International Journal of Geometric Methods in Modern Physics
Vol. 13, No. 3 (2016) 1650025 (20 pages)

DOI: 10.1142/S0219887816500250

Geometric probability theory and Jaynes's methodology

Federico Holik

Center Leo Apostel for Interdisciplinary Studies
Brussels Free University, Krijgskundestraaat 33, 1160 Brussels, Belgium

Department of Mathematics, Brussels Free University
Krijgskundestraat 33, 1160 Brussels, Belgium

National University La Plata, C.C. 727 - 1900 La Plata, Argentina

CONICET IFLP-CCT, C.C. 727 - 1900 La Plata, Argentina
holik@fisica.unlp.edu.ar

Cesar Massri

Department of Mathematics
University of Buenos Aires, Buenos Aires, Argentina
CONICET IMAS, Buenos Aires, Argentina
cmassri@dm.uba.ar

A. Plastino

National University La Plata, C.C. 727 - 1900 La Plata, Argentina

CONICET IFLP-CCT, C.C. 727 - 1900 La Plata, Argentina
angeloplastino@gmail.com

Received 29 January 2015
Accepted 24 November 2015
Published 19 January 2016

We provide a generalization of the approach to geometric probability advanced by the great mathematician Gian Carlo Rota, in order to apply it to generalized probabilistic physical theories. In particular, we use this generalization to provide an improvement of the Jaynes' MaxEnt method. The improvement consists in providing a framework for the introduction of symmetry constraints. This allows us to include group theory within MaxEnt. Some examples are provided.

Keywords: Maximum entropy principle; generalized probabilistic theories; geometric probability; symmetries in quantum mechanics.

Mathematics Subject Classification 2010: 81P16, 81P45, 81P50, 94A17

1. Introduction

Jaynes' MaxEnt approach is a statistical approach in which probability notions become essential [1–3]. Thus, new viewpoints regarding probability are potentially capable of modifying the MaxEnt approach. We center our present efforts on the notion of geometric probability, characterized by Gian Carlo Rota as the study of invariant measures [4, 5]. This idea has led to interesting mathematical problems, which have defined a rich field of study. *In this work, we provide a generalization of the Rota's axioms in order to find a physical characterization of the problem of looking for generalized probabilities in the spirit of Jaynes's MaxEnt approach.* As it is well known, this technique relies on the determination of the less unbiased distribution compatible with the known data, by appealing to the maximization of the entropy [1, 2], and has manifold applications in diverse fields of research [6–22] (see [3] for a complete review). Our methodology can be used to find a derivation of both classical and quantum statistical mechanics as well.

The study of the algebraic and geometric properties of different physical theories showed that their formal structure can be very different. Indeed, as Birkhoff and von Neumann showed [38], the empirical propositions associated to a classical system can be naturally represented as a Boolean lattice which is orthocomplemented and distributive (a lattice is a partially ordered set with unique least upper bounds and greatest lower bounds. For details see, for instance, [23, 26]). This is related to the fact that classical observables are represented as functions over the phase space and form a commutative algebra. But this is no longer the case for a quantum system. Due to the non-commutative nature of the algebra of observables of quantum theory (which is generated by the set of bounded operators $\mathcal{B}(\mathcal{H})$ acting on a Hilbert space $\mathcal{H}$), empirical propositions regarding quantum systems are represented by projection operators, which are in one-to-one correspondence to closed subspaces related to a projective geometry [25, 24]. In this way, empirical propositions regarding a quantum system form a lattice which is not distributive, and thus it is not Boolean: this fact expresses the structural difference between classical and quantum theories.

As the story progressed, in a series of papers Murray and von Neumann searched for algebras more general than $\mathcal{B}(\mathcal{H})$ [39–42]. They actually found more general algebras that can be classified. The dimensionality function allows to place them in three types of groups, nowadays known as Type I, Type II and Type III factors, together with their associated orthomodular lattices. Then, it turns out that the Boolean lattice associated to classical mechanics and the projection lattice of a Hilbert spaces are particular cases of orthomodular lattices. The first one is the well-known Boolean algebra, and the latter belongs to a Type I factor (Type I_n for finite-dimensional Hilbert spaces and Type I_∞ for infinite-dimensional models), which are algebras isomorphic to the set of bounded operators on a Hilbert space. This is not just a mathematical curiosity! A rigorous approach to the study of quantum systems with infinite degrees of freedom shows that Type III factors must be used in the axiomatic formulation of relativistic quantum mechanics (QM) [44, 45], and a similar situation holds in algebraic quantum statistical mechanics [32, 46]. Thus, the use of the more general algebras found by Murray and von Neumann is unavoidable in a rigorous approach to physical theories of interest. States can be represented as measures over lattices. In this way, different kinds of measures over orthomodular lattices correspond to different physical theories:

- States of classical systems are measures over Boolean lattices. Notice that, in this case, states are mathematically equivalent to probability measures written in the form of the Kolmogorov's axioms, which can be considered in turn, as a particular case of classical measure theory.
- States of non-relativistic quantum systems are measures over lattices originated in Type I factors (i.e. lattices of projection operators over a Hilbert space). These lattices are non-distributive.
- States of relativistic quantum systems are measures over lattices originated in local regions of space-time. These lattices are non-distributive and can be non-Hilbertian, in the sense of being Type III factors [43–45, 32].
- States in algebraic quantum statistical mechanics can be measures over orthomodular lattices originating in algebras which are not Type I factors [46, 32].
- States of a general theory can be described using measures over orthomodular lattices. All the above examples fall into this category. Notice that all the models described by the classical probability calculus (in the form of the Kolmogorov's axioms) fall into this setting as special cases too.

Thus, states of physical theories of interest can be represented as measures over orthomodular lattices. But these measures are not necessarily defined over Boolean algebras (as is the case of the standard measure theory, in which measures are functions over a sigma-algebra). The algebras involved can be non-Boolean (because of being non-distributive). Thus, these measures are sometimes called *generalized measures* [32]. Besides, because of its resemblance to the axioms of Kolmogorov, these generalized states are said to define a *non-commutative probability calculus*. For the standard quantum mechanical case, they are called *quantum probabilities* [50–52, 49, 53–55]. While this terminology is quite natural and it is well established in the literature specialized in the foundations of QM, some researchers reject it because it leads them to confusion. Thus, in this work we will refer to the probability measures defined by states appearing in QM and other theories as *generalized measures*.

