

which by the conformal invariance of the measure $(1-|w|^2)^{-2}\,\mathrm{d}A(w)$, is bounded by a constant times K_3. Using that $\log x^{-2} \lesssim (1-x^2)$ for $1/2 < x < 1$, the second integral can be estimated by

$$\int_{\mathbb{D}\setminus D_z} \left(\frac{|V(w)|}{1-|w|} + |\partial V(w)|\right) \frac{(1-|z|^2)(1-|w|^2)}{|1-\bar z w|^2}\, \mathrm{d}A(w),$$

which by (a) and (b) is bounded by an absolute constant times $K_1 + K_2$, since by Lemma 3.3, the fact that $|V(z)|^2(1-|z|^2)^p\,\mathrm{d}A(z)$ is a p-Carleson measure implies that $|V(z)|\,\mathrm{d}A(z)$ is a 1-Carleson measure. Hence, we deduce that

$$\sup_{z\in\mathbb{D}} |b_n(z)| \le C.$$

Now we may consider a weak-star limit $b \in L^\infty(\mathbb{D})$ of a suitable subsequence of b_n. Then $\|b\|_{L^\infty(\mathbb{D})} \le C$, has boundary values in $Q_p(\mathbb{T})$ and $\bar\partial b = V$ in the sense of distributions. Since $V \in C^\infty(\mathbb{D})$, we obtain that $b \in C^\infty(\mathbb{D})$ by the hypoellipticity of the operator $\bar\partial$ (see [9, p. 270]). This completes the proof. ∎

4. *p*-interpolating Blaschke products

Let $0 < p \le 1$. We say that a Blaschke product B is a p-interpolating Blaschke product (p-IBP for short) if its zero sequence $Z(B)$ is an interpolating sequence for $Q_p \cap H^\infty$. By a result from [8], this holds if and only if $Z(B)$ is separated and the measure

$$\mu_B = \sum_{z\in Z(B)} (1-|z|^2)^p\, \delta_z$$

is a p-Carleson measure. It is clear that a p-IBP is also a 1-IBP. The following lemma appears in the proof of Lemma 4.1 of [6].

Lemma 4.1 *Let B be a 1-IBP. For each $a \in Z(B)$ there is a curve $\Gamma_a = H_a \cup V_a$ from a to $\mathbb{T}$ consisting of an angular arc H_a and a radial arc V_a with lengths $|H_a|$ and $|V_a|$ majorized by $(1-|a|)$, such that*

$$\rho(\Gamma_a, \Gamma_b) \geq \alpha,$$

where $\alpha > 0$ depends only on $\|\mu_B\|_1$ and the separation constant of $Z(B)$.

Lemma 4.2 *Let B be a 1-IBP and $g \in H^\infty$ with $\|g\|_\infty \leq 1$. Suppose that there are δ, $\gamma > 0$ such that*

$$|B(z)| > \gamma \quad \text{if } |g(z)| < \delta.$$

Consider the curves Γ_a, $a \in Z(B)$, of Lemma 4.1. There is $0 < \delta' \leq \delta$ such that if Ω is any connected component of $\{z : |g(z)| < \delta'\}$, then the number of curves Γ_a which meet Ω is bounded by a constant C independent of Ω.

Proof This result is in the proof of Lemma 4.1 of [6], but we sketch it here for our purposes. By Lemma 2.1 there is $0 < \delta' \leq \delta$ such that if Q is a Carleson box and any of the sets

$$\{r : \exists re^{it} \in Q, |g(re^{it})| < \delta'\}, \quad \{e^{it} : \exists re^{it} \in Q, |g(re^{it})| < \delta'\},$$

has length bigger than $|Q|/8$ then

$$|g(z)| < \delta \quad \text{ on } \{z \in Q : 1 - |z| \geq |Q|/4\}. \tag{3}$$

Let Ω be any connected component of $\{z : |g(z)| < \delta'\}$, and let Q be a minimal Carleson box containing Ω. We note that if $a \in Z(B)$ is such that $1 - |a| \geq |Q|/100$ and $\Gamma_a \cap \Omega \neq \emptyset$, then Γ_a must meet

$$\{z \in Q : 1 - |z| \geq |Q|/100\}.$$

Since this set has pseudohyperbolic diameter bounded away from 1, Lemma 4.1 implies that the number of such zeros is bounded by a constant C.