As states define probabilities which can be tested empirically, the above discussion shows that there exists a deep connection between states of the most important physical theories, measure theory (in a generalized sense), and probability. As a result, if we want to build a systematic and rigorous approach to the study of the actions of groups in the MaxEnt approach, we are forced to specify how these notions can be described in the general setting of measures over orthomodular lattices. The word "rigorous" here is used in the following sense: to establish a methodology on solid mathematical and formal grounds. This ground is provided in our case by the theory of generalized measures in orthomodular lattices [23, 33, 35]. It is important to mention that the problem of developing a method for performing inference in situations involving additional symmetries that was studied in previous works, focusing on the validity, justification and rigor of the MaxEnt approach (see for example [57, 58, 56, 59–69]). With regards to the action of groups representing general symmetries, these references *do not go further than analyzing some examples*, and while it is suggested (as in [56]) that the method could be extended in general, the technicalities involved are not specified. This is not enough for building a systematic method, because, for example, it does not allow one to pose the problem of the existence of solutions on a solid mathematical ground. As we show in this work, this can be done with our theoretic constructs.

In geometric probability theory, it is usually assumed that measures are defined over Boolean algebras: the algebra of subsets of a certain space (for example, Euclidean space). But as we have seen above, if we want to apply it to physical theories, we cannot restrict ourselves to Boolean lattices but to more general orthomodular ones. Thus, in this work, we propose a generalization of the Rotta problem to measures over general orthomodular lattices and group actions acting as automorphisms of these lattices. This is done in order to cope with physical theories which depart from the classical probability calculus. Thus, *our treatment reformulates the MaxEnt approach in geometric probability terms, allowing for the inclusion of group actions representing physical symmetries*. Within this framework, states of a physical system are regarded as *invariant measures* over general orthomodular lattices. The determination of invariant measures under the action of groups representing physical symmetries is of interest in many research fields, as for example, in the problem of the determination of equilibrium states in equilibrium statistical mechanics [28–30]. We also provide an improvement on the treatment of constraints by formulating the problem in the rigorous basis of measure theory. We allow for them to exhibit a more general character than mere mean values. We show as well that the introduction of group actions reduces the dimensionality of the mathematical variety on which the maximization process takes place. This economizes computational resources. We demonstrate that this economization can be estimated for certain examples. Finally, we provide some examples and specify conditions under which solutions for our method exist.

The paper is organized as follows. In Sec. 2, we introduce the elementary notions of geometric probability theory following Rota [4].[a] In Sec. 3, we review (i) event structures appearing in both quantum and classical mechanics and (ii) their associated probabilities. This is done more general probabilistic settings as well. In Sec. 4, we propose a generalization of geometric probability theory which allows one to describe physical systems. In Sec. 5, we explain how covariance conditions and physical symmetries can be accommodated by our conceptual framework. In Secs. 6 and 7, we show how to describe, within our framework, quantum coherent states and the correlations appearing in the no-signal polytope. Finally, we draw some conclusions in Sec. 8.

[a]The reader familiarized with this theory can skip this section.

2. Geometric Probability

In his classical approach to geometric probability [4, 5], Gian Carlo Rota introduces the problem of invariant measures as follows. First, one looks for a measure $\mu : \Sigma \to \mathbb{R}_{\geq 0}$, defined on a sigma algebra $\Sigma \subseteq \mathcal{P}(\mathbb{R}^n)$, satisfying the following axioms:

Axiom 1 (R1).

$$\mu(\emptyset) = 0,$$

where $\emptyset$ denotes the empty set. If A and B are measurable sets:

Axiom 2 (R2).

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B).$$

For Boolean algebras, the above axiom is equivalent to the sum rule

$$\mu(A \cup B) = \mu(A) + \mu(B) \tag{1}$$

for A and B disjoint. The following axiom has to do with the invariance of measures (therefore, the name *invariant measures*):

Axiom 3 (R3). The measure of a set A does not depend on the position of A; in other words, if A can be rigidly transformed into B, then, B and A have the same measure.

Notice that the last axiom involves the action of a group, namely, the Euclidean group E_0 of rotations and translations in Euclidean space. The last axiom specifies a normalization for a given measure; we must pick a special subset and establish its measure. Let us choose the set of parallelotopes P with orthogonal side lengths $x_1, \ldots, x_n$ and impose the constraint:

Axiom 4 (R4).

$$\mu(P) = x_1 x_2 \cdots x_n.$$

The above axioms yield the usual Lebesgue measure on $\mathbb{R}^n$. Rota poses the question of what happens if the normalization Axiom 4 is changed. Instead of Axiom 4, one could use one of the following polynomials

$$e_1(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n, \tag{2a}$$

$$e_2(x_1, x_2, \ldots, x_n) = x_1 x_2 + x_1 x_3 + \cdots + x_{n-1} x_n, \tag{2b}$$

$$\vdots$$

$$\begin{aligned} e_{n-1}(x_1, x_2, \ldots, x_n) &= x_2 x_3 \cdots x_n + x_1 x_3 x_4 \cdots x_n \\ &\quad + \cdots + x_1 x_2 \cdots x_{n-1}, \end{aligned} \tag{2c}$$

$$e_n(x_1, x_2, \ldots, x_n) = x_1 x_2 \cdots x_n. \tag{2d}$$

Indeed, the symmetric polynomial e_n is coincident with the normalization of Axiom 4. *Geometric probability studies the conditions under which these measures exist, and how they can be used to generate more general ones* [4, 5].

Geometric probability theory can be also used for studying invariant measures in Grassmannians. A complete introduction to the subject can be found in [5]. In Sec. 3, we will review the formulation of the axioms of a non-commutative probability calculus, i.e. probabilities which generalize Kolmogorov's [31] axioms to non-Boolean settings [32, 33].

3. Event Structures

When faced with a concrete physical problem, we are interested in determining the probabilities of certain events of interest. An event will be the definite outcome of a certain experiment for which we can determine the answer with certainty. As an example, we can think about the detection of a particle (classical or quantal) in a certain region of space-time, and the probability for this event to occur.

It happens that events of a physical system can be endowed with definite mathematical structures [34, 35, 33]; if the particle is classical, events may be represented as measurable subsets of the phase space Γ. Measurable subsets of phase space form a well-known structure, namely, a *Boolean algebra* [36, 34] that we will denote by $\mathcal{P}(\Gamma)$.[b]

On the other hand, as shown by Birkhoff and von Neumann [38], events associated to a quantum particle will be naturally represented by *projection operators*, specifically those associated to the spectral decomposition of self-adjoint operators representing physical observables. Unlike the classical Boolean case, projections of a Hilbert space form an *orthomodular lattice* $\mathcal{P}(\mathcal{H})$, which can be shown to be non-distributive [34, 35, 38] (and thus, not Boolean).[c] This important mathematical difference between classical and quantum theories is the direct consequence of the incompatibility of complementary observables in QM.

3.1. *Classical case*

To illustrate these ideas, let us start by considering the example of the phase space $\mathbb{R}^6$ of a classical particle moving in Euclidean space-time. If f represents an observable quantity, the proposition "the value of f lies in the interval Δ" defines an *event* f_Δ, which can be represented as the measurable set $f^{-1}(\Delta)$ (the set of all states

[b]A Boolean lattice will be a partially ordered set for which (i) the least upper bound (disjunction) and maximum lower bound (conjunction) exist for every pair of elements; (ii) it is orthocomplemented; (iii) it is distributive. A typical model for a Boolean lattice will be that of the subsets of a given set, with set intersection as conjunction, set union as disjunction, and set theoretical complement as orthocomplementation [34] (see also [37] for a study of the algebraic symmetries of Boolean lattices).

[c]An orthomodular lattice will be an orthocomplemented lattice for which a condition weaker than distributivity holds (see for example [23, 32, 26]). Boolean algebras are always orthomodular lattices, but the converse is not true [34]. For $\mathcal{P}(\mathcal{H})$, conjunction is given intersection, disjunction by closure of direct sum, and orthocomplementation by orthogonal complement of the closed subspaces associated to each projection operator (projection operators can be put in one-to-one correspondence with closed subspaces of $\mathcal{H}$).

which make the proposition true). If the probabilistic state of the system is given by μ, the corresponding probability of occurrence of f_Δ will be given by $\mu(f^{-1}(\Delta))$. As an example, consider the energy of an harmonic oscillator. The proposition "the energy of the oscillator equals ε" corresponds to an ellipse in phase space for each possible value of ε.

The situation is analogous for more general classical probabilistic systems. There is a strict correspondence between a classical probabilistic state and the axioms of classical probability theory. Indeed, the axioms of Kolmogorov [31] define a probability function as a measure μ on a sigma-algebra Σ such that

$$\mu : \Sigma \to [0, 1] \tag{3a}$$

which satisfies

$$\mu(\emptyset) = 0, \tag{3b}$$

$$\mu(A^c) = 1 - \mu(A), \tag{3c}$$

where $(\ldots)^c$ means set-theoretical-complement. For any pairwise disjoint denumerable family $\{A_i\}_{i\in I}$,

$$\mu\left(\bigcup_{i\in I} A_i\right) = \sum_i \mu(A_i). \tag{3d}$$

A state of a classical probabilistic theory will be defined as a Kolmogorovian measure with $\Sigma = \mathcal{P}(\Gamma)$. The reader will also notice the analogy between the first two Rota's Axioms 1 and 2 and the axioms of Kolmogorovian probability theory.

3.2. *Quantum case*

The quantum case can be described in an analogous way. If $\mathbf{A}$ represents the self-adjoint operator of an observable associated to a quantum particle, the proposition "the value of $\mathbf{A}$ lies in the interval Δ" will define an event represented by the projection operator $\mathbf{P_A}(\Delta) \in \mathcal{P}(\mathcal{H})$, i.e. the projection that the spectral measure of $\mathbf{A}$ assigns to the Borel set Δ. The probability assigned to the event $\mathbf{P_A}(\Delta)$, given that the system is prepared in the state ρ, is computed using the Born's rule: $p(\mathbf{P_A}(\Delta)) = \mathrm{tr}(\rho \mathbf{P_A}(\Delta))$. Born's rule defines a measure on $\mathcal{P}(\mathcal{H})$ with which it is possible to compute all probabilities and mean values for all physical observables [34, 38]. As an example, consider the energy of a quantum harmonic oscillator. The proposition "the energy of the oscillator equals ε_i" corresponds to the projection operator associated to the eigenspace of the eigenvalue ε_i.

It is well known that, due to Gleason's theorem [47], a quantum state will be defined by a measure s over the orthomodular lattice of projection operators $\mathcal{P}(\mathcal{H})$ as follows [32]:

$$s : \mathcal{P}(\mathcal{H}) \to [0; 1] \tag{4a}$$

such that:

$$s(\mathbf{0}) = 0 \quad (\mathbf{0} \text{ is the null subspace}), \tag{4b}$$

$$s(P^{\perp}) = 1 - s(P), \tag{4c}$$

and, for a denumerable and pairwise orthogonal family of projections P_j

$$s\left(\sum_j P_j\right) = \sum_j s(P_j). \tag{4d}$$

3.3. *General case*

Notice that despite their similarities, the difference between (3) and (4) is that Σ is replaced by $\mathcal{P}(\mathcal{H})$, and the other conditions are the natural generalizations of the classical event structure to the non-Boolean setting. A general probabilistic framework — encompassing the Kolmogorovian and the quantal cases — can be described by the following equations

$$s : \mathcal{L} \to [0; 1], \tag{5a}$$

($\mathcal{L}$ standing for the lattice of all events) such that:

$$s(\mathbf{0}) = 0, \tag{5b}$$

$$s(E^{\perp}) = 1 - s(E), \tag{5c}$$

and, for a denumerable and pairwise orthogonal family of events E_j

$$s\left(\sum_j E_j\right) = \sum_j s(E_j). \tag{5d}$$

where $\mathcal{L}$ is a general orthomodular lattice (with $\mathcal{L} = \Sigma$ and $\mathcal{L} = \mathcal{P}(\mathcal{H})$ for the Kolmogorovian and quantum cases respectively). Equation (5a) defines what is known as a *non-commutative probability theory* [32, 74].[d] We will speak of *generalized measures*. Discussing the conditions under which the measure s in Eq. (5a) is well defined (for very general orthomodular lattices) lies outside the scope of this paper; for a detailed discussion see [35, Chap. 11]. It will suffice for us to notice that many examples of interest in physics, including non-relativistic and relativistic QM, and many examples of classical and quantum statistical physics, can be described using orthomodular lattices of projections arising from factors of Types I, II, and III, for which measures such as those defined by Eq. (5a) are well defined [33, 32].

[d]It is important to mention that this axioms can be derived for distributive lattices (see for example [37]) and for general *atomic* orthomodular lattices [27]. It is not known to us whether it is possible to make a similar derivation for continuous geometries such as the ones originated in Type II_1 factors. Thus, we will restrict to the axiomatic approach here and study this problem elsewhere.

In the following sections, we will develop a theoretical framework which combines geometric probability theory, generalized probability theory, and the Jayne's MaxEnt method.

4. A New Set of Axioms for Physical Problems

4.1. *Classical states as invariant measures*

Suppose that we are faced with the problem of determining the particular probabilistic state μ of a classical system S. To fix ideas, we will study the example of a particle moving in Euclidean space-time.[e] In order to determine μ, we must use the fact that it is a probability measure over the event space $\mathcal{P}(\Gamma)$. Thus, it will obey Eqs. (3), which are equivalent to the first two axioms of Rota (Eqs. (1) and (2)) plus the sigma-additivity condition (3d). Imposing Axiom 3 entails that our system would be in a state which possesses the symmetry of being invariant under the whole group E_0 of translations and rotations of $\mathcal{P}(\Gamma)$. Define E to be the group of all possible Galilean transformations acting on the system (notice that $E_0 \subseteq E$). In the general case, the state will not be invariant under all the elements of E_0, but will be invariant under a subgroup $G \subseteq E$ (which could be just the identity group, $\{\mathbf{1}\}$). For example, equilibrium states of a system with cylindric symmetry will typically be invariant under rotations and translations along $\hat{z}$ axis, but not for all possible rotations and translations. We will use these observations to generalize Rota's axioms.