If $a \in Z(B)$ with $1 - |a| < |Q|/100$, an argument of Treil [4] will show that Γ_a cannot meet Ω. Indeed, if $\Gamma_a \cap \Omega \neq \emptyset$, taking the Carleson box R_a whose base has the centre at $a/|a|$ and length $|R_a| = 4(1 - |a|)$, the angular or the radial projection of $R_a \cap \Omega$ must have length $\geq |R_a|/8$. Hence, by (3), $|g(z)| < \delta$ on $\{z \in R_a : 1 - |z| \geq |R_a|/4\}$, and therefore $|g(a)| < \delta$ in contradiction with the fact that a is a zero of B. ■

Proposition 4.3 *Let $0 < p < 1$, B be a p-IBP and $g \in Q_p \cap H^\infty$, with $\|g\|_\infty \leq 1$. Suppose that there are δ, $\gamma > 0$ such that*

$$|B(z)| > \gamma \quad \text{if } |g(z)| < \delta. \tag{4}$$

Then there is $0 < \delta' \leq \delta$, a function $h \in Q_p$ and a suitable branch of $\log B$ on $\{|g| < \delta'\}$ such that

$$|\operatorname{Re} h(z)| \leq C_1, \quad z \in \mathbb{D}, \quad \text{and}$$
$$|\log B(z) - h(z)| \leq C_2 \quad \text{if } |g(z)| < \delta',$$

for some positive constants C_1, C_2 depending only on δ, γ and $\beta := \|\mu_B\|_p$.

Proof For each $a \in Z(B)$, consider the slits Γ_a given by Lemma 4.1. Fix $\tau = \tau(\gamma) < \min\{\alpha/4, \gamma/2\}$ and let

$$\tilde{\Gamma}_a = \{z \in \mathbb{D} : \rho(z, \Gamma_a) \le \tau\}.$$

Since $\tau < \alpha/4$, Lemma 4.1 implies that $\tilde{\Gamma}_a \cap \tilde{\Gamma}_b = \emptyset$ if a, $b \in Z(B)$, $a \neq b$. Since $\tau < \gamma/2$, then (4) and the Schwarz–Pick lemma gives

$$\{z\colon \rho(z, Z(B)) \le \tau\} \cap \{z\colon |g(z)| < \delta\} = \emptyset. \tag{5}$$

For each $a \in Z(B)$ take a branch of $\log \varphi_a(z)$ defined in $\mathbb{D} \backslash \Gamma_a$ that jumps $2\pi i$ when z crosses $\Gamma_a \backslash \{a\}$. By regularization we can obtain a smooth function ψ_a on $\mathbb{D}$ with

(A) $\psi_a \equiv \log \varphi_a$ in $\mathbb{D} \setminus \tilde{\Gamma}_a$,
(B) $\operatorname{Re} \psi_a \equiv \log |\varphi_a|$ in $\{z\colon \rho(z, a) \ge \tau/4\}$,
(C) $|\psi_a| \le C$ and $0 \le \operatorname{Im} \psi_a \le 2\pi$,
(D) $(1 - |z|)|\nabla \psi_a(z)| \le C$,
(E) $(1 - |z|)^2 |\Delta \psi_a(z)| \le C$,

where $C = C(\tau)$ is a constant depending only on τ.

The Blaschke condition implies that the sum $\psi(z) := \sum_{a \in Z(B)} \psi_a(z)$ converges uniformly on compact subsets of the disc. Also, $|\operatorname{Re} \psi(z)| \le C(\beta)$ for any $z \in \mathbb{D}$.