Thus, a classical system will have probabilities obeying an alteration of the Rota axioms. In it, (i) E_0 in Axiom (3) is replaced by a general subgroup $G \subseteq E$, and (ii) Axiom 4 is replaced by a series of conditions of the form

$$\langle f_i \rangle = r_i, \tag{6}$$

which represent the mean values of observables that are available as empirical data. The group G and conditions (6) represent the *a priori* information that we have regarding the system (notice that, to the traditional prior information of the Jaynes's method expressed as mean values, we are adding the possibility of symmetry constraints).

Thus, in order to determine the state μ of the system, we must first solve the problem of determining the measures which satisfy the usual probability axioms, plus (i) the condition of being invariant under the group G and (ii) satisfying the condition given by Eq. (6). *In this way, the problem of handling geometric probability can be transformed into a physical one.*

[e] While we discuss this simple example here, it is important to stress that our general approach will not be restricted to this space-time model. Indeed, as shown below, we study arbitrary automorphisms of orthomodular lattices (and thus, the groups involved can be different than the rotations and translations ones acting in space-time). In this way, our approach contains all possible classical probabilistic models which can be described by Kolmogorov's axioms.

4.2. *Quantum states as invariant measures*

Let us concentrate now on the quantum case before we turn to the general setting. (Continuous) symmetry transformations in QM are represented by the elements of the group of unitary operators $\mathcal{U}$ [48]. If we know in advance that the state that we are looking for possesses a certain symmetry, this condition will be represented by the invariance of the state under the action of a subgroup $\mathcal{G} \subseteq \mathcal{U}$. Next, a series of conditions on mean values of observables can be added. These can be either mean values of operators or more general ones, but which are insufficient on their own to fully determine the state. These conditions can be cast in the form

$$\langle \mathbf{A}_i \rangle = a_i. \tag{7}$$

A state will be represented by a *measure* s over the event structure $\mathcal{P}(\mathcal{H})$. In other words, we are looking for a measure s which (i) satisfies Eqs. (4), (ii) that is *invariant* under the action of the group $\mathcal{G}$, and (iii) satisfies Eq. (7). Thus, in order to determine a quantum state compatible with the prior knowledge about symmetries and mean values, we must determine a measure such that the Axioms (3) and (4) be adequately modified.

We see that, as in the classical case, the Rota's problem can be extended to the problem of determining the state of a physical system, provided we generalize subsets of Euclidean space to the lattice of projections in a Hilbert space, replace the roto-translational group by the corresponding quantum one, and replace the normalization condition by known mean values of a given set of observables. These conditions restrict the possible states to a subset of the space of quantum states. Following Jaynes [2] now, the least biased probability distribution can be determined by maximizing von Neumann's entropy in this subset. It is nice that these observations are susceptible of an even greater degree of generalization.

4.3. *Invariant measures in generalized theories*

Now we pass to a systematic generalization of the above procedure for quite arbitrary statistical theories, which will provide a new ground for the MaxEnt principle. In this vein, we are led to formulate the following set of axioms for a general physical system, incorporating prior knowledge about symmetries and conditions on expectation values (or even more general conditions). The objective is to determine the unknown state s of given system as an invariant measure obeying Eqs. (4).

Symmetries: Knowledge about symmetries of the physical system will be represented by the existence of a subgroup $\mathfrak{G}$ of the group automorphisms of $\mathcal{L}$, $\mathrm{Aut}(\mathcal{L})$, such that for all $g \in \mathfrak{G}$, and for all $E \in \mathcal{L}$,

$$s(g \cdot E) = s(E). \tag{8}$$

Normalization condition: There exists a set of equations $\{e_i\}_I$ in the values $\{s(E_j)\}_J$,

$$e_i(s(E_1), s(E_2), \ldots) = 0, \tag{9}$$

where $\{E_j\}_J \subseteq \mathcal{L}$ is some subset of events.

To summarize, we set down all the axioms that the unknown state ν — now considered as a generalized invariant measure $\nu : \mathcal{L} \to [0; 1]$ over an arbitrary orthomodular lattice $\mathcal{L}$ — must satisfy:

Axiom 5 (G1).

$$\nu(\mathbf{0}) = 0.$$

Axiom 6 (G2).

$$\nu(E^{\perp}) = 1 - s(E).$$

Axiom 7 (G3). For a denumerable and pairwise orthogonal family of events E_j,

$$\nu\left(\sum_j E_j\right) = \sum_j \nu(E_j).$$

Axiom 8 (G4). For all $g \in \mathfrak{G}$

$$\nu(g \cdot E) = \nu(E).$$

Axiom 9 (G5). There exists a family of events $\{E_j\}$ which satisfy the equations defined by functions e_i

$$e_i(\nu(E_1), \nu(E_2), \ldots, \nu E_{m_i}) = 0.$$

The above axioms represent our generalization of geometric probability to the non-commutative case. The fact that we are dealing with general orthomodular lattices allows one to include within our framework the models emerging from Kolmogorov's axioms and those described by non-relativistic QM. Also more general ones, such as the ones described by Types II and III factors. In this way, *our formulation can be considered as a generalization of geometric probability theory to the framework of generalized measure theory. Symmetries of the system under study are represented as lattice automorphisms and possible states are regarded as invariant measures under the action of these groups.*

Axioms (5)–(7) univocally determine a convex set $\mathcal{S}$ (provided that ν be well defined, cf. [35, Chap. 11]). It is important to remark that the introduction of Axiom (8) yields a smaller set $\mathcal{S}_{\mathfrak{G}} \subseteq \mathcal{S}$ which is also convex. The addition of Axiom (9) determines a manifold $\mathcal{M}$, which, when intersected with $\mathcal{S}_{\mathfrak{G}}$, will not necessarily yield a convex set. However, it can be shown that if the constraints are mean values imposed on observables, or more generally, on effects, the set determined by $\mathcal{S}_{\mathfrak{G}} \cap \mathcal{M}$ will be convex [70]. Thus, the set of states compatible with the prior knowledge about symmetries and measured quantities will be the intersection $\mathcal{S}_{\mathfrak{G}} \cap \mathcal{M}$.

Once this set is determined, Jaynes's entropic maximization process singles out the less unbiased state which will rule the probabilities of the system. In Sec. 4.4, we discuss which entropic measures are to be used for this purpose. Notice that if $\mathcal{S}$ is compact, then $\mathcal{S}_{\mathfrak{G}}$ and $\mathcal{S}_{\mathfrak{G}} \cap \mathcal{M}$ will be also compact, and we can ensure the existence of a solution for the maximization procedure (provided that the entropic measure that we use be continuous). Many physical examples comply with these assumptions (for example, in non-relativistic QM, the state space is compact and the symmetry groups are locally compact).

4.4. *Entropies*

We wish to define a meaningful notion of entropy for using it in several frameworks, in the sense of being applicable to QM, classical mechanics, and to general theories. Thus, we need an appropriate notion of information measure to be applied to general statistical theories. One possibility is to use the so-called *measurement entropy*, which reduces to Shannon's measure for classical models and to von Neumann's in the quantum case [71–73]. Let s be a state in a generalized probability theory. Then, following [71], we define

$$H_E(\nu) := -\sum_{x\in E} \nu(x)\ln(\nu(x)), \tag{10}$$

$$H(\nu) := \inf_{E\in\mathcal{L}} H_E(\nu). \tag{11}$$

We show a comparison of the different cases in Table 1.

4.5. *Frame functions and group actions*

Assume that a group $\mathfrak{G}$ is acting by automorphisms on a lattice of events $\mathcal{L}$, $G \subseteq \mathrm{Aut}(\mathcal{L})$ [48, 75]. Consider the convex set $\mathcal{S}$ of Sec. 4.3. Axiom (8) states that invariant states are constant along the orbits of the action,

$$s(g\cdot E) = s(E), \quad g\in\mathfrak{G},\ E\in\mathcal{L},$$

and an invariant state in $\mathcal{L}$ defines in a canonical way a state in $\mathcal{L}/\mathfrak{G}$, where $\mathcal{L}/\mathfrak{G}$ is the quotient lattice.

Table 1. Comparison of the differences between the classical, quantal, and general cases.

	Classical	Quantum	General
Lattice	$\mathcal{P}(\Gamma)$	$\mathcal{P}(\mathcal{H})$	$\mathcal{L}$
Group	$G\subseteq E$	$\mathcal{G}\subseteq\mathcal{U}$	$\mathfrak{G}\subseteq \mathrm{Aut}(\mathcal{L})$
Entropy	$-\sum_i p(i)\ln(p(i))$	$-\mathrm{tr}\rho\ln(\rho)$	$\inf_{E\in\mathcal{L}} H_E(\nu)$

Assume now that the lattice $\mathcal{L}$ is atomic, where the set of atoms is an n-dimensional compact manifold $\mathcal{A}$. According to Gleason [47], a state in $\mathcal{L}$ is determined by a *frame function* in $\mathcal{A}$, that is,

$$f : \mathcal{A} \to \mathbb{R}, \quad \sum_{i=1}^{r} f(x_i) = 1,$$

where $\{x_1, \ldots, x_r\}$ is a set in $\mathcal{L}$ such that $x_i \perp x_j$ $(i \neq j)$ and $x_1 \vee \cdots \vee x_r = 1$. Call $\mathcal{F}$ to the set of frame functions. The full group of automorphisms of the atomic lattice, $\mathrm{Aut}(\mathcal{L})$, induces an action in $\mathcal{A}$ and $\mathcal{F}$ is stable under this action. If $f \in \mathcal{F}$ and $g \in \mathrm{Aut}(\mathcal{L})$, then $g \cdot f$ is also a frame function. Note that the continuous frame functions $\mathcal{F}_{\mathrm{cont}} \subseteq \mathcal{F}$ are a subset of all the bounded continuous functions in $\mathcal{A}$, $F_{\mathrm{cont}} \subseteq L^\infty(\mathcal{A})$, and that the polynomial frame functions are dense in F_{cont}.

The action of the group G in $\mathcal{L}$ restricts itself to an action on $\mathcal{A}$, and a frame function determines an invariant state if and only if the frame function is invariant, $g \cdot f = f$, for all $g \in \mathfrak{G}$. Thus, the invariant states are characterized by the frame functions in $\mathcal{A}/\mathfrak{G}$. Recall that the dimension of $\mathcal{A}/\mathfrak{G}$ is equal to the dimension of $\mathcal{A}$ minus the dimension of an orbit.

As an example, consider an $(n+1)$-dimensional Hilbert space and its lattice of subspaces, $\mathcal{L}$. The set of atoms (the rays in the Hilbert space) is a projective space $\mathbb{P}^n$. It is a compact variety of dimension n, $\mathcal{A} = \mathbb{P}^n$. The full group of automorphisms of $\mathcal{L}$ is the Lie group $\mathcal{U}$. In [47], the fact that the set of frame functions $\mathcal{F}$ is stable under $\mathcal{U}$ is used to characterize frame functions as density matrices (positive semi-definite self-adjoint operators of the trace class).

Consider now a group $\mathcal{G} \subseteq \mathcal{U}$, acting on $\mathbb{P}^n$, and let us consider states invariant under the group $\mathcal{G}$. Given that states are characterized by density matrices, the invariant states are density matrices stable under $\mathcal{G}$

$$\rho = g \cdot \rho, \quad \forall\, g \in \mathcal{G},$$

or equivalently, frame functions in $\mathbb{P}^n/\mathcal{G}$. Note that we are reducing the dimension of the convex set of states and the reduction will depend on the nature of the action of $\mathcal{G}$.

5. Covariance and Symmetries

A space-time symmetry will have an action on the observables of the system and on the state space. But this implies at the same time that it will have an action on the associated operational logic. As an example, consider the Galilei group in non-relativistic QM. Any operator of the group acts on the variety of space-time observables (position, momentum) but at the same time there exists a representation of this group in the set of unitary operators of Hilbert space. Indeed, the content of Wigner's theorem asserts that symmetry transformation preserving probabilities will have a representation as a unitary or anti-unitary operator in Hilbert space. This means that for each symmetry, say, a rotation, there exists an automorphism acting on the logic of projection operators.

Thus, symmetries are usually generalized as follows.[f] Suppose that we have a group $\mathfrak{G}$ representing symmetries of a physical system. Call $\mathcal{S}$ the set of all probability measures. The elements of $\mathfrak{G}$ will also induce transformations in $\mathcal{S}$ as convex automorphisms. As it is well known [48, 75], this group will also have a representation in $\mathrm{Aut}(\mathcal{L})$. Thus, for any element $g \in \mathfrak{G}$, any event $E \in \mathcal{L}$ and any $\nu \in \mathcal{S}$, a symmetry of the system will satisfy the covariance condition

$$\nu(E) = \nu'(E'), \tag{12}$$

where $E' = g \cdot E$ and $\nu' = g \cdot \nu$.