Let Ω be any connected component of $\{|g| < \delta'\}$, and fix $z_0 \in \Omega$. Let $\log B$ be a suitable branch of the logarithm of B on Ω with $\log B(z_0) = \psi(z_0)$. By (5) and (C),

$$|\log \varphi_a - \psi_a| \le 4\pi \quad \text{on} \quad \{|g| < \delta'\},$$

and since Ω is an arbitrary component, by Lemma 4.2 we have that

$$|\log B(z) - \psi(z)| \le C_1(\tau) \qquad \text{if } |g(z)| < \delta'. \tag{6}$$

Since $\{\tilde{\Gamma}_a : a \in Z(B)\}$ are pairwise disjoints, it follows from (D) and (E) that

$$(1 - |z|)\,|\nabla \psi(z)| \le C = C(\tau) \quad \text{and} \quad (1 - |z|)^2\,|\Delta \psi(z)| \le C = C(\tau). \tag{7}$$

Since the support of $\bar{\partial}\psi$ is contained in $\cup \tilde{\Gamma}_a$ and $\sum_{a \in Z(B)} (1 - |a|)^p \delta_a$ is a p-Carleson measure, then $|\bar{\partial}\psi(z)|^2 (1 - |z|^2)^p \,\mathrm{d}A(z)$ is a p-Carleson measure. In fact, let $S(I)$ be a Carleson box. It is easy to see that for $0 \le s \le 1$ one has

$$\int_{\tilde{\Gamma}_a} \frac{\mathrm{d}A(z)}{(1 - |z|^2)^{2-s}} \lesssim (1 - |a|^2)^s, \quad a \in Z(B).$$

Also, since $\tilde{\Gamma}_a \cap \tilde{\Gamma}_b = \emptyset$ for $a \neq b$, there are at most M points $a \in Z(B) \backslash S(2I)$ with $\tilde{\Gamma}_a \cap S(I) \neq \emptyset$. Therefore, by (7) we have

$$\begin{aligned}
\int_{S(I)} |\bar{\partial}\psi(z)|^2 (1 - |z|^2)^p \,\mathrm{d}A(z) &= \sum_{a \in Z(B)} \int_{S(I) \cap \tilde{\Gamma}_a} |\bar{\partial}\psi(z)|^2 (1 - |z|^2)^p \,\mathrm{d}A(z) \\
&\le \sum_{a \in Z(B) \cap S(2I)} \int_{\tilde{\Gamma}_a} \frac{\mathrm{d}A(z)}{(1 - |z|^2)^{2-p}} + \sum_{a \in Z(B) \backslash S(2I)} |I|^p \int_{S(I) \cap \tilde{\Gamma}_a} \frac{\mathrm{d}A(z)}{(1 - |z|^2)^2} \\
&\lesssim \sum_{a \in Z(B) \cap S(2I)} (1 - |a|^2)^p + M|I|^p \lesssim |I|^p.
\end{aligned}$$

In a similar way, since $\sum_{a\in Z(B)}(1-|a|)\delta_a$ is a 1-Carleson measure, it follows that $|\Delta\psi(z)|$ $(1-|z|)\mathrm{d}A(z)$ is a 1-Carleson measure. Hence $\bar{\partial}\psi$ satisfies the hypothesis of Lemma 3.4, which gives us a function b such that $\bar{\partial}b=\bar{\partial}\psi$,

$$\sup_{z\in\mathbb{D}}|b(z)|\le C_1(\beta,\tau)\quad\text{and}\quad \|b\|_{Q_p(\mathbb{T})}\le C_2(\beta,\tau).$$