The above equation is important for two main reasons:

- It allows us to incorporate into our system the very important notion of representation of groups, acting as convex automorphisms on $\mathcal{S}$ and automorphisms of $\mathcal{L}$. The action of these groups represents the actions of symmetry transformations (including the spatiotemporal ones) and imposes conditions on the geometry of $\mathcal{S}$ and observable algebras.
- We will use this approach to define coherent states in the general setting. First, because the introduction of symmetries obeying the covariance condition (12) allows for the definition of a base state (as is the case for the vacuum state of the electromagnetic field). Second, because the group axiom allows us to pick up only those measures which satisfy the condition of being coherent states.

6. Coherent States

Given the Heisenberg uncertainty relation in a state ρ

$$\Delta\mathbf{P}\Delta\mathbf{Q} \geq \frac{\hbar}{2}, \tag{13}$$

where for an operator $\mathbf{O}$, $\Delta\mathbf{O} = \sqrt{\langle\mathbf{O}^2\rangle - \langle\mathbf{O}\rangle^2}$, coherent states [79–81] are defined as those which saturate (13) with equal mean values, i.e.:

$$\Delta\mathbf{P}\Delta\mathbf{Q} = \frac{\hbar}{2}, \tag{14a}$$

$$\Delta\mathbf{P} = \sqrt{\frac{\hbar}{2}} = \Delta\mathbf{Q}. \tag{14b}$$

Thus, we can easily incorporate such states into our conceptual framework by replacing (9) by Eqs. (14). Note that Eq. (14b) produces a real algebraic variety $\mathcal{M}$ in the real vector space of Hermitian operators (it is given by the zero locus of the two polynomial equations of degree two, $\Delta\mathbf{P} = \sqrt{\frac{\hbar}{2}}$ and $\Delta\mathbf{Q} = \sqrt{\frac{\hbar}{2}}$).

In arbitrary dimension ($n \leq \infty$) the states satisfying Eq. (14b) are given by the intersection of two quadrics. Recall that any quadric can be parameterized and

[f] This methodology can be traced back to [76, 77, 48, 75, 78].

thus the intersection $\mathcal{C} \cap \mathcal{M}$ can be computed in finite dimensions. If the convex set $\mathcal{C}$ is a compact set, then the intersection $\mathcal{C} \cap \mathcal{M}$ is also compact.

We can also define coherent states using group theory. This has the advantage of being easily applicable to general statistical theories.[g] While the choice of a reference state s_0 is, in principle, arbitrary [81], the use of physical symmetries could be useful for its determination. These will be represented by a group action $\mathfrak{G}$ which, as mentioned above, induces actions in $\mathcal{L}$ and $\mathcal{S}$. This procedure singles out the correct reference state s_0 [81] by using the generalization of geometric probability described in previous sections. Once s_0 is specified, we invoke the action of a given dynamical group G, determine its maximum stability subgroup H [81], and construct the set of all coherent states $\mathcal{S}_G \subseteq \mathcal{S}$ as follows

$$s_g := g \cdot s_0, \tag{15}$$

where g ranges over all the elements of G/H [81].

7. Bell Inequalities, No-Signal Polytope and Local Polytope

Immense interest generates in the study of correlations in QM. For two separate observers, A and B, both of them having available two observables $\{a_0, a_1\}$ and $\{b_0, b_1\}$, with two possible outcomes for each, the correlations will be governed by probability distributions of the form $P(a_i, b_j|x, y)$. It can be shown that the following inequalities can be violated by QM

$$S = |\langle a_0 b_0\rangle + \langle a_1 b_0\rangle + \langle a_0 b_1\rangle - \langle a_1 b_1\rangle| \leq 2. \tag{16}$$

These are known as the Clauser–Horne–Shimony–Holt (CHSH) inequalities [82, 83]. The no-signal polytope, formed by all possible correlations respecting the no-signal condition of special relativity, is defined by the following conditions [83]

$$\sum_j P(a_i, b_j|x, y) = \sum_j P(a_i, b_j|x, y'), \quad \forall\ y, y', \tag{17a}$$

$$\sum_i P(a_i, b_j|x, y) = \sum_i P(a_i, b_j|x', y), \quad \forall\ y, y'. \tag{17b}$$

Quantum correlations can violate the CHSH inequalities, but at the same time, they respect the no-signal condition (the distributions $P(a, b|x, y)$ lie inside the no-signal polytope). One may ask which is the characteristic trait of QM that distinguishes it from general statistical theories which are also no-signal, but do not produce the correlations predicted by QM [83]. This issue can be studied within our theoretical framework by setting conditions (16) and (17) as axioms in the event space. By replacing condition (9) by (16), we obtain the local polytope, and by replacing it

[g]It is important to remark here that both the definition of coherent states that uses Eqs. (14) and the group theoretical one are equivalent for the case of the electromagnetic field, but will not be equivalent in general (as is the case for finite-dimensional Hilbert spaces) [81]. Thus, it is not expected that these definitions will be equivalent in arbitrary statistical theories neither.

by (17), we obtain the no-signal polytope. The reformulation of these geometrical objects within our framework could permit the study of the action of suitable groups of space-time symmetries (by introducing these groups through Axiom (8)).

8. Conclusions

It is important to remark that a systematic presentation of the Jaynes's method, as we have done here, has not yet be advanced in the literature, as far as we know. We summarize our conclusions as follows:

- We accomplish the merging of a generalization of geometric probability theory with the axiomatic method of a generalized measure theory. This allows for a rigorous approach to the action of groups representing symmetries and constraints in the MaxEnt methodology. While some applications of the action of groups representing symmetries were studied in previous literature, no general prior formal method existed thus far. Our formalism allows for this possibility because it is based on a solid mathematical ground. As an example, we have provided general conditions for the existence of solutions, something which was not clear in previous approaches. In this way, the MaxEnt method is strengthened.
- We have exhibited many circumstances that can be accommodated within the axiomatic framework presented here (coherent states, no-signal polytopes, local polytopes). When our group symmetry is reduced to the identity and the constraints are expressed as mean values, our method reduces to previous generalizations of the Jaynes's methodology [70, 72].
- When $\mathcal{L} = \mathcal{L}_{v\mathcal{N}}$ or $\mathcal{L} = \mathcal{P}(\Gamma)$, and the constraints are expressed as mean values, our method reduces to the pioneer Jaynes's one for the quantum and classical cases, respectively.
- Our rigorous formulation allows us to establish precise conditions for the existence of solutions to the MaxEnt problem for very general constraints (including group theory, nonlinear conditions on the mean values of observables, and inequalities as well).
- At the same time, we provide an intrinsic geometric characterization for the different mathematical objects defined within our theoretical framework (quadrics for coherent states, a convex set for the local polytope, etc.). Notice that this may be of help in studying the geometrical properties of the non-signal and local polytopes for the most general case (a continuous range of observables with possibly continuous spectra, as in models of QM with infinite dimensional Hilbert spaces). Our formulation may help to extrapolate, in the future, results from Geometric Probability Theory to physics.
- By reformulating the problem in terms of the determination of invariant measures, we provide a natural framework for the introduction of group theory. We have explicitly shown that the introduction of groups reduces the dimensionality of the mathematical variety in which the maximization process takes place.

Thus, our proposal may be useful to economize computational resources. Our axioms allow one to incorporate into the Jaynes's framework the symmetries of the physical system under study. For example, one could insert a group representing a spacial symmetry of a system. This method yields a powerful resource for deriving laws of physics out of general physical principles.

- The facts that (i) probability theory is a well-established theory and (ii) explicit solutions to our problem can be found (as in the examples studied in this work) show that our mathematical problem is meaningful. This fact constitutes a clear improvement on the MaxEnt method, constituting a step forward in its axiomatization.

Acknowledgments

This work has been supported by CONICET.

References

[1] E. Jaynes, *Phys. Rev.* **106** (1957) 620.
[2] E. Jaynes, *Phys. Rev.* **108** (1957) 171.
[3] S. Pressé, K. Ghosh, J. Lee and K. Dill, *Rev. Mod. Phys.* **85** (2013) 1115.
[4] G. C. Rota, *Math. Intell.* **20** (1998) 11.
[5] D. A. Klain and G. C. Rota, *Introduction to Geometric Probability* (Cambridge University Press, Cambridge, 1997).
[6] A. Cavagna, I. Giardina, F. Ginelli, T. Mora, D. Piovani, R. Tavarone and A. M. Walczak, *Phys. Rev. E* **89** (2014) 042707.
[7] G. P. Beretta, *Phys. Rev. E* **90** (2014) 042113.
[8] R. Sinatra, J. Gómez-Gardeñes, R. Lambiotte, V. Nicosia and V. Latora, *Phys. Rev. E* **83** (2011) 030103.
[9] A. Dirks, P. Werner, M. Jarrell and T. Pruschke, *Phys. Rev. E* **82** (2010) 026701.
[10] M. Trovato and L. Reggiani, *Phys. Rev. E* **81** (2010) 021119.
[11] L. Diambra and A. Plastino, *Phys. Rev. E* **52** (1995) 4557.
[12] L. Rebollo-Neira and A. Plastino, *Phys. Rev. E* **65** (2001) 011113.
[13] L. Diambra and A. Plastino, *Phys. Rev. E* **53** (1996) 1021.
[14] G. Goswami and J. Prasad, *Phys. Rev. D* **88** (2013) 023522.
[15] R. Cofré and B. Cessac, *Phys. Rev. E* **89** (2014) 052117.
[16] N. Lanatà, H. U. R. Strand, Y. Yao and G. Kotliar, *Phys. Rev. Lett.* **113** (2014) 036402.
[17] M. Trovato and L. Reggiani, *Phys. Rev. Lett.* **110** (2013) 020404.
[18] D. Gonçalves, C. Lavor, M. Gomes-Ruggiero, A. Cesário, R. Vianna and T. Maciel, *Phys. Rev. A* **87** (2013) 052140.
[19] N. Canosa, A. Plastino and R. Rossignoli, *Phys. Rev. A* **40** (1989) 519.
[20] L. Arrachea, N. Canosa, A. Plastino, M. Portesi and R. Rossignoli, *Phys. Rev. A* **45** (1992) 7104.
[21] G. Tkačik, O. Marre, T. Mora, D. Amodei, M. J. Berry II and W. Bialek, *J. Statist. Mech.: Theory Exp.* **2013** (2013) P03011.
[22] K. H. Knuth, arXiv: 1411.1854 [quant-ph].
[23] G. Kalmbach, *Orthomodular Lattices*, London Mathematical Society Monographs, Vol. 18 (Academic Press, London, 1983).

[24] I. Stubbe and B. V. Steirteghem, Propositional systems, Hilbert lattices and generalized Hilbert spaces, in *Handbook of Quantum Logic Quantum Structures: Quantum Structures*, eds. K. Engesser, D. M. Gabbay and D. Lehmann (Elesiver, 2007).

[25] J. Bub, Quantum computation from a quantum logical perspective, *Quant. Inform. Process.* **7** (2007) 281–296.

[26] F. Holik, C. Massri, A. Plastino and L. Zuberman, *Int. J. Theoret. Phys.* **52** (2013) 1836.

[27] F. Holik, M. Sáenz and A. Plastino, A discussion on the origin of quantum probabilities, *Ann. Phys.* **340**(1) (2014) 293–310.

[28] O. E. Lanford and D. W. Robinson, *J. Math. Phys.* **9** (1968) 1120.

[29] I. Lanford, E. Oscar and D. Robinson, *Commun. Math. Phys.* **9** (1968) 327.

[30] O. E. Lanford and D. Ruelle, *Commun. Math. Phys.* **13** (1969) 194.

[31] A. N. Kolmogorov, On logical foundations of probability theory, in *Probability Theory and Mathematical Statistics*, Lecture Notes in Mathematics, Vol. 1021 (Springer, Berlin, 1983), pp. 1–5.

[32] M. Rédei and S. J. Summers, *Stud. Hist. Philos. Sci. Part B*: *Stud. Hist. Philos. Mod. Phys.* **38** (2007) 390.

[33] M. Rédei, *Quantum Logic in Algebraic Approach*, Fundamental Theories of Physics, Vol. 91 (Kluwer Academic Publishers Group, Dordrecht, 1998).

[34] M. Dalla Chiara, R. Giuntini and R. Greechie, *Reasoning in Quantum Theory*: *Sharp and Unsharp Quantum Logics*, Trends in Logic — Studia Logica Library, Vol. 22 (Kluwer Academic Publishers, Dordrecht, 2004).

[35] E. G. Beltrametti and G. Cassinelli, *The Logic of Quantum Mechanics*, Encyclopedia of Mathematics and Its Applications, Vol. 15 (Addison-Wesley, Reading, MA, 1981). With a foreword by Peter A. Carruthers.

[36] G. Boole, *An Investigation of the Laws of Thought*: *On which are Founded the Mathematical Theories of Logic and Probabilities*, Cambridge Library Collection (Cambridge University Press, Cambridge, 2009).

[37] K. H. Knuth and J. Skilling, *Axioms* **1** (2012) 38.

[38] G. Birkhoff and J. von Neumann, *Ann. of Math.* (2) **37** (1936) 823.

[39] F. J. Murray and J. von Neumann, On rings of operators, *Ann. of Math.* **37** (1936) 116–229.

[40] F. J. Murray and J. von Neumann, On rings of operators. II, *Trans. Amer. Math. Soc.* **41** (1937) 208–248.