Hence, the function $h=\psi-b$ is analytic. Since $\|b\|_{Q_p(\mathbb{T})}<\infty$ and $|\nabla\psi(z)|^2\,(1-|z|^2)^p\,\mathrm{d}A(z)$ is a p-Carleson measure, Lemma 3.1 tells us that h has boundary values in $Q_p(\mathbb{T})$. Since h is analytic, h is in Q_p. Finally, (6) yields

$$|\log B(z)-h(z)|\le C_2(\beta,\tau)\ \text{ if }\ |g(z)|<\delta'.$$

■

5. Proof of Theorem 1.1

Recall that a bounded analytic function in the unit disc is called *inner* if it has radial limits of modulus 1 along almost every radius. Let $\mathcal{B}_p$ denote the class of Blaschke products B for which the measure

$$\sum_{z\in Z(B)}(1-|z|^2)^p\,\delta_z$$

is a p-Carleson measure. By Theorem 5.2.1 of [5] we have that the functions in $\mathcal{B}_p$ are just the inner functions that are in Q_p. It can be noted that any Blaschke product B in $\mathcal{B}_p$ is a finite product of p-interpolating Blaschke products. Indeed, if $\sum_{z\in Z(B)}(1-|z|^2)^p\,\delta_z$ is a p-Carleson measure, then $\sum_{z\in Z(B)}(1-|z|^2)\delta_z$ is a 1-Carleson measure, and therefore B is a finite product of interpolating Blaschke products b_i (see, for example, [10]). But it is clear that the measure $\sum_{z\in Z(b_i)}(1-|z|^2)^p\,\delta_z$ is p-Carleson, and then b_i is actually a p-interpolating Blaschke product.

It is enough to prove the theorem when f_1 is a p-interpolating Blaschke product. Indeed, suppose that $f_1\in\mathcal{B}_p$ is such that (f_1,f_2) is a corona pair. Then $f_1=\prod_{i=1}^N b_i$, where N is some positive integer and each b_i is a p-IBP. If the theorem holds for each pair (b_i,f_2), $1\le i\le N$, we can find functions $k_i\in Q_p\cap H^\infty$ such that $b_i+f_2k_i\in(Q_p\cap H^\infty)^{-1}$ for $1\le i\le N$. Therefore, there is some $k\in Q_p\cap H^\infty$ such that

$$\left(\prod_{i=1}^N b_i\right)+kf_2=\prod_{i=1}^N(b_i+k_if_2)\in\left(Q_p\cap H^\infty\right)^{-1}.$$

So, let $f_1=B$ be a p-IBP and $f_2\in Q_p\cap H^\infty$ with

$$\inf_{z\in\mathbb{D}}(|B(z)|+|f_2(z)|)>2\delta>0,$$

and we can assume that $\|f_2\|_\infty\le 1$. Observe that

$$|B(z)|>\delta\quad\text{if } |f_2(z)|<\delta. \tag{8}$$

By Proposition 4.3, there is $0<\delta'\le\delta$ and $h\in Q_p$, such that $|\mathrm{Re}\,h|\le C_1$ and

$$|\log B(z)-h(z)|\le C_2\quad\ \text{if } |f_2(z)|<\delta'. \tag{9}$$

The p-Carleson measure characterization of Q_p shows that $e^{-h} \in Q_p \cap H^\infty$. Now, in order to find a function $g \in Q_p \cap H^\infty$ with $B + f_2 g$ invertible in $Q_p \cap H^\infty$, it is enough to find $k \in Q_p \cap H^\infty$ with

$$g = \frac{(e^k - Be^{-h})}{f_2} \in Q_p \cap H^\infty.$$

To do this, take a radial C^∞ function φ in a neighbourhood of $\overline{\mathbb{D}}$ such that $0 \le \varphi \le 1$, $\varphi \equiv 0$ on $|z| \ge \delta'$ and $\varphi \equiv 1$ on $|z| < \delta'/2$ with $|\nabla \varphi| \le K_1$ and $|\Delta\varphi| \le K_2$ for some positive constants K_1 and K_2 depending only on δ'. Let $\Psi(z) = \varphi(f_2(z))$. Then Ψ is of class C^∞ in $\mathbb{D}$, $0 \le \Psi \le 1$, $\Psi(z) = 0$ on $\{|f_2| \ge \delta'\}$, and $\Psi(z) = 1$ on $\{|f_2| < \delta'/2\}$. Also, note that

$$|\nabla\Psi| \le |(\nabla\varphi) \circ f_2| \, |f_2'| \le K_1 |f_2'| \tag{10}$$

and

$$|\Delta\Psi| \le |(\nabla\varphi) \circ f_2| \, |f_2'|^2 + |(\Delta\varphi) \circ f_2| \, |f_2''| \le K_1 |f_2'|^2 + K_2 |f_2''|. \tag{11}$$