[41] J. Von Neumann, On rings of operators. III, *Ann. of Math.* **41** (1940) 94–161.

[42] J. Von Neumann and F. J. Murray, On rings of operators IV, *Ann. of Math.* **44** (1943) 716–808.

[43] J. Yngvason, The role of type III factors in quantum field theory, *Rep. Math. Phys.* **55** (2005) 135–147.

[44] H. Halvorson and M. Müger, Algebraic quantum field theory, in *Philosophy of Physics*, eds. J. B. Butterfield and J. E. Earman (Elsevier, Amsterdam, The Netherlands, 2006), pp. 731–922.

[45] R. Clifton, J. Bub and H. Halvorson, Characterizing quantum theory in terms of information-theoretic constraints, *Found. Phys.* **33** (2003) 1561–1591.

[46] O. Bratteli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics*, Vols. 1 and 2 (Springer, Berlin, 2012).

[47] A. Gleason, Measures on the closed subspaces of a Hilbert space, in *The Logico-algebraic Approach to Quantum Mechanics*, ed. C. Hooker, The University of Western Ontario Series in Philosophy of Science, Vol. 5a (Springer, Netherlands, 1975), pp. 123–133.

[48] V. S. Varadarajan, *Geometry of Quantum Theory*, Vol. 1 (D. Van Nostrand, Princeton, NJ, 1968).

[49] M. Rédei, Some historical and philosophical aspects of quantum probability theory and its interpretation, in *Probabilities*, *Laws*, *and Structures* (Springer, Berlin, 2012), pp. 497–506.

[50] E. B. Davies and J. T. Lewis, An operational approach to quantum probability, *Commun. Math. Phys.* **17** (1970) 239–260.

[51] M. Srinivas, Foundations of a quantum probability theory, *Commun. Math. Phys.* **16** (1975) 1672–1685.

[52] S. P. Gudder, *Stochastic Methods in Quantum Mechanics*, ed. A. T. Bharucha, Series in Probability and Applied Mathematics (Dover Publications, Mineola, NY, USA, 1979).

[53] L. Accardi, Topics in quantum probability, *Physics Rep.* (Review Section of *Phys. Lett.*) **77**(3) (1981) 169–192.

[54] L. Accardi and F. Fagnola (eds.), *Quantum Probability and Related Topics* (World Scientific, 2013).

[55] L. Accardi, Some trends and problems in quantum probability, in *Proc. 2nd Conf. Quantum Probability and Applications to the Theory of Irreversible Processes* (Springer, Berlin, 1982), pp. 1–19.

[56] A. Caticha, Entropic inference and the foundations of physics, Brazilian Chapter of the International Society for Bayesian Analysis-ISBrA, Sao Paulo, Brazil (2012).

[57] A. Caticha, Entropic inference: Some pitfalls and paradoxes we can avoid, preprint (2012), arXiv: 1212.6967.

[58] A. Caticha, Towards an informational pragmatic realism, *Minds Mach.* **24** (1) (2014) 37–70.

[59] P. Cheeseman and J. Stutz, On the relationship between Bayesian and maximum entropy inference, in *Bayesian Inference and Maximum Entropy Methods in Science and Engineering*, Vol. 735 (American Istitute of Physics, 2004), pp. 445–461.

[60] A. Giffin, Maximum entropy: The universal method for inference (ProQuest, Umi Dissertation Publishing, Michigan, USA, 2011).

[61] M. Grendár and M. Grendár, Maximum entropy: Clearing up mysteries, *Entropy* **3**(2) (2001) 58–63.

[62] P. Harremoës and F. Topsøe, Maximum entropy fundamentals, *Entropy* **3**(3) (2001) 191–226.

[63] J. Landes and J. Williamson, Objective Bayesianism and the maximum entropy principle, *Entropy* **15**(9) (2013) 3528–3591.

[64] S. Pressé, K. Ghosh, J. Lee and K. A. Dill, Principles of maximum entropy and maximum caliber in statistical physics, *Rev. Mod. Phys.* **85**(3) (2013) 1115.

[65] J. Skilling, Foundations and geometry, in *MaxEnt* 2014 *Proc.*

[66] W. Stephan, Continuity of the maximum-entropy inference, *Commun. Math. Phys.* **330**(3) (2014) 1263–1292.

[67] A. A. Toda, Axiomatization of maximum entropy without the Bayes rule, in *Bayesian Inference and Maximum Entropy Methods in Science and Engineering*, Vol. 1443, No. 1 (American Institute of Physics, 2012), pp. 7–13.

[68] A. A. Toda, Unification of maximum entropy and Bayesian inference via plausible reasoning, preprint (2011), arXiv: 1103.2411.

[69] S. J. van Enk, The Brandeis Dice problem and statistical mechanics, *Stud. Hist. Philos. Sci. Part B*: *Stud. Hist. Philos. Mod. Phys.* **48** (2014) 1–6.

[70] F. Holik and A. Plastino, *J. Math. Phys.* **53** (2012) 073301.

[71] H. Barnum, J. Barrett, L. O. Clark, M. Leifer, R. Spekkens, N. Stepanik, A. Wilce and R. Wilke, *New J. Phys.* **12** (2010) 033024.

[72] C. Hein, *Found. Phys.* **9** (1979) 751.

[73] A. Short and S. Wehner, Entropy in general physical theories, *New J. Physics* **12**(3) (2010) 033023.

[74] F. Holik, G. M. Bosyk and G. Bellomo, Quantum information as a non-Kolmogorovian generalization of Shannon's theory, *Entropy* **17** (2015) 7349–7373, doi:10.3390/e17117349.

[75] V. S. Varadarajan, *Geometry of Quantum Theory*, Vol. 2, The University Series in Higher Mathematics (Van Nostrand Reinhold, Princeton, NJ, 1970).

[76] G. Ludwig, *Foundations of Quantum Mechanics. I*, Texts and Monographs in Physics (Springer-Verlag, New York, 1983).

[77] G. Ludwig, *Foundations of Quantum Mechanics. II*, Texts and Monographs in Physics (Springer-Verlag, New York, 1985).

[78] G. W. Mackey, *Mathematical Foundations of Quantum Mechanics* (Dover Publications, Mineola, NY, 2004). With a foreword by A. S. Wightman.

[79] R. Glauber, *Phys. Rev.* **130** (1963) 2529.

[80] R. Glauber, *Phys. Rev.* **131** (1963) 2766.

[81] W.-M. Zhang, D. H. Feng and R. Gilmore, *Rev. Mod. Phys.* **62** (1990) 867.

[82] N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani and S. Wehner, *Rev. Mod. Phys.* **86** (2014) 419.

[83] S. Popescu, *Nature* **10** (2014) 264.