Since $f_2 \in H^\infty$, these two inequalities say that

$$|\nabla\Psi(z)| \, (1 - |z|^2) \le K_3 \quad \text{and} \quad |\Delta\Psi(z)| \, (1 - |z|^2)^2 \le K_3, \tag{12}$$

where $K_3 > 0$ depends only on δ' and $\|f_2\|_\infty$. Since $|f_2'(z)|^2 \, (1 - |z|^2)^p \, \mathrm{d}A(z)$ is a p-Carleson measure, (10) tells us that

$$|\nabla\Psi(z)|^2 \, (1 - |z|^2)^p \, \mathrm{d}A(z) \tag{13}$$

is a p-Carleson measure.

Furthermore, since $f_2 \in Q_p$, Theorem 1.4.1 of [5] says that $|f_2''|^2 \, (1 - |z|^2)^{2+p} \, \mathrm{d}A$ is a p-Carleson measure, which by Lemma 3.3 implies that $|f_2''| \, (1 - |z|^2) \mathrm{d}A$ is a 1-Carleson measure. In addition, since $f_2 \in H^\infty$, $|f_2'|^2 \, (1 - |z|^2) \mathrm{d}A$ is a 1-Carleson measure [7, VI, Theorem 3.4], and consequently (11) yields

$$|\Delta\Psi|(1 - |z|^2)\mathrm{d}A \tag{14}$$

is a 1-Carleson measure.

Consider the function

$$V(z) = \frac{\log\big(B(z)e^{-h(z)}\big)}{f_2(z)} \, \bar\partial\Psi(z).$$

Now we are going to check that V satisfies the assumptions of Lemma 3.4. Observe that

$$\Delta\Psi = \bar\partial\Psi = 0 \;\; \text{on} \;\; \{|f_2| \ge \delta'\} \cup \{|f_2| < \delta'/2\}, \tag{15}$$

which together with (9) gives

$$\left| \frac{\log(Be^{-h})}{f_2} \, \bar\partial\Psi \right| \le \frac{2C_2}{\delta'} \, \big|\bar\partial\Psi\big|. \tag{16}$$

By (13) then $|V(z)|^2 \, (1 - |z|^2)^p \, \mathrm{d}A(z)$ is a p-Carleson measure, which is (a) of Lemma 3.4. To see that $|\partial V(z)|$, $(1 - |z|^2) \, \mathrm{d}A(z)$ is a 1-Carleson measure we first compute ∂V,

$$\partial V = -\frac{f_2'}{f_2^2} \log(Be^{-h}) \, \bar\partial\Psi + \frac{\partial\big(\log(Be^{-h})\big)}{f_2} \, \bar\partial\Psi + \frac{\log(Be^{-h})}{f_2} \, \Delta\Psi.$$

Hence, by (8), (9) and (15),

$$|\partial V| \leq \frac{4C_2}{(\delta')^2}\,|f_2'|\,|\bar{\partial}\Psi| + \frac{2C_2}{(\delta')^2}\,(|B'| + |h'|)\,|\bar{\partial}\Psi| + \frac{2C_2}{\delta'}\,|\Delta\Psi|,$$

which together with (12) gives

$$(1-|z|^2)|\partial V| \lesssim |f_2'| + |B'| + |h'| + (1-|z|^2)|\Delta\Psi|. \tag{17}$$

Since f_2, B and h are in Q_p, using Lemma 3.3 for the first three summands in the above sum, and (14), it follows that $(1-|z|^2)|\partial V|\mathrm{d}A$ is a 1-Carleson measure. Also, since Q_p is contained in the Bloch space, (17), (16) and (12) yield

$$(1-|z|^2)|V(z)| \leq C \quad \text{and} \quad (1-|z|^2)^2|\partial V(z)| \leq C.$$

Then the function V satisfies the assumptions of Lemma 3.4, and hence there is a function $u \in C^2(\mathbb{D}) \cap L^\infty(\mathbb{D})$ with boundary values in $Q_p(\mathbb{T})$ such that

$$\bar{\partial}u = \frac{\log(Be^{-h})}{f_2}\,\bar{\partial}\Psi.$$

Therefore, the function $k := \Psi\,\log(Be^{-h}) - uf_2$ is analytic. Since

$$\|k\|_\infty \leq \sup_{|f_2|<\delta'} |\log B - h| + \|u\|_\infty\,\|f_2\|_\infty,$$

we have $k \in H^\infty$, and to see that k is also in Q_p, it suffices to check that $k|_\mathbb{T}$ defined by

$$k|_\mathbb{T}(e^{it}) := \lim_{r\to 1} k(re^{it})$$

is in $Q_p(\mathbb{T})$. Clearly $uf_2 \in Q_p(\mathbb{T})$, and this implies that $\Psi\,\log(Be^{-h})$ has radial limits almost everywhere. Therefore, by Lemma 3.1 it is enough to prove that

$$|\nabla(\Psi\log(Be^{-h}))|^2\,(1-|z|^2)^p\,\mathrm{d}A(z) \tag{18}$$

is a p-Carleson measure. But since Ψ and $\nabla\Psi$ are supported on $\{|f_2|<\delta'\}$, (8) and (9) give

$$\begin{aligned} |\nabla(\Psi\log(Be^{-h}))| &\lesssim |\nabla\Psi|\,|\log(Be^{-h})| + \Psi\left|\frac{B' - Bh'}{B}\right| \\ &\lesssim |\nabla\Psi|\,C_2 + \frac{(|B'|+|h'|)}{\delta}, \end{aligned}$$

and (18) follows from (13). Therefore, k is in $Q_p \cap H^\infty$ and only remains to prove that

$$g = \frac{e^k - Be^{-h}}{f_2} \in Q_p \cap H^\infty.$$

It is clear that g is bounded on $\{|f_2| \geq \delta'/2\}$, and since

$$g = \frac{(e^{-uf_2} - 1)}{uf_2}\,uBe^{-h} \ \text{ on } \{|f_2| < \delta'/2\},$$

the boundedness of g follows from the inequality $|(e^{-x}-1)/x| \leq e^{|x|}$. To see that g is also in Q_p, let $k_1 = \Psi\log(Be^{-h}) - \tilde{u}f_2$, where $\tilde{u}$ is the Poisson integral of the boundary values of u, and consider the function

$$g_1 = \frac{e^{k_1} - Be^{-h}}{f_2}.$$

Since g and g_1 have the same boundary values and g is analytic, by Lemma 3.1 it is enough to show that $|\nabla g_1(z)|^2(1-|z|^2)^p\,\mathrm{d}A(z)$ is a p-Carleson measure.

Since by (2), $|\nabla\tilde{u}(z)|^2\,(1-|z|^2)^p\,\mathrm{d}A(z)$ is a p-Carleson measure, (18) and the fact that $f_2\in Q_p\cap H^\infty$ imply that so is $|\nabla k_1(z)|^2\ (1-|z|^2)^p\,\mathrm{d}A(z)$. Therefore, if $S(I)$ is a Carleson box,

$$\begin{aligned}
&\int_{S(I)\cap\{|f_2|\geq(\delta'/2)\}}|\nabla g_1(z)|^2\,(1-|z|^2)^p\,\mathrm{d}A(z)\\
&\quad\leq\frac{C}{(\delta')^2}\int_{S(I)}(|\nabla k_1(z)|^2+|(Be^{-h})'(z)|^2+|f_2'(z)|^2)\,(1-|z|^2)^p\,\mathrm{d}A(z)\\
&\quad\leq\frac{C}{(\delta')^2}\,|I|^p,
\end{aligned}$$

since f_2 and Be^{-h} are in Q_p. When $|f_2|<\delta'/2$, using the inequality

$$\left|\left(\frac{e^{-x}-1}{x}\right)'\right|=\left|\frac{1-e^{-x}-xe^{-x}}{x^2}\right|\leq e^{|x|},$$

and rewriting $g_1=((e^{-\tilde{u}f_2}-1)/\tilde{u}f_2)\tilde{u}Be^{-h}$, we obtain

$$\int_{S(I)\cap\{|f_2|<\frac{\delta'}{2}\}}|\nabla g_1(z)|^2\,(1-|z|^2)^p\,\mathrm{d}A(z)\leq C|I|^p,$$

which completes the proof. ■

6. Final remarks

To determine whether $\mathrm{sr}(Q_p\cap H^\infty)=1$ is a subtler problem than in the case of H^∞, mainly because the inner factor of a function in $Q_p\cap H^\infty$ does not need to be in Q_p. For this reason, the usual methods to show that it is enough to consider f_1 an inner function in the algebra, or even a finite Blaschke product, do not work in this setting. Probably, the key obstruction to prove $\mathrm{sr}(Q_p\cap H^\infty)=1$ is the problem that we pose below.

If A is a commutative Banach algebra with identity, it is shown in [11] that, for every $g\in A$, the set $\{f:(f,g)$ is reducible$\}$ is closed in the set $\{f:(f,g)$ is a corona pair$\}$. Using this fact for the Banach algebra $Q_p\cap H^\infty$, together with Theorem 1.1, this immediately says that if (f,g) is a corona pair and f is in the closure of

$$\mathcal{B}_p\mathcal{I}:=\{bh:b\in\mathcal{B}_p,\quad h\in(Q_p\cap H^\infty)^{-1}\}$$

then (f,g) is reducible. This leads to the following question.

Question: Is $\mathcal{B}_p\mathcal{I}$ dense in $Q_p\cap H^\infty$?

Note that an affirmative answer to that question would imply that the stable rank of the algebra $Q_p\cap H^\infty$ is one.

Acknowledgements

Both authors share Grant 2005SGR00774. Also, J. Pau is partially supported by DGICYT Grant MTM2005-08984-C02-02, and D. Suárez is supported by the Ramón y Cajal program and DGICYT Grant MTM2005-00544.

References

[1] H. Bass, *K-theory and stable algebra*, Publ. Math. Inst. Hautes Études Sci. 22 (1964), pp. 5–60.
[2] L.N. Vaserstein, *Stable range of rings and dimensionality of topological spaces*, Funct. Anal. Appl. 5 (1971), pp. 102–110.
[3] P.W. Jones, D. Marshall, and T. Wolff, *Stable rank of the disc algebra*, Proc. Amer. Math. Soc. 96 (1986), pp. 603–604.
[4] S. Treil, *The stable rank of the algebra H^∞ equals 1*, J. Funct. Anal. 109 (1992), pp. 130–154.
[5] J. Xiao, *Holomorphic Q-classes*, *Lecture Notes in Mathematics 1767*, Springer-Verlag, Berlin, 2001.
[6] A. Nicolau and D. Suárez, *Approximation by invertible functions of H^∞*, Math. Scand. 99 (2006), pp. 287–319.
[7] J.B. Garnett, *Bounded Analytic Functions*, Academic Press, New York, 1981.
[8] A. Nicolau and J. Xiao, *Bounded functions in Mobius invariant Dirichlet spaces*, J. Funct. Anal. 150 (1997), pp. 383–425.
[9] G.B. Folland, *Introduction to Partial Differential Equations*, Princeton University Press, Princeton, 1976.
[10] P.J. McKenna, *Discrete Carleson measures and some interpolating problems*, Michigan Math. J. 24 (1977), pp. 311–319.
[11] G. Corach and D. Suárez, *On the stable range of uniform algebras and H^∞*, Proc. Amer. Math. Soc. 98 (1986), pp. 607–610